Stochastic Finance (FIN 519)
Midterm Exam

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2022-23 Module 3 (2023. 3. 21.)

BM stands for Brownian motion. Assume that B_t is a standard **BM**. **RN** and **RV** stand for random number and random variable, respectively. The PDF and CDF of the standard normal distribution are denoted by $n(z)$ and $N(z)$, respectively. You can use $n(z)$ and $N(z)$ in your answers without further evaluation.

1. (8 points) **[Poisson Distribution]** The RV, N, follows a Poisson distribution with rate λ. The Poisson distribution is a discrete probability distribution (i.e., $N = 0, 1, 2, \cdots$) of the number of the events occurring in a unit time interval $T = 1$. The probability function is given by

$$P(N = k) = f_\lambda(k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

   (a) (3 points) Find the moment generating function (MGF) of N:

$$M_N(t) = E(e^{tN}) = \sum_{k=0}^{\infty} e^{tk} f_\lambda(k).$$

   (Hint: use that $\sum_{k=0}^{\infty} f_\lambda(k) = 1$ for any $\lambda > 0$.)

   (b) (3 points) Prove that $E(N) = \lambda$ and $\mathrm{Var}(N) = \lambda$. (If you obtained the MGF from (a), use it. You may still be able to prove them without MGF.)

   (c) (2 points) Find the skewness and ex-kurtosis of N.

Solution:

(a) Although you don't have to show this, $\sum_{k=0}^{\infty} f_\lambda(k)$ because of the Taylor's expansion of e^λ:

$$e^\lambda = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}\ .$$

The MGF of N is

$$M_N(t) = E(e^{tN}) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} = e^{\eta-\lambda} \sum_{k=0}^{\infty} \frac{\eta^k}{k!} e^{-\eta} = e^{\lambda(e^t-1)},$$

where $\eta = \lambda e^t$.

(b) From the expansions of the MGF in (a),

$$M_N(t) = e^{\lambda(e^t-1)} = \exp\big(\lambda(t + t^2/2 + \cdots)\big) = \lambda t + \lambda\frac{t^2}{2} + \cdots + \lambda^2\frac{t^2}{2} + \cdots,$$

we prove that

$$E(N) = \lambda, \quad \mathrm{Var}(N) = \lambda + \lambda^2 - E(N)^2 = \lambda.$$

(c) The skewness and ex-kurtosis (the mean and variance as well) can be easily obtained from the 3rd and 4th terms of the cumulant generating function:

$$K_N(t) = \log M_N(t) = \lambda(e^t - 1) = \lambda t + \lambda\frac{t^2}{2} + \lambda\frac{t^3}{6} + \lambda\frac{t^4}{4!} + \cdots.$$

Therefore, skewness and ex-kurtosis are

$$s = \frac{E((N-\lambda)^3)}{\mathrm{Var}(N)^{1.5}} = \frac{\lambda}{\lambda^{1.5}} = \frac{1}{\sqrt{\lambda}} \quad \text{and} \quad \kappa = \frac{E((N-\lambda)^4)}{\mathrm{Var}(N)^2} - 3 = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}.$$

2. (4 points) **[Gambler's ruin with BM]** Assume that a stock price follows a BM with volatility σ and initial price S_0:

$$S_t = S_0 + \sigma B_t,$$

and that the stopping time τ is the first time S_t hits $S_0 + A$ or $S_0 - B$ for $A, B > 0$. Find $E(\tau)$.

Solution: From class, we know

$$E(\tau) = AB \quad \text{where} \quad \tau = \min\{t : B_t = A \quad \text{or} \quad B_t = -B\}.$$

The stopping time in the problem is

$$\begin{aligned}\tau &= \min\{t : S_t = S_0 + A \quad \text{or} \quad S_t = S_0 - B\}\\ &= \min\{t : \sigma B_t = A \quad \text{or} \quad \sigma B_t = -B\}\\ &= \min\{t : B_t = A/\sigma \quad \text{or} \quad B_t = -B/\sigma\}.\end{aligned}$$

Therefore,

$$E(\tau) = \frac{AB}{\sigma^2}.$$

3. (6 points) **[Brownian Motion]**

(a) (3 points) For a standard BM, B_t, find the probability that

$$B_1 + B_2 + B_3 - 3B_4 \geq 2.$$

(b) (3 points) When a stock price follows $S_t = S_0 + \sigma B_t$, consider the following option payout at the expiry $T = 4$:

$$\max\left(S_4 - \frac{S_1 + S_2 + S_3}{3}, 0\right).$$

The payout is similar to that of the regular call option, except that the strike price K is determined by the average stock price at $t = 1$, 2, and 3. What is the price of this option (i.e., the expectation of the payout)?

Solution: Let $x = B_1$, $y = B_2 - B_1$, $z = B_3 - B_2$, and $w = B_4 - B_3$, then x, y, z and w follow independent standard normal variables. It follows that

$$B_1+B_2+B_3-3B_4 = x+(x+y)+(x+y+z)-3(x+y+z+w) = -(y+2z+3w) \sim \mathcal{N}(0, 14).$$

(a) The probability is

$$P(B_1 + B_2 + B_3 - 3B_4 \geq 2) = 1 - N(2/\sqrt{14}) \approx 0.2965.$$

(b) The payout is normally distributed as

$$S_4 - \frac{S_1 + S_2 + S_3}{3} = \frac{\sigma}{3}(y + 2z + 3w) \sim \mathcal{N}\left(0, \frac{14}{9}\sigma^2\right).$$

From the extended Bachelier model, the call option price is given by

$$C \approx 0.4\frac{\sqrt{14}}{3}\sigma.$$

4. (2×3 points) **[Martingale related to BM]** If B_t is a standard BM, determine whether the following is a martingale or not. Give a brief reason.

(a) $Y_t = B_{\lambda t}^2 - \lambda t^2$

(b) $Y_t = \exp(2\sigma B_t - \sigma^2 t)$

(c) $S_t = S_0 \exp(\sigma B_{t\wedge\tau} - \frac{\sigma^2\,(t\wedge\tau)}{2})$ where $\tau = \min\{t : |B_t - B_{t-1}| > A\}$ for some $A > 0$.

Solution:

(a) **No**. $Y_t = B_{\lambda t}^2 - \lambda t$ is a martingale.

(b) **No**. $Y_t = \exp(2\sigma B_t - 2\sigma^2 t)$ is a martingale.

(c) **Yes**. τ is a proper stopping time and $M_t = S_0 \exp(\sigma B_t - \sigma^2\, t/2)$ is a martingale. Therefore, $M_{t\wedge\tau}$ is a martingale.

5. (6 points) **[Knock-out (up-and-out) digital option]** We are going to derive the price of the binary call option with knock-out (up-and-out) feature under the Bachelier model. Assume that the underlying stock follows the process $S_t = S_0 + \sigma B_t$. The option will pay you \$1 at the expiry T if $S_T > K$ for a strike price K **and** the stock price S_t has never gone above $H > \max(S_0, K)$ anytime $0 \leq t \leq T$. In other words, this option knocks out (expires worthless) if S_t goes above H any time before the expiry T. Let the running maximum of B_t

$$B_T^M = \max_{0\leq t\leq T} B_t.$$

(a) (3 points) In the class (and in the textbook), we derived the joint CDF for B_T and B_T^M,

$$P(B_T^M < v,\ B_T < u) = N\Big(\frac{u}{\sqrt{T}}\Big) - N\Big(\frac{u-2v}{\sqrt{T}}\Big) \quad \text{for} \quad v \geq \max(0, u)$$

Using this result, derive the joint CDF for BM with volatility, σB_t,

$$P(\sigma B_T^M < v,\ \sigma B_T > u).$$

(b) (3 points) Finally find the price of the binary call option with the up-and-out feature? Assume that interest rate and divided rate are zero. How much is this derivative cheaper (or more expensive) than the regular binary call option **without** knock-out feature?

Solution:

(a) We first derive $P(B_T^M < v,\ B_T > u)$:

$$P(B_T^M < v,\ B_T > u) = P(B_T^M < v) - P(B_T^M < v,\ B_T < u).$$

Since (or you can just use the result from class)

$$P(B_T^M < v) = P(B_T^M < v,\ B_T < v) = N\Big(\frac{v}{\sqrt{T}}\Big) - N\Big(\frac{-v}{\sqrt{T}}\Big) = 2N\Big(\frac{v}{\sqrt{T}}\Big) - 1,$$

Therefore,

$$\begin{aligned} P(B_T^M < v,\ B_T > u) &= 2N\Big(\frac{v}{\sqrt{T}}\Big) - 1 - N\Big(\frac{u}{\sqrt{T}}\Big) + N\Big(\frac{u-2v}{\sqrt{T}}\Big) \\ &= N\Big(\frac{-u}{\sqrt{T}}\Big) - 2N\Big(\frac{-v}{\sqrt{T}}\Big) + N\Big(\frac{u-2v}{\sqrt{T}}\Big). \end{aligned}$$

Finally

$$P(\sigma B_T^M < v,\ \sigma B_T > u) = N\Big(\frac{-u}{\sigma\sqrt{T}}\Big) - 2N\Big(\frac{-v}{\sigma\sqrt{T}}\Big) + N\Big(\frac{u-2v}{\sigma\sqrt{T}}\Big).$$

(b) The price of the up-and-out binary call option is obtained by plugging in $u = K - S_0$ and $v = H - S_0$:

$$D(K, H) = N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) - 2N\Big(\frac{S_0 - H}{\sigma\sqrt{T}}\Big) + N\Big(\frac{K + S_0 - 2H}{\sigma\sqrt{T}}\Big).$$

This price makes sense because

$$D(K, H = K) = N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) - 2N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big) + N\Big(\frac{K + S_0 - 2K}{\sigma\sqrt{T}}\Big) = 0.$$

The price of the regular binary call option is

$$D(K) = D(K, \infty) = N\Big(\frac{S_0 - K}{\sigma\sqrt{T}}\Big).$$

Therefore, the price is cheaper by

$$D(K) - D(K, H) = 2N\Big(\frac{S_0 - H}{\sigma\sqrt{T}}\Big) - N\Big(\frac{K + S_0 - 2H}{\sigma\sqrt{T}}\Big).$$