Stochastic Finance (FIN 519)
Midterm Exam

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BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable, respectively. The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$, respectively. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (10 points) [**Cumulants**] The cumulants (and cumulant generating function) of an RV X provide interesting alternatives to the moments (and moment generating function) of X. The cumulant generating function $K_X(t)$ is defined as the log of MGF, and the cumulants κ_n are defined as the coefficients of Taylor's expansion of $K_X(t)$:

$$\log M_X(t) = K_X(t) = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!} = \kappa_1\, t + \kappa_2 \frac{t^2}{2} + \kappa_3 \frac{t^3}{6} + \kappa_4 \frac{t^4}{24} + \cdots .$$

The MGF, $M_X(t)$, and the moments, μ_n, are defined as usual:

$$M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} \mu_n \frac{t^n}{n!} \quad \text{where} \quad \mu_n = E(X^n).$$

As you will see, the cumulants are particularly useful when $M_X(t)$ is in an exponential form. In the following questions, clearly show your derivation.

(a) (2 points) Express κ_1 and κ_2 in terms of μ_n. What are the statistical meanings of κ_1 and κ_2?

(b) (3 points) Express κ_3 and κ_4 in terms of μ_n. How are κ_3 and κ_4 related to the skewness and ex-kurtosis of X?

(c) (2 points) The MGF of a normal RV, $X \sim N(\mu, \sigma^2)$, is $\exp(\mu t + \sigma^2 t^2/2)$ (see **2018HW 2-1**). What are the cumulants of X?

(d) (3 points) In **HW 1-1**, we derived the MGF of the gamma RV with parameter (a, b). But the calculation for variance, skewness, and ex-kurtosis was very tedious. Derive them again by using $K_X(t)$.

Solution:

(a) We will rely on the expansion of $\log(1+x)$ for small x:

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \cdots .$$

Comparing the coefficients,

$$\begin{aligned}\kappa_1 t + \kappa_2 \frac{t^2}{2} + \cdots &= \log\left(1 + \mu_1 t + \mu_2 \frac{t^2}{2} + \cdots\right) \\ &= \mu_1 t + (\mu_2 - \mu_1^2)\frac{t^2}{2} + \cdots,\end{aligned}$$

we obtain

$$\kappa_1 = \mu_1 \quad \text{and} \quad \kappa_2 = \mu_2 - \mu_1^2.$$

Therefore, κ_1 and κ_2 are the mean and variance, respectively, of the distribution.

(b) Equation the t^3 and t^4 terms, we find

$$\begin{aligned}\kappa_3 &= \mu_3 - 3\mu_2\mu_1 + 2\mu_1^3 \\ \kappa_4 &= \mu_4 - 4\mu_3\mu_1 - 3\mu_2^2 + 12\mu_2\mu_1^2 - 6\mu_1^4.\end{aligned}$$

In turn, they can be expressed as

$$\begin{aligned}\kappa_3 &= E((X-\mu_1)^3) = \text{Skewness} \times \kappa_2^{3/2} \\ \kappa_4 &= E((X-\mu_1)^4) - 3\kappa_2^2 = \text{Ex-kurtosis} \times \kappa_2^2.\end{aligned}$$

(c) Because $K_X(t) = \mu t + \sigma^2 t^2/2$,

$$\kappa_1 = \mu, \quad \kappa_2 = \sigma^2, \quad \text{and} \quad \kappa_3 = \kappa_4 = \cdots = 0.$$

This is consistent with what we already know!

(d) The cumulant generating function of the gamma distribution is

$$\log E(e^{tX}) = -a \log\left(1 - \frac{t}{b}\right) = a\left(\frac{t}{b} + \frac{t^2}{2b^2} + \frac{t^3}{3b^3} + \frac{t^4}{4b^4} + \cdots\right)$$

The mean and variance are

$$\mu_1 = \kappa_1 = \frac{a}{b} \quad \text{and} \quad \kappa_2 = \frac{a}{b^2}.$$

From $\kappa_3 = 2a/b^3$ and $\kappa_4 = 6a/b^4$, the skewness and ex-kurtosis are

$$\text{Skewness} = \frac{2a/b^3}{(a/b^2)^{3/2}} = \frac{2}{\sqrt{a}} \quad \text{and} \quad \text{Ex-kurtosis} = \frac{6a/b^4}{(a/b^2)^2} = \frac{6}{a}.$$

2. (10 points) **[Coin toss for two heads in a row (HH)]** From **Chapter 1, 2016HW**, we know it takes on average 6 times to get 2 heads (**H**) in a row. We are going to solve this again, but using a martingale similar to the gambler's ruin.

Suppose you bet A_{n-1} ($A_0 = 1$) for head in the n-th toss. Therefore, your wealth after the n-th toss is

$$M_n = A_0 X_1 + \cdots + A_{n-1} X_n + \cdots \quad (M_0 = 0) \quad \text{where} \quad X_n = +1(\mathbf{H}) \text{ or } -1(\mathbf{T}).$$

You stop this gamble when you get **HH**. That is, the stopping time τ is defined as the first n such that $X_n = X_{n-1} = +1$ (this is obviously a stopping time).

(a) (2 points) You want to determine the bet A_{n-1} such that $M_n = -n$ if $X_n = -1(\mathbf{T})$. For example, you want $M_1 = -1$ if $X_n = -1$. Therefore $A_0 = 1$ is correct. You can also satisfy $M_n = -n$ by betting the same amount $A_{n-1} = 1$ until you get $X_n = 1(\mathbf{H})$ for the first time. What is M_n when you get $X_n = 1(\mathbf{H})$ for the first time? What should be your next bet A_n to make sure $M_{n+1} = -(n+1)$ if $X_{n+1} = -1$?

(b) (2 points) What is M_n when you get two **H**'s in a row? In other words, what is M_τ?

(c) (2 points) What is $E(\tau)$? Hint: The process M_n is martingale and so is $M_{n\wedge\tau}$. Use $M_0 = E(M_\tau)$, assuming that all mathematical conditions are satisfied.

(d) (4 points) What is the expected number of coin toss until you get 3 **H**'s in a row? Use martingale property. You can get the same answer using the recurrence relation, but you will get only 3 points.

Solution:

(a) Because $M_{n-1} = -(n-1)$, $M_n = 2 - n$ when you get **H** for the first time. For the $(n+1)$-th toss, you should bet $A_n = 3$ because, when you get **T** next time, $M_{n+1} = 2 - n - 3 = -(n+1)$.

(b) You get two **H**'s (for the first time) at $t = n$ means that you had **T** at $t = n - 2$, so $M_{n-2} = 2 - n$. Now you had **H** at $t = n - 1$, so $M_{n-1} = 3 - n$. From (a), you bet $A_{n-1} = 3$ and won it (because you get the second **H**), so $M_n = 6 - n$.

(c) We arrive at the same answer, $E(\tau) = 6$, because

$$0 = M_0 = E(6 - \tau) = 6 - E(\tau).$$

(d) From (b), $M_n = 6 - n$ when you get two **H**'s in a row for the first time. To have $M_{n+1} = -(n+1)$ in case you get **T**, you need to bet $A_n = 7$ ($M_{n+1} = 6 - n - 7 = -(n+1)$). Having 3 **H**'s in a row at $t = n$ means that from $M_{n-3} = -(n-3)$, you won 1, 3, and 7 to arrive at $M_n = 14 - n$. From the martingale property,

$$0 = M_0 = E(14 - \tau) = 14 - E(\tau),$$

it takes 14 coin toss until you get 3 **H**'s in a row.

3. (4 points) **[Brownian Motion]** If B_t is a standard BM, determine whether each of the followings is a standard BM or not. Explain briefly why.

(a) $\frac{1}{2}B_{4t}$

(b) $\frac{3}{5}B_t + \frac{4}{5}X_t$ where X_t is another BM independent from B_t.

(c) $B_t^2 - t$

(d) $X_t = B_t - B_{t/2}$

Solution:

(a) Yes from the scaling property: $\mathrm{Var}(\frac{1}{2}B_{4t}) = \frac{1}{4}4t = t$.

(b) Yes. $\mathrm{Var}(\frac{3}{5}B_t + \frac{4}{5}X_t) = \frac{9t}{25} + \frac{16t}{25} = t$.

(c) No. At $t = 1$, $B_1^2 - 1 \sim Z^2 - 1$ for $Z \sim N(0,1)$ is not a Gaussian distribution.

(d) No. At $t = 1$, $\mathrm{Var}(X_1) = 1/2 \neq 1$.

4. (6 points) **[Knock-out (up-and-out) option under the Bachelier model]**
In **Chapter 5, 2016HW 3-3**, we derived the price of the up-and-out call option with strike price K and knock-out barrier H ($> K, > S_0$). In the problem, the option expires worthless when the stock price S_t hits H. Instead, suppose that the option holder receive a payout of $H - K$ when $S_t = H$ (and the option expires).

(a) (3 points) How should you modify the price obtained in **Chapter 5, 2016HW 3-3**?

(b) (3 points) The knock-out payout $H - K$ makes this option very similar to the regular European option in the sense that the option pays European option's payout $(S_t - K)^+ = H - K$ at an early expiry $t < T$. Therefore, the price from (a) is similar, but not equal, to the European option price. Is the price from (a) more expensive or cheaper than the European option price? Intuitively explain why.

Solution:

(a) From **Chapter 5, 2016HW 3-3**, we obtained the knock-out option price as

$$\begin{aligned} C(K,H) =&(S_0 - K)\big(N(d) - 2N(d^*) + N(2d^* - d)\big) \\ &+ \sigma\sqrt{T}\big(n(d) - n(2d^* - d) + 2d^*N(d^*) - 2d^*N(2d^* - d)\big), \end{aligned}$$

where $d = (S_0 - K)/\sigma\sqrt{T}$ and $d^* = (S_0 - H)/\sigma\sqrt{T}$. In the old problem, the option knocks out worthless when $S_T^* > H$, where S_T^* is the running maximum, $S_T^* = \max_{0\le t\le T} S_t$. Therefore, the new knock-out option (with the payout at knock-out) is more valuable by the payout $(H-K)$ times the knock-out probability. The probability is given by

$$P(S_T^* > H) = P\left(B_T^* > \frac{H - S_0}{\sigma}\right) = 2 - 2N\left(\frac{H - S_0}{\sigma\sqrt{T}}\right) = 2N(d^*).$$

Therefore, the new option price is adjusted to

$$C'(K,H) = C(K,H) + 2(H - K)N(d^*)$$

(b) The price obtained in (a) is further simplified to

$$\begin{aligned} C'(K,H) &= (S_0 - K)\big(N(d) + N(2d^* - d)\big) + \sigma\sqrt{T}\big(n(d) - n(2d^* - d) - 2d^*N(2d^* - d)\big) \\ &= C(K) - \sigma\sqrt{T}\big(n(2d^* - d) + (2d^* - d)N(2d^* - d)\big), \end{aligned}$$

where $C(K)$ is the regular call option price under the Bachelier model. The last term,

$$\sigma\sqrt{T}\big(n(2d^* - d) + (2d^* - d)N(2d^* - d)\big)$$

is understood as the call option price with strike $2H - K$ because

$$2d^* - d = \frac{S_0 - (2H - K)}{\sigma\sqrt{T}}$$

Since option price is positive, we conclude that $C(K) \geq C'(K, H)$

Another intuitive method of comparing the two option values is to consider the knock-out moment (i.e., $S_t = H$). (i) if you hold the knock-out option, you receive $H - K$. (ii) if you hold the regular European option, you may receive the option value (by selling the option at the moment). Or course, the European option value is larger then the payout. Therefore, $C(K) \geq C'(K, H)$.

5. (10 points) **[Stochastic integral]** We want to solve the stochastic integral:

$$I_T = \int_0^T e^{B_t} dB_t.$$

(a) (3 points) A naive guess from the traditional calculus is

$$I_T = [e^x]_0^{B_T} = e^{B_T} - 1.$$

Show that this is <u>not</u> the correct answer.

(b) (2 points) From class, we learned that

$$\int_0^T B_t dB_t = \frac{1}{2}(B_T^2 - T) \neq \frac{1}{2}B_T^2,$$

and that the unexpected term $T/2$ ensures that the expectation is zero. Using a similar logic, can you guess the correct I_T?

(c) (2 points) Imagine a trading strategy where you long 1 share of stock from start ($t = 0$) and continuously change the position to $e^{\mu(S_t - S_0)}$ for $\mu > 0$. Also assume that the stock price follows $S_t = S_0 + \sigma B_t$. What is the profit (or loss) of the strategy at $t = T$? Hint: Check your answer with $\mu = 0$. If $\mu = 0$, you long 1 share all time, therefore, the answer should be σB_T.

(d) (3 points) What is the standard deviation (e.g., risk) of the profit & loss from (b)? Is this strategy more or less risky than just holding 1 share?

Solution:

(a) One way to show that it is not a correct answer is

$$E(e^{B_T} - 1) = e^{T/2} - 1 \neq 0.$$

(b) To ensure $E(I_T) = 0$, we guess

$$I_T = e^{B_T} - e^{T/2}.$$

Note: There was a mistake in this problem. This is not the correct answer either. There is no analytic answer for this problem. The grading will be based on your logic.

(c) Generalizing (a),

$$I_T = \int_0^T e^{\mu B_t} dB_t = \frac{1}{\mu}\left(e^{\mu B_T} - e^{\mu^2 T/2}\right).$$

Therefore the P&L of the trading strategy is

$$P\&L = \int_0^T e^{\mu(S_t - S_0)} dS_t = \int_0^T e^{\mu\sigma B_t} \sigma dB_t = \frac{1}{\mu}\left(e^{\mu\sigma B_T} - e^{\mu^2\sigma^2 T/2}\right).$$

In the limit of $\mu \to 0$,

$$P\&L = \frac{1}{\mu}\left(1 + \mu\sigma B_T + O(\mu^2) - (1 + O(\mu^2))\right) \to \sigma B_T.$$

(d) The P&L in (c) has a lognormal distribution:

$$P\&L = \frac{1}{\mu}\left(e^{\mu\sigma\sqrt{T}Z} - e^{\mu^2\sigma^2 T/2}\right) = \frac{e^{\mu^2\sigma^2 T/2}}{\mu}\left(e^{\mu\sigma\sqrt{T}Z - \mu^2\sigma^2 T/2} - 1\right),$$

where $Z \sim N(0,1)$. The standard deviation of the log-normal distribution is

$$\sigma(P\&L) = \frac{e^{\mu^2\sigma^2 T/2}}{\mu}\sqrt{\left(e^{\mu^2\sigma^2 T} - 1\right)}.$$