Stochastic Finance (FIN 519) Final Exam

Instructor: Jaehyuk Choi

2021-22 Module 3 (2021. 4. 30.)

BM stands for Brownian motion. **RN** and **RV** stand for random number and random variable respectively. Assume that B_t is a standard **BM**. The PDF and CDF for standard normal distribution is denoted by $n(z)$ and $N(z)$. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (8 points) Calculate the following stochastic derivatives.
   (a) (2 points) $d\left((T-t)B_t^2\right)$ (with respect to the time variable t)
   (b) (2 points) $d\,(B_t^3 - 3t\,B_t)$
   (c) (2 points) $d\left(\frac{1}{B_t}\right)$
   (d) (2 points) $d\left(e^{-aB_t}B_t\right)$

Solution: Applying Itô's lemma,

$$df(t,B_t) = \left(f_t(t,B_t) + \frac{1}{2}f_{xx}(t,B_t)\right)dt + f_x(t,B_t)dB_t,$$

we obtain the following:

(a)

$$d\left((T-t)B_t^2\right) = (T-t-B_t^2)dt + 2(T-t)B_t dB_t$$

(b)

$$d\,(B_t^3 - 3t\,B_t) = 3(B_t^2 - t)dB_t$$

(c)

$$d\left(\frac{1}{B_t}\right) = -\frac{dB_t}{B_t^2} + \frac{dt}{B_t^3}$$

(d)

$$d\left(e^{-aB_t}B_t\right) = (1-aB_t)e^{-aB_t}dB_t - \frac{1}{2}(2a - a^2B_t)e^{-aB_t}dt$$

2. (6 points) For a given stochastic process X_t, find a transformation $Y_t = f(X_t)$ that makes the SDE on Y_t in the form of

$$d\,Y_t = \sigma dB_t + \mu(t,Y_t)\,dt,$$

and also find $\mu(t,Y_t)$.
   (a) (2 points) $dX_t = \sigma\sqrt{X_t}\,dB_t$.
   (b) (2 points) $dX_t = \sigma\,e^{-\lambda X_t}\,dB_t$
   (c) (2 points) $dX_t = \sigma\tanh(X_t)\,dB_t$ $\left(\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}\right)$.
   You may use other hyperbolic functions, such as $\sinh x$, $\cosh x$, $\sinh^{-1} x$, and $\cosh^{-1} x$.

Solution: When the SDE for X_t is given by

$$dX_t = \sigma h(X_t) dB_t,$$

The transformation, $f(x) = \int 1/h(x)\, dx$ ($f'(x) = 1/h(x)$), makes the required result:

$$dY_t = \frac{dX_t}{h(X_t)} - \frac{h'(X_t)}{2h^2(X_t)}(dX_t)^2 = \sigma d\, B_t - \frac{\sigma^2}{2} h'(X_t)dt$$

(a)

$$f(x) = \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} \quad \Rightarrow \quad Y_t = 2\sqrt{X_t}$$

$$dY_t = d\left(2\sqrt{X_t}\right) = \sigma dB_t - \frac{\sigma^2\, dt}{4\sqrt{X_t}} = \sigma dB_t - \frac{\sigma^2}{2Y_t}dt \quad \left(\mu(t, Y_t) = -\frac{\sigma^2}{2Y_t}\right)$$

(b)

$$f(x) = \int e^{\lambda x} dx = \frac{1}{\lambda} e^{\lambda x} \quad \Rightarrow \quad Y_t = \frac{1}{\lambda} e^{\lambda X_t}$$

$$dY_t = d\left(\frac{e^{\lambda X_t}}{\lambda}\right) = \sigma dB_t + \frac{\lambda \sigma^2}{2} e^{-\lambda X_t} dt = \sigma dB_t + \frac{\sigma^2}{2Y_t}dt \quad \left(\mu(t, Y_t) = \frac{\sigma^2}{2Y_t}\right)$$

(c) The derivative of $h(x) = \tanh(x)$ is derived as

$$h'(x) = \tanh'(x) = 1 - \tanh^2(x) = \frac{1}{\cosh^2(x)} = \frac{1}{1 + \sinh^2(x)}.$$

Therefore,

$$f(x) = \int \frac{dx}{\tanh x} = \int \frac{\cosh x}{\sinh x} dx = \log(\sinh x) \quad \Rightarrow \quad Y_t = \log(\sinh X_t)$$

$$dY_t = d \log(\sinh X_t) = \sigma dB_t - \frac{\sigma^2\, dt}{2(1 + \sinh^2 X_t)} = \sigma dB_t - \frac{\sigma^2\, dt}{2(1 + e^{2Y_t})}$$

$$\left(\mu(t, Y_t) = -\frac{\sigma^2}{2(1 + e^{2Y_t})}\right)$$

3. (3 points) A stochastic process X_t starts from $X_0 = 1$ and follows the process,

$$dX_t = \alpha(\theta - X_t)dt + \sigma dB_t.$$

You made two observations about X_t:

(i) At $t = 0$, the average speed of X_t was 4. That is, $E(X_t) = 1 + 4t$ for very short time after $t = 0$.
(ii) After long enough time t, X_t is normally distributed with mean 2 and variance 1/2.

Based on these observations, can you estimate the parameters (i.e., θ, α, and σ)?

Solution: At $t = 0$,

$$E(X_t) = X_0 + \alpha(\theta - X_0)t.$$

At $t \gg 1$,

$$X_t \sim N\left(\theta,\, \frac{\sigma^2}{2\alpha}\right).$$

Therefore, the parameters are estimated as $\alpha = 4$, $\theta = 2$, and $\sigma = 2$.

4. (6 points) **[Black-Scholes Greeks]** Remind that the call option price under the Black–Scholes model (assuming $r = q = 0$) is

$$C = S_0 N(d_1) - K N(d_2) \quad \text{where} \quad d_{1,2} = \frac{\log(S_0/K)}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}.$$

For the Greeks, consider the following expressions.

$$(1)\ N(d_1), \quad (2)\ Kn(d_2)\sqrt{T}, \quad (3)\ S_0\, n(d_1)\sqrt{T}, \quad (4)\ S_0\, n(d_2)\sqrt{T}$$

$$(5)\ N(d_2), \quad (6)\ \frac{n(d_1)}{S_0\,\sigma\sqrt{T}}, \quad (7)\ -\frac{\sigma\, S_0\, n(d_1)}{2\sqrt{T}}, \quad (8)\ \frac{\sigma\, S_0\, n(d_1)}{2\sqrt{T}}$$

(a) (1 point) From above, choose Delta, $D = \frac{\partial C}{\partial S_0}$.

(b) (1 point) From above, choose Vega, $V = \frac{\partial C}{\partial \sigma}$.

(c) (1 point) From above, choose Gamma, $G = \frac{\partial^2 C}{\partial S_0^2}$.

(d) (1 point) From above, choose Theta, $\Theta = -\frac{\partial C}{\partial T}$.

(e) (2 points) Obtain the corresponding Greeks for the put option by properly modifying your answers in (a)–(d). (The answers are not necessarily in the list above.)

Solution:

(a) (1)

(b) (2) or (3)

(c) (6)

(d) (7)

(e) From the Greeks of put options, we use the put–call parity:

$$P = C - (S_0 - K).$$

Regarding Delta, we know $\frac{\partial P}{\partial S_0} = \frac{\partial C}{\partial S_0} - 1$. Therefore, the Delta of put option is

$$D = N(d_1) - 1 = -N(-d_1).$$

Vega, Gamma, and Theta are same as those of call option.

5. (6 points) **[Displaced Black-Scholes (DBS) model]** For a stock price S_t, assume that the 'displaced' price,

$$D(S_t) = \beta S_t + (1-\beta)A \quad \text{for some constants} \quad A > 0 \text{ and } 0 \le \beta \le 1,$$

follows a geometric BM with volatility $\beta\sigma_{\mathrm{D}}$:

$$\frac{dD(S_t)}{D(S_t)} = \beta\sigma_{\mathrm{D}} dB_t \quad \text{or} \quad \frac{dS_t}{D(S_t)} = \sigma_{\mathrm{D}} dB_t.$$

$D(S_t)$ can be understood as a liner interpolation between S_t and A as β changes. The DBS model offers a "bridge" between the Bachelier and BSM models. For the questions, assume $r = q = 0$.

(a) (2 points) Express S_T as a function of B_T and other parameters.

(b) (2 points) Derive the price of the call option with strike price K and time to maturity T under the DBS model. (Hint: you should recover the BS formula as $\beta \to 1$.)

(c) (2 points) In the limit $\beta \to 0$, show that the price from (b) converges to the Bachelier model price. What is the Bachelier volatility σ_{N} in the limit?

Solution:

(a) Because $D(S_t)$ follows a geometric BM, $D(S_T)$ is given by

$$D(S_T) = D(S_0)\exp\left(\beta\sigma_{\text{D}} B_T - \beta^2\sigma_{\text{D}}^2 T/2\right).$$

Therefore, S_T is

$$S_T = \left(S_0 + \frac{1-\beta}{\beta}A\right)\exp\left(\beta\sigma_{\text{D}} B_T - \frac{\beta^2\sigma_{\text{D}}^2 T}{2}\right) - \frac{1-\beta}{\beta}A.$$

(b) The call option price is given by

$$C_{\text{D}}(K) = E((S_T-K)^+) = \frac{E((D(S_T)-D(K))^+)}{\beta} = \frac{D(S_0)N(d_1) - D(K)N(d_2)}{\beta}$$
$$\text{where}\quad d_{1,2} = \frac{\log\left(D(S_0)/D(K)\right)}{\beta\sigma_{\text{D}}\sqrt{T}} \pm \frac{\beta\sigma_{\text{D}}\sqrt{T}}{2}.$$

Note $D(S_t) = S_t$ when $\beta = 1$. Therefore, it is obvious that $C_{\text{D}}(K)$ converges to the Black–Scholes option price with volatility σ_{D} when $\beta = 1$.

(c) For small β, we have the following approximations:

$$\log\left(\frac{D(S_0)}{D(K)}\right) = \frac{\beta(S_0-K)}{(1-\beta)A}\left(1 + \frac{\beta(S_0+K)}{2(1-\beta)A}\right) + O(\beta^2),$$
$$d_{1,2} = \frac{S_0-K}{(1-\beta)A\sigma_{\text{D}}\sqrt{T}}\left(1 + \frac{\beta(S_0+K)}{2(1-\beta)A}\right) \pm \frac{\beta\sigma_{\text{D}}\sqrt{T}}{2} + O(\beta).$$

The DBS model option price converges to the Bachelier model price as $\beta \downarrow 0$:

$$\begin{aligned}
C_{\text{D}}(K) &= \frac{D(S_0)-D(K)}{\beta}N(d_2) + \frac{D(S_0)}{\beta}\left(N(d_1)-N(d_2)\right) \\
&= (S_0-K)N(d_2) + \frac{D(S_0)}{\beta}(d_1-d_2)\,n(d_1) + O(\beta) \\
&\to (S_0-K)N(d_{\text{N}}) + \sigma_{\text{N}}\sqrt{T}\,n(d_{\text{N}}) = C_{\text{N}}(K) \quad \text{with} \quad \sigma_{\text{N}} = A\sigma_{\text{D}}.
\end{aligned}$$

6. (4 points) In a **cash-settled** forward contact, you receive or pay (i.e., settle) the value of the contact in cash, instead of buying the underlying asset. If you hold a long forward contact at price F, for example, you receive (or pay) $S_T - F$ in cash at the maturity $t = T$. The cash-settled forward contact is usually equivalent to the regular forward contact. The present value of the forward contract is

$$P(0,T)\,E_T\left(\frac{S_T - F}{P(T,T)=1}\right),$$

where $E^T(\cdot)$ is the expectation under the T-forward measure and $P(t,T)$ is the time t price of the zero coupon bond maturing at T. Therefore, the fair forward price F (that makes the present value zero) observed at $t=0$ is

$$F = E^T(S_T) = S_0/P(0,T).$$

Now, consider a new rule: Although, the payout, $S_T - F$, is determined at $t = T$, the cash settlement happens (i.e, you receive or pay) at a **delayed time** $t = T + \Delta$.

(a) (2 points) What is the fair forward price F' under this new rule? Express the answer using $E^T(\cdot)$, $P(t, T)$, and S_T.

(b) (2 points) How does this new forward price F' compare (e.g., higher or lower) to the original price F? Make your answer based on the three scenarios: the correlation between the asset price S_T and the interest rate is positive, zero, and negative.

Solution:

(a) The present value of the new forward contract is

$$P(0,T)\, E_T\left(P(T, T+\Delta)(S_T - F')\right),$$

Since the bond price $P(0, T+\Delta)$ is expressed by

$$P(0, T+\Delta) = P(0,T)E^T(P(T, T+\Delta)),$$

Therefore, the forward price with delayed settlement is give by

$$F' = \frac{E^T(P(T,T+\Delta)S_T)}{E^T(P(T,T+\Delta))} = \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta)S_T).$$

(b) Since the future bond price and the interest rate are inversely related as $P(T, T+\Delta) \approx e^{-r_T\Delta}$,

$$E^T(P(T,T+\Delta)\,S_T) \begin{cases} < E^T(P(T,T+\Delta))E^T(S_T) & \text{for positive correlation} \\ = E^T(P(T,T+\Delta))E^T(S_T) & \text{for zero correlation} \\ > E^T(P(T,T+\Delta))E^T(S_T) & \text{for negative correlation} \end{cases}.$$

Therefore,

1. When the asset price S_T and the interest rate have zero correlation,

$$F' = \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta))E^T(S_T) = \frac{S_0}{P(0,T)} = F.$$

2. If they have positive correlation,

$$F' < \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta))E^T(S_T) = F.$$

3. If they have negative correlation,

$$F' > \frac{P(0,T)}{P(0,T+\Delta)}E^T(P(T,T+\Delta))E^T(S_T) = F.$$