ECO5002 Introduction to Economics

Lecture 7: The Real Economy in the Long Run

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I. Production and Growth

The Variety of Growth Experiences

		Real GDP per Person		
Country	Period	At Beginning of Period[a]	At End of Period[a]	Growth Rate (per year)
Brazil	1900–2014	$ 828	$15,590	2.61%
Japan	1890–2014	1,600	37,920	2.59
China	1900–2014	762	13,170	2.53
Mexico	1900–2014	1,233	16,640	2.31
Germany	1870–2014	2,324	46,850	2.11
Indonesia	1900–2014	948	10,190	2.10
Canada	1870–2014	2,527	43,360	1.99
India	1900–2014	718	5,630	1.82
United States	1870–2014	4,264	55,860	1.80
Pakistan	1900–2014	785	5,090	1.65
Argentina	1900–2014	2,440	12,510	1.44
Bangladesh	1900–2014	663	3,330	1.43
United Kingdom	1870–2014	5,117	39,040	1.42

[a]Real GDP is measured in 2014 dollars.

Source: Robert J. Barro and Xavier Sala-i-Martin, *Economic Growth* (New York: McGraw-Hill, 1995), Tables 10.2 and 10.3; *World Bank* online data; and author's calculations. To account for international price differences, data are PPP-adjusted when available.

I. Production and Growth

Facts about Economic Growth (The Kaldor Facts):

1. Output per worker grows at a sustained, roughly constant, rate over long periods of time.
2. Capital per worker grows at a sustained, approximately constant, rate over long periods of time.
3. The capital to output ratio is roughly constant over long periods of time.
4. Labor's share of income is roughly constant over long periods of time.
5. The rate of return on capital is relatively constant.
6. Real wages grow at a sustained, approximately constant, rate.

Cross Country Facts:

1. There are enormous variations in income across countries.
2. There are growth miracles (China) and growth disasters (Madagascar).
3. There is a strong, positive correlation between income per capita and human capital.

I. Production and Growth

- **Productivity:** the quantity of goods and services produced from each unit of labor input.
  1. **physical capital**: the stock of equipment and structures that are used to produce goods and services.
  2. **human capital**: the knowledge and skills that workers acquire through education, training, and experience.
  3. **natural resources**: the inputs into the production of goods and services that are provided by nature, such as land, rivers, and mineral deposits.
  4. **technological knowledge**: society's understanding of the best ways to produce goods and services.

- **Production function**:

$$Y_t = A_t F(L_t, K_t, H_t, N_t)$$

where Y_t is output, L_t is labor, K_t is physical capital, H_t is human capital, N is natural resources, and A_t is technology.

I. Production and Growth

- There are so many factors that can affect economic growth:
  - saving and investment: trade-off between today and tomorrow.
  - diminishing returns and catch-up effect.
  - investment from abroad.
  - education.
  - health and nutrition.
  - property rights and political stability.
  - free trade.
  - research and development (R&D).
  - population growth.
  - ...
- Once you start thinking about [economic] growth, it's hard to think about anything else. (Lucas)

II. The Basic Solow Model

- The Solow model is designed to show how (i) growth in the capital stock, (ii) growth in the labor force, and (iii) advances in technology interact in an economy as well as how they affect a nation's total output of goods and services.
- **Supply side**: a production function $Y_t = F(K_t, L_t)$ that satisfies the constant return to scale (CRS), i.e., $zY_t = F(zK_t, zL_t)$.
  - the intuition is that if both capital and labor are multiplied by z, the amount of output is also multiplied by z.
  - the output per capita is $y_t \equiv \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1\right) = f(k_t)$.
  - the slope of $f(k_t)$ shows how much extra output a worker produces when given an extra unit of capital, i.e., $MPK = f(k_t + 1) - f(k_t)$.
- **Demand side**: output per worker y is divided between consumption per worker c and investment per worker i, i.e., $y_t = c_t + i_t = (1 - s)y_t + i_t$ where s_t is saving rate. $(i_t = sy_t)$

II. The Basic Solow Model

- **Law of motion of capital**:

$$\dot{k}_t = i_t - \delta k_t$$

where $\dot{k}_t \equiv \lim_{\Delta t \to 0} \frac{\Delta k_t}{\Delta t} = \frac{\mathrm{d}k_t}{\mathrm{d}t}$, and $\delta \in (0,1)$ represents the depreciation rate of capital.

- Combine these conditions together, and then we have:

$$\dot{k}_t = sf(k_t) - \delta k_t$$

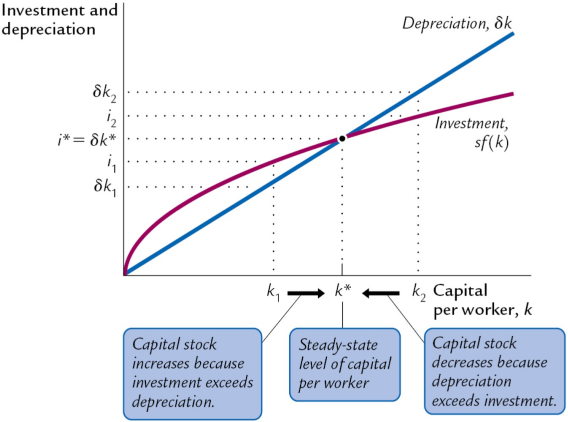
which is the core equation of the Solow model.

  - steady state: the amount of capital will not change over time, i.e.,

$$\dot{k}_t = 0 \quad \Rightarrow \quad sf(k^*) = \delta k^*$$

which captures the long-run equilibrium of the economy.

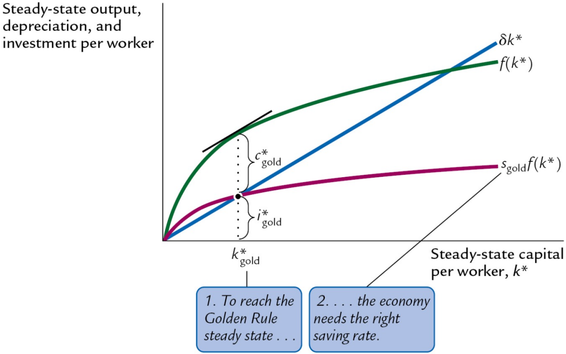
II. The Basic Solow Model



- what if s increases?
- what if δ increases?

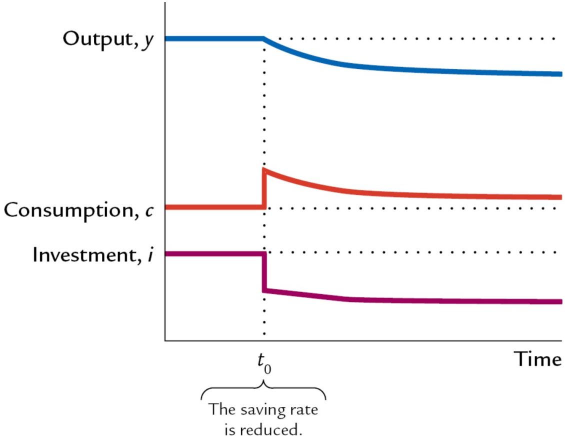
II. The Basic Solow Model

- More saving = More output. More is better?
- What steady state should the policymaker choose?
- The steady-state value of k that maximizes consumption is called the **Golden Rule** level of capital and is denoted k^*_{gold}.
  - $c^* = y^* - i^* = f(k^*) - \delta k^*$.
  - k^*_{gold} should satisfy $\frac{\partial f(k^*)}{\partial k^*} = \delta$.



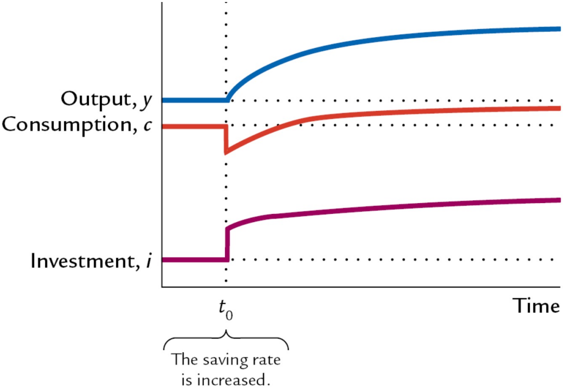
II. The Basic Solow Model

- Reducing saving when starting with more capital than in the Golden Rule steady state.



II. The Basic Solow Model

- Increasing saving when starting with less capital than in the Golden Rule steady state.



III. Population Growth and Technology Progress

- Now, suppose that the production function is

$$Y_t = F(K_t, A_t L_t)$$

- Population growth follows $\dot{L}_t = nL_t$.
- Technology progress follows $\dot{A}_t = gA_t$.
- Law of motion of total capital is still $\dot{K}_t = sY_t - \delta K_t$.
- In the previous section, we define the per capita level of variables, e.g., $k_t = K_t/L_t$, $y_t = Y_t/L_t$. Now we define the "per effective labor" level of variables, i.e.,

$$\tilde{y}_t \equiv \frac{Y_t}{A_t L_t}, \quad \tilde{k}_t \equiv \frac{K_t}{A_t L_t}, \quad \ldots$$

III. Population Growth and Technology Progress

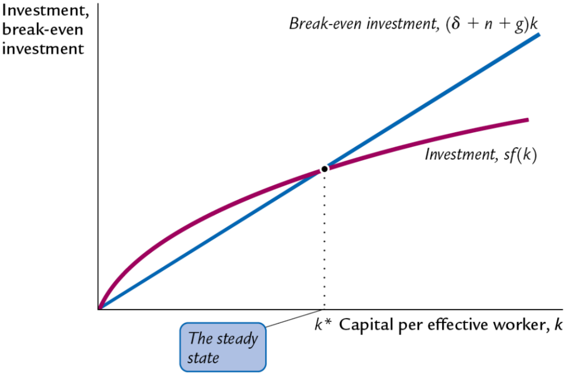
- Do some derivations:

$$\dot{\tilde{k}}_t = \frac{\dot{K}_t A_t L_t - K_t \left(\dot{A}_t L_t + A_t \dot{L}_t\right)}{(A_t L_t)^2}$$
$$= \frac{\dot{K}_t}{K_t}\frac{K_t}{A_t L_t} - \frac{K_t}{A_t L_t}\frac{\dot{A}_t}{A_t} - \frac{K_t}{A_t L_t}\frac{\dot{L}_t}{L_t}$$
$$= sf(\tilde{k}_t) - (\delta + g + n)\tilde{k}_t$$

- The "per effective labor" variables have a steady state:

$$sf(\tilde{k}^*) = (\delta + g + n)\tilde{k}^*$$

III. Population Growth and Technology Progress



- Think about:
  - what if n increases?
  - what if g increases?

III. Population Growth and Technology Progress

- Although the "per effective labor" level of variables do not change at the steady state, per capita variables and the original variables do change.
- Think about K_t (when $\dot{\tilde{k}}_t = 0$):

$$g_K = \frac{\dot{K}_t}{K_t} = g + n$$

- Think about $k_t = K_t/L_t$ (when $\dot{\tilde{k}}_t = 0$):

$$g_k = \frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t L_t - K_t \dot{L}_t}{L_t^2} \cdot \frac{L_t}{K_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{L}_t}{L_t} = g$$

- Balanced Growth Path (BGP): $g_y = g_k = g_c = g$.

III. Population Growth and Technology Progress

- The following table summarize the results:

Variable	**Symbol**	**Steady-State Growth Rate**
Capital per effective worker	$k = K/(E \times L)$	0
Output per effective worker	$y = Y/(E \times L) = f(k)$	0
Output per worker	$Y/L = y \times E$	g
Total output	$Y = y \times (E \times L)$	$n + g$

IV. Solow Model with Human Capital

- Production function (a general case):

$$Y_t = F(K_t, H_t, z_t N_t),$$

  which satisfies (i) continuity and differentiability; (ii) positive and diminishing marginal products w.r.t. K_t, N_t, H_t; (iii) CRS; (iv) Inada conditions. H_t denotes the human capital at time t.
- Population and technology growth: $\dot{N}_t = nN_t$ and $\dot{z}_t = gz_t$.
- Law of motion of capital and human capital:

$$\dot{K}_t = s_K Y_t - \delta_K K_t,$$
$$\dot{H}_t = s_H Y_t - \delta_H H_t.$$

- Resource constraint is given by
  $Y_t = C_t + I_{K,t} + I_{H,t} = C_t + (s_K + s_H) Y_t.$

IV. Solow Model with Human Capital

- As before, we define that

$$k_t \equiv \frac{K_t}{z_t N_t}, \quad h_t \equiv \frac{H_t}{z_t N_t},$$

$$y_t \equiv \frac{Y_t}{z_t N_t} = F\left(\frac{K_t}{z_t N_t}, \frac{H_t}{z_t N_t}, 1\right) \equiv f(k_t, h_t).$$

- Two key equations in this economy are

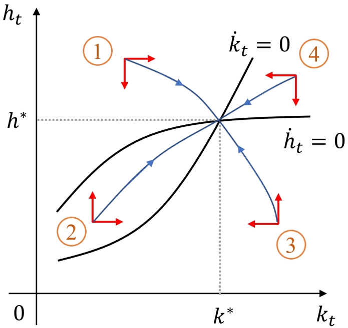
$$\dot{k}_t = s_K \cdot f(k_t, h_t) - (\delta_K + g + n)k_t,$$
$$\dot{h}_t = s_H \cdot f(k_t, h_t) - (\delta_H + g + n)h_t.$$

IV. Solow Model with Human Capital

- The steady state (k^*, h^*) is then given by $\dot{k}_t = 0$ and $\dot{h}_t = 0$:

$$\frac{f(k^*, h^*)}{k^*} = \frac{\delta_K + g + n}{s_K},$$
$$\frac{f(k^*, h^*)}{h^*} = \frac{\delta_H + g + n}{s_H},$$

- The illustrative phase diagram is shown below.



V. Optimal Growth

- In this part, I will show you a growth model with richer micro-foundations.
- Consider the following program with infinite horizon:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t), \quad \beta \in (0,1)$$

$$\text{s.t.} \quad c_t + k_{t+1} = k_t^{\alpha} + (1-\delta)k_t, \quad \forall t,$$

where c_t is consumption, k_{t+1} is saving (next-period capital), k_t is capital stock today. $k_0 > 0$ is given and the transversality condition is:

$$\lim_{t\to\infty} \left[\beta^t (\alpha k_t^{\alpha-1} + 1 - \delta) u'(c_t) k_t\right] = 0.$$

V. Optimal Growth

- k_t = state variable. c_t and k_{t+1} = choice (control) variables.
- To solve this dynamic program, one way is to write down the Lagrangian equation first:

$$\mathcal{L}_t = \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) - \lambda_t \left[c_t + k_{t+1} - k_t^{\alpha} - (1-\delta)k_t \right] \right\},$$

- Then, take the FOC w.r.t. c_t and k_{t+1}:

$$\lambda_t = u'(c_t),$$
$$\lambda_t = \beta \lambda_{t+1}(\alpha k_{t+1}^{\alpha-1} - 1 + \delta).$$

- Combining these two conditions gives us the consumption Euler equation:

$$u'(c_t) = \beta u'(c_{t+1})(\alpha k_{t+1}^{\alpha-1} - 1 + \delta).$$

V. Optimal Growth

- Another way to solve this problem is changing this sequential form into a recursive form (Bellman equation):

$$V(k) = \max \left[u(c) + \beta V(k') \right],$$

$$\text{s.t.} \quad c + k' = k^{\alpha} + (1-\delta)k, \quad \forall t,$$

- Plug in the constraint, and we have

$$V(k) = \max \left[u(k^{\alpha} + (1-\delta)k - k') + \beta V(k') \right],$$

- FOC w.r.t. k' gives $u'(c) = \beta \frac{\partial V(k')}{\partial k'}$
- Envelop theorem gives $\frac{\partial V(k)}{\partial k} = u'(c) \left[\alpha k^{\alpha-1} + (1-\delta) \right]$.
- Then, we could get the same consumption Euler equation.

V. Optimal Growth

Example: Solve for An Optimal Consumption Path

- For simplicity, we assume $\alpha = 1$ and $u(c) = \ln c$. Then, by assuming $\xi = 2 - \delta$ all conditions we have are, given k_0, $\forall t$,

$$
\begin{aligned}
c_{t+1} &= \beta \xi c_t, \\
c_t + k_{t+1} &= \xi k_t, \\
\lim_{t \to \infty} \left[\beta^t \xi \frac{k_t}{c_t} \right] &= 0.
\end{aligned}
$$

- From above, we know that

$$
c_t = (\beta \xi)^t \cdot c_0
$$

V. Optimal Growth

Example: Solve for An Optimal Consumption Path

- We have:

$$
\begin{aligned}
c_{t-1} + k_t &= \xi k_{t-1} \\
\xi c_{t-2} + \xi k_{t-1} &= \xi^2 k_{t-2} \\
&\vdots \\
\xi^{t-1} c_0 + \xi^{t-1} k_1 &= \xi^t k_0
\end{aligned}
$$

- Sum them up, then we have

$$
(1 + \beta + \beta^2 + \cdots + \beta^{t-1}) \cdot \xi^{t-1} \cdot c_0 + k_t = \xi^t k_0
$$

$$
\frac{1 - \beta^t}{1 - \beta} \cdot \xi^{t-1} \cdot c_0 + k_t = \xi^t k_0
$$

V. Optimal Growth

Example: Solve for An Optimal Consumption Path

- We cannot solve for c_0 because k_∞ is unknown. But if we impose the transversality condition, we will have a solution for c_0. From two equations mentioned above, we have

$$\frac{1-\beta^t}{1-\beta} \cdot \frac{1}{\beta^t \xi} + \frac{k_t}{c_t} = \frac{k_0}{\beta^t c_0}.$$

Multiply both sides by $\beta^t \xi$ and take the limit when t goes to infinity, then

$$\lim_{t\to\infty}\left[\frac{1-\beta^t}{1-\beta}\right] + \lim_{t\to\infty}\left[\beta^t \xi \frac{k_t}{c_t}\right] = \frac{\xi k_0}{c_0} \Longrightarrow c_0 = (1-\beta)\xi k_0.$$

Reading

- Chapter 25, *Principles of Economics* by Mankiw.
- Chapter $8 \sim 9$, *Macroeconomics* by Mankiw.
- Chapter 4, *Intermediate Macroeconomics* by GLS.