Stochastic Finance (FIN 519) Final Exam

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BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**. The PDF and CDF for standard normal distribution is denoted by $n(z)$ and $N(z)$. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (6 points) Calculate the following stochastic derivatives.
   (a) (2 points) $d\,(B_t^4 - 6t\,B_t^2 + 3t^2)$
   (b) (2 points) $d\left(\frac{1}{1+e^{-B_t}}\right)$
   (c) (2 points) $d\left(W_t\, e^{\lambda B_t}\right)$ (W_t is another BM correlated with B_t by $dB_t\, dW_t = \rho dt$.)

Solution: Applying Itô's lemma, we obtain the following stochastic derivatives:

(a)

$$d\,(B_t^4 - 6t\,B_t^2 + 4t^2) = 4(B_t^3 - 3tB_t)dB_t$$

(b)

$$\begin{aligned} d\left(\frac{1}{1+e^{-B_t}}\right) &= \frac{e^{-B_t}\,dB_t}{(1+e^{-B_t})^2} + \frac{e^{-2B_t}\,dt}{(1+e^{-B_t})^3} - \frac{e^{-B_t}\,dt}{2(1+e^{-B_t})^2} \\ &= \frac{e^{-B_t}}{(1+e^{-B_t})^2}\left(dB_t - \frac{(1-e^{-B_t})}{2(1+e^{-B_t})}\,dt\right) \end{aligned}$$

(c)

$$\begin{aligned} d\left(W_t\, e^{\lambda B_t}\right) &= e^{\lambda B_t}\,dW_t + \lambda W_t\, e^{\lambda B_t}\,dB_t + \frac{\lambda^2}{2} W_t\, e^{\lambda B_t}\,(dB_t)^2 + \lambda e^{\lambda B_t}(dB_t\,dW_t) \\ &= e^{\lambda B_t}(dW_t + \lambda W_t\,dB_t) + \lambda e^{\lambda B_t}\left(\frac{\lambda}{2}W_t + \rho\right) dt \end{aligned}$$

2. (5 points) Consider the following integral over time t:

$$I(T) = \int_0^T e^{-B_t} dt.$$

   (a) (2 points) Apply Itô's lemma to derive the stochastic differentiation of e^{-B_t}:
   (b) (3 points) Based on (a), find the mean and variance of $I(T)$.

Solution:

See **2016HW 4-1, Itô's isometry** and **2020ME, Stochastic integral**.

(a)

$$d\left(e^{-B_t}\right) = -e^{-B_t}dB_t + \frac{1}{2}e^{-B_t}dt$$

(b) From (a), $I(T)$ can be alternatively expressed by

$$I(T) = \int_0^T e^{-B_t}dt = 2\left(e^{-B_T} - 1\right) + 2\int_0^T e^{-B_t}dB_t$$

Therefore, we have the mean:

$$E(I(T)) = 2E\left(e^{-B_T} - 1\right) = 2\left(e^{T/2} - 1\right).$$

For variance, we apply a trick similar to Gaussian integral:

$$\begin{aligned} E(I(T)^2) = E(I(T)\cdot I(T)) &= E\left(\int_0^T e^{-B_t}dt \cdot \int_0^T e^{-B_s}ds\right) \\ &= \int_0^T\int_0^T E\left(e^{-(B_t+B_s)}\right)dtds. \end{aligned}$$

The integral on (s,t) can be divided into the two regions: (i) $s > t$ and (ii) $t > s$. If (i) $s > t$ holds,

$$E\left(e^{-(B_t+B_s)}\right) = E\left(e^{-(B_s-B_t)-2B_t}\right) = E\left(e^{-(B_s-B_t)}\right)E\left(e^{-2B_t}\right) = \exp\left(\frac{s-t}{2} + 2t\right).$$

Because of the symmetry, we perform the integral by using (i) only and multiplying 2:

$$\begin{aligned} E(I(T)^2) &= 2\int_0^T\int_0^s \exp\left(\frac{s-t}{2} + 2t\right)dtds \\ &= 2\int_0^T e^{s/2}\int_0^s e^{3t/2}dtds = \frac{2}{3}e^{2T} - \frac{8}{3}e^{T/2} + 2. \end{aligned}$$

Finally, the variance is obtained by

$$\mathrm{Var}(I(T)) = E(I(T)^2) - E(I(T))^2 = \frac{2}{3}\left(e^{2T} - 6e^T + 8e^{T/2} - 3\right).$$

3. (5 points) Consider the following martingale:

$$X_t = E\left(e^{\lambda B_T}\,\middle|\,\mathcal{F}_t\right).$$

(a) (2 points) Directly calculate X_t as a function of t, T, and B_t.

(b) (3 points) Find the martingale representation of X_t. In other words, find ϕ_t that satisfies

$$dX_t = \phi_t\,dB_t \quad \text{or} \quad X_T = X_0 + \int_0^T \phi_t\,dB_t.$$

Solution: This question was inspired from this quant finance StackExchange post.

(a) X_t is calculated as

$$X_t = E\left(e^{\lambda B_T} \,\middle|\, \mathcal{F}_t\right) = e^{\lambda B_t} E\left(e^{\lambda(B_T - B_t)} \,\middle|\, \mathcal{F}_t\right) = e^{\lambda B_t} e^{\lambda^2(T-t)/2} = e^{\lambda B_t + \lambda^2(T-t)/2}.$$

Basically, X_t is a geometric BM: $X_t = e^{\lambda^2 T/2}\, e^{\lambda B_t - \lambda^2 t/2}$.

(b) Applying Itô's lemma, we get the SDE:

$$dX_t = \lambda e^{\lambda B_t + \lambda^2(T-t)/2} dB_t \quad (= \lambda\, X_t\, dB_t).$$

Therefore, $\phi_t = \lambda\, X_t$:

$$\phi_t = \lambda e^{\lambda B_t + \lambda^2(T-t)/2}.$$

4. (4 points) The stochastic variance V_t in the GARCH diffusion model is given by

$$dV_t = \alpha(V_\infty - V_t)dt + \sigma V_t\, dB_t.$$

This stochastic process has both geometric BM and mean-reversion, and it is known as the inhomogeneous geometric Brownian motion (IGBM).

Derive $E(V_t \,|\, \mathcal{F}_0)$ under this model. (Hint: The transformation, $y_t = e^{\alpha t}(V_t - V_\infty)$, used in the Ornstein-Uhlenbeck process is also useful in this problem. Then, use the martingale property.) What is $E(V_t \,|\, \mathcal{F}_0)$ as $t \to \infty$.

Solution:

(a) The transformation y_t is a martingale because

$$dy_t = \alpha e^{\alpha t}(V_t - V_\infty) + e^{\alpha t} dV_t = e^{\alpha t} \sigma V_t dB_t.$$

Therefore, $y_0 = E(y_t | \mathcal{F}_0)$:

$$V_0 - V_\infty = E(e^{\alpha t}(V_t - V_\infty) \,|\, \mathcal{F}_0)$$
$$E(V_t|\mathcal{F}_0) = V_\infty + e^{-\alpha t}(V_0 - V_\infty) = (1 - e^{-\alpha t})V_\infty + e^{-\alpha t} V_0.$$

$E(V_t|\mathcal{F}_0)$ goes to V_∞ as $t \to \infty$.

5. (6 points) A multi-variate function F is *homogeneous* if

$$F(\lambda x_1, \cdots, \lambda x_n) = \lambda\, F(x_1, \cdots, x_n).$$

Euler's homogeneous function theorem states that, if F is a homogeneous function,

$$F(x_1, \cdots, x_n) = \sum_{k=1}^{n} x_k \frac{\partial}{\partial x_k} F(x_1, \cdots, x_n).$$

(a) (2 points) Show that the Black–Scholes (call or put) option price is a homogeneous function of the strike price K and spot price S_0:

$$C(\lambda K, \lambda S_0) = \lambda\, C(K, S_0).$$

Show it using that S_t follows a geometric BM, but **do not** use the Black–Scholes formula.

(b) (2 points) Is Bachelier option price a homogeneous function of K and S_0? Explain why. For this question, you may use the Bachelier formula.

(c) (2 points) If (a) is true, the BS option price satisfies:

$$C(K, S_0) = S_0 \frac{\partial C}{\partial S_0} + K \frac{\partial C}{\partial K}.$$

By comparing to the BS call option price formula (you can use it), find the two partial derivatives, $\frac{\partial C}{\partial S_0}$ and $\frac{\partial C}{\partial K}$. ($\frac{\partial C}{\partial S_0}$ is the option delta. This is another way of deriving $\frac{\partial C}{\partial S_0}$ and $\frac{\partial C}{\partial K}$.)

Solution:

(a) The stock price follows a geometric BM, so the process is multiplicative. If the spot price S_0 is multiplied by λ, so is the terminal price S_T:

$$(\lambda S_T) = (\lambda S_0)\, e^{\sigma B_T - \sigma^2 T/2}.$$

Therefore,

$$C(\lambda K, \lambda S_0) = e^{-rT} E\left((\lambda S_T - \lambda K)^+\right) = \lambda\, e^{-rT} E\left((S_T - K)^+\right) = \lambda\, C(K, S_0).$$

(b) The stock price under the Bachelier model,

$$S_T = S_0 + \sigma T,$$

is not multiplicative. The spot price λS_0 does not guarantee λS_T. It can be verified from the Bachelier formula:

$$C_{\text{N}}(\lambda K, \lambda S_0) = (\lambda S_0 - \lambda K) N(d_{\text{N}}) + \sigma \sqrt{T}\, n(d_{\text{N}}) \neq \lambda\, C(K, S_0) \quad \text{for} \quad d_{\text{N}} = \frac{\lambda S_0 - \lambda K}{\sigma_{\text{N}} \sqrt{T}}.$$

Instead, it is translative; $S_0 + \lambda \;\Rightarrow\; S_T + \lambda$:

$$C_{\text{N}}(K + \lambda, S_0 + \lambda) = (S_0 - K) N(d_{\text{N}}) + \sigma \sqrt{T}\, n(d_{\text{N}}) = C(K, S_0),$$

where d_{N} remains the same.

(c) By comparing to the BS call option formula, we have

$$\frac{\partial C}{\partial S_0} = N(d_1) \quad \text{and} \quad \frac{\partial C}{\partial K} = -e^{-rT} N(d_2),$$

where

$$d_{1,2} = \frac{\log(S_0\, e^{rT}/K)}{\sigma \sqrt{T}} \pm \frac{\sigma \sqrt{T}}{2}.$$

6. (4 points) Derive the current price of a modified call option that pays

$$h(S_T) = \frac{S_T}{S_0} (S_T - K)^+$$

at maturity T. The stock price S_t follows the Black–Scholes assumption under the risk-neutral measure:

$$\frac{dS_t}{S_t} = r\, dt + \sigma\, dB_t.$$

Hint: Evaluate this option with the numeraire $N_t = S_t$.

Solution: This question is from this quant finance StackExchange post.

Using S_t as a numeraire, the price of the modified option is expressed using the corresponding equivalent martingale measure Q^S:

$$C_0 = S_0\, E^{Q^S}\left(\frac{h(S_T)}{S_T}\right) = E^{Q^S}\left((S_T - K)^+\right).$$

This is the same as the price of the regular call option except that we use the Q^S measure, instead of the risk neutral measure Q. Since BMs in the two measures are related by

$$B_t^{Q^S} + \sigma t = B_t^Q,$$

the terminal price S_T is express in terms of $B_t^{Q^S}$:

$$\begin{aligned} S_T &= S_0 e^{rT} \exp\left(\sigma B_T^Q - \frac{\sigma^2 T}{2}\right) = S_0 e^{rT} \exp\left(\sigma B_T^{Q^S} + \frac{\sigma^2 T}{2}\right) \\ &= S_0 e^{(r+\sigma^2)T} \exp\left(\sigma B_T^{Q^S} - \frac{\sigma^2 T}{2}\right). \end{aligned}$$

Therefore, the call price is modified from the BS formula for undiscounted call option by replacing $S_0 e^{rT}$ with $S_0 e^{(r+\sigma^2)T}$:

$$C_0 = S_0 e^{(r+\sigma^2)T} N(d_1') - K N(d_2'),$$

where

$$\begin{aligned} d_1' &= \frac{\log(S_0 e^{(r+\sigma^2)T}/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = \frac{\log(S_0 e^{rT}/K)}{\sigma\sqrt{T}} + \frac{3\sigma\sqrt{T}}{2} = d_1 + \sigma\sqrt{T} \\ d_2' &= \frac{\log(S_0 e^{(r+\sigma^2)T}/K)}{\sigma\sqrt{T}} - \frac{\sigma\sqrt{T}}{2} = \frac{\log(S_0 e^{rT}/K)}{\sigma\sqrt{T}} + \frac{\sigma\sqrt{T}}{2} = d_2 + \sigma\sqrt{T} = d_1. \end{aligned}$$

Here d_1 and d_2 are from the regular BS formula.