Stochastic Finance (FIN 519)
Homework Solutions

Jaehyuk Choi

2016-17 Module 3 (Spring 2018)

HW 1 (A popular interview quiz) Imagine that you keep tossing a fair coin (50% for head and 50% for tail) until you get two heads in a row. On average, how many times do you need to toss the coin?

Answer Let X be the answer (expected number of tosses) and branch on the following three cases based on the outcomes in the beginning. The head is denoted by **H** and tail by **T**.

1. **T** (Prob $= 1/2$): You start from the scratch with 1 toss wasted. So the expected number of tosses in the branch is 1+X.
2. **HT** (Prob $= 1/4$): You start from the scratch with 1 toss wasted. So the expected number of tosses in the branch is 2+X.
3. **HH** (Prob $= 1/4$): You get two heads in a row in 2 tosses.

Therefore, we obtain the following equation on X

$$X = \frac{1}{2}(1+X) + \frac{1}{4}(2+X) + \frac{1}{4}\cdot 2$$

and conclude that $X = 6$.

HW 2-1. Related to the last statement of Chapter 1 of **SCFA**, prove that the expected time for the gambler to become first positive is infinite: $E(\tau) = \infty$. [Hint: consider the derivative of $\phi(z)$ with respect to z]

Answer

$$\begin{aligned} E(\tau) &= \sum_{k=1}^{\infty} k\, P(\tau = k | S_0)\, z^{k-1} \Big|_{z=1} = \frac{d}{dz}\phi(z)\Big|_{z=1} \\ &= -\frac{1-\sqrt{1-z^2}}{z^2} + \frac{1}{\sqrt{1-z^2}}\Big|_{z=1} = \infty \end{aligned}$$

HW 2-2. Exercise 2.1 of SCFA

Answer The roots of the equation qualify for the x

$$E(x_1^X) = \frac{0.52}{x} + 0.45x + 0.03x^2 = 1.$$

Form

$$\frac{0.52}{x} + 0.45x + 0.03x^2 - 1 = \frac{x-1}{x}\big(0.03x^2 + 0.48x - 0.52\big)$$

we have the following three values.

$$x = 1,\ \frac{-0.24 \pm \sqrt{0.24^2 + 0.03 \times 0.24}}{0.03} = 1,\ 1.01850,\ -17.0185.$$

We pick $x = 1.01850$ since the change under high powers are reasonable. If we let τ be the first time Gambler's wealth is either 100, 101 or -100, we have the equation from the Martingale property,

$$1 = E(M_\tau) = x^{100} P(S_\tau = 100) + x^{101} P(S_\tau = 101) + x^{-100} P(S_\tau = -100).$$

Letting $p = P(S_\tau = 100) + P(S_\tau = 101)$ and using the fact that $x > 1$,

$$x^{100} p + x^{-100}(1-p) < 1 < x^{101} p + x^{-100}(1-p)$$

$$\frac{1 - x^{-100}}{x^{101} - x^{-100}} < p < \frac{1 - x^{-100}}{x^{100} - x^{-100}} \quad \Rightarrow \quad 0.13531 < x < 0.13788.$$

HW 3-1. (digital option) Derive the (forward) price of the digital(binary) call/put option struck at K at maturity T. The digital(binary) call/put option pays \$1 if S_T is above/below the strike K, i.e. $1_{S_T \geq K} / 1_{S_T \leq K}$.

Answer Similarly following the derivation of the call price, the digital call option price is

$$C_D(K) = E(1_{S_T \geq K}) = \text{Prob}(S_T \geq K) = \int_{-d}^{\infty} n(z) dz = 1 - N(-d) = N(d).$$

and the digital put option price is

$$P_D(K) = E(1_{S_T \leq K}) = \text{Prob}(S_T \leq K) = \int_{-\infty}^{-d} n(z) dz = N(-d) = 1 - N(d),$$

where d is given as

$$d = \frac{F - K}{\sigma \sqrt{T}}.$$

Notice that $N(d)$ has another meaning as the probability of the stock price ends up in-the-money in addition the delta of the call option.

HW 3-2. (asser-or-nothing option) The payoff of the call option, $\max(S_T - K, 0)$ can be decomposed into two parts,

$$S_T \cdot 1_{S_T \geq K} \quad - \quad K \cdot 1_{S_T \geq K}.$$

The first payout is the payout of the **asset-or-nothing** call option and the second payout if the binary call option multiplied with $-K$. Under normal model, what is the price of the asset-or-nothing call option?

Answer From the binary call option price above and the (regular) call option price from the class,

$$C(K) = (F - K) N(d) + \sigma \sqrt{T}\, n(d),$$

we conclude that

$$C_{\text{A-or-N}} = F\, N(d) + \sigma \sqrt{T}\, n(d).$$

HW 3-3. (Knock-out call option under normal model) Using the joint distribution of B_t and B_t^*, derive the price of the call option struck at K and knock-out at K^* ($> K$). First, generalize the joint CDF function $P(u < B_t, v < B_t^*)$ to σB_t. Next, derive the pdf on u by taking derivative on u. Then, integrate the payoff $(S_T - K)^+$ from K to K^*. (Assume that the risk-free rate is zero, $r = 0$, so that $S_0 = F$. Otherwise the problem is too complicated.)

Answer

$$\begin{aligned}P(S_T^* < v, S_T < u) &= P(\sigma B_T^* < v - F,\ \sigma B_T < u - F)\\ &= P(B_T^* < (v - F)/\sigma,\ B_T < (u - F)/\sigma)\\ &= N\Big(\frac{u-F}{\sigma\sqrt{T}}\Big) - N\Big(\frac{u-2v+F}{\sigma\sqrt{T}}\Big)\end{aligned}$$

The probability density function on u conditional on $S_T^* < K^*$ is obtained from the partial derivative w.r.t. u,

$$f(u) = \frac{1}{\sigma\sqrt{T}}\left(n\Big(\frac{u-F}{\sigma\sqrt{T}}\Big) - n\Big(\frac{u-2K^*+F}{\sigma\sqrt{T}}\Big)\right) \quad \text{for} \quad -\infty < u \le v$$

$z = (u - F)/\sigma\sqrt{T}$. $d = (F - K)/\sigma\sqrt{T}$ $d^* = (F - K^*)/\sigma\sqrt{T}$

The knock-out call option price is given as

$$\begin{aligned}C(K, K^*) = &\int_K^{K_1} (u - K)\, f(u) du = \int_{-d}^{-d^*} (F - K + \sigma\sqrt{T}\, z)\big(n(z) - n(z + 2d^*)\big) dz\\ =&(F-K)\int_{-d}^{-d^*} \big(n(z) - n(z+2d^*)\big) dz\\ &+ \sigma\sqrt{T}\int_{-d}^{-d^*} \big(z\, n(z) - (z+2d^*)\, n(z+2d^*) + 2d^*\, n(z+2d^*)\big) dz\\ =&(F-K)\big(N(-d^*) - N(-d) - N(d^*) + N(-d+2d^*)\big)\\ &+ \sigma\sqrt{T}\big(-n(-d^*) + n(-d) + n(d^*) - n(-d+2d^*) + 2d^* N(d^*) - 2d^* N(-d+2d^*)\big)\\ =&(F-K)\big(N(d) - 2N(d^*) + N(2d^* - d)\big)\\ &+ \sigma\sqrt{T}\big(n(d) - n(2d^* - d) + 2d^* N(d^*) - 2d^* N(2d^* - d)\big)\end{aligned}$$

We can verify two trivial cases:

1. If $K = K^*$ ($d = d^*$), the option price should be zero
$$C(K, K^* = K) = 0.$$
2. If $K^* = \infty$ ($d^* = -\infty$), the price is same as the price of the regular call option
$$C(K, K^* = \infty) = (F - K)N(d) + \sigma\sqrt{T}\, n(d).$$

We also note that the difference in the prices of the knock-out option and the regular option is given as

$$C(K) - C(K, K^*) = (F-K)\big(2N(d^*) - N(2d^*-d)\big) + \sigma\sqrt{T}\big(n(2d^*-d) - 2d^* N(d^*) + 2d^* N(2d^*-d)\big).$$

HW 4-1. Itô's isometry Find the mean and variance of the following stochastic integral

$$\int_0^t e^{B_s} dB_s$$

Answer The mean is zero because an Itô's integral is a martingale. Alternatively, if $Y_t = \int_0^t e^{B_s} dB_s$, then $dY_t = e^{B_t} dB_t$ and there is no drift term, i.e., dt.

The variance can be calculated using Itô's isometry:

$$\mathrm{Var} = E\Big[\Big(\int_0^t e^{B_s} dB_s\Big)^2\Big] = E\Big[\int_0^t e^{2B_s} ds\Big] = \int_0^t E(e^{2B_s}) ds = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1)$$

HW 4-2. Exercise 7.1 of SCFA

Answer

$$\tau_t = \mathrm{Var}(Y_t) = \mathrm{Var}(X_t) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1)$$

$$E(X_t^2) = \int_0^t e^{2s} ds = \frac{1}{2}(e^{2t} - 1), \quad E(Y_t^2) = \tau_t = \frac{1}{2}(e^{2t} - 1)$$

$$E(X_t^4) = E(Y_t^2 = B_{\tau_t}^4) = 3\tau_t^2 = \frac{3}{4}(e^{2t} - 1)^2$$

Note the difference between this problem and **Corollary 7.1** (Brownian motion time change). Given B_t is a standard BM,

$$X_t = \int_0^t f(s) dB_s, \quad \text{and} \quad \tau(t) = v = \mathrm{Var}(X_t) = \int_0^t f^2(s) ds,$$

this exercise problem is effectively stating that X_t and $B_{\tau(t)}$ are same processes. Whereas, the Corollary 7.1 states that $X_{\tau^{-1}(v)}$ and B_v are same processes where $\tau^{-1}(\cdot)$ is the inverse function of $\tau(\cdot)$, i.e., $t = \tau^{-1}(v)$. Although they look different in forms, the intuitions behind them are same in that the variance of X_t can be used as a new *time scale* of a standard BM.

HW 4-3. SDE. For the following functions $f(t, x)$, find the stochastic differential equation (SDE) of the stochastic process $Y_t = f(t, B_t)$ where B_t is a standard BM. If $f(x) = x^2$, for example, the SDE is

$$dY_t = d\,f(B_t) = d\,B_t^2 = 2B_t dB_t + dt.$$

1. $f(t, x) = x^3 - 3tx$
2. $f(t, x) = e^{t/2} \sin(x)$

Answer I constructed $f(t, x)$ such that $dY_t = df(t, B_t)$ becomes a martingale.

1.

$$df(t, x = B_t) = (3x^2 - 3t) dB_t + \frac{1}{2}(6x)(dB_t)^2 - 3x dt = 3(B_t^2 - t) dB_t$$

2.

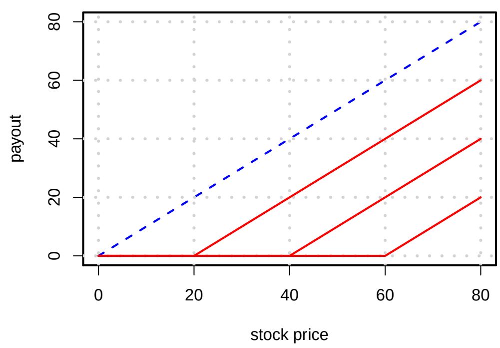
$$
\begin{aligned}
df(t, x = B_t) &= e^{t/2}\cos(x)dB_t - \frac{1}{2}e^{t/2}\sin(x)(dB_t)^2 + \frac{1}{2}e^{t/2}\cos(x)dt \\
&= e^{t/2}\cos(B_t)dB_t
\end{aligned}
$$

HW 4-4. Call option value

1. Pleased with the outstanding performance of the Stochastic Finance class, Professor is going to give each student a gift of **EITHER** one share of Tencent stock **OR** one unit of the call option on Tencent stock struck just at 1 yuan (so the call option is deep in-the-money) with maturity at the end of this module. Which gift has more financial value? (Assume that Tencent pays no dividend. No calculation required. Use your common sense.)
2. What is the upper limit of the call option value under the normal model? In normal model, under which circumstance the option is more valuable than the underling stock itself? How does it affect your choice of the gift in the previous question?

Answer

1. Given that a stock price can not go below zero, the payoff of a stock is always greater than that of a call option with any strike K. Therefore, the underlying stock is more valuable. See the plot below for the payoff of a stock (dashed blue) versus that of the options with $K = 20, 40$ and 60.



2. The price of a call option (and put option as well) is unbounded (can go to infinity) under the normal model. This happens when volatility is very high. As $\sigma \to \infty$, $d \to 0$, $N(d) \to 0$ and $n(d) \to 1/\sqrt{2\pi}$. Therefore $C \approx 0.4\sigma\sqrt{T} \to \infty$. Intuitively, it is because a call option gives you a protection against the negative underlying price (underlying asset becoming liability), which is possible under the normal model. On the other hand, under Black-Scholes-Merton model the call option price is always bounded by the price of the underlying asset as mentioned in the class. Since a stock value can not be negative, you still better of by choosing a stock rather than a call option.