2

System Modelling

The task of mathematical modelling is an important step in the analysis and design of control systems. In this chapter, we will develop mathematical models for the mechanical, electrical, hydraulic and thermal systems which are used commonly in everyday life. The mathematical models of systems are obtained by applying the fundamental physical laws governing the nature of the components making these systems. For example, Newton's laws are used in the mathematical modelling of mechanical systems. Similarly, Kirchhoff's laws are used in the modelling and analysis of electrical systems.

Our mathematical treatment will be limited to linear, time-invariant ordinary differential equations whose coefficients do not change in time. In real life many systems are nonlinear, but they can be linearized around certain operating ranges about their equilibrium conditions. Real systems are usually quite complex and exact analysis is often impossible. We shall make approximations and reduce the system components to idealized versions whose behaviours are similar to the real components.

In this chapter we shall look only at the passive components. These components are of two types: those storing energy (e.g. the capacitor in an electrical system), and those dissipating energy (e.g. the resistor in an electrical system).

The mathematical model of a system is one or more differential equations describing the dynamic behaviour of the system. The Laplace transformation is applied to the mathematical model and then the model is converted into an algebraic equation. The properties and behaviour of the system can then be represented as a block diagram, with the transfer function of each component describing the relationship between its input and output behaviour.

2.1 MECHANICAL SYSTEMS

Models of mechanical systems are important in control engineering because a mechanical system may be a vehicle, a robot arm, a missile, or any other system which incorporates a mechanical component. Mechanical systems can be divided into two categories: translational systems and rotational systems. Some systems may be purely translational or rotational, whereas others may be hybrid, incorporating both translational and rotational components.

2.1.1 Translational Mechanical Systems

The basic building blocks of translational mechanical systems are masses, springs, and dashpots (Figure 2.1). The input to a translational mechanical system may be a force, F, and the output the displacement, y.

Springs store energy and are used in most mechanical systems. As shown in Figure 2.2, some springs are hard, some are soft, and some are linear. A hard or a soft spring can be linearized for small deviations from its equilibrium condition. In the analysis in this section, a spring is assumed massless, or of negligible mass, i.e. the forces at both ends of the spring are assumed to be equal in magnitude but opposite in direction.

For a linear spring, the extension y is proportional to the applied force F and we have

$$F = ky, \tag{2.1}$$

where k is known as the *stiffness* constant. The spring when stretched stores energy given by

$$E = \frac{1}{2}ky^2. \tag{2.2}$$

This energy is released when the spring contracts back to its original length.

In some applications springs can be in parallel or in series. When n springs are in parallel, then the equivalent stiffness constant k_{eq} is equal to the sum of all the individual spring stiffnesses k_i:

$$k_{eq} = k_1 + k_2 + \cdots + k_n. \tag{2.3}$$

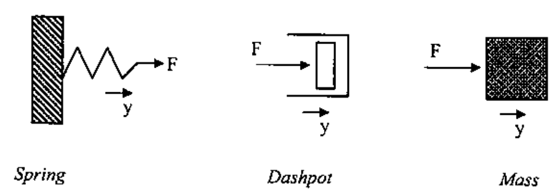

Figure 2.1 Translational mechanical system components

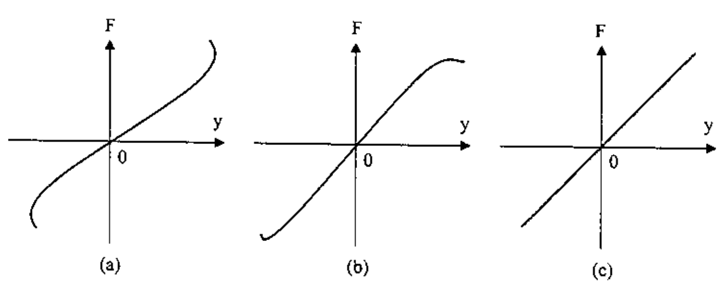

Figure 2.2 (a) Hard spring, (b) soft spring, (c) linear spring

Similarly, when n springs are in series, then the reciprocal of the equivalent stiffness constant k_{eq} is equal to the sum of all the reciprocals of the individual spring stiffnesses k_i:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n}. \tag{2.4}$$

As an example, if there are two springs k_1 and k_2 in series, then the equivalent stiffness constant is given by

$$k_{eq} = \frac{k_1 k_2}{k_1 + k_2}. \tag{2.5}$$

A *dashpot* element is a form of damping and can be considered to be represented by a piston moving in a viscous medium in a cylinder. As the piston moves the liquid passes through the edges of the piston, damping to the movement of the piston. The force F which moves the piston is proportional to the velocity of the piston movement. Thus,

$$F = c\frac{dy}{dt}. \tag{2.6}$$

A dashpot does not store energy.

When a force is applied to a *mass*, the relationship between the force F and the acceleration a of the mass is given by Newton's second law as $F = ma$. Since acceleration is the rate of change of velocity and the velocity is the rate of change of displacement, we can write

$$F = m\frac{d^2y}{dt^2}. \tag{2.7}$$

The energy stored in a mass when it is moving is the kinetic energy which is dependent on the velocity of the mass and is given by

$$E = \frac{1}{2}mv^2. \tag{2.8}$$

This energy is released when the mass stops.

Some examples of translational mechanical system models are given below.

Example 2.1

Figure 2.3 shows a simple mechanical translational system with a mass, spring and dashpot. A force F is applied to the system. Derive a mathematical model for the system.

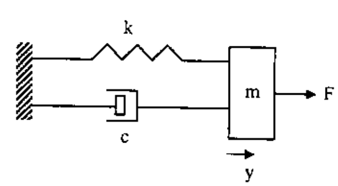

Figure 2.3 Mechanical system with mass, spring and dashpot

Solution

As shown in Figure 2.3, the net force on the mass is the applied force minus the forces exerted by the spring and the dashpot. Applying Newton's second law, we can write

$$F - ky - c\frac{dy}{dt} = m\frac{d^2y}{dt^2} \tag{2.9}$$

or

$$F = m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky. \tag{2.10}$$

Equation (2.10) is usually written in the form

$$F = m\ddot{y} + c\dot{y} + ky. \tag{2.11}$$

Taking the Laplace transform of (2.11), we can derive the transfer function of the system as

$$F(s) = ms^2Y(s) + csY(s) + kY(s)$$

or

$$\frac{Y(s)}{F(s)} = \frac{1}{ms^2 + cs + k}. \tag{2.12}$$

The transfer function in (2.12) is represented by the block diagram shown in Figure 2.4.

Example 2.2

Figure 2.5 shows a mechanical system with two masses and two springs. Drive an expression for the mathematical model of the system.

Solution

Applying Newton's second law to the mass m_1,

$$-k_2(y_1 - y_2) - c\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) - k_1y_1 = m_1\frac{d^2y_1}{dt^2}, \tag{2.13}$$

and for the mass m_2,

$$F - k_2(y_2 - y_1) - c\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) = m_2\frac{d^2y_2}{dt^2}, \tag{2.14}$$

we can write (2.13) and (2.14) as

$$m_1\ddot{y}_1 + c\dot{y}_1 - c\dot{y}_2 + (k_1 + k_2)y_1 - k_2y_2 = 0, \tag{2.15}$$

$$m_2\ddot{y}_2 + c\dot{y}_2 - c\dot{y}_1 + k_2y_2 - k_2y_1 = F. \tag{2.16}$$

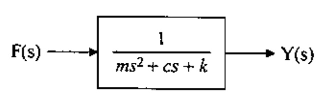

Figure 2.4 Block diagram of the simple mechanical system

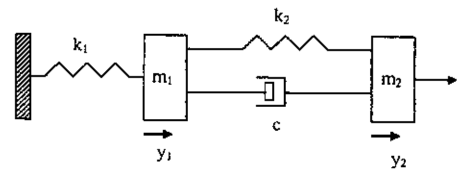

Figure 2.5 Example mechanical system

Equations (2.15) and (2.16) can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F \end{bmatrix}.$$

Example 2.3

Figure 2.6 shows a mechanical system with two masses, and forces applied to each mass. D an expression for the mathematical model of the system.

Solution

Applying Newton's second law to the mass m_1,

$$F_1 - k(y_1 - y_2) - c\left(\frac{dy_1}{dt} - \frac{dy_2}{dt}\right) = m_1\frac{d^2y_1}{dt^2},$$

and to the mass m_2,

$$F_2 - k(y_2 - y_1) - c\left(\frac{dy_2}{dt} - \frac{dy_1}{dt}\right) = m_2\frac{d^2y_2}{dt^2},$$

we can write (2.13) and (2.14) as

$$m_1\ddot{y}_1 + c\dot{y}_1 - c\dot{y}_2 + ky_1 - ky_2 = F_1,$$

$$m_2\ddot{y}_2 + c\dot{y}_2 - c\dot{y}_1 + ky_2 - ky_1 = F_2.$$

Equations (2.20) and (2.21) can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}.$$

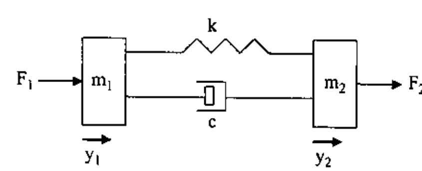

Figure 2.6 Example mechanical system

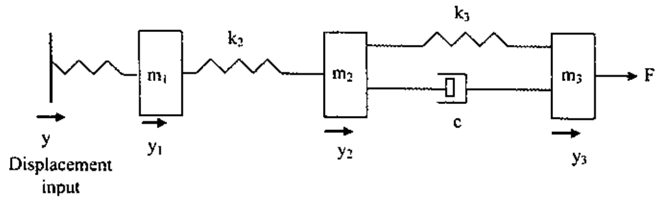

Figure 2.7 Example mechanical system

Example 2.4

Figure 2.7 shows a mechanical system with three masses, two springs and a dashpot. A force is applied to mass m_3 and a displacement is applied to spring k_1. Drive an expression for the mathematical model of the system.

Solution

Applying Newton's second law to the mass m_1,

$$k_1 y - k_1 y_1 - k_2(y_1 - y_2) + k_2 y_2 = m_1 \frac{d^2 y_1}{dt^2} \tag{2.23}$$

to the mass m_2,

$$-c\left(\frac{dy_2}{dt} - \frac{dy_3}{dt}\right) - k_2(y_2 - y_1) - k_3(y_2 - y_3) = m_2 \frac{d^2 y_2}{dt^2}, \tag{2.24}$$

and to the mass m_3,

$$F - c\left(\frac{dy_3}{dt} - \frac{dy_2}{dt}\right) - k_3(y_3 - y_2) = m_3 \frac{d^2 y_3}{dt^2}, \tag{2.25}$$

we can write (2.23)–(2.25) as

$$m_1 \ddot{y}_1 + (k_1 + k_2) y_1 - k_2 y_2 = k_1 y, \tag{2.26}$$

$$m_2 \ddot{y}_2 + c\dot{y}_2 - c\dot{y}_3 - k_2 y_1 + (k_2 + k_3) y_2 - k_3 y_3 = 0, \tag{2.27}$$

$$m_3 \ddot{y}_3 + c\dot{y}_3 - c\dot{y}_2 + k_3 y_3 - k_3 y_2 = F. \tag{2.28}$$

The above equations can be written in matrix form as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & c & -c \\ 0 & -c & c \end{bmatrix} \begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{y}_3 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} k_1 y \\ 0 \\ F \end{bmatrix}. \tag{2.29}$$

2.1.2 Rotational Mechanical Systems

The basic building blocks of rotational mechanical systems are the moment of inertia, torsion spring (or rotational spring) and rotary damper (Figure 2.8). The input to a rotational mechanical system may be the torque, T, and the output the rotational displacement, or angle, θ.

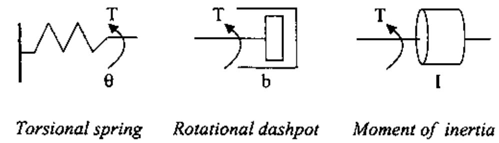

Figure 2.8 Rotational mechanical system components

A *rotational spring* is similar to a translational spring, but here the spring is twisted. T relationship between the applied torque, T, and the angle θ rotated by the spring is given b

$$T = k\theta, \quad (2.3$$

where θ is known as the rotational *stiffness* constant. In our modelling we are assuming tl the mass of the spring is negligible and the spring is linear.

The energy stored in a torsional spring when twisted by an angle θ is given by

$$E = \frac{1}{2}k\theta^2. \quad (2.3$$

A *rotary damper* element creates damping as it rotates. For example, when a disk rotates ir fluid we get a rotary damping effect. The relationship between the applied torque, T, and t angular velocity of the rotary damper is given by

$$T = c\omega = c\frac{d\theta}{dt}. \quad (2.3$$

In our modelling the mass of the rotary damper will be neglected, or will be assumed to negligible. A rotary damper does not store energy.

Moment of inertia refers to a rotating body with a mass. When a torque is applied to a bo with a moment of inertia we get an angular acceleration, and this acceleration rotates the boc The relationship between the applied torque, T, angular acceleration, a, and the moment inertia, I, I is given by

$$T = Ia = I\frac{d\omega}{dt} \quad (2.3$$

or, since $\omega = d\theta/dt$,

$$T = I\frac{d^2\theta}{dt^2}. \quad (2.3$$

The energy stored in a mass rotating with an angular velocity ω is given by

$$E = \frac{1}{2}I\omega^2. \quad (2.3$$

Some examples of rotational system models are given below.

Example 2.5

A disk of moment of inertia I is rotated (see Figure 2.9) with an applied torque of T. The di is fixed at one end through an elastic shaft. Assuming that the shaft can be modelled witl

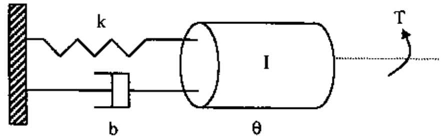

Figure 2.9 Rotational mechanical system

›tational dashpot and a rotational spring, derive an equation for the mathematical model of ıis system.

›lution

he damper torque and spring torque oppose the applied torque. If θ is the angular displacement om the equilibrium, we can write

$$T - b\frac{d\theta}{dt} - k\theta = I\frac{d^2\theta}{dt^2} \tag{2.36}$$

$$I\frac{d^2\theta}{dt^2} + b\frac{d\theta}{dt} + k\theta = T. \tag{2.37}$$

quation (2.37) is normally written in the form

$$I\ddot{\theta} + b\dot{\theta} + k\theta = T. \tag{2.38}$$

xample 2.6

igure 2.10 shows a rotational mechanical system with two moments of inertia and a torque ›plied to each one. Derive a mathematical model for the system.

olution

or the system shown in Figure 2.10 we can write the following equations: for disk 1,

$$T_1 - k(\theta_1 - \theta_2) - b\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right) = I_1\frac{d^2\theta_1}{dt^2}; \tag{2.39}$$

ıd for disk 2,

$$T_2 - k(\theta_2 - \theta_1) - b\left(\frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}\right) = I_2\frac{d^2\theta_2}{dt^2}. \tag{2.40}$$

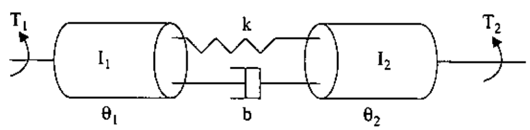

Figure 2.10 Rotational mechanical system

Equations (2.39) and (2.40) can be written as

$$I_1\ddot{\theta}_1 + b\dot{\theta}_1 - b\dot{\theta}_2 + k\theta_1 - k\theta_2 = T_1 \tag{2.41}$$

and

$$I_2\ddot{\theta}_2 - b\dot{\theta}_1 + b\dot{\theta}_2 - k\theta_1 + k\theta_2 = T_2. \tag{2.42}$$

Writing the equations in matrix form, we have

$$\begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} b & -b \\ -b & b \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \tag{2.43}$$

2.1.2.1 Rotational Mechanical Systems with Gear-Train

Gear-train systems are very important in many mechanical engineering systems. Figure 2.11 shows a simple gear-train, consisting of two gears, each connected to two masses with moments of inertia I_1 and I_2. Suppose that gear 1 has n_1 teeth and radius r_1, and that gear 2 has n_2 teeth and radius r_2. In this analysis we assume that the gears have no backlash, they are rigid bodies, and the moment of inertia of the gears is assumed to be negligible.

The rotational displacement of the two gears depends on their radii and is given by the relationship

$$r_1\theta_1 = r_2\theta_2 \tag{2.44}$$

or

$$\theta_2 = \frac{r_1}{r_2}\theta_1, \tag{2.45}$$

where θ_1 and θ_2 are the rotational displacements of gear 1 and gear 2, respectively.

The ratio of the teeth numbers is equal to the ratio of the radii and is given by

$$\frac{r_1}{r_2} = \frac{n_1}{n_2} = n, \tag{2.46}$$

where n is the gear teeth ratio.

Assuming that a torque T is applied to the system, we can write

$$I_1\frac{d^2\theta_1}{dt^2} = T - T_1 \tag{2.47}$$

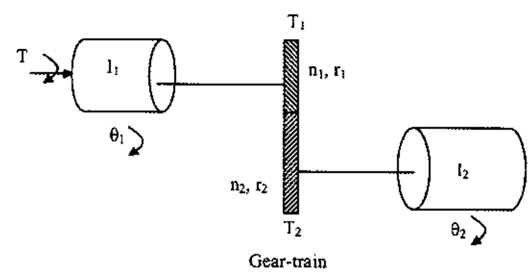

Figure 2.11 A two gear-train system

nd

$$I_2\frac{d^2\theta_2}{dt^2} = T_2. \tag{2.48}$$

quating the power transmitted by the gear-train,

$$T_1\frac{d\theta_1}{dt} = T_2\frac{d\theta_2}{dt} \quad \text{or} \quad \frac{T_1}{T_2} = \frac{d\theta_2/dt}{d\theta_1/dt} = n. \tag{2.49}$$

ubstituting (2.49) into (2.47), we obtain

$$I_1\frac{d^2\theta_1}{dt^2} = T - nT_2 \tag{2.50}$$

r

$$I_1\frac{d^2\theta_1}{dt^2} = T - n\left(I_2\frac{d^2\theta_2}{dt^2}\right); \tag{2.51}$$

ien, since $\theta_2 = n\theta_1$, we obtain

$$(I_1 + n^2 I_2)\frac{d^2\theta_1}{dt^2} = T. \tag{2.52}$$

is clear from (2.52) that the moment of inertia of the load, I_2, is reflected to the other side of ie gear-train as $n^2 I_2$.

An example of a system coupled with a gear-train is given below.

xample 2.7

igure 2.12 shows a rotational mechanical system coupled with a gear-train. Derive an expres-on for the model of the system.

olution

ssuming that a torque T is applied to the system, we can write

$$I_1\frac{d^2\theta_1}{dt^2} + b_1\frac{d\theta_1}{dt} + k_1\theta_1 = T - T_1 \tag{2.53}$$

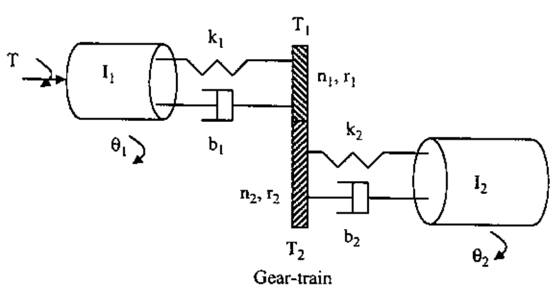

Figure 2.12 Mechanical system with gear-train

and

$$I_2\frac{d^2\theta_2}{dt^2} + b_2\frac{d\theta_2}{dt} + k_2\theta_2 = T_2. \tag{2.54}$$

Equating the power transmitted by the gear-train,

$$T_1\frac{d\theta_1}{dt} = T_2\frac{d\theta_2}{dt} \quad \text{or} \quad \frac{T_1}{T_2} = \frac{d\theta_2/dt}{d\theta_1/dt} = n. \tag{2.55}$$

Substituting (2.55) into (2.53), we obtain

$$I_1\frac{d^2\theta_1}{dt^2} + b_1\frac{d\theta_1}{dt} + k_1\theta_1 = T - nT_2 \tag{2.56}$$

or

$$I_1\frac{d^2\theta_1}{dt^2} + b_1\frac{d\theta_1}{dt} + k_1\theta_1 = T - n\left(I_2\frac{d^2\theta_2}{dt^2} + b_2\frac{d\theta_2}{dt} + k_2\theta_2\right). \tag{2.57}$$

Since $\theta_2 = n\theta_1$, this gives

$$(I_1 + n^2 I_2)\frac{d^2\theta_1}{dt^2} + (b_1 + n^2 b_2)\frac{d\theta_1}{dt} + (k_1 + n^2 k_2)\theta_1 = T. \tag{2.58}$$

2.2 ELECTRICAL SYSTEMS

The basic building blocks of electrical systems are the resistor, inductor and capacitor (Figure 2.13). The input to an electrical system may be the voltage, V, and current, i.

The relationship between the voltage across a *resistor* and the current through it is given by

$$V_r = Ri, \tag{2.59}$$

where R is the resistance.

For an *inductor*, the potential difference across the inductor depends on the rate of change of current through the inductor, given by

$$v_L = L\frac{di}{dt}, \tag{2.60}$$

where L is the inductance. Equation (2.60) can also be written as

$$i = \frac{1}{L}\int v_L dt. \tag{2.61}$$

The energy stored in an inductor is given by

$$E = \frac{1}{2}Li^2. \tag{2.62}$$

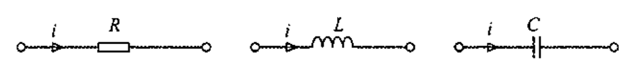

Figure 2.13 Electrical system components

The potential difference across a *capacitor* depends on the charge the plates hold, and is given by

$$v_C = \frac{q}{C}. \tag{2.63}$$

The relationship between the current through the capacitor and the voltage across it is given by

$$i = C\frac{dv_C}{dt} \tag{2.64}$$

or

$$v_C = \frac{1}{C}\int i dt. \tag{2.65}$$

The energy stored in a capacitor depends on the capacitance and the voltage across the capacitor and is given by

$$E = \frac{1}{2}Cv_C^2. \tag{2.66}$$

Electrical circuits are modelled using Kirchhoff's laws. There are two laws: Kirchhoff's current aw and Kirchhoff's voltage law. To apply these laws effectively, a sign convention should be ɜmployed.

Kirchhoff's current law The sum of the currents at a node in a circuit is zero, i.e. the total :urrent flowing into any junction in a circuit is equal to the total current leaving the junction.

Figure 2.14 shows the sign convention that can be employed when using Kirchhoff's current aw. We can write

$$i_1 + i_2 + i_3 = 0$$

or the circuit in Figure 2.14(a),

$$-(i_1 + i_2 + i_3) = 0$$

or the circuit in Figure 2.14(b) and

$$i_1 + i_2 - i_3 = 0$$

or the circuit in Figure 2.14(c).

Kirchhoff's voltage law The sum of voltages around any loop in a circuit is zero, i.e. in a rcuit containing a source of electromotive force (e.m.f.), the algebraic sum of the potential ·ops across each circuit element is equal to the algebraic sum of the applied e.m.f.s.

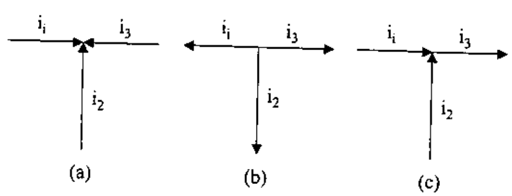

Figure 2.14 Applying Kirchhoff's current law

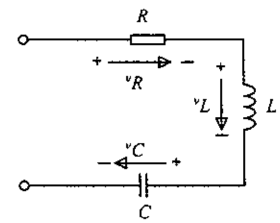

Figure 2.15 Applying Kirchhoff's voltage law

It is important to observe the sign convention when applying Kirchhoff's voltage law. An example circuit is given in Figure 2.15. For this circuit we can write.

$$v_R + v_L + v_C = 0.$$

Some examples of the modelling of electrical circuits are given below.

Example 2.8

Figure 2.16 shows a simple electrical circuit consisting of a resistor, an inductor and a capacitor. A voltage V_a is applied to the circuit. Derive an expression for the mathematical model for this system.

Solution

Applying Kirchhoff's voltage law, we can write

$$v_R + v_L + v_C = V_a$$

or

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i dt = V_a. \tag{2.67}$$

For the capacitor we can write

$$i = C\frac{dv_C}{dt}.$$

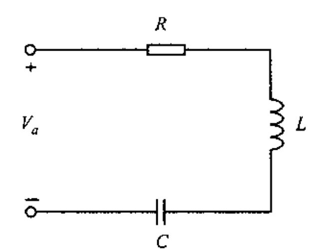

Figure 2.16 Simple electrical circuit

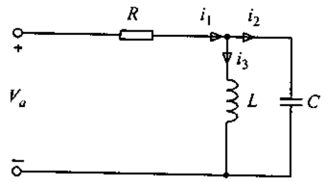

Figure 2.17 Electrical circuit for the Example 2.9

ubstituting this into (2.67), we obtain

$$RC\frac{dv_C}{dt} + LC\frac{d^2v_C}{dt^2} + v_C = V_a \tag{2.68}$$

hich can also be written as

$$LC\ddot{v}_C + RC\,v_C \dot{+} v_C = V_a. \tag{2.69}$$

xample 2.9

igure 2.17 shows an electrical circuit consisting of a capacitor, an inductor and a resistor. he inductor and the capacitor are connected in parallel. A voltage V_a is applied to the circuit. erive a mathematical model for the system.

olution

pplying Kirchhoff's current law, we can write

$$i_1 = i_2 + i_3. \tag{2.70}$$

ow, the potential difference across the inductor and also across the capacitor is v_C. Similarly, e potential difference across the resistor is $V_a - v_C$. Thus,

$$i_i = \frac{V_a - v_C}{R}, \tag{2.71}$$

$$i_2 = C\frac{dv_C}{dt}, \tag{2.72}$$

$$i_3 = \frac{1}{L}\int v_C dt. \tag{2.73}$$

ombining (2.70)–(2.73) we obtain,

$$\frac{V_a - v_C}{R} = C\frac{dv_C}{dt} + \frac{1}{L}\int v_C dt$$

$$\frac{R}{L}\int v_C dt + RC\frac{dv_C}{dt} + v_C = V_a.$$

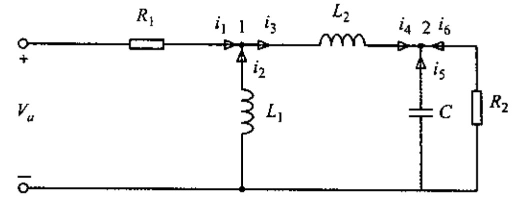

Figure 2.18 Circuit for Example 2.18

Example 2.10

Figure 2.18 shows an electrical circuit consisting of two inductors, two resistors and a capacitor. A voltage V_a is applied to the circuit. Derive an expression for the mathematical model for the circuit.

Solution

The circuit consists of two nodes and two loops. We can apply Kirchhoff's current law to the nodes. For node 1,

$$i_1 + i_2 + i_3 = 0$$

or

$$\frac{V_a - v_1}{R_1} + \frac{1}{L_1}\int (0 - v_1)dt + \frac{1}{L_2}\int (v_2 - v_1)dt = 0. \tag{2.74}$$

Differentiating (2.74) with respect to time, we obtain

$$\frac{\dot{V}_a}{R_1} - \frac{\dot{V}_1}{R_1} - \frac{v_1}{L_1} + \frac{v_2}{L_2} - \frac{v_1}{L_2} = 0$$

or

$$\frac{\dot{V}_a}{R_1} = \frac{\dot{V}_1}{R_1} + \left(\frac{1}{L_1} + \frac{1}{L_2}\right) v_1 - \frac{v_2}{L_2}. \tag{2.75}$$

For node 2,

$$i_4 + i_5 + i_6 = 0$$

or

$$\frac{1}{L_2}\int (v_1 - v_2)dt + C\frac{d(0 - v_2)}{dt} + \frac{0 - v_2}{R_2}. \tag{2.76}$$

Differentiating (2.76) with respect to time, we obtain

$$\frac{v_1 - v_2}{L_2} - C\ddot{v}_2 - \frac{\dot{v}_2}{R_2} = 0$$

which can be written as

$$C\ddot{v}_2 + \frac{\dot{v}_2}{R_2} - \frac{v_1}{L_2} + \frac{v_2}{L_2} = 0. \tag{2.77}$$

Equations (2.75) and (2.76) describe the operation of the circuit. These two equations can be represented in matrix form as

$$\begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix}\begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \end{bmatrix} + \begin{bmatrix} 1/R_1 & 0 \\ 0 & 1/R_2 \end{bmatrix}\begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} + \begin{bmatrix} 1/L_1 + 1/L_2 & -1/L_2 \\ -1/L_2 & 1/L_2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \dot{V}_a \\ R_1 \\ 0 \end{bmatrix}.$$

2.3 ELECTROMECHANICAL SYSTEMS

Electromechanical systems such as electric motors and electric pumps are used in most industrial and commercial applications. Figure 2.19 shows a simple d.c. motor circuit. The torque produced by the motor is proportional to the applied current and is given by

$$T = k_t i, \tag{2.78}$$

where T is the torque produced, k_t is the torque constant and i is the motor current. Assuming there is no load connected to the motor, the motor torque can be expressed as

$$T = I\frac{d\omega}{dt}$$

or

$$I\frac{d\omega}{dt} = k_t i. \tag{2.79}$$

As the motor armature coil is rotating in a magnetic field there will be a *back e.m.f.* induced in the coil in such a way as to oppose the change producing it. This e.m.f. is proportional to the angular speed of the motor and is given by:

$$v_b = k_e \omega, \tag{2.80}$$

where v_b is the back e.m.f., k_e is the back e.m.f. constant, and ω is the angular speed of the motor.

Using Kirchhoff's voltage law, we can write the following equation for the motor circuit:

$$V_a - v_b = L\frac{di}{dt} + Ri, \tag{2.81}$$

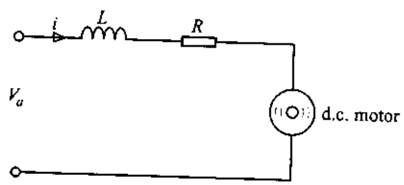

Figure 2.19 Simple d.c. motor

where V_a is the applied voltage, and L and R are the inductance and the resistance of the armature circuit, respectively. From (2.79),

$$i = \frac{I}{k_t}\frac{d\omega}{dt} \tag{2.82}$$

Combining (2.80)–(2.82), we obtain

$$\frac{LI}{k_t}\frac{d^2\omega}{dt^2} + \frac{RI}{k_t}\frac{d\omega}{dt} + k_e\omega = V_a. \tag{2.83}$$

Equation (2.83) is the model for a simple d.c. motor, describing the change of the angular velocity with the applied voltage. In many applications the motor inductance is small and can be neglected. The model then becomes

$$\frac{RI}{k_t}\frac{d\omega}{dt} + k_e\omega = V_a.$$

Models of more complex d.c. motor circuits are given in the following examples.

Example 2.11

Figure 2.20 shows a d.c. motor circuit with a load connected to the motor shaft. Assume that the shaft is rigid, has negligible mass and has no torsional spring effect or rotational damping associated with it. Derive an expression for the mathematical model for the system.

Solution

Since the shaft is assumed to be massless, the moments of inertia of the rotor and the load can be combined into I, where

$$I = I_M + I_L$$

where I_M is the moment of inertia of the motor and I_L is the moment of inertia of the load.

Using Kirchhoff's voltage law, we can write the following equation for the motor circuit:

$$V_a - v_b = L\frac{di}{dt} + Ri, \tag{2.84}$$

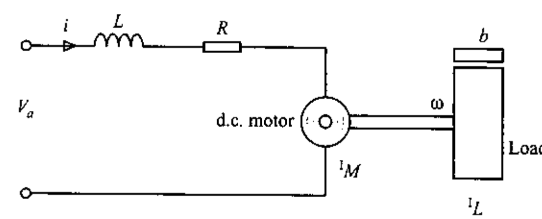

Figure 2.20 Direct current motor circuit for Example 2.11

where V_a is the applied voltage and L and R are the inductance and the resistance of the armature circuit, respectively. Substituting (2.80), we obtain

$$V_a = L\frac{di}{dt} + Ri + k_e\omega$$

or

$$V_a = L\dot{i} + Ri + k_e\dot{\theta}. \tag{2.85}$$

We can also write the torque equation as

$$T + T_L - b\omega = I\frac{d\omega}{dt}.$$

Using (2.78),

$$I\frac{d\omega}{dt} + b\omega - k_t i = T_L$$

or

$$I\ddot{\theta} + b\dot{\theta} - k_t i = T_L. \tag{2.86}$$

Equations (2.85) and (2.86) describe the model of the circuit. These two equations can be represented in matrix form as

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \ddot{\theta} \\ \dot{i} \end{bmatrix} + \begin{bmatrix} b & 0 \\ k_e & L \end{bmatrix}\begin{bmatrix} \dot{\theta} \\ i \end{bmatrix} + \begin{bmatrix} 0 & -k_t \\ 0 & R \end{bmatrix}\begin{bmatrix} \theta \\ i \end{bmatrix} = \begin{bmatrix} T_L \\ V_a \end{bmatrix}.$$

2.4 FLUID SYSTEMS

Gases and liquids are collectively referred to as fluids. Fluid systems are used in many industrial as well as commercial applications. For example, liquid level control is a well-known application of liquid systems. Similarly, gas systems are used in robotics and in industrial movement control applications.

In this section, we shall look at the models of simple liquid systems (or hydraulic systems).

2.4.1 Hydraulic Systems

The basic elements of hydraulic systems are *resistance*, *capacitance* and *inertance* (see Figure 2.21). These elements are similar to their electrical equivalents of resistance, capacitance and inductance. Similarly, electrical current is equivalent to volume flow rate, and the potential difference in electrical circuits is similar to pressure difference in hydraulic systems.

Hydraulic resistance

Hydraulic resistance occurs whenever there is a pressure difference, such as liquid flowing from a pipe of one diameter to to one of a different diameter. If the pressures at either side of a hydraulic resistance are p_1 and p_2, then the hydraulic resistance R is defined as

$$p_1 - p_2 = Rq$$

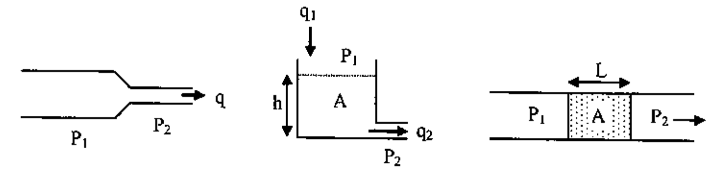

Figure 2.21 Hydraulic system elements

where q is the volumetric flow rate of the fluid.

Hydraulic capacitance

Hydraulic capacitance is a measure of the energy storage in a hydraulic system. An example of hydraulic capacitance is a tank which stores energy in the form of potential energy. Consider the tank shown in Figure 2.21(b). If q_1 and q_2 are the inflow and outflow, respectively, and V is the volume of the fluid inside the tank, we can write

$$q_1 - q_2 = \frac{dV}{dt} = A\frac{dh}{dt}. \tag{2.87}$$

Now, the pressure difference is given by

$$p_1 - p_2 = h\rho g = p$$

or

$$h = \frac{p}{\rho g}. \tag{2.88}$$

Substituting in (2.87), we obtain

$$q_1 - q_2 = \frac{A}{\rho g}\frac{dp}{dt}. \tag{2.89}$$

Writing (2.89) as

$$q_1 - q_2 = C\frac{dp}{dt}, \tag{2.90}$$

we then arrive at the definition of hydraulic capacitance:

$$C = \frac{A}{\rho g}. \tag{2.91}$$

Note that (2.90) is similar to the expression for a capacitor and can be written as

$$p = \frac{1}{C}\int (q_1 - q_2)dt. \tag{2.92}$$

Hydraulic inertance

Hydraulic inertance is similar to the inductance in electrical systems and is derived from the inertia force required to accelerate fluid in a pipe.

Let $p_1 - p_2$ be the pressure drop that we want to accelerate in a cross-sectional area of A, where m is the fluid mass and v is the fluid velocity. Applying Newton's second law, we can write

$$m\frac{dv}{dt} = A(p_1 - p_2). \tag{2.93}$$

If the pipe length is L, then the mass is given by

$$m = L\rho A.$$

We can now write (2.93) as

$$L\rho A\frac{dv}{dt} = A(p_1 - p_2)$$

or

$$p_1 - p_2 = L\rho\frac{dv}{dt}, \tag{2.94}$$

but the rate of flow is given by $q = Av$, so (2.94) can be written as

$$p_1 - p_2 = \frac{L\rho}{A}\frac{dq}{dt}. \tag{2.95}$$

The inertance I is then defined as

$$I = \frac{L\rho}{A},$$

and thus the relationship between the pressure difference and the flow rate is similar to the relationship between the potential difference and the current flow in an inductor, i.e.

$$p_1 - p_2 = I\frac{dq}{dt}. \tag{2.96}$$

Models of some hydraulic systems are given below.

Example 2.12

Figure 2.22 shows a liquid level system where liquid enters a tank at the rate of q_i and leaves at the rate of q_o through an orifice. Derive the mathematical model for the system, showing the relationship between the height h of the liquid and the input flow rate q_i.

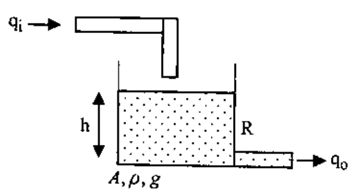

Figure 2.22 Liquid level system

Solution

From (2.89),

$$q_i - q_o = \frac{A}{\rho g}\frac{dp}{dt}$$

or

$$q_i = \frac{A}{\rho g}\frac{dp}{dt} + q_o. \tag{2.97}$$

Recalling that

$$p = h\rho g,$$

(2.97) becomes

$$q_i = A\frac{dh}{dt} + q_o. \tag{2.98}$$

Since

$$p_1 - p_2 = Rq_o,$$

so that

$$q_o = \frac{p_1 - p_2}{R} = \frac{h\rho g}{R},$$

substituting in (2.98) gives

$$q_i = A\frac{dh}{dt} + \frac{\rho g}{R}h. \tag{2.99}$$

Equation (2.99) shows the variation of the height of the water with the inflow rate. If we take the Laplace transform of both sides, we obtain

$$q_i(s) = Ash(s) + \frac{\rho g}{R}h(s)$$

and the transfer function of the system can be written as

$$\frac{h(s)}{q_i(s)} = \frac{1}{As + \rho g/R};$$

the block diagram is shown in Figure 2.23.

Example 2.13

Figure 2.24 shows a two-tank liquid level system where liquid enters the first tank at the rate of q_i and then flows to the second tank at the rate of q_1 through an orifice R_1. Water then leaves

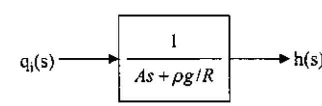

Figure 2.23 Block diagram of the liquid level system

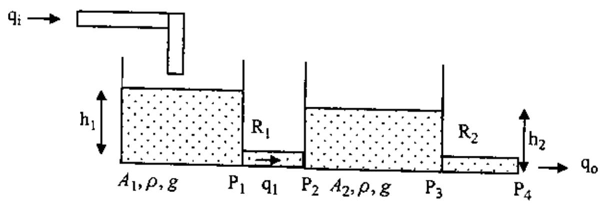

Figure 2.24 Two tank liquid level system

the second tank at the rate of q_o through an orifice of R_2. Derive the mathematical model for the system.

Solution

The solution is similar to Example 2.12, but we have to consider both tanks.
For tank 1,

$$q_i - q_1 = \frac{A_1}{\rho g}\frac{dp}{dt}$$

or

$$q_i = \frac{A_1}{\rho g}\frac{dp}{dt} + q_1. \tag{2.100}$$

But

$$p = h\rho g,$$

thus (2.100) becomes

$$q_i = A_1\frac{dh_1}{dt} + q_1 \tag{2.101}$$

Since

$$p_1 - p_2 = R_1 q_1$$

or

$$q_1 = \frac{p_1 - p_2}{R_1} = \frac{h_1\rho g - h_2\rho g}{R_1},$$

we have

$$q_i = A_1\frac{dh_1}{dt} + \frac{\rho g h_1}{R_1} - \frac{\rho g h_2}{R_1}. \tag{2.102}$$

For tank 2,

$$q_1 - q_0 = \frac{A_2}{\rho g}\frac{dp}{dt}, \tag{2.103}$$

and with

$$p = h\rho g$$

(2.103) becomes

$$q_1 - q_o = A_2\frac{dh_2}{dt}. \tag{2.104}$$

But

$$q_1 = \frac{p_1 - p_2}{R_1} \quad \text{and} \quad q_o = \frac{p_2 - p_3}{R_2}$$

so

$$q_1 - q_0 = \frac{p_1 - p_2}{R1} - \frac{p_2 - p_3}{R_2} = \frac{h_1\rho g - h_2\rho g}{R_1} - \frac{h_2\rho g}{R_2}.$$

Substituting in (2.104), we obtain

$$A_2\frac{dh_2}{dt} - \frac{\rho g h_1}{R_1} + \left(\frac{1}{R_1} + \frac{1}{R_2}\right)\rho g h_2 = 0. \tag{2.105}$$

Equations (2.102) and (2.105) describe the behaviour of the system. These two equations can be represented in matrix form as

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} \dot{h}_1 \\ \dot{h}_2 \end{bmatrix} + \begin{bmatrix} \rho g/R_1 & -\rho g/R_1 \\ -\rho g/R_1 & \rho g/R_1 + \rho g/R_2 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} q_i \\ 0 \end{bmatrix}.$$

2.5 THERMAL SYSTEMS

Thermal systems are encountered in chemical processes, heating, cooling and air conditioning systems, power plants, etc. Thermal systems have two basic components: thermal resistance and thermal capacitance. Thermal resistance is similar to the resistance in electrical circuits. Similarly, thermal capacitance is similar to the capacitance in electrical circuits. The across variable, which is measured across an element, is the temperature, and the through variable is the heat flow rate. In thermal systems there is no concept of inductance or inertance. Also, the product of the across variable and the through variable is not equal to power. The mathematical modelling of thermal systems is usually complex because of the complex distribution of the temperature. Simple approximate models can, however, be derived for the systems commonly used in practice.

Thermal resistance, R, is the resistance offered to the heat flow, and is defined as:

$$R = \frac{T_2 - T_1}{q}, \tag{2.106}$$

where T_1 and T_2 are the temperatures, and q is the heat flow rate.

Thermal capacitance is a measure of the energy storage in a thermal system. If q_1 is the heat flowing into a body and q_2 is the heat flowing out then the difference $q_1 - q_2$ is stored by the body, and we can write

$$q_1 - q_2 = mc\frac{dT}{dt}, \tag{2.107}$$

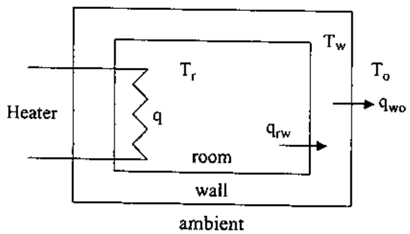

Figure 2.25 Simple thermal system

ıere m is the mass and c is the specific heat capacity of the body. If we let the heat capacity denoted by C, then

$$q_1 - q_2 = C\frac{dT}{dt}, \tag{2.108}$$

ıere $C = mc$.
An example thermal system model is given below.

:ample 2.14

ʒure 2.25 shows a room heated with an electric heater. The inside of the room is at temperature and the walls are assumed to be at temperature T_w. If the outside temperature is T_o, develop nodel of the system to show the relationship between the supplied heat q and the room nperature T_r.

lution

e heat flow from inside the room to the walls is given by

$$q_{rw} = \frac{T_r - T_w}{R_r}, \tag{2.109}$$

ere R_r is the thermal resistance of the room.
Similarly, the heat flow from the walls to the outside is given by

$$q_{wo} = \frac{T_w - T_o}{R_w}, \tag{2.110}$$

ere R_w is the thermal resistance of the walls.
Jsing (2.108) and (2.109), we can write

$$q - \left(\frac{T_r - T_w}{R_r}\right) = C_1\frac{dT_r}{dt}$$

$$C_1\dot{T}_r + \frac{T_r}{R_r} - \frac{T_w}{R_r} = q. \tag{2.111}$$

Also, using (2.108) and (2.110), we can write

$$\left(\frac{T_r - T_w}{R_r}\right) - \left(\frac{T_w - T_o}{R_w}\right) = C_2\frac{dT_w}{dt}$$

or

$$C_2\dot{T}_w - \frac{T_r}{R_r} + \left(\frac{1}{R_r} + \frac{1}{R_w}\right)T_w = \frac{T_o}{R_w}. \tag{2.112}$$

Equations (2.111) and (2.112) describe the behaviour of the system and they can be written in matrix form as

$$\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}\begin{bmatrix} \dot{T}_r \\ \dot{T}_w \end{bmatrix} + \begin{bmatrix} 1/R_r & -1/R_r \\ -1/R_r & 1/R_r + 1/R_w \end{bmatrix}\begin{bmatrix} T_r \\ T_w \end{bmatrix} = \begin{bmatrix} q \\ T_o/R_w \end{bmatrix}.$$

Example 2.15

Figure 2.26 shows a heated stirred tank thermal system. Liquid enters the tank at the temperature T_i with a flow rate of W. The water is heated inside the tank to temperature T. The temperature leaves the tank at the same flow rate of W. Derive a mathematical model for the system, assuming that there is no heat loss from the tank.

Solution

The following equation can be written for the conservation of energy:

$$Q_p + Q_i = Q_l + Q_o, \tag{2.113}$$

where Q_p is the heat supplied by the heater, Q_i is the heat flow via the liquid entering the tank, Q_l is the heat flow into the liquid and Q_o is the heat flow via the liquid leaving the tank.
Now,

$$Q_i = WC_pT_i \tag{2.114}$$

where W is the flow rate (kg/s), and C_p is the specific heat capacity of the liquid. Also

$$Q_o = WC_pT \tag{2.115}$$

and

$$Q_l = C\frac{dT}{dt}, \tag{2.116}$$

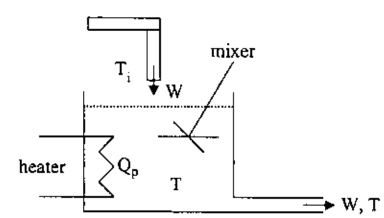

Figure 2.26 Heated stirred tank for Example 2.15

where C is the thermal capacity, i.e. $C = \rho V C_p$ and V is the volume of the tank. Substituting (2.114)–(2.116) into (2.113) gives

$$Q_p + WC_pT_i = C\frac{dT}{dt} + WC_pT$$

or

$$\frac{dT}{dt} = \frac{WC_p(T_i - T) + Q_p}{\rho V C_p}.$$

2.6 EXERCISES

1. Figure 2.27 shows a simple mechanical system consisting of a mass, spring and damper. Derive a mathematical model for the system, determine the transfer function, and draw the block diagram.
2. Consider the system of two massless springs shown in Figure 2.28. Derive a mathematical model for the system.
3. Three massless springs with the same stiffness constant are connected in series. Derive an expression for the equivalent spring stiffness constant.
4. Figure 2.29 shows a simple mechanical system. Derive an expression for the mathematical model for the system.
5. Figure 2.30 shows a rotational mechanical system. Derive an expression for the mathematical model for the system.
6. Two rotational springs are connected in parallel. Derive an expression for the equivalent spring stiffness constant.
7. Figure 2.31 shows a simple system with a gear-train. Derive an expression for the mathematical model for the system.
8. A simple electrical circuit is shown in Figure 2.32. Derive an expression for the mathematical model for the system.
9. Figure 2.33 shows an electrical circuit. Use Kirchhoff's laws to derive the mathematical model for the system.
10. A liquid level system is shown in Figure 2.34, where q_i and q_o are the inflow and outflow rates, respectively. The system has two fluid resistances, R_1 and R_2, in series. Derive an expression for the mathematical model for the system.
11. Figure 2.35 shows a liquid level system with three tanks. Liquid enters the first tank at the rate q_i and leaves the third tank at the rate q_o. Assume that all tanks have the same dimensions. Derive an expression for the mathematical model for this system.

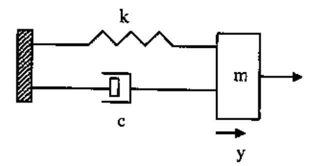

Figure 2.27 Simple mechanical system for Exercise 1

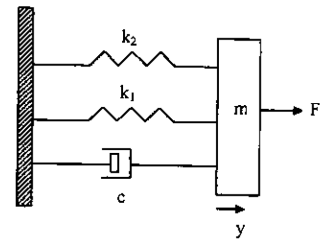

Figure 2.28 System of two massless springs for Exercise 2

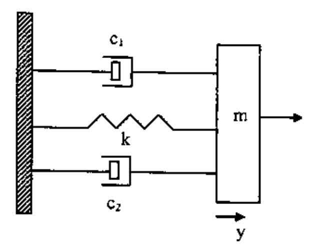

Figure 2.29 Simple mechanical system for Exercise 4

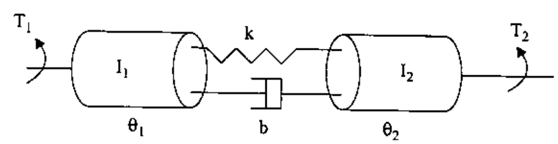

Figure 2.30 Simple mechanical system for Exercise 5

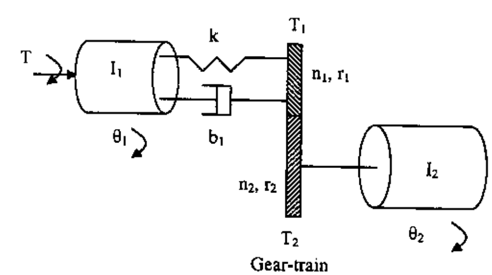

Figure 2.31 Simple system with a gear-train for Exercise 7

[Bell et al., 1992] Bell, G., Morrey, I., and Pugh, J. Software Engineering. Prentice Hall, Englewood Cliffs, NJ, 1992.

[Bennett, 1994] Bennett, S. Real-Time Computer Control: An Introduction. Prentice Hall, Englewood Cliffs, NJ, 1994.

[Bibbero, 1977] Bibbero, R.J. Microprocessors in Instruments and Control. John Wiley & Sons, Inc., New York, 1977.

6

Sampled Data Systems and the *z*-Transform

A sampled data system operates on discrete-time rather than continuous-time signals. A digital computer is used as the controller in such a system. A D/A converter is usually connected to the output of the computer to drive the plant. We will assume that all the signals enter and leave the computer at the same fixed times, known as the sampling times.

A typical sampled data control system is shown in Figure 6.1. The digital computer performs the controller or the compensation function within the system. The A/D converter converts the error signal, which is a continuous signal, into digital form so that it can be processed by the computer. At the computer output the D/A converter converts the digital output of the computer into a form which can be used to drive the plant.

6.1 THE SAMPLING PROCESS

A sampler is basically a switch that closes every T seconds, as shown in Figure 6.2. When a continuous signal $r(t)$ is sampled at regular intervals T, the resulting discrete-time signal is shown in Figure 6.3, where q represents the amount of time the switch is closed.

In practice the closure time q is much smaller than the sampling time T, and the pulses can be approximated by flat-topped rectangles as shown in Figure 6.4.

In control applications the switch closure time q is much smaller than the sampling time T and can be neglected. This leads to the ideal sampler with output as shown in Figure 6.5.

The ideal sampling process can be considered as the multiplication of a pulse train with a continuous signal, i.e.

$$r^*(t) = P(t)r(t), \tag{6.1}$$

where $P(t)$ is the delta pulse train as shown in Figure 6.6, expressed as

$$P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT); \tag{6.2}$$

thus,

$$r^*(t) = r(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{6.3}$$

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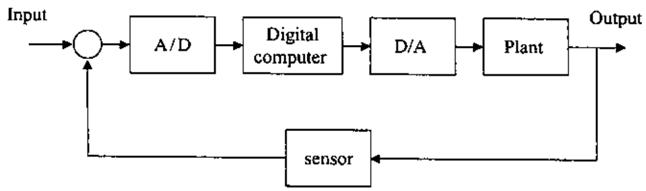

Figure 6.1 Sampled data control system

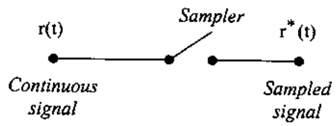

Figure 6.2 A sampler

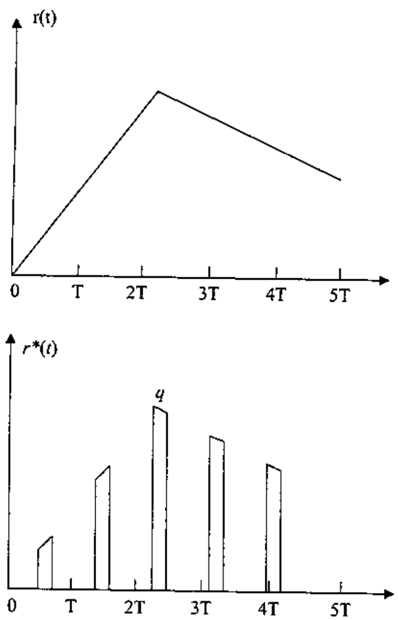

Figure 6.3 The signal $r(t)$ after the sampling operation

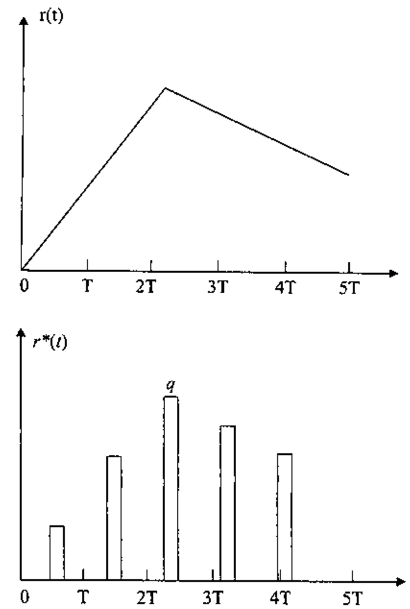

Figure 6.4 Sampled signal with flat-topped pulses

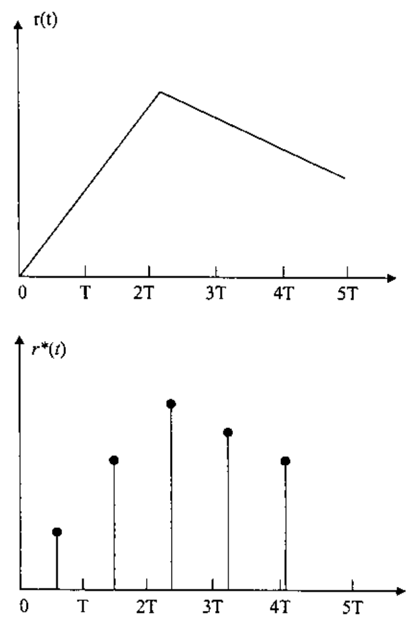

Figure 6.5 Signal $r(t)$ after ideal sampling

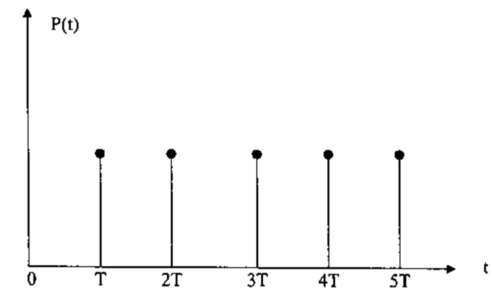

Figure 6.6 Delta pulse train

or

$$r^*(t) = \sum_{n=-\infty}^{\infty} r(nT)\delta(t - nT). \tag{6.4}$$

Now

$$r(t) = 0, \quad \text{for } t < 0, \tag{6.5}$$

and

$$r^*(t) = \sum_{n=0}^{\infty} r(nT)\delta(t - nT). \tag{6.6}$$

Taking the Laplace transform of (6.6) gives

$$R^*(s) = \sum_{n=0}^{\infty} r(nT)e^{-snT}. \tag{6.7}$$

Equation (6.7) represents the Laplace transform of a sampled continuous signal $r(t)$.

A D/A converter converts the sampled signal $r^*(t)$ into a continuous signal $y(t)$. The D/A can be approximated by a zero-order hold (ZOH) circuit as shown in Figure 6.7. This circuit remembers the last information until a new sample is obtained, i.e. the zero-order hold takes the value $r(nT)$ and holds it constant for $nT \le t < (n+1)T$, and the value $r(nT)$ is used during the sampling period.

The impulse response of a zero-order hold is shown in Figure 6.8. The transfer function of a zero-order hold is given by

$$G(t) = H(t) - H(t - T), \tag{6.8}$$

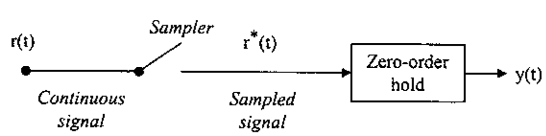

Figure 6.7 A sampler and zero-order hold

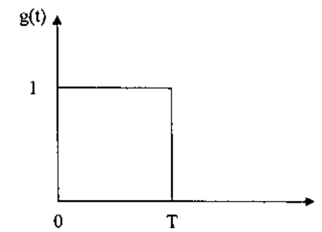

Figure 6.8 Impulse response of a zero-order hold

where $H(t)$ is the step function, and taking the Laplace transform yields

$$G(s) = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}. \tag{6.9}$$

A sampler and zero-order hold can accurately follow the input signal if the sampling time T is small compared to the transient changes in the signal. The response of a sampler and a zero-order hold to a ramp input is shown in Figure 6.9 for two different values of sampling period.

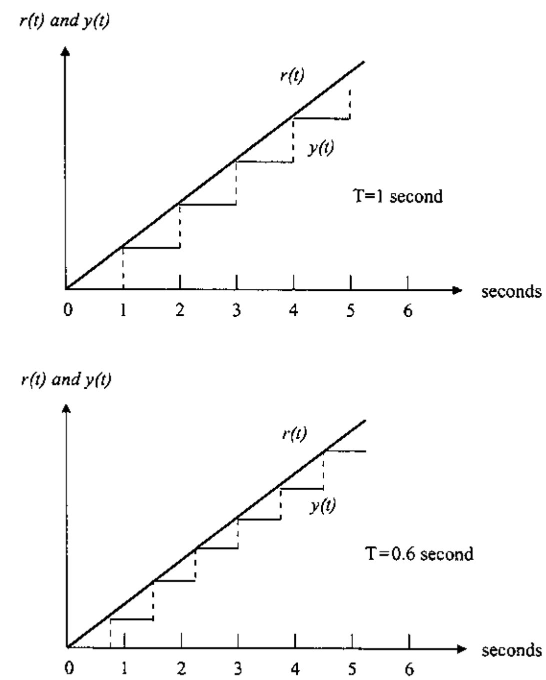

Figure 6.9 Response of a sampler and a zero-order hold for a ramp input

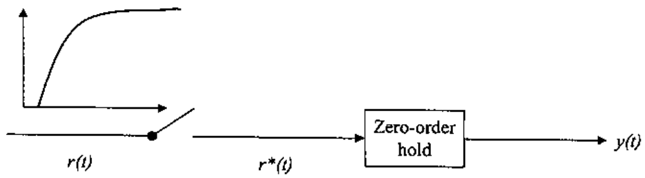

Figure 6.10 Ideal sampler and zero-order hold for Example 6.1

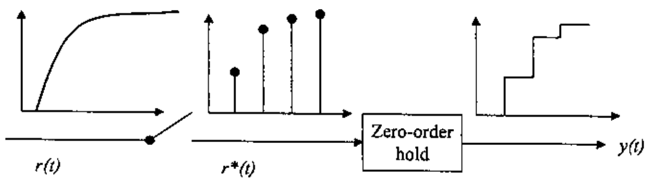

Figure 6.11 Solution for Example 6.1

Example 6.1

Figure 6.10 shows an ideal sampler followed by a zero-order hold. Assuming the input signal $r(t)$ is as shown in the figure, show the waveforms after the sampler and also after the zero-order hold.

Solution

The signals after the ideal sampler and the zero-order hold are shown in Figure 6.11.

6.2 THE z-TRANSFORM

Equation (6.7) defines an infinite series with powers of e^{-snT}. The z-transform is defined so that

$$Z = e^{sT}; \tag{6.10}$$

the z-transform of the function $r(t)$ is $Z[r(t)] = R(z)$ which, from (6.7), is given by

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n}. \tag{6.11}$$

Notice that the z-transform consists of an infinite series in the complex variable z, and

$$R(z) = r(0) + r(T)z^{-1} + r(2T)z^{-2} + r(3T)z^{-3} + \ldots,$$

i.e. the $r(nT)$ are the coefficients of this power series at different sampling instants.

The z-transformation is used in sampled data systems just as the Laplace transformation is used in continuous-time systems. The response of a sampled data system can be determined easily by finding the z-transform of the output and then calculating the inverse z-transform, just like the Laplace transform techniques used in continuous-time systems. We will now look at how we can find the z-transforms of some commonly used functions.

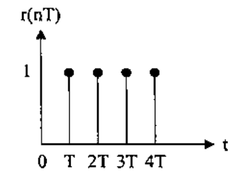

Figure 6.12 Unit step function

6.2.1 Unit Step Function

Consider a unit step function as shown in Figure 6.12, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ 1, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + \ldots$$

or

$$R(z) = \frac{z}{z-1}, \quad \text{for } |z| > 1.$$

6.2.2 Unit Ramp Function

Consider a unit ramp function as shown in Figure 6.13, defined by

$$r(nT) = \begin{cases} 0, & n < 0, \\ nT, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} nTz^{-n} = Tz^{-1} + 2Tz^{-2} + 3Tz^{-3} + 4Tz^{-4} + \ldots$$

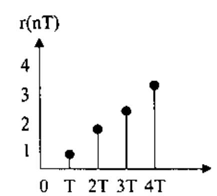

Figure 6.13 Unit ramp function

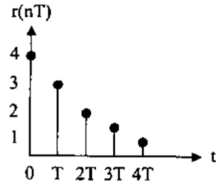

Figure 6.14 Exponential function

or

$$R(z) = \frac{Tz}{(z-1)^2}, \qquad \text{for } |z| > 1.$$

6.2.3 Exponential Function

Consider the exponential function shown in Figure 6.14, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ e^{-anT}, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} e^{-anT}z^{-n} = 1 + e^{-aT}z^{-1} + e^{-2aT}z^{-2} + e^{-3aT}z^{-3} + \ldots$$

or

$$R(z) = \frac{1}{1 - e^{-aT}z^{-1}} = \frac{z}{z - e^{-aT}}, \qquad \text{for } |z| < e^{-aT}. \tag{6.12}$$

6.2.4 General Exponential Function

Consider the general exponential function

$$r(n) = \begin{cases} 0, & n < 0, \\ p^n, & n \geq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} p^n z^{-n} = 1 + pz^{-1} + p^2z^{-2} + p^3z^{-3} + \ldots$$

or

$$R(z) = \frac{z}{z-p}, \qquad \text{for } |z| < |p|.$$

Similarly, we can show that

$$R(p^{-k}) = \frac{z}{z - p^{-1}}.$$

6.2.5 Sine Function

Consider the sine function, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ \sin n\omega T. & n \geq 0. \end{cases}$$

Recall that

$$\sin x = \frac{e^{jx} - e^{-jx}}{2j},$$

so that

$$r(nT) = \frac{e^{jn\omega T} - e^{-jn\omega T}}{2j} = \frac{e^{jn\omega T}}{2j} - \frac{e^{-jn\omega T}}{2j}. \tag{6.13}$$

But we already know from (6.12) that the z-transform of an exponential function is

$$R(e^{-anT}) = R(z) = \frac{z}{z - e^{-aT}}.$$

Therefore, substituting in (6.13) gives

$$R(z) = \frac{1}{2j}\left(\frac{z}{z - e^{j\omega T}} - \frac{z}{z - e^{-j\omega T}}\right) = \frac{1}{2j}\left(\frac{z(e^{j\omega T} - e^{-j\omega T}}{z^2 - z(e^{j\omega T} + e^{-j\omega T}) + 1}\right)$$

or

$$R(z) = \frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}.$$

6.2.6 Cosine Function

Consider the cosine function, defined as

$$r(nT) = \begin{cases} 0, & n < 0, \\ \cos n\omega T, & n \geq 0. \end{cases}$$

Recall that

$$\cos x = \frac{e^{jx} + e^{-jx}}{2},$$

so that

$$r(nT) = \frac{e^{jn\omega T} + e^{-jn\omega T}}{2} = \frac{e^{jn\omega T}}{2} + \frac{e^{-jn\omega T}}{2}. \tag{6.14}$$

But we already know from (6.12) that the z-transform of an exponential function is

$$R(e^{-anT}) = R(z) = \frac{z}{z - e^{-aT}}.$$

Therefore, substituting in (6.14) gives

$$R(z) = \frac{1}{2}\left(\frac{z}{z - e^{j\omega T}} + \frac{z}{z - e^{-j\omega T}}\right)$$

or

$$R(z) = \frac{z(z - \cos \omega T)}{z^2 - 2z\cos \omega T + 1}.$$

6.2.7 Discrete Impulse Function

Consider the discrete impulse function defined as

$$\delta(n) = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = 1.$$

6.2.8 Delayed Discrete Impulse Function

The delayed discrete impulse function is defined as

$$\delta(n-k) = \begin{cases} 1, & n = k > 0, \\ 0, & n \neq k. \end{cases}$$

From (6.11),

$$R(z) = \sum_{n=0}^{\infty} r(nT)z^{-n} = \sum_{n=0}^{\infty} z^{-n} = z^{-n}.$$

6.2.9 Tables of z-Transforms

A table of z-transforms for the commonly used functions is given in Table 6.1 (a bigger table is given in Appendix A). As with the Laplace transforms, we are interested in the output response $y(t)$ of a system and we must find the inverse z-transform to obtain $y(t)$ from $Y(z)$.

6.2.10 The z-Transform of a Function Expressed as a Laplace Transform

It is important to realize that although we denote the z-transform equivalent of $G(s)$ by $G(z)$, $G(z)$ is *not* obtained by simply substituting z for s in $G(s)$. We can use one of the following methods to find the z-transform of a function expressed in Laplace transform format:

- Given $G(s)$, calculate the time response $g(t)$ by finding the inverse Laplace transform of $G(s)$. Then find the z-transform either from the first principles, or by looking at the z-transform tables.
- Given $G(s)$, find the z-tranform $G(z)$ by looking at the tables which give the Laplace transforms and their equivalent z-transforms (e.g. Table 6.1).
- Given the Laplace transform $G(s)$, express it in the form $G(s) = N(s)/D(s)$ and then use the following formula to find the z-transform $G(z)$:

$$G(z) = \sum_{n=1}^{p} \frac{N(x_n)}{D'(x_n)} \frac{1}{1 - e^{x_n T} z^{-1}}, \tag{6.15}$$

Table 6.1 Some commonly used z-transforms

$f(kT)$	$F(z)$
$\delta(t)$	1
1	$\dfrac{z}{z-1}$
kT	$\dfrac{Tz}{(z-1)^2}$
e^{-akT}	$\dfrac{z}{z-e^{-aT}}$
kTe^{-akT}	$\dfrac{Tze^{-aT}}{(z-e^{-aT})^2}$
a^k	$\dfrac{z}{z-a}$
$1-e^{-akT}$	$\dfrac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$
$\sin akT$	$\dfrac{z\sin aT}{z^2-2z\cos aT+1}$
$\cos akT$	$\dfrac{z(z-\cos aT)}{z^2-2z\cos aT+1}$

where $D' = \partial D/\partial s$ and the x_n, $n = 1, 2, \ldots, p$, are the roots of the equation $D(s) = 0$. Some examples are given below.

Example 6.2

Let

$$G(s) = \frac{1}{s^2 + 5s + 6}.$$

Determine $G(z)$ by the methods described above.

Solution

Method 1: By finding the inverse Laplace transform. We can express $G(s)$ as a sum of its partial fractions:

$$G(s) = \frac{1}{(s+3)(s+2)} = \frac{1}{s+2} - \frac{1}{s+3}. \tag{6.16}$$

The inverse Laplace transform of (6.16) is

$$g(t) = L^{-1}[G(s)] = e^{-2t} - e^{-3t}. \tag{6.17}$$

From the definition of the z-transforms we can write (6.17) as

$$G(z) = \sum_{n=0}^{\infty} (e^{-2nT} - e^{-3nT}) z^{-n}$$
$$= (1 + e^{-2T}z^{-1} + e^{-4T}z^{-2} + \ldots) - (1 + e^{-3T}z^{-1} + e^{-6T}z^{-2} + \ldots)$$
$$= \frac{z}{z - e^{-2T}} - \frac{z}{z - e^{-3T}}$$

or

$$G(z)=\frac{z(e^{-2T}-e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}.$$

Method 2: By using the z-transform transform tables for the partial product. From Table 6.1, the z-transform of $1/(s+a)$ is $z/(z-e^{-aT})$. Therefore the z-transform of (6.16) is

$$G(z)=\frac{z}{z-e^{-2T}}-\frac{z}{z-e^{-3T}}$$

or

$$G(z)=\frac{z(e^{-2T}-e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}.$$

Method 3: By using the z-transform tables for G(S). From Table 6.1, the z-transform of

$$G(s)=\frac{b-a}{(s+a)(s+b)} \tag{6.18}$$

is

$$G(z)=\frac{z(e^{-aT}-e^{-bT})}{(z-e^{-aT})(z-e^{-bT})}. \tag{6.19}$$

Comparing (6.18) with (6.16) we have, $a=2$, $b=3$. Thus, in (6.19) we get

$$G(z)=\frac{z(e^{-2T}-e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}.$$

Method 4: By using equation (6.15). Comparing our expression

$$G(s)=\frac{1}{s^2+5s+6}$$

with (6.15), we have $N(s)=1$, $D(s)=s^2+5s+6$ and $D'(s)=2s+5$, and the roots of $D(s)=0$ are $x_1=-2$ and $x_2=-3$. Using (6.15),

$$G(z)=\sum_{n=1}^{2}\frac{N(x_n)}{D'(x_n)}\frac{1}{1-e^{x_nT}z^{-1}}$$

or, when $x_1=-2$,

$$G_1(z)=\frac{1}{1}\frac{1}{1-e^{-2T}z^{-1}}$$

and when $x_1=-3$,

$$G_2(z)=\frac{1}{-1}\frac{1}{1-e^{-3T}z^{-1}}.$$

Thus,

$$G(z)=\frac{1}{1-e^{-2T}z^{-1}}-\frac{1}{1-e^{-3T}z^{-1}}=\frac{z}{z-e^{-2T}}-\frac{z}{z-e^{-3T}}$$

or

$$G(z)=\frac{z(e^{-2T}-e^{-3T})}{(z-e^{-2T})(z-e^{-3T})}.$$

6.2.11 Properties of z-Transforms

Most of the properties of the z-transform are analogs of those of the Laplace transforms. Important z-transform properties are discussed in this section.

1. Linearity property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and the z-transform of $g(nT)$ is $G(z)$. Then
   $$Z[f(nT)\pm g(nT)]=Z[f(nT)]\pm Z[g(nT)]=F(z)\pm G(z) \tag{6.20}$$
   and for any scalar a
   $$Z[af(nT)]=aZ[f(nT)]=aF(z) \tag{6.21}$$
2. Left-shifting property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT)=f(nT+mT)$. Then
   $$Y(z)=z^mF(z)-\sum_{i=0}^{m-1}f(iT)z^{m-i}. \tag{6.22}$$
   If the initial conditions are all zero, i.e. $f(iT)=0$, $i=0,1,2,\ldots,m-1$, then,
   $$Z[f(nT+mT)]=z^mF(z). \tag{6.23}$$
3. Right-shifting property
   Suppose that the z-transform of $f(nT)$ is $F(z)$ and let $y(nT)=f(nT-mT)$. Then
   $$Y(z)=z^{-m}F(z)+\sum_{i=0}^{m-1}f(iT-mT)z^{-i}. \tag{6.24}$$
   If $f(nT)=0$ for $k<0$, then the theorem simplifies to
   $$Z[f(nT-mT)]=z^{-m}F(z). \tag{6.25}$$
4. Attenuation property
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then,
   $$Z[e^{-anT}f(nT)]=F[ze^{aT}]. \tag{6.26}$$
   This result states that if a function is multiplied by the exponential e^{-anT} then in the z-transform of this function z is replaced by ze^{aT}.
5. Initial value theorem
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the initial value of the time response is given by
   $$\lim_{n\to 0}f(nT)=\lim_{z\to\infty}F(z). \tag{6.27}$$

6. Final value theorem
   Suppose that the z-transform of $f(nT)$ is $F(z)$. Then the final value of the time response is given by

$$\lim_{n\to\infty} f(nT) = \lim_{z\to 1}(1 - z^{-1})F(z). \tag{6.28}$$

Note that this theorem is valid if the poles of $(1 - z^{-1})F(z)$ are inside the unit circle or at $z = 1$.

Example 6.3

The z-transform of a unit ramp function $r(nT)$ is

$$R(z) = \frac{Tz}{(z-1)^2}.$$

Find the z-transform of the function $5r(nT)$.

Solution

Using the linearity property of z-transforms,

$$Z[5r(nT)] = 5R(z) = \frac{5Tz}{(z-1)^2}.$$

Example 6.4

The z-transform of trigonometric function $r(nT) = \sin nwT$ is

$$R(z) = \frac{z \sin wT}{z^2 - 2z\cos wT + 1}.$$

find the z-transform of the function $y(nT) = e^{-2T} \sin nWT$.

Solution

Using property 4 of the z-transforms,

$$Z[y(nT)] = Z[e^{-2T} r(nT)] = R[ze^{2T}].$$

Thus,

$$Z[y(nT)] = \frac{ze^{2T} \sin wT}{(ze^{2T})^2 - 2ze^{2T}\cos wT + 1} = \frac{ze^{2T} \sin wT}{z^2e^{4T} - 2ze^{2T}\cos wT + 1}$$

or, multiplying numerator and denominator by e^{-4T},

$$Z[y(nT)] = \frac{ze^{-2T} \sin wT}{z^2 - 2ze^{-2T} + e^{-4T}}.$$

Example 6.5

Given the function

$$G(z) = \frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)},$$

find the final value of $g(nT)$.

Solution

Using the final value theorem,

$$\lim_{n\to\infty} g(nT) = \lim_{z\to 1}(1 - z^{-1})\frac{0.792z}{(z-1)(z^2 - 0.416z + 0.208)}$$

$$= \lim_{z\to 1}\frac{0.792}{z^2 - 0.416z + 0.208}$$

$$= \frac{0.792}{1 - 0.416 + 0.208} = 1.$$

6.2.12 Inverse z-Transforms

The inverse z-transform is obtained in a similar way to the inverse Laplace transforms. Generally, the z-transforms are the ratios of polynomials in the complex variable z, with the numerator polynomial being of order no higher than the denominator. By finding the inverse z-transform we find the sequence associated with the given z-transform polynomial. As in the case of inverse Laplace transforms, we are interested in the output time response of a system. Therefore, we use an inverse transform to obtain $y(t)$ from $Y(z)$. There are several methods to find the inverse z-transform of a given function. The following methods will be described here:

- power series (long division);
- expanding $Y(z)$ into partial fractions and using z-transform tables to find the inverse transforms;
- obtaining the inverse z-transform using an inversion integral.

Given a z-transform function $Y(z)$, we can find the coefficients of the associated sequence $y(nT)$at the sampling instants by using the inverse z-transform. The time function $y(t)$ is then determined as

$$y(t) = \sum_{n=0}^{\infty} y(nT)\delta(t - nT).$$

Method 1: Power series. This method involves dividing the denominator of $Y(z)$ into the numerator such that a power series of the form

$$Y(z) = y_0 + y_1z^{-1} + y_2z^{-2} + y_3z^{-3} + \ldots$$

is obtained. Notice that the values of $y(n)$ are the coefficients in the power series.

Example 6.6

Find the inverse z-transform for the polynomial

$$Y(z) = \frac{z^2 + z}{z^2 - 3z + 4}.$$

Solution

Dividing the denominator into the numerator gives

$$
\begin{array}{r|l}
 & 1 + 4z^{-1} + 8z^{-2} + 8z^{-3} \\
\hline
z^2 - 3z + 4 & z^2 + z \\
 & \underline{z^2 - 3z + 4} \\
 & \quad 4z - 4 \\
 & \quad \underline{4z - 12 + 16z^{-1}} \\
 & \qquad 8 - 16z^{-1} \\
 & \qquad \underline{8 - 24z^{-1} + 32z^{-2}} \\
 & \qquad\quad 8z^{-1} - 32z^{-2} \\
 & \qquad\quad 8z^{-1} - 24z^{-2} + 32z^{-3} \\
 & \qquad\quad \ldots
\end{array}
$$

and the coefficients of the power series are

$$
\begin{aligned}
y(0) &= 1, \\
y(T) &= 4, \\
y(2T) &= 8, \\
y(3T) &= 8, \\
&\ldots.
\end{aligned}
$$

The required sequence is

$$y(t) = \delta(t) + 4\delta(t - T) + 8\delta(t - 2T) + 8\delta(t - 3T) + \ldots$$

Figure 6.15 shows the first few samples of the time sequence $y(nT)$.

Example 6.7

Find the inverse z-transform for $Y(z)$ given by the polynomial

$$Y(z) = \frac{z}{z^2 - 3z + 2}.$$

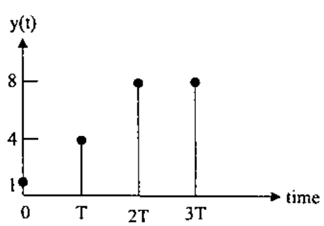

Figure 6.15 First few samples of $y(t)$

Solution

Dividing the denominator into the numerator gives

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} + 15z^{-4} \\
\hline
z^2 - 3z + 2 & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & \quad 3 - 2z^{-1} \\
 & \quad \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & \qquad 7z^{-1} - 6z^{-2} \\
 & \qquad \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & \qquad\quad 15z^{-2} - 14z^{-3} \\
 & \qquad\quad 15z^{-2} - 45z^{-3} + 30z^{-4} \\
 & \qquad\quad \ldots
\end{array}
$$

and the coefficients of the power series are

$$
\begin{aligned}
y(0) &= 0 \\
y(T) &= 1 \\
y(2T) &= 3 \\
y(3T) &= 7 \\
y(4T) &= 15 \\
&\ldots.
\end{aligned}
$$

The required sequence is thus

$$y(t) = \delta(t - T) + 3\delta(t - 2T) + 7\delta(t - 3T) + 15\delta(t - 4T) + \ldots.$$

Figure 6.16 shows the first few samples of the time sequence $y(nT)$.

The disadvantage of the power series method is that it does not give a closed form of the resulting sequence. We often need a closed-form result, and other methods should be used when this is the case.

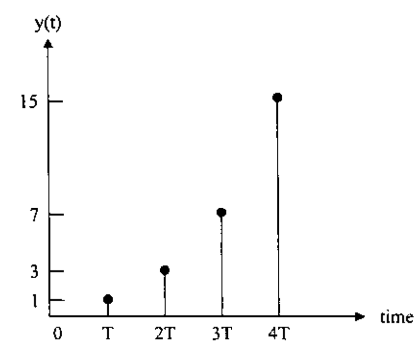

Figure 6.16 First few samples of $y(t)$

Method 2: Partial fractions. Similar to the inverse Laplace transform techniques, a partial fraction expansion of the function $Y(z)$ can be found, and then tables of known z-transforms can be used to determine the inverse z-transform. Looking at the z-transform tables, we see that there is usually a z term in the numerator. It is therefore more convenient to find the partial fractions of the function $Y(z)/z$ and then multiply the partial fractions by z to obtain a z term in the numerator.

Example 6.8

Find the inverse z-transform of the function

$$Y(z) = \frac{z}{(z-1)(z-2)}$$

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}.$$

The values of A and B can be found by equating like powers in the numerator, i.e.

$$A(z-2) + B(z-1) \equiv 1.$$

We find $A = -1$, $B = 1$, giving

$$\frac{Y(z)}{z} = \frac{-1}{z-1} + \frac{1}{z-2}$$

or

$$Y(z) = \frac{-z}{z-1} + \frac{z}{z-2}$$

From the z-transform tables we find that

$$y(nT) = -1 + 2^n$$

and the coefficients of the power series are

$$\begin{aligned} y(0) &= 0, \\ y(T) &= 1, \\ y(2T) &= 3, \\ y(3T) &= 7, \\ y(4T) &= 15, \\ &\dots \end{aligned}$$

so that the required sequence is

$$y(t) = \delta(t-T) + 3\delta(t-2T) + 7\delta(t-3T) + 15\delta(t-4T) + \dots.$$

Example 6.9

Find the inverse z-transform of the function

$$Y(z) = \frac{1}{(z-1)(z-2)}.$$

Solution

The above expression can be written as

$$\frac{Y(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z-2}.$$

The values of A, B and C can be found by equating like powers in the numerator, i.e.

$$A(z-1)(z-2) + Bz(z-2) + Cz(z-1) \equiv 1$$

or

$$A(z^2 - 3z + 2) + Bz^2 - 2Bz + Cz^2 - Cz \equiv 1,$$

giving

$$\begin{aligned} A + B + C &= 0, \\ -3A - 2B - C &= 0, \\ 2A \qquad &= 1. \end{aligned}$$

The values of the coefficients are found to be $A = 0.5$, $B = -1$ and $C = 0.5$. Thus,

$$\frac{Y(z)}{z} = \frac{1}{2z} - \frac{1}{z-1} + \frac{1}{2(z-2)}$$

or

$$Y(z) = \frac{1}{2} - \frac{z}{z-1} + \frac{z}{2(z-2)}.$$

Using the inverse z-transform tables, we find

$$y(nT) = a - 1 + \frac{2^n}{2} = a - 1 + 2^{n-1}$$

where

$$a = \begin{cases} 1/2, & n = 0, \\ 0, & n \neq 0, \end{cases}$$

the coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 0 \\ y(2T) &= 1 \\ y(3T) &= 3 \\ y(4T) &= 7 \\ y(5T) &= 15 \\ &\dots, \end{aligned}$$

and the required sequence is

$$y(t) = \delta(t-2T) + 3\delta(t-3T) + 7\delta(t-4T) + 15\delta(t-5T) + \dots.$$

The process of finding inverse z-transforms is aided by considering what form is taken by the roots of $Y(z)$. It is useful to distinguish the case of distinct real roots and that of multiple order roots.

Case I: Distinct real roots. When $Y(z)$ has distinct real roots in the form

$$Y(z) = \frac{N(z)}{(z-p_1)(z-p_2)(z-p_3)\dots(z-p_n)},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{A_1}{z-p_1} + \frac{A_2}{z-p_2} + \frac{A_3}{z-p_3} + \dots + \frac{A_n}{z-p_n}$$

and the coefficients A_i can easily be found as

$$A_i = (z-p_i)\,Y(z)|_{z=p_i} \quad \text{for } i = 1,2,3,\dots,n.$$

Example 6.10

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z}{(z-1)(z-2)}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-2},$$

we find that

$$A = (z-1)\left.\frac{1}{(z-1)(z-2)}\right|_{z=1} = -1,$$

$$B = (z-2)\left.\frac{1}{(z-1)(z-2)}\right|_{z=2} = 1.$$

Thus,

$$Y(z) = \frac{z}{z-1} + \frac{z}{z-2}$$

and the inverse z-transform is obtained from the tables as

$$y(nT) = -1 + 2^n,$$

which is the same answer as in Example 6.7.

Example 6.11

Using the partial expansion method described above, find the inverse z-transform of

$$Y(z) = \frac{z^2+z}{(z-0.5)(z-0.8)(z-1)}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{A}{z-0.5} + \frac{B}{z-0.8} + \frac{C}{z-1}$$

we find that

$$A = (z-0.5)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=0.5} = 10,$$

$$B = (z-0.8)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=0.8} = -30,$$

$$C = (z-1)\left.\frac{z+1}{(z-0.5)(z-0.8)(z-1)}\right|_{z=1} = 20.$$

Thus,

$$Y(z) = \frac{10z}{z-0.5} - \frac{30z}{z-0.8} + \frac{20z}{z-1}$$

The inverse transform is found from the tables as

$$y(nT) = 10(0.5)^n - 30(0.8)^n + 20$$

The coefficients of the power series are

$$\begin{aligned} y(0) &= 0 \\ y(T) &= 1 \\ y(2T) &= 3.3 \\ y(3T) &= 5.89 \\ &\dots \end{aligned}$$

and the required sequence is

$$y(t) = \delta(t-T) + 3.3\delta(t-2T) + 5.89\delta(t-3T) + \dots.$$

Case II: Multiple order roots. When $Y(z)$ has multiple order roots of the form

$$Y(z) = \frac{N(z)}{(z-p_1)(z-p_1)^2(z-p_1)^3\dots(z-p_1)^r},$$

then the partial fraction expansion can be written as

$$Y(z) = \frac{\lambda_1}{z-p_1} + \frac{\lambda_2}{(z-p_2)^2} + \frac{\lambda_3}{(z-p_1)^3} + \dots + \frac{\lambda_r}{(z-p_1)^r}$$

and the coefficients λ_i can easily be found as

$$\lambda_{r-k} = \frac{1}{k!}\left.\frac{d^k}{dz^k[(z-p_i)^r(X(z)/z)]}\right|_{z=p_i}. \tag{6.29}$$

Example 6.12

Using (6.29), find the inverse z-transform of

$$Y(z) = \frac{z^2 + 3z - 2}{(z+5)(z-0.8)(z-2)^2}.$$

Solution

Rewriting the function as

$$\frac{Y(z)}{z} = \frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2} = \frac{A}{z} + \frac{B}{z+5} + \frac{C}{z-0.8} + \frac{D}{(z-2)} + \frac{E}{(z-2)^2}$$

we obtain

$$A = z\,\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0} = \frac{-2}{5 \times (-0.8) \times 4} = 0.125,$$

$$B = (z+5)\,\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=-5} = \frac{8}{-5 \times (-5.8) \times 49} = 0.0056,$$

$$C = (z-0.8)\,\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=0.8} = \frac{1.04}{0.8 \times 5.8 \times 1.14} = 0.16,$$

$$E = (z-2)^2\,\frac{z^2+3z-2}{z(z+5)(z-0.8)(z-2)^2}\bigg|_{z=2} = \frac{8}{2 \times 7 \times 1.2} = 0.48,$$

$$D = \frac{d}{dz}\left[\frac{z^2+3z-2}{z(z+5)(z-0.8)}\right]\bigg|_{z=2}$$

$$= \frac{[z(z+5)(z-0.8(2z+3) - (z^2+3z-2)(3z^2+8.4z-4)]}{(z^3+4.2z^2-4z)^2}\bigg|_{z=2} = -0.29.$$

We can now write $Y(z)$ as

$$Y(z) = 0.125 + \frac{0.0056z}{z+5} + \frac{0.016z}{z-0.8} - \frac{0.29z}{(z-2)} + \frac{0.48z}{(z-2)^2}$$

The inverse transform is found from the tables as

$$y(nT) = 0.125a + 0.0056(-5)^n + 0.016(0.8)^n - 0.29(2)^n + 0.24n(2)^n,$$

where

$$a = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

Method 3: Inversion formula method. The inverse z-transform can be obtained using the inversion integral, defined by

$$y(nT) = \frac{1}{2\pi j}\oint_{r} Y(z)z^{n-1}dz. \tag{6.30}$$

Using the theorem of residues, the above integral can be evaluated via the expression

$$y(nT) = \sum_{\substack{\text{at poles of} \\ [Y(z)z^{k-1}]}} [\text{residues of } Y(z)z^{n-1}]. \tag{6.31}$$

If the function has a simple pole at $z = a$, then the residue is evaluated as

$$[\text{residue}]|_{z=a} = [(z-a)Y(z)z^{n-1}]. \tag{6.32}$$

Example 6.13

Using the inversion formula method, find the inverse z-transform of

$$Y(z) = \frac{z}{(z-1)(z-2)}.$$

Solution

Using (6.31) and (6.32):

$$y(nT) = \frac{z^n}{z-2}\bigg|_{z=1} + \frac{z^n}{z-1}\bigg|_{z=2} = -1 + 2^n$$

which is the same answer as in Example 6.9.

Example 6.14

Using the inversion formula method, find the inverse z-transform of

$$Y(z) = \frac{z}{(z-1)(z-2)(z-3)}.$$

Solution

Using (6.31) and (6.32),

$$y(nT) = \frac{z^n}{(z-2)(z-3)}\bigg|_{z=1} + \frac{z^n}{(z-1)(z-3)}\bigg|_{z=2} + \frac{z^n}{(z-1)(z-2)}\bigg|_{z=3} = \frac{1}{2} - 2^n + \frac{3^n}{2}.$$

6.3 PULSE TRANSFER FUNCTION AND MANIPULATION OF BLOCK DIAGRAMS

The pulse transfer function is the ratio of the z-transform of the sampled output and the input at the sampling instants.

Suppose we wish to sample a system with output response given by

$$y(s) = e^*(s)G(s). \tag{6.33}$$

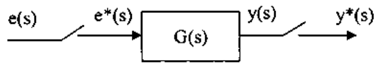

Figure 6.17 Sampling a system

as illustrated in Figure 6.17. We sample the output signal to obtain

$$y^*(s) = [e^*(s)G(s)]^* = e^*(s)G^*(s) \tag{6.34}$$

and

$$y(z) = e(z)G(z). \tag{6.35}$$

Equations (6.34) and (6.35) tell us that if at least one of the continuous functions has been sampled, then the z-transform of the product is equal to the product of the z-transforms of each function (note that $[e^*(s)]^* = [e^*(s)]$, since sampling an already sampled signal has no further effect). $G(z)$ is the transfer function between the sampled input and the output at the sampling instants and is called the *pulse transfer function*. Notice from (6.35) that we have no information about the output $y(z)$ between the sampling instants.

6.3.1 Open-Loop Systems

Some examples of manipulating open-loop block diagrams are given in this section.

Example 6.15

Figure 6.18 shows an open-loop sampled data system. Derive an expression for the z-transform of the output of the system.

Solution

For this system we can write

$$y(s) = e^*(s)KG(s)$$

or

$$y^*(s) = [e^*(s)KG(s)]^* = e^*(s)KG^*(s)$$

and

$$y(z) = e(z)KG(z).$$

Example 6.16

Figure 6.19 shows an open-loop sampled data system. Derive an expression for the z-transform of the output of the system.

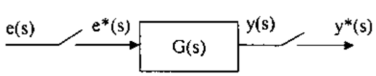

Figure 6.18 Open-loop system

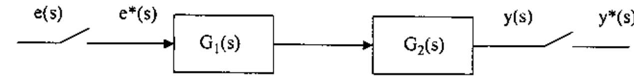

Figure 6.19 Open-loop system

Solution

The following expressions can be written for the system:

$$y(s) = e^*(s)G_1(s)G_2(s)$$

or

$$y^*(s) = [e^*(s)G_1(s)G_2(s)]^* = e^*(s)[G_1G_2]^*(s)$$

and

$$y(z) = e(z)G_1G_2(z),$$

where

$$G_1G_2(z) = Z\{G_1(s)G_2(s)\} \neq G_1(z)G_2(z).$$

For example, if

$$G_1(s) = \frac{1}{s}$$

and

$$G_2(s) = \frac{a}{s+a},$$

then from the z-transform tables,

$$Z\{G_1(s)G_2(s)\} = Z\left\{\frac{a}{s(s+a)}\right\} = \frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$$

and the output is given by

$$y(z) = e(z)\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}.$$

Example 6.17

Figure 6.20 shows an open-loop sampled data system. Derive an expression for the z-transform of the output of the system.

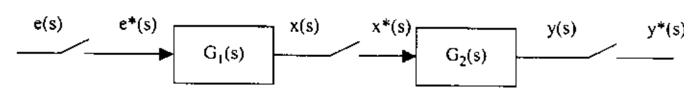

Figure 6.20 Open-loop system

Solution

The following expressions can be written for the system:

$$x(s) = e^*(s)G_1(s)$$

or

$$x^*(s) = e^*(s)G_1^*(s), \tag{6.36}$$

and

$$y(s) = x^*(s)G_2(s)$$

or

$$y^*(s) = x^*(s)G_2^*(s). \tag{6.37}$$

From (6.37) and (6.38),

$$y^*(s) = e^*(s)G_1^*(s)G_2^*(s),$$

which gives

$$y(z) = e(z)G_1(z)G_2(z).$$

For example, if

$$G_1(s) = \frac{1}{s} \quad \text{and} \quad G_2(s) = \frac{a}{s+a},$$

then

$$Z\{G_1(s)\} = \frac{z}{z-1} \quad \text{and} \quad Z\{G_2(s)\} = \frac{az}{z - ze^{-aT}},$$

and the output function is given by

$$y(z) = e(z)\frac{z}{z-1}\frac{az}{z-ze^{-aT}}$$

or

$$y(z) = e(z)\frac{az}{(z-1)(1-e^{-aT})}.$$

6.3.2 Open-Loop Time Response

The open-loop time response of a sampled data system can be obtained by finding the inverse z-transform of the output function. Some examples are given below.

Example 6.18

A unit step signal is applied to the electrical RC system shown in Figure 6.21. Calculate and draw the output response of the system, assuming a sampling period of $T = 1$ s.

Solution

The transfer function of the RC system is

$$G(s) = \frac{1}{s+1}.$$

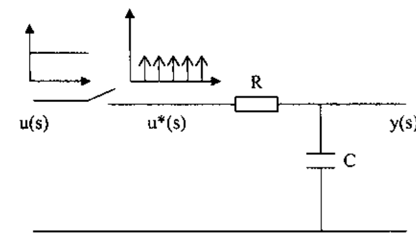

Figure 6.21 RC system with unit step input

For this system we can write

$$y(s) = u^*(s)G(s)$$

and

$$y^*(s) = u^*(s)G^*(s),$$

and taking z-transforms gives

$$y(z) = u(z)G(z).$$

The z-transform of a unit step function is

$$u(z) = \frac{z}{z-1}$$

and the z-transform of $G(s)$ is

$$G(z) = \frac{z}{z-e^{-T}}.$$

Thus, the output z-transform is given by

$$y(z) = u(z)G(z) = \frac{z}{z-1}\frac{z}{z-e^{-T}} = \frac{z^2}{(z-1)(z-e^{-T})};$$

since $T = 1$ s and $e^{-1} = 0.368$, we get

$$y(z) = \frac{z^2}{(z-1)(z-0.368)}.$$

The output response can be obtained by finding the inverse z-transform of $y(z)$. Using partial fractions,

$$\frac{y(z)}{z} = \frac{A}{z-1} + \frac{B}{z-0.368}.$$

Calculating A and B, we find that

$$\frac{y(z)}{z} = \frac{1.582}{z-1} - \frac{0.582}{z-0.368}$$

or

$$y(z) = \frac{1.582z}{z-1} - \frac{0.582z}{z-0.368}.$$

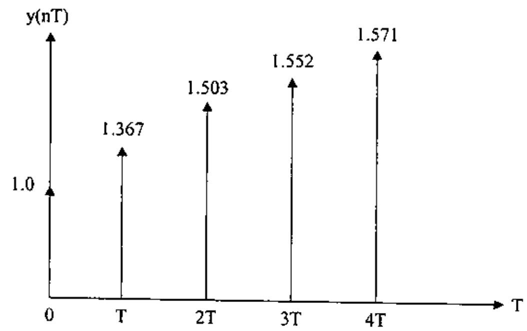

Figure 6.22 RC system output response

From the z-transform tables we find

$$y(nT) = 1.582 - 0.582(0.368)^n.$$

The first few output samples are

$$\begin{aligned} y(0) &= 1, \\ y(1) &= 1.367, \\ y(2) &= 1.503, \\ y(3) &= 1.552, \\ y(4) &= 1.571, \end{aligned}$$

and the output response (shown in Figure 6.22) is given by

$$y(nT) = \delta(T) + 1.367\delta(t - T) + 1.503\delta(t - 2T) + 1.552\delta(t - 3T) + 1.571\delta(t - 4T) + \dots.$$

It is important to notice that the response is only known at the sampling instants. For example, in Figure 6.22 the capacitor discharges through the resistor between the sampling instants, and this causes an exponential decay in the response between the sampling intervals. But this behaviour between the sampling instants cannot be determined by the z-transform method of analysis.

Example 6.19

Assume that the system in Example 6.17 is used with a zero-order hold (see Figure 6.23). What will the system output response be if (i) a unit step input is applied, and (ii) if a unit ramp input is applied.

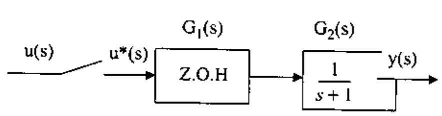

Figure 6.23 RC system with a zero-order hold

Solution

The transfer function of the zero-order hold is

$$G_1(s) = \frac{1 - e^{-Ts}}{s}$$

and that of the RC system is

$$G(s) = \frac{1}{s + 1}.$$

For this system we can write

$$y(s) = u^*(s)G_1G_2(s)$$

and

$$y^*(s) = u^*(s)[G_1G_2]^*(s)$$

or, taking z-transforms,

$$y(z) = u(z)G_1G_2(z).$$

Now, $T = 1$ s and

$$G_1G_2(s) = \frac{1 - e^{-s}}{s}\frac{1}{s + 1},$$

and by partial fraction expansion we can write

$$G_1G_2(s) = (1 - e^{-s})\left(\frac{1}{s} - \frac{1}{s + 1}\right).$$

From the z-transform tables we then find that

$$G_1G_2(z) = (1 - z^{-1})\left(\frac{z}{z - 1} - \frac{z}{z - e^{-1}}\right) = \frac{0.63}{z - 0.37}.$$

(i) For a unit step input,

$$u(z) = \frac{z}{z - 1}$$

and the system output response is given by

$$y(z) = \frac{0.63z}{(z - 1)(z - 0.37)}.$$

Using the partial fractions method, we can write

$$\frac{y(z)}{z} = \frac{A}{z - 1} + \frac{B}{z - 0.37},$$

where $A = 1$ and $B = -1$; thus,

$$y(z) = \frac{z}{z - 1} - \frac{z}{z - 0.37}.$$

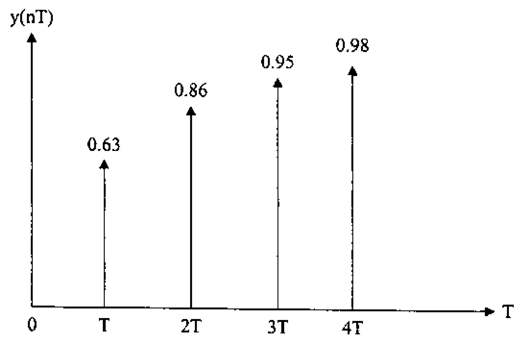

Figure 6.24 Step input time response of Example 6.19

From the inverse z-transform tables we find that the time response is given by

$$y(nT) = a - (0.37)^n,$$

where a is the unit step function; thus

$$y(nT) = 0.63\delta(t-1) + 0.86\delta(t-2) + 0.95\delta(t-3) + 0.98\delta(t-4) + \ldots.$$

The time response in this case is shown in Figure 6.24.

ii) For a unit ramp input,

$$u(z) = \frac{Tz}{(z-1)^2}$$

and the system output response (with $T = 1$) is given by

$$y(z) = \frac{0.63z}{(z-1)^2(z-0.37)} = \frac{0.63z}{z^3 - 2.37z^2 + 1.74z - 0.37}.$$

Using the long division method, we obtain the first few output samples as

$$y(z) = 0.63z^{-2} + 1.5z^{-3} + 2.45z^{-4} + 3.43z^{-5} + \ldots$$

and the output response is given as

$$y(nT) = 0.63\delta(t-2) + 1.5\delta(t-3) + 2.45\delta(t-4) + 3.43\delta(t-5) + \ldots,$$

as shown in Figure 6.25.

xample 6.20

he open-loop block diagram of a system with a zero-order hold is shown in Figure 6.26. alculate and plot the system response when a step input is applied to the system, assuming at $T = 1$ s.

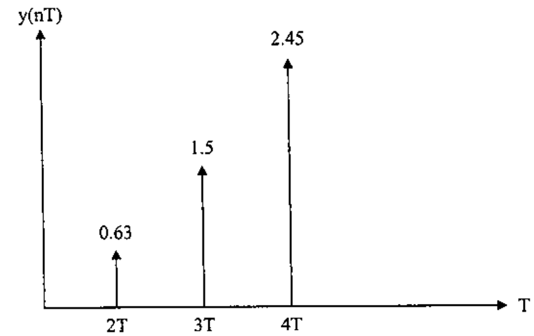

Figure 6.25 Ramp input time response of Example 6.19

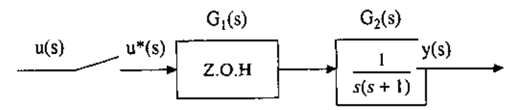

Figure 6.26 Open-loop system with zero-order hold

Solution

The transfer function of the zero-order hold is

$$G_1(s) = \frac{1 - e^{-Ts}}{s}$$

and that of the plant is

$$G(s) = \frac{1}{s(s+1)}.$$

For this system we can write

$$y(s) = u^*(s)G_1G_2(s)$$

and

$$y^*(s) = u^*(s)[G_1G_2]*(s)$$

or, taking z-transforms,

$$y(z) = u(z)G_1G_2(z).$$

Now, $T = 1$ s and

$$G_1G_2(s) = \frac{1 - e^{-s}}{s^2(s+1)}$$

or, by partial fraction expansion,

$$G_1G_2(s) = (1 - e^{-s})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

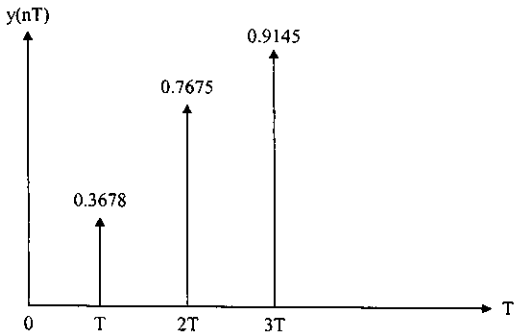

Figure 6.27 Output response

and the z-transform is given by

$$G_1G_2(z) = (1 - z^{-1})Z\left[\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right].$$

From the z-transform tables we obtain

$$G_1G_2(z) = (1 - z^{-1})\left[\frac{z}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-1}}\right] = \frac{ze^{-1} + 1 - 2e^{-1}}{(z-1)(z-e^{-1})}$$
$$= \frac{0.3678z + 0.2644}{z^2 - 1.3678z + 0.3678}.$$

After long division we obtain the time response

$$y(nT) = 0.3678\delta(t-1) + 0.7675\delta(t-2) + 0.9145\delta(t-3) + \ldots,$$

shown in Figure 6.27.

6.3.3 Closed-Loop Systems

Some examples of manipulating the closed-loop system block diagrams are given in this section.

Example 6.21

The block diagram of a closed-loop sampled data system is shown in Figure 6.28. Derive an expression for the transfer function of the system.

Solution

For the system in Figure 6.28 we can write

$$e(s) = r(s) - H(s)y(s) \tag{6.38}$$

and

$$y(s) = e^*(s)G(s). \tag{6.39}$$

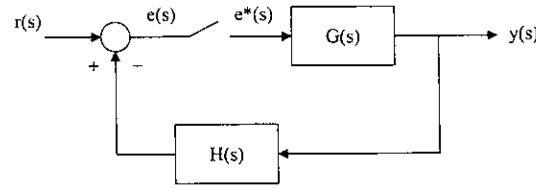

Figure 6.28 Closed-loop sampled data system

Substituting (6.39) into (6.38),

$$e(s) = r(s) - G(s)H(s)e^*(s) \tag{6.40}$$

or

$$e^*(s) = r^*(s) - GH^*(s)e^*(s)$$

and, solving for $e^*(s)$, we obtain

$$e^*(s) = \frac{r^*(s)}{1 + GH^*(s)} \tag{6.41}$$

and, from (6.39),

$$y(s) = G(s)\frac{r^*(s)}{1 + GH^*(s)}. \tag{6.42}$$

The sampled output is then

$$y^*(s) = \frac{r^*(s)G^*(s)}{1 + GH^*(s)} \tag{6.43}$$

Writing (6.43) in z-transform format,

$$y(z) = \frac{r(z)G(z)}{1 + GH(z)} \tag{6.44}$$

and the transfer function is given by

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + GH(z)}. \tag{6.45}$$

Example 6.22

The block diagram of a closed-loop sampled data system is shown in Figure 6.29. Derive an expression for the output function of the system.

Solution

For the system in Figure 6.29 we can write

$$y(s) = e(s)G(s) \tag{6.46}$$

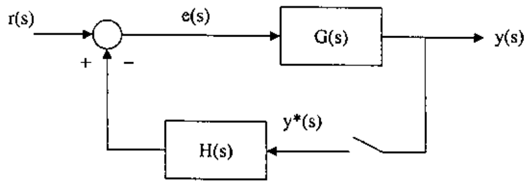

Figure 6.29 Closed-loop sampled data system

nd

$$e(s) = r(s) - H(s)y^*(s). \tag{6.47}$$

ubstituting (6.47) into (6.46), we obtain

$$y(s) = G(s)r(s) - G(s)H(s)y^*(s) \tag{6.48}$$

r

$$y^*(s) = Gr^*(s) - GH^*(s)y^*(s). \tag{6.49}$$

olving for $y^*(s)$, we obtain

$$y^*(s) = \frac{Gr^*(s)}{1 + GH^*(s)} \tag{6.50}$$

nd

$$y(z) = \frac{Gr(z)}{1 + GH(z)}. \tag{6.51}$$

xample 6.23

he block diagram of a closed-loop sampled data control system is shown in Figure 6.30. erive an expression for the transfer function of the system.

olution

he A/D converter can be approximated with an ideal sampler. Similarly, the D/A converter the output of the digital controller can be approximated with a zero-order hold. Denoting

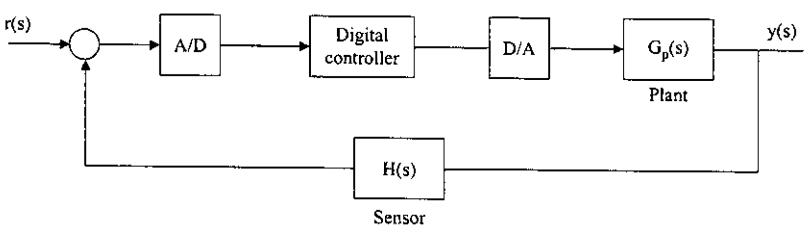

Figure 6.30 Closed-loop sampled data system

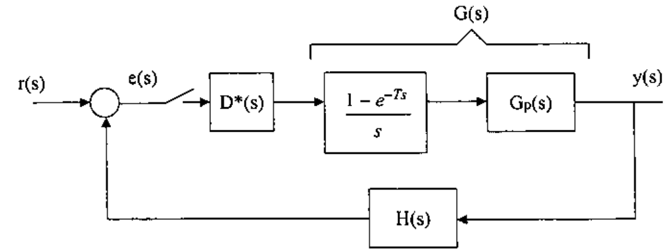

Figure 6.31 Equivalent diagram for Example 6.23

the digital controller by $D(s)$ and combining the zero-order hold and the plant into $G(s)$, the block diagram of the system can be drawn as in Figure 6.31. For this system can write

$$e(s) = r(s) - H(s)y(s) \tag{6.52}$$

and

$$y(s) = e^*(s)D^*(s)G(s). \tag{6.53}$$

Note that the digital computer is represented as $D^*(s)$. Using the above two equations, we can write

$$e(s) = r(s) - D^*(s)G(s)H(s)e^*(s) \tag{6.54}$$

or

$$e^*(s) = r^*(s) - D^*(s)GH^*(s)e^*(s)$$

and, solving for $e^*(s)$, we obtain

$$e^*(s) = \frac{r^*(s)}{1 + D^*(s)GH^*(s)} \tag{6.55}$$

and, from (6.53),

$$y(s) = D^*(s)G(s)\frac{r^*(s)}{1 + D^*(s)GH^*(s)}. \tag{6.56}$$

The sampled output is then

$$y^*(s) = \frac{r^*(s)D^*(s)G^*(s)}{1 + D^*(s)GH^*(s)}, \tag{6.57}$$

Writing (6.57) in z-transform format,

$$y(z) = \frac{r(z)D(z)G(z)}{1 + D(z)GH(z)} \tag{6.58}$$

and the transfer function is given by

$$\frac{y(z)}{r(z)} = \frac{D(z)G(z)}{1 + D(z)GH(z)}. \tag{6.59}$$

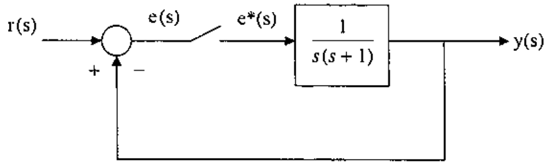

Figure 6.32 Closed-loop system

6.3.4 Closed-Loop Time Response

The closed-loop time response of a sampled data system can be obtained by finding the inverse z-transform of the output function. Some examples are given below.

Example 6.24

A unit step signal is applied to the sampled data digital system shown in Figure 6.32. Calculate and plot the output response of the system. Assume that $T = 1$ s.

Solution

The output response of this system is given in (6.44) as

$$y(z) = \frac{r(z)G(z)}{1 + GH(z)}.$$

where

$$r(z) = \frac{z}{z-1}, \quad G(z) = \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}, \quad H(z) = 1;$$

thus,

$$y(z) = \frac{z/z-1}{1 + (z(1-e^{-T})/(z-1)(z-e^{-T}))} \frac{z(1-e^{-T})}{(z-1)(z-e^{-T})}.$$

Simplifying,

$$y(z) = \frac{z^2(1-e^{-T})}{(z^2 - 2ze^{-T} + e^{-T})(z-1)}.$$

Since $T = 1$,

$$y(z) = \frac{0.632z^2}{z^3 - 1.736z^2 + 1.104z - 0.368}.$$

After long division we obtain the first few terms

$$y(z) = 0.632z^{-1} + 1.096z^{-2} + 1.25z^{-3} + \ldots.$$

The first 10 samples of the output response are shown in Figure 6.33.

6.4 EXERCISES

1. A function $y(t) = 2\sin 4t$ is sampled every $T = 0.1$ s. Find the z-transform of the resultant number sequence.

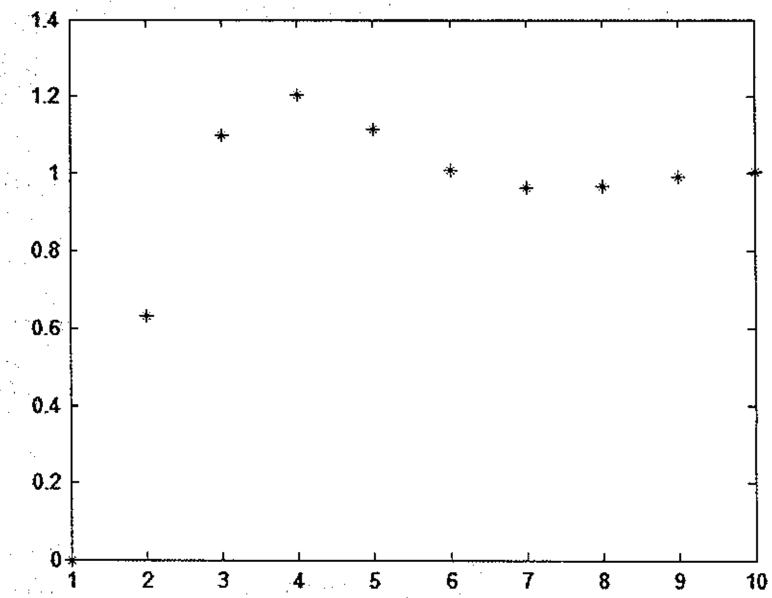

Figure 6.33 First 10 output samples

2. Find the z-transform of the function $y(t) = 3t$.
3. Find the inverse z-transform of the function

$$y(z) = \frac{z}{(z+1)(z-1)}.$$

4. The output response of a system is described with the z-transform

$$y(z) = \frac{z}{(z+0.5)(z-0.2)}.$$

   (i) Apply the final value theorem to calculate the final value of the output when a unit step input is applied to the system.
   (ii) Check your results by finding the inverse z-transform of $y(z)$.

5. Find the inverse z-transform of the following functions using both long division and the method of partial fractions. Compare the two methods.

   (i) $y(z) = \dfrac{0.2z}{(z-1)(z-0.5)}$ (ii) $y(z) = \dfrac{0.1(z+1)}{(z-0.2)(z-1)}$

   (iii) $y(z) = \dfrac{0.2}{(z-3)(z-1)}$ (iv) $y(z) = \dfrac{z(z-1)}{(z-2)^2}$

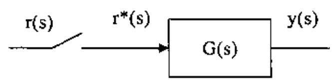

Figure 6.34 Open-loop system for Exercise 6

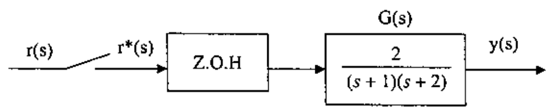

Figure 6.35 Open-loop system with zero-order hold for Exercise 10

6. Consider the open-loop system given in Figure 6.34. Find the output response when a unit step is applied, if

$$G(s) = \frac{0.2}{s(s+1)}.$$

7. Draw the output waveform of Exercise 6.

8. Find the z-transform of the following function, assuming that $T = 0.5$ s:

$$y(s) = \frac{s+1}{(s-1)(s+3)}.$$

9. Find the z-transforms of the following functions, using z-transform tables:

(i) $y(s) = \dfrac{s+1}{s(s+2)}$ (ii) $y(s) = \dfrac{s}{(s+1)^2}$

(iii) $y(s) = \dfrac{s^2}{(s+1)^2(s+2)}$ (iv) $y(s) = \dfrac{0.4}{s(s+1)(s+2)}$

0. Figure 6.35 shows an open-loop system with a zero-order hold. Find the output response when a unit step input is applied. Assume that $T = 0.1$ s and

$$G(s) = \frac{2}{(s+1)(s+2)}.$$

1. Repeat *Exercise 10* for the case where the plant transfer function is given by

(i) $G(s) = \dfrac{0.1}{s(s+2)}$ (ii) $G(s) = \dfrac{2s}{(s+1)(s+4)}$

2. Derive an expression for the transfer function of the closed-loop system whose block diagram is shown in Figure 6.36.

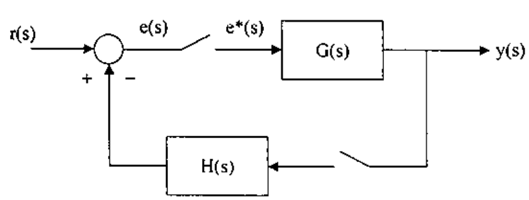

Figure 6.36 Closed-loop system for Exercise 12

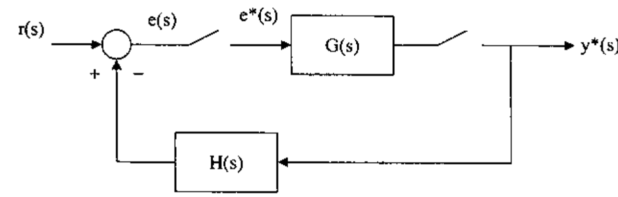

Figure 6.37 Closed-loop system for Exercise 13

13. Derive an expression for the output function of the closed-loop system whose block diagram is shown in Figure 6.37.

FURTHER READING

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7

System Time Response Characteristics

In this chapter we investigate the time response of a sampled data system and compare it with the response of a similar continuous system. In addition, the mapping between the s-domain and the z-domain is examined, the important time response characteristics of continuous systems are revised and their equivalents in the discrete domain are discussed.

7.1 TIME RESPONSE COMPARISON

An example closed-loop discrete-time system with a zero-order hold is shown in Figure 7.1(a). The continuous-time equivalent of this system is also shown in Figure 7.1(b), where the sampler (A/D converter) and the zero-order hold (D/A converter) have been removed. We shall now derive equations for the step responses of both systems and then plot and compare them.

As described in Chapter 6, the transfer function of the above discrete-time system is given by

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + G(z)}, \tag{7.1}$$

where

$$r(z) = \frac{z}{z-1} \tag{7.2}$$

and the z-transform of the plant is given by

$$G(s) = \frac{1 - e^{-sT}}{s^2(s+1)}.$$

Expanding by means of partial fractions, we obtain

$$G(s) = (1 - e^{-sT})\left(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right)$$

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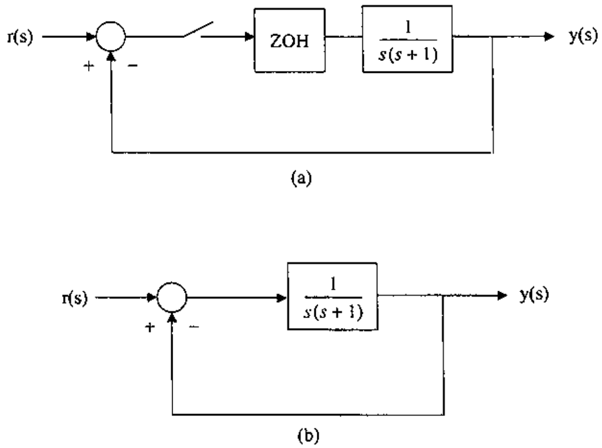

Figure 7.1 (a) Discrete system and (b) its continuous-time equivalent

and the z-transform is

$$G(z) = (1 - z^{-1})Z\left\{\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1}\right\}.$$

From z-transform tables we obtain

$$G(z) = (1 - z^{-1})\left[\frac{Tz}{(z-1)^2} - \frac{z}{z-1} + \frac{z}{z-e^{-T}}\right].$$

Setting $T = 1$s and simplifying gives

$$G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}.$$

Substituting into (7.1), we obtain the transfer function

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1 + G(z)} = \frac{0.368z + 0.264}{z^2 - z + 0.632},$$

and then using (7.2) gives the output

$$y(z) = \frac{z(0.368z + 0.264)}{(z-1)(z^2 - z + 0.632)}.$$

The inverse z-transform can be found by long division: the first several terms are

$$y(z) = 0.368z^{-1} + z^{-2} + 1.4z^{-3} + 1.4z^{-4} + 1.15z^{-5} + 0.9z^{-6} + 0.8z^{-7} + 0.87z^{-8}$$
$$+0.99z^{-9} + \ldots$$

and the time response is given by

$$y(nT) = 0.368\delta(t-1) + \delta(t-2) + 1.4\delta(t-3) + 1.4\delta(t-4) + 1.15\delta(t-5)$$
$$+0.9\delta(t-6) + 0.8\delta(t-7) + 0.87\delta(t-8) + \ldots.$$

From Figure 7.1(b), the equivalent continuous-time system transfer function is

$$\frac{y(s)}{r(s)} = \frac{G(s)}{1+G(s)} = \frac{1/(s(s+1))}{1 + (1/s(s+1))} = \frac{1}{s^2 + s + 1}.$$

Since $r(s) = 1/s$, the output becomes

$$y(s) = \frac{1}{s(s^2 + s + 1)}.$$

To find the inverse Laplace transform we can write

$$y(s) = \frac{1}{s} - \frac{s+1}{s^2+s+1} = \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2 - 0.5^2} - \frac{0.5}{(s+0.5)^2 - 0.5^2}.$$

From inverse Laplace transform tables we find that the time response is

$$y(t) = 1 - e^{-0.5t}(\cos 0.5t + 0.577 \sin 0.5t).$$

Figure 7.2 shows the time responses of both the discrete-time system and its continuous-time equivalent. The response of the discrete-time system is accurate only at the sampling instants. As shown in the figure, the sampling process has a destabilizing effect on the system.

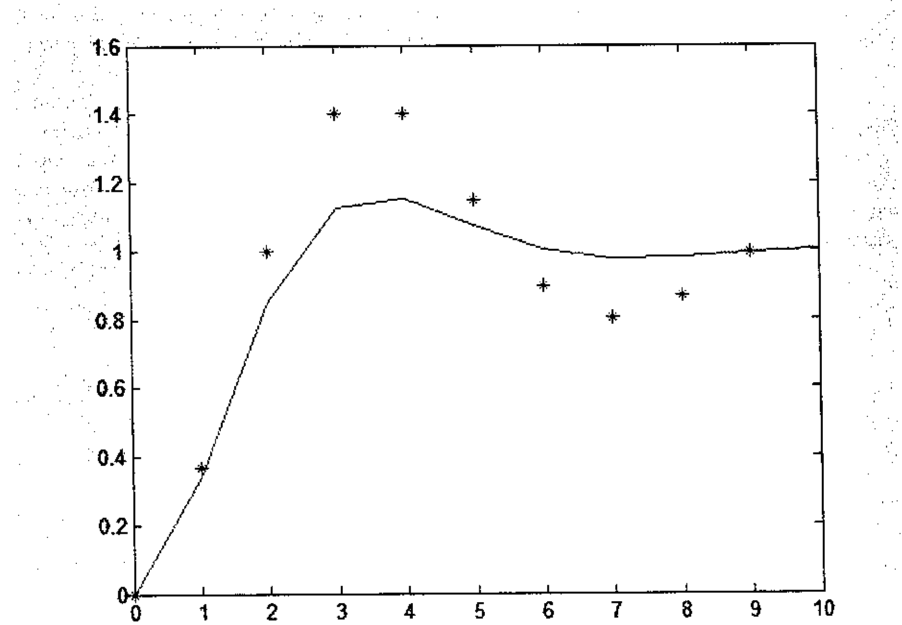

Figure 7.2 Step response of the system shown in Figure 7.1

'.2 TIME DOMAIN SPECIFICATIONS

he performance of a control system is usually measured in terms of its response to a step ıput. The step input is used because it is easy to generate and gives the system a nonzero eady-state condition, which can be measured.

Most commonly used time domain performance measures refer to a second-order system ith the transfer function:

$$\frac{y(s)}{r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

here ω_n is the undamped natural frequency of the system and ζ is the damping ratio of the /stem.

When a second-order system is excited with a unit step input, the typical output response as shown in Figure 7.3. Based on this figure, the following performance parameters are sually defined: maximum overshoot; peak time; rise time; settling time; and steady-state ror.

The maximum overshoot, M_p, is the peak value of the response curve measured from unity. his parameter is usually quoted as a percentage. The amount of overshoot depends on the ımping ratio and directly indicates the relative stability of the system.

The peak time, T_p, is defined as the time required for the response to reach the first peak of ıe overshoot. The system is more responsive when the peak time is smaller, but this gives rise a higher overshoot.

The rise time, T_r, is the time required for the response to go from 0 % to 100 % of its final ılue. It is a measure of the responsiveness of a system, and smaller rise times make the system ore responsive.

The settling time, T_s, is the time required for the response curve to reach and stay within a nge about the final value. A value of 2–5 % is usually used in performance specifications.

The steady-state error, E_{ss}, is the error between the system response and the reference input ılue (unity) when the system reaches its steady-state value. A small steady-tate error is a quirement in most control systems. In some control systems, such as position control, it is ıe of the requirements to have no steady-state error.

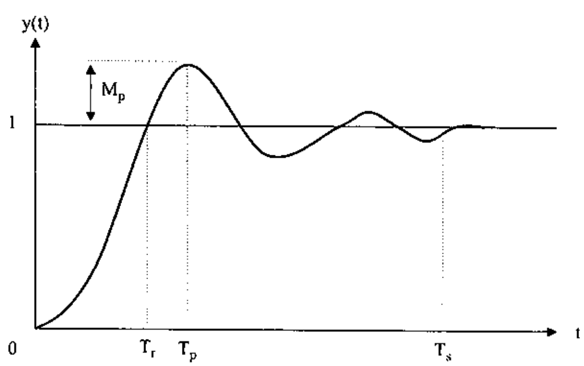

Figure 7.3 Second-order system unit step response

Having introduced the parameters, we are now in a position to give formulae for them (readers who are interested in the derivation of these formulae should refer to books on control theory). The maximum overshoot occurs at at peak time ($t = T_p$) and is given by

$$M_p = e^{-(\zeta\pi/\sqrt{1-\zeta^2})},$$

i.e. overshoot is directly related to the system damping ratio – the lower the damping ratio, the higher the overshoot. Figure 7.4 shows the variation of the overshoot (expressed as a percentage) with the damping ratio.

The peak time is obtained by differentiating the output response with respect to time, letting this equal zero. It is given by

$$T_p = \frac{\pi}{\omega_d},$$

where

$$\omega_d = \omega_n^2\sqrt{1-\zeta^2}$$

is the damped natural frequency.

The rise time is obtained by setting the output response to 1 and finding the time. It is given by

$$T_r = \frac{\pi - \beta}{\omega_d},$$

where

$$\beta = \tan^{-1}\frac{w_d}{\zeta\omega_n}.$$

The settling time is usually specified for a 2 % or 5 % tolerance band, and is given by

$$T_s = \frac{4}{\zeta\omega_n} \quad \text{(for 2\% settling time)},$$

$$T_s = \frac{3}{\zeta\omega_n} \quad \text{(for 5\% settling time)}.$$

The steady-state error can be found by using the final value theorem, i.e. if the Laplace transform of the output response is $y(s)$, then the final value (steady-state value) is given by

$$\lim_{s\to 0} sy(s),$$

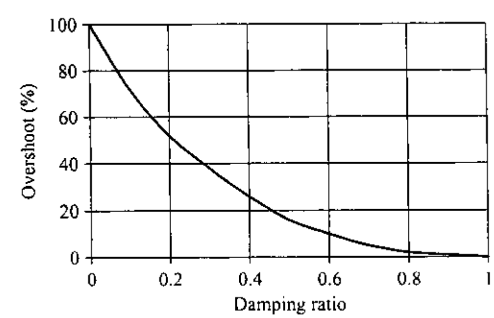

Figure 7.4 Variation of overshoot with damping ratio

d the steady-state error when a unit step input is applied can be found from

$$E_{ss} = 1 - \lim_{s\to 0} s\, y(s).$$

ample 7.1

etermine the performance parameters of the system given in Section 7.1 with closed-loop nsfer function

$$\frac{y(s)}{r(s)} = \frac{1}{s^2 + s + 1}.$$

lution

mparing this system with the standard second-order system transfer function

$$\frac{y(s)}{r(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

find that $\zeta = 0.5$ and $\omega_n = 1$ rad/s. Thus, the damped natural frequency is

$$\omega_d = \omega_n^2 \sqrt{1 - \zeta^2} = 0.866\text{rad/s}.$$

e peak overshoot is

$$M_p = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} = 0.16$$

16 %. The peak time is

$$T_p = \frac{\pi}{\omega_d} = 3.627\,\text{s}$$

e rise time is

$$T_r = \frac{\pi - \beta}{\omega_n};$$

ice

$$\beta = \tan^{-1} \frac{\omega_d}{\zeta\omega_n} = 1.047,$$

have

$$T_r = \frac{\pi - \beta}{\omega_n} = \frac{\pi - 1.047}{1} = 2.094\,\text{s}$$

e settling time (2 %) is

$$T_s = \frac{4}{\zeta\omega_n} = 8\,\text{s},$$

d the settling time (5 %) is

$$T_s = \frac{3}{\zeta\omega_n} = 6\,\text{s}.$$

nally, the steady state error is

$$E_{ss} = 1 - \lim_{s\to 0} s\, y(s) = 1 - \lim_{s\to 0} s \frac{1}{s(s^2 + s + 1)} = 0.$$

7.3 MAPPING THE s-PLANE INTO THE z-PLANE

The pole locations of a closed-loop continuous-time system in the s-plane determine the behaviour and stability of the system, and we can shape the response of a system by positioning its poles in the s-plane. It is desirable to do the same for the sampled data systems. This section describes the relationship between the s-plane and the z-plane and analyses the behaviour of a system when the closed-loop poles are placed in the z-plane.

First of all, consider the mapping of the left-hand side of the s-plane into the z-plane. Let $s = \sigma + j\omega$ describe a point in the s-plane. Then, along the $j\omega$ axis,

$$z = e^{sT} = e^{\sigma T} e^{j\omega T}.$$

But $\sigma = 0$ so we have

$$z = e^{j\omega T} = \cos\omega T + j\sin\omega T = 1\angle\omega T.$$

Hence, the pole locations on the imaginary axis in the s-plane are mapped onto the unit circle in the z-plane. As ω changes along the imaginary axis in the s-plane, the angle of the poles on the unit circle in the z-plane changes.

If ω is kept constant and σ is increased in the left-hand s-plane, the pole locations in the z-plane move towards the origin, away from the unit circle. Similarly, if σ is decreased in the left-hand s-plane, the pole locations in the z-plane move away from the origin in the z-plane. Hence, the entire left-hand s-plane is mapped into the interior of the unit circle in the z-plane. Similarly, the right-hand s-plane is mapped into the exterior of the unit circle in the z-plane. As far as the system stability is concerned, a sampled data system will be stable if the closed-loop poles (or the zeros of the characteristic equation) lie within the unit circle. Figure 7.5 shows the mapping of the left-hand s-plane into the z-plane.

As shown in Figure 7.6, lines of constant σ in the s-plane are mapped into circles in the z-plane with radius $e^{\sigma T}$. If the line is on the left-hand side of the s-plane then the radius of the circle in the z-plane is less than 1. If on the other hand the line is on the right-hand side of the s-plane then the radius of the circle in the z-plane is greater than 1. Figure 7.7 shows the corresponding pole locations between the s-plane and the z-plane.

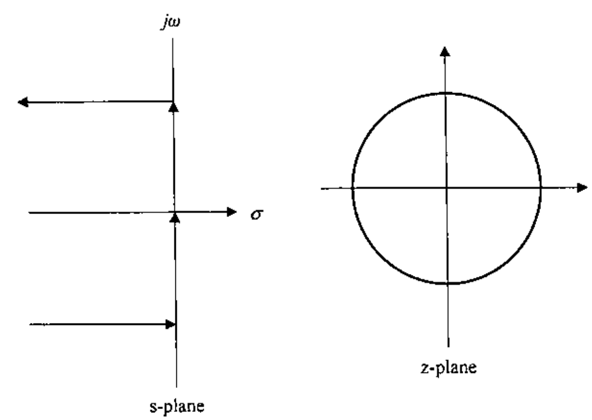

Figure 7.5 Mapping the left-hand s-plane into the z-plane

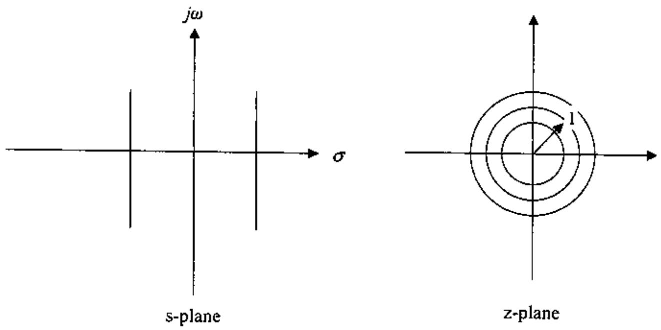

Figure 7.6 Mapping the lines of constant σ

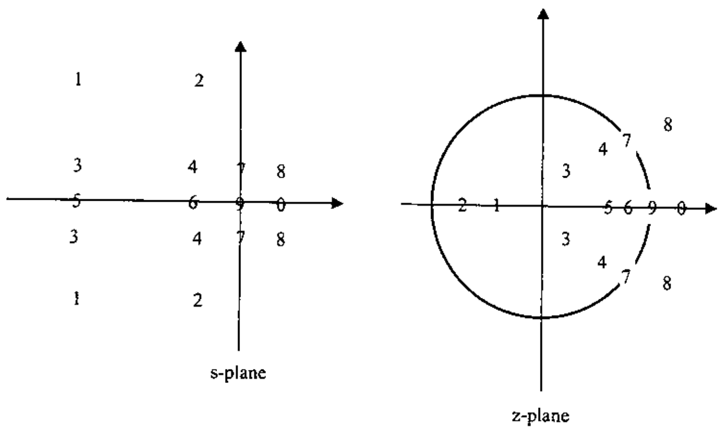

Figure 7.7 Poles in the s-plane and their corresponding z-plane locations

The time responses of a sampled data system based on its pole positions in the z-plane are shown in Figure 7.8. It is clear from this figure that the system is stable if all the closed-loop poles are within the unit circle.

7.4 DAMPING RATIO AND UNDAMPED NATURAL FREQUENCY IN THE z-PLANE

7.4.1 Damping Ratio

As shown in Figure 7.9(a), lines of constant damping ratio in the s-plane are lines where $\zeta = \cos\alpha$ for a given damping ratio. The locus in the z-plane can then be obtained by the substitution $z = e^{sT}$. Remembering that we are working in the third and fourth quadrants in

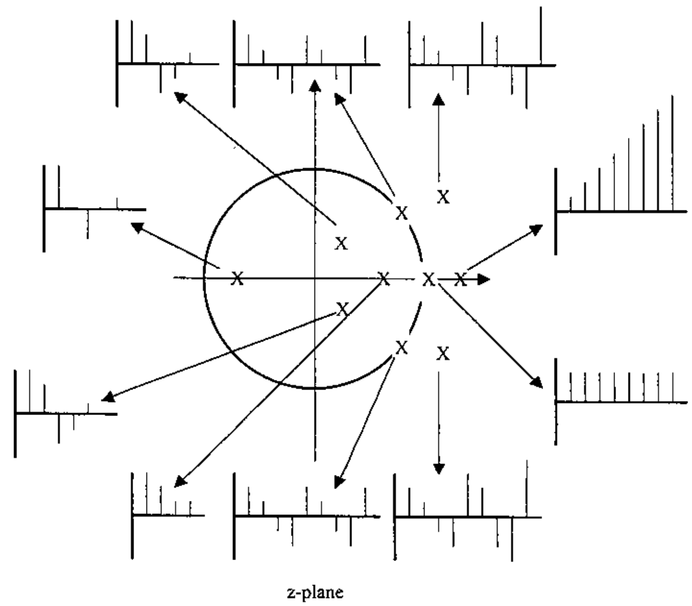

Figure 7.8 Time response of z-plane pole locations

the s-plane where s is negative, we get

$$z = e^{-\sigma\omega T} e^{j\omega T}. \tag{7.3}$$

Since, from Figure 7.9(a),

$$\sigma = \tan\left(\frac{\pi}{2} - \cos^{-1}\zeta\right), \tag{7.4}$$

substituting in (7.3) we have

$$z = \exp\left[-\omega T \tan\left(\frac{\pi}{2} - \cos^{-1}\zeta\right)\right] e^{j\omega T}. \tag{7.5}$$

Equation (7.5) describes a logarithmic spiral in the z-plane as shown in Figure 7.9(b). The spiral starts from $z = 1$ when $\omega = 0$. Figure 7.10 shows the lines of constant damping ratio in the z-plane for various values of ζ.

7.4.2 Undamped Natural Frequency

As shown in Figure 7.11, the locus of constant undamped natural frequency in the s-plane is a circle with radius ω_n. From this figure, we can write

$$\omega^2 + \sigma^2 = \omega_n^2 \quad \text{or} \quad \sigma = \sqrt{\omega_n^2 - \omega^2}. \tag{7.6}$$

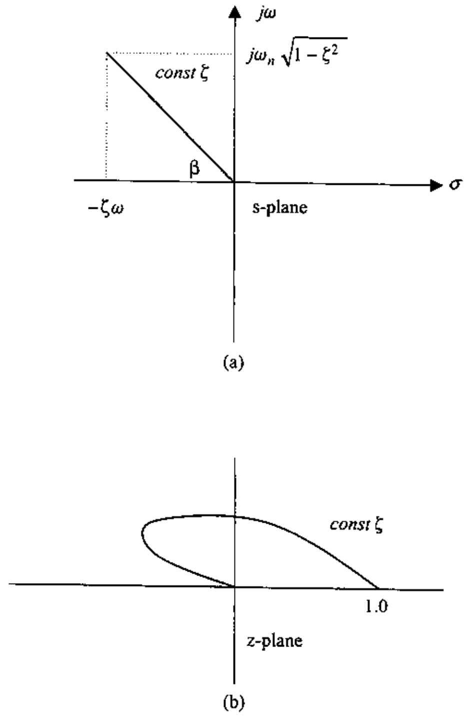

Figure 7.9 (a) Line of constant damping ratio in the s-plane, and (b) the corresponding locus in the z-plane

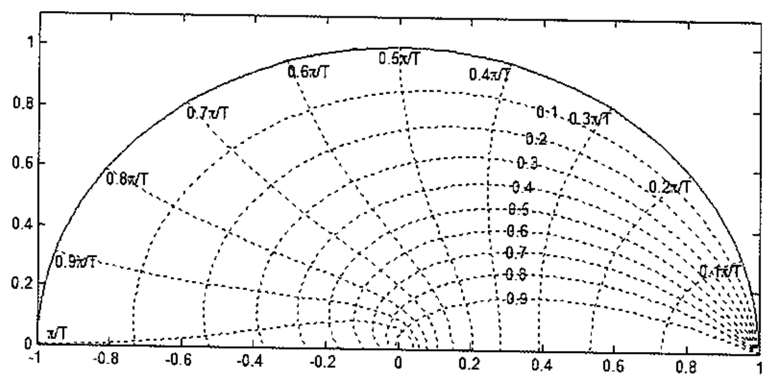

Figure 7.10 Lines of constant damping ratio for different ζ. The vertical lines are the lines of constant ω_n

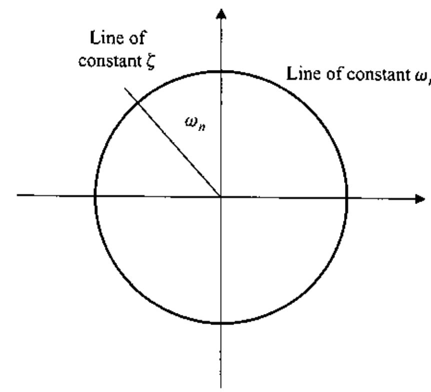

Figure 7.11 Locus of constant ω_n in the s-plane

Thus, remembering that s is negative, we have

$$z = e^{-sT} = e^{-\sigma T} e^{-j\omega T} = \exp\left[-T\left(\sqrt{\omega_n^2 - \omega^2}\right)\right] e^{-j\omega T} \tag{7.7}$$

The locus of constant ω_n in the z-plane is given by (7.7) and is shown in Figure 7.10 as the vertical lines. Notice that the curves are given for values of ω_n ranging from $\omega_n = \pi/10T$ to $\omega_n = \pi/T$.

Notice that the loci of constant damping ratio and the loci of undamped natural frequency are usually shown on the same graph.

7.5 DAMPING RATIO AND UNDAMPED NATURAL FREQUENCY USING FORMULAE

In Section 7.4 above we saw how to find the damping ratio and the undamped natural frequency of a system using a graphical technique. Here, we will derive equations for calculating the damping ratio and the undamped natural frequency.

The damping ratio and the natural frequency of a system in the z-plane can be determined if we first of all consider a second-order system in the s-plane:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{7.8}$$

The poles of this system are at

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}. \tag{7.9}$$

We can now find the equivalent z-plane poles by making the substitution $z = e^{sT}$, i.e.

$$z = e^{sT} = e^{-\zeta\omega_n T} \angle \pm \omega_n T\sqrt{1-\zeta^2}. \tag{7.10}$$

which we can write as

$$z = r\angle \pm \theta, \tag{7.11}$$

where

$$r = e^{-\zeta\omega_n T} \quad \text{or} \quad \zeta\omega_n T = -\ln r \tag{7.12}$$

and

$$\theta = \omega_n T\sqrt{1-\zeta^2}. \tag{7.13}$$

From (7.12) and (7.13) we obtain

$$\frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{-\ln r}{\theta}$$

or

$$\zeta = \frac{-\ln r}{\sqrt{(\ln r)^2+\theta^2}}, \tag{7.14}$$

and from (7.12) and (7.14) we obtain

$$\omega_n = \frac{1}{T}\sqrt{(\ln r)^2+\theta^2}. \tag{7.15}$$

Example 7.2

Consider the system described in Section 7.1 with closed-loop transfer function

$$\frac{y(z)}{r(z)} = \frac{G(z)}{1+G(z)} = \frac{0.368z+0.264}{z^2-z+0.632}.$$

Find the damping ratio and the undamped natural frequency. Assume that $T = 1$ s.

Solution

We need to find the poles of the closed-loop transfer function. The system characteristic equation is $1 + G(z) = 0$,
i.e.

$$z^2 - z + 0.632 = (z - 0.5 - j0.618)(z - 0.5 + j0.618) = 0,$$

which can be written in polar form as

$$z_{1,2} = 0.5 \pm j0.618 = 0.795\angle \pm 0.890 = r\angle \pm \theta$$

(see (7.11)). The damping ratio is then calculated using (7.14) as

$$\zeta = \frac{-\ln r}{\sqrt{(\ln r)^2+\theta^2}} = \frac{-\ln 0.795}{\sqrt{(\ln 0.795)^2+0.890^2}} = 0.25,$$

and from (7.15) the undamped natural frequency is, taking $T = 1$,

$$\omega_n = \frac{1}{T}\sqrt{(\ln r)^2+\theta^2} = \sqrt{(\ln 0.795)^2+0.890^2} = 0.92.$$

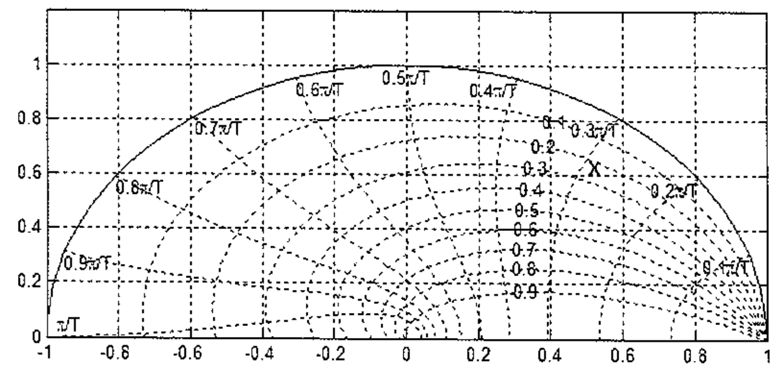

Figure 7.12 Finding ζ and ω_n graphically

Example 7.3

Find the damping ratio and the undamped natural frequency for Example 7.2 using the graphical method.

Solution

The characteristic equation of the system is found to be

$$z^2 - z + 0.632 = (z - 0.5 - j0.618)(z - 0.5 + j0.618) = 0$$

and the poles of the closed-loop system are at

$$z_{1,2} = 0.5 \pm j0.618.$$

Figure 7.12 shows the loci of the constant damping ratio and the loci of the undamped natural frequency with the poles of the closed-loop system marked with an $\times$ on the graph. From the graph we can read the damping ratio as 0.25 and the undamped natural frequency as

$$\omega_n = \frac{0.29\pi}{T} = 0.91.$$

7.6 EXERCISES

1. Find the damping ratio and the undamped natural frequency of the sampled data systems whose characteristic equations are given below
   (a) $z^2 - z + 2 = 0$
   (b) $z^2 - 1 = 0$
   (c) $z^2 - z + 1 = 0$
   (d) $z^2 - 0.81 = 0$

8
System Stability

This chapter is concerned with the various techniques available for the analysis of the stability of discrete-time systems.

Suppose we have a closed-loop system transfer function

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} = \frac{N(z)}{D(z)},$$

where $1 + GH(z) = 0$ is also known as the characteristic equation. The stability of the system depends on the location of the poles of the closed-loop transfer function, or the roots of the characteristic equation $D(z) = 0$. It was shown in Chapter 7 that the left-hand side of the s-plane, where a continuous system is stable, maps into the interior of the unit circle in the z-plane. Thus, we can say that a system in the z-plane will be stable if all the roots of the characteristic equation, $D(z) = 0$, lie inside the unit circle.

There are several methods available to check for the stability of a discrete-time system:

- Factorize $D(z) = 0$ and find the positions of its roots, and hence the position of the closed-loop poles.
- Determine the system stability without finding the poles of the closed-loop system, such as Jury's test.
- Transform the problem into the s-plane and analyse the system stability using the well-established s-plane techniques, such as frequency response analysis or the Routh–Hurwitz criterion.
- Use the root-locus graphical technique in the z-plane to determine the positions of the system poles.

The various techniques described in this section will be illustrated with examples.

8.1 FACTORIZING THE CHARACTERISTIC EQUATION

The stability of a system can be determined if the characteristic equation can be factorized. This method has the disadvantage that it is not usually easy to factorize the characteristic equation. Also, this type of test can only tell us whether or not a system is stable as it is. It does not tell us about the margin of stability or how the stability is affected if the gain or some other parameter is changed in the system.

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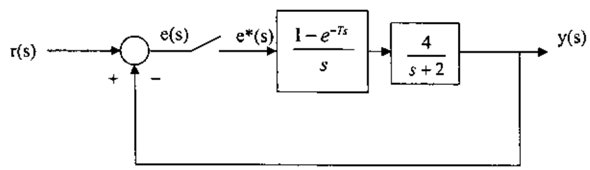

Figure 8.1 Closed-loop system

Example 8.1

The block diagram of a closed-loop system is shown in Figure 8.1. Determine whether or not the system is stable. Assume that $T = 1$ s.

Solution

The closed-loop system transfer function is

$$\frac{Y(z)}{R(z)} = \frac{G(z)}{1 + G(z)}, \tag{8.1}$$

where

$$G(z) = Z\left\{\left[\frac{1 - e^{-Ts}}{s}\frac{4}{s+2}\right]\right\} = (1 - z^{-1})Z\left\{\left[\frac{4}{s(s+2)}\right]\right\} = (1 - z^{-1})\frac{2z(1 - e^{-2T})}{(z-1)(z - e^{-2T})}$$

$$= \frac{2(1 - e^{-2T})}{z - e^{-2T}}. \tag{8.2}$$

For $T = 1$ s,

$$G(z) = \frac{1.729}{z - 0.135}.$$

The roots of the characteristic equation are $1 + G(z) = 0$, or $1 + 1.729/(z - 0.135) = 0$, the solution of which is $z = -1.594$ which is outside the unit circle, i.e. the system is not stable.

Example 8.2

For the system given in Example 8.1, find the value of T for which the system is stable.

Solution

From (8.2),

$$G(z) = \frac{2(1 - e^{-2T})}{z - e^{-2T}}.$$

The roots of the characteristic equation are $1 + G(z) = 0$, or $1 + 2(1 - e^{-2T})/(z - e^{-2T}) = 0$, giving

$$z - e^{-2T} + 2(1 - e^{-2T}) = 0$$

or

$$z = 3e^{-2T} - 2.$$

The system will be stable if the absolute value of the root is inside the unit circle, i.e.

$$|3e^{-2T} - 2| < 1,$$

from which we get

$$2T < \ln\left(\tfrac{1}{3}\right) \quad \text{or} \quad T < 0.549.$$

Thus, the system will be stable as long as the sampling time $T < 0.549$.

8.2 JURY'S STABILITY TEST

Jury's stability test is similar to the Routh–Hurwitz stability criterion used for continuous-time systems. Although Jury's test can be applied to characteristic equations of any order, its complexity increases for high-order systems.

To describe Jury's test, express the characteristic equation of a discrete-time system of order n as

$$F(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0 = 0, \tag{8.3}$$

where $a_n > 0$. We now form the array shown in Table 8.1. The elements of this array are defined as follows:

- The elements of each of the even-numbered rows are the elements of the preceding row, in reverse order.
- The elements of the odd-numbered rows are defined as:

$$b_k = \begin{vmatrix} a_0 & a_{n-k} \\ a_n & a_k \end{vmatrix}, \quad c_k = \begin{vmatrix} b_0 & b_{n-k-1} \\ n_{n-1} & b_k \end{vmatrix}, \quad d_k = \begin{vmatrix} c_0 & c_{n-2-k} \\ c_{n-2} & c_k \end{vmatrix}, \quad \ldots$$

Table 8.1 Array for Jury's stability tests

z^0	z^1	z^2	...	z^{n-k}	...	z^{n-1}	z^n
a_0	a_1	a_2	...	a_{n-k}	...	a_{n-1}	a_n
a_n	a_{n-1}	a_{n-2}	...	a_k	...	a_1	a_0
b_0	b_1	b_2	...	b_{n-k}	...	b_{n-1}	
b_{n-1}	b_{n-2}	b_{n-3}	...	b_{k-1}	...	b_0	
c_0	c_1	c_2	...	c_{n-k}	...		
c_{n-2}	c_{n-3}	c_{n-4}	...	c_{k-2}	...		
...	...	...	...	...			
...	...	...	...	...			
l_0	l_1	l_2	l_3				
l_3	l_2	l_1	l_0				
m_0	m_1	m_2					

The necessary and sufficient conditions for the characteristic equation (8.3) to have roots inside the unit circle are given as

$$F(1) > 0, \quad (-1)^n F(-1) > 0, \quad |a_0| < a_n, \tag{8.4}$$

$$\begin{aligned} |b_0| &> b_{n-1} \\ |c_0| &> c_{n-2} \\ |d_0| &> d_{n-3} \\ &\cdots \\ &\cdots \\ |m_0| &> m_2. \end{aligned} \tag{8.5}$$

Jury's test is then applied as follows:

- Check the three conditions given in (8.4) and stop if any of these conditions is not satisfied.
- Construct the array given in Table 8.1 and check the conditions given in (8.5). Stop if any condition is not satisfied.

Jury's test can become complex as the order of the system increases. For systems of order 2 and 3 the test reduces to the following simple rules. Given the second-order system characteristic equation

$$F(z) = a_2 z^2 + a_1 z + a_0 = 0, \quad \text{where } a_2 > 0,$$

no roots of the system characteristic equation will be on or outside the unit circle provided that

$$F(1) > 0, \quad F(-1) > 0, \quad |a_0| < a_2..$$

Given the third-order system characteristic equation

$$F(z) = a_3 z^3 + a_2 z^2 + a_1 z + a_0 = 0, \quad \text{where } a_3 > 0,$$

no roots of the system characteristic equation will be on or outside the unit circle provided that

$$F(1) > 0, \quad F(-1) < 0, \quad |a_0| < a_3,$$

$$\left| \det \begin{bmatrix} a_0 & a_3 \\ a_3 & a_0 \end{bmatrix} \right| > \left| \det \begin{bmatrix} a_0 & a_1 \\ a_3 & a_2 \end{bmatrix} \right|.$$

Examples are given below.

Example 8.3

The closed-loop transfer function of a system is given by

$$\frac{G(z)}{1 + G(z)},$$

where

$$G(z) = \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2}.$$

Determine the stability of this system using Jury's test.

Solution

The characteristic equation is

$$1 + G(z) = 1 + \frac{0.2z + 0.5}{z^2 - 1.2z + 0.2} = 0$$

or

$$z^2 - z + 0.7 = 0.$$

Applying Jury's test,

$$F(1) = 0.7 > 0, \quad F(-1) = 2.7 > 0, \quad 0.7 < 1.$$

All the conditions are satisfied and the system is stable.

Example 8.4

The characteristic equation of a system is given by

$$1 + G(z) = 1 + \frac{K(0.2z + 0.5)}{z^2 - 1.2z + 0.2} = 0.$$

Determine the value of K for which the system is stable.

Solution

The characteristic equation is

$$z^2 + z(0.2K - 1.2) + 0.5K = 0, \quad \text{where } K > 0.$$

Applying Jurys's test,

$$F(1) = 0.7K - 0.2 > 0, \quad F(-1) = 0.3K + 2.2 > 0, \quad 0.5K < 1.$$

Thus, the system is stable for $0.285 < K < 2$.

Example 8.5

The characteristic equation of a system is given by

$$F(z) = z^3 - 2z^2 + 1.4z - 0.1 = 0.$$

Determine the stability of the system.

Solution

Applying Jury's test, $a_3 = 1$, $a_2 = -2$, $a_1 = 1.4$, $a_0 = -0.1$ and

$$F(1) = 0.3 > 0, \quad F(-1) = -4.5 < 0, \quad 0.1 < 1.$$

The first conditions are satisfied. Applying the other condition,

$$\left| \begin{bmatrix} -0.1 & 1 \\ 1 & -0.1 \end{bmatrix} \right| = -0.99 \quad \text{and} \quad \left| \begin{bmatrix} -0.1 & 1.4 \\ 1 & -2 \end{bmatrix} \right| = -1.2;$$

since $|0.99| < |-1.2|$, the system is not stable.

8.3 ROUTH–HURWITZ CRITERION

The stability of a sampled data system can be analysed by transforming the system characteristic equation into the s-plane and then applying the well-known Routh–Hurwitz criterion.

A bilinear transformation is usually used to transform the left-hand s-plane into the interior of the unit circle in the z-plane. For this transformation, z is replaced by

$$z = \frac{1+w}{1-w}. \tag{8.6}$$

Given the characteristic equation in w,

$$F(w) = b_n w^n + b_{n-1} w^{n-1} + \ldots + b_1 w + b_0 = 0,$$

then the Routh–Hurwitz array is formed as follows:

$$\begin{array}{c|cccc}
w^n & b_n & b_{n-2} & b_{n-4} & \ldots \\
w^{n-1} & b_{n-1} & b_{n-3} & b_{n-5} & \ldots \\
w^{n-2} & c_1 & c_2 & c_3 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
w^1 & j_1 & & & \\
w^0 & k_1 & & &
\end{array}$$

The first two rows are obtained from the equation directly and the other rows are calculated as follows:

$$c_1 = \frac{b_{n-1}b_{n-2} - b_n b_{n-3}}{b_{n-1}},$$
$$c_2 = \frac{b_{n-1}b_{n-4} - b_n b_{n-5}}{b_{n-1}},$$
$$c_3 = \frac{b_{n-1}b_{n-6} - b_n b_{n-7}}{b_{n-1}},$$
$$d_1 = \frac{c_1 b_{n-3} - b_{n-1} c_2}{c_1},$$
$$\ldots.$$

The Routh–Hurwitz criterion states that the number of roots of the characteristic equation in the right hand s-plane is equal to the number of sign changes of the coefficients in the first column of the array. Thus, for a stable system all coefficients in the first column must have the same sign.

Example 8.6

The characteristic equation of a sampled data system is given by

$$z^2 - z + 0.7 = 0.$$

Determine the stability of the system using the Routh–Hurwitz criterion.

Solution

Transforming the characteristic equation into the w-plane gives

$$\left(\frac{1+w}{1-w}\right)^2 - \frac{1+w}{1-w} + 0.7 = 0,$$

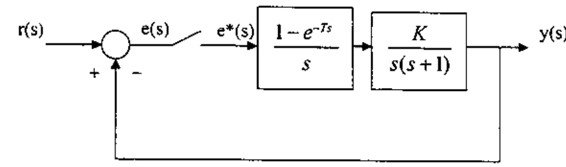

Figure 8.2 Closed-loop system

or

$$2.7w^2 + 0.6w + 0.7 = 0.$$

Forming the Routh–Hurwitz array,

$$\begin{array}{c|cc}
w^2 & 2.7 & 0.7 \\
w^1 & 0.6 & 0 \\
w^0 & 0.7 &
\end{array}$$

there are no sign changes in the first column and thus the system is stable.

Example 8.7

The block diagram of a sampled data system is shown in Figure 8.2. Use the Routh–Hurwitz criterion to determine the value of K for which the system is stable. Assume that $K > 0$ and $T = 1$ s.

Solution

The characteristic equation is $1 + G(z) = 0$, where

$$G(s) = \frac{1-e^{-Ts}}{s}\frac{K}{s(s+1)}.$$

The z-transform is given by

$$G(z) = (1 - z^{-1})Z\left\{\frac{K}{s^2(s+1)}\right\},$$

which gives

$$G(z) = \frac{K(0.368z + 0.264)}{(z-1)(z-0.368)}.$$

The characteristic equation is

$$1 + \frac{K(0.368z + 0.264)}{(z-1)(z-0.368)} = 0,$$

or

$$z^2 - z(1.368 - 0.368K) + 0.368 + 0.264K = 0.$$

Transforming into the w-plane gives

$$\left(\frac{1+w}{1-w}\right)^2 - \left(\frac{1+w}{1-w}\right)(1.368 - 0.368K) + 0.368 + 0.264K = 0$$

or

$$w^2(2.736 - 0.104K) + w(1.264 - 0.528K) + 0.632K = 0.$$

We can now form the Routh–Hurwitz array

$$\begin{array}{c|cc} w^2 & 2.736 - 0.104K & 0.632K \\ w^1 & 1.264 - 0.528K & 0 \\ w^0 & 0.632K & \end{array}$$

The system is stable if there is no sign change in the first column. Thus, for stability,

$$1.264 - 0.528K > 0$$

or

$$K < 2.4.$$

8.4 ROOT LOCUS

The root locus is one of the most powerful techniques used to analyse the stability of a closed-loop system. This technique is also used to design controllers with required time response characteristics. The root locus is a plot of the locus of the roots of the characteristic equation as the gain of the system is varied. The rules of the root locus for discrete-time systems are identical to those for continuous systems. This is because the roots of an equation $Q(z) = 0$ in the z-plane are the same as the roots of $Q(s) = 0$ in the s-plane. Even though the rules are the same, the interpretation of the root locus is quite different in the s-plane and the z-plane. For example, a continuous system is stable if the roots are in the left-hand s-plane. A discrete-time system, on the other hand, is stable if the roots are inside the unit circle. The construction and the rules of the root locus for continuous-time systems are described in many textbooks. In this section only the important rules for the construction of the discrete-time root locus are given, with worked examples.

Given the closed-loop system transfer function

$$\frac{G(z)}{1 + GH(z)},$$

we can write the characteristic equation as $1 + kF(z) = 0$, and the root locus can then be plotted as k is varied. The rules for constructing the root locus can be summarized as follows:

1. The locus starts on the poles of $F(z)$ and terminate on the zeros of $F(z)$.
2. The root locus is symmetrical about the real axis.
3. The root locus includes all points on the real axis to the left of an odd number of poles and zeros.

4. If $F(z)$ has zeros at infinity, the root locus will have asymptotes as $k \to \infty$. The number of asymptotes is equal to the number of poles n_p, minus the number of zeros n_z. The angles of the asymptotes are given by

$$\theta = \frac{180r}{n_p - n_z}, \quad \text{where } r = \pm 1, \pm 3, \pm 5, \ldots.$$

   The asymptotes intersect the real axis at σ, where

$$\sigma = \frac{\sum \text{poles of } F(z) - \sum \text{zeros of } F(z)}{n_p - n_z}.$$

5. The breakaway points on the real axis of the root locus are at the roots of

$$\frac{dF(z)}{dz} = 0.$$

6. If a point is on the root locus, the value of k is given by

$$1 + kF(z) = 0 \quad \text{or} \quad k = -\frac{1}{F(z)}.$$

Example 8.8

A closed-loop system has the characteristic equation

$$1 + GH(z) = 1 + K\frac{0.368(z + 0.717)}{(z-1)(z-0.368)} = 0.$$

Draw the root locus and hence determine the stability of the system.

Solution

Applying the rules:

1. The above equation is in the form $1 + kF(z) = 0$, where

$$F(z) = \frac{0.368(z + 0.717)}{(z-1)(z-0.368)}.$$

   The system has two poles at $z = 1$ and at $z = 0.368$. There are two zeros, one at $z = -0.717$ and the other at minus infinity. The locus will start at the two poles and terminate at the two zeros.

2. The section on the real axis between $z = 0.368$ and $z = 1$ is on the locus. Similarly, the section on the real axis between $z = -\infty$ and $z = -0.717$ is on the locus.

3. Since $n_p - n_z = 1$, there is one asymptote and the angle of this asymptote is

$$\theta = \frac{180r}{n_p - n_z} = \pm 180^\circ \quad \text{for } r = \pm 1.$$

   Note that since the angles of the asymptotes are $\pm 180^\circ$ it is meaningless to find the real axis intersection point of the asymptotes.

4. The breakaway points can be found from

$$\frac{dF(z)}{dz} = 0,$$

or

$$0.368(z-1)(z-0.368) - 0.368(z+0.717)(2z-1.368) = 0,$$

which gives

$$z^2 + 1.434z - 1.348 = 0$$

and the roots are at

$$z = -2.08 \text{ and } z = 0.648.$$

5. The value of k at the breakaway points can be calculated from

$$k = -\frac{1}{F(z)}\bigg|_{z=-2.08,0.648}$$

which gives $k = 15$ and $k = 0.196$.

The root locus of the system is shown in Figure 8.3. The locus is a circle starting from the poles, breaking away at $z = 0.648$ on the real axis, and then joining the real axis at $z = -2.08$.

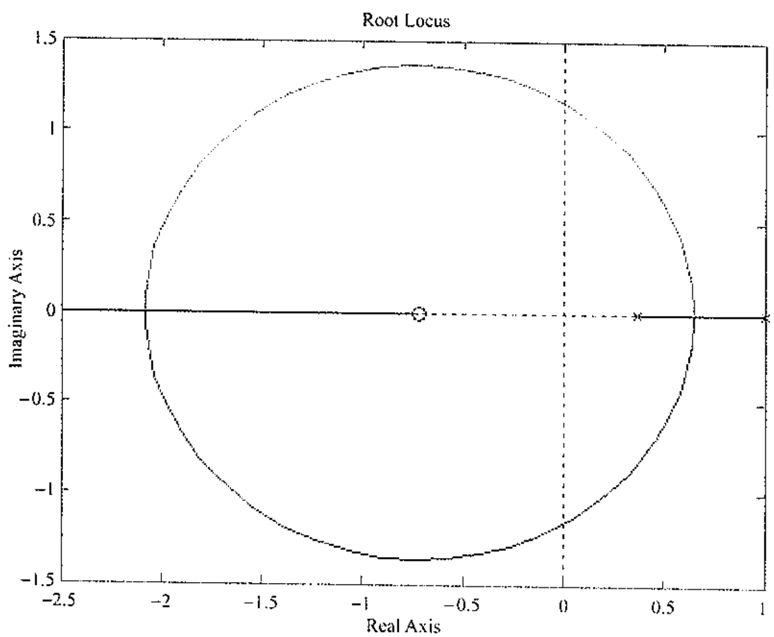

Figure 8.3 Root locus for Example 8.8

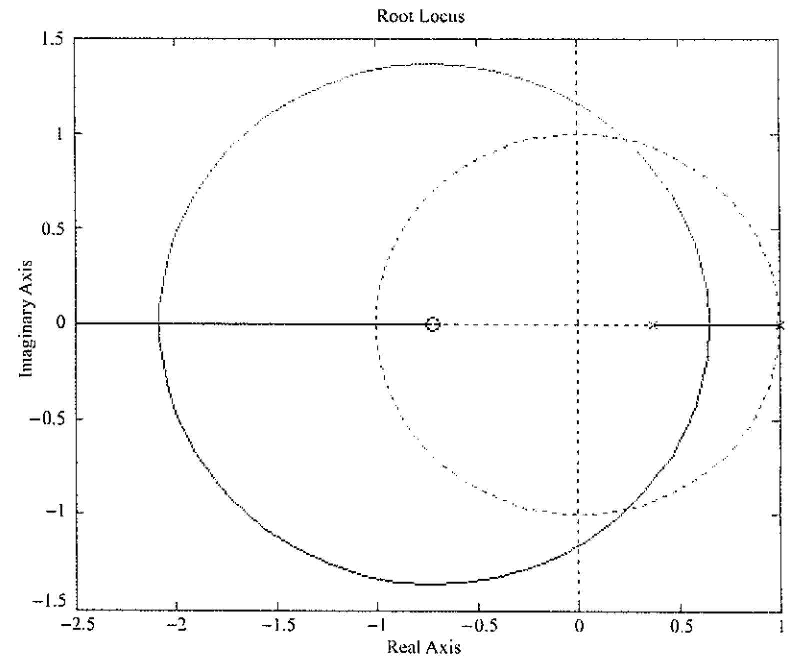

Figure 8.4 Root locus with unit circle

At this point one part of the locus moves towards the zero at $z = -0.717$ and the other moves towards the zero at $-\infty$.

Figure 8.4 shows the root locus with the unit circle drawn on the same axis. The system will become marginally stable when the locus is on the unit circle. The value of k at these points can be found either from Jury's test or by using the Routh–Hurwitz criterion.

Using Jury's test, the characteristic equation is

$$1 + K\frac{0.368(z+0.717)}{(z-1)(z-0.368)} = 0,$$

or

$$z^2 - z(1.368 - 0.368K) + 0.368 + 0.263K = 0.$$

Applying Jury's test

$$F(1) = 0.631 \quad \text{for } K > 0.$$

Also,

$$|0.263K + 0.368| < 1$$

which gives $K = 2.39$ for marginal stability of the system.

Example 8.9

For Example 8.8, calculate the value of k for which the damping factor is $\zeta = 0.7$.

Solution

In Figure 8.5 the root locus of the system is redrawn with the lines of constant damping factor and constant natural frequency.

From the figure, the roots when $\zeta = 0.7$ are read as $s_{1,2} = 0.61 \pm j0.25$ (see Figure 8.6). The value of k can now be calculated as

$$k = -\frac{1}{F(z)}\bigg|_{z=0.61\pm j0.25}$$

which gives $k = 0.324$.

Example 8.10

A closed-loop system has the characteristic equation

$$1 + GH(z) = 1 + K\frac{(z-0.2)}{z^2 - 1.5z + 0.5} = 0.$$

Draw the root locus and hence determine the stability of the system. What will be the value of K for a damping factor $\zeta > 0.6$ and a natural frequency of $\omega_n > 0.6$ rad/s?

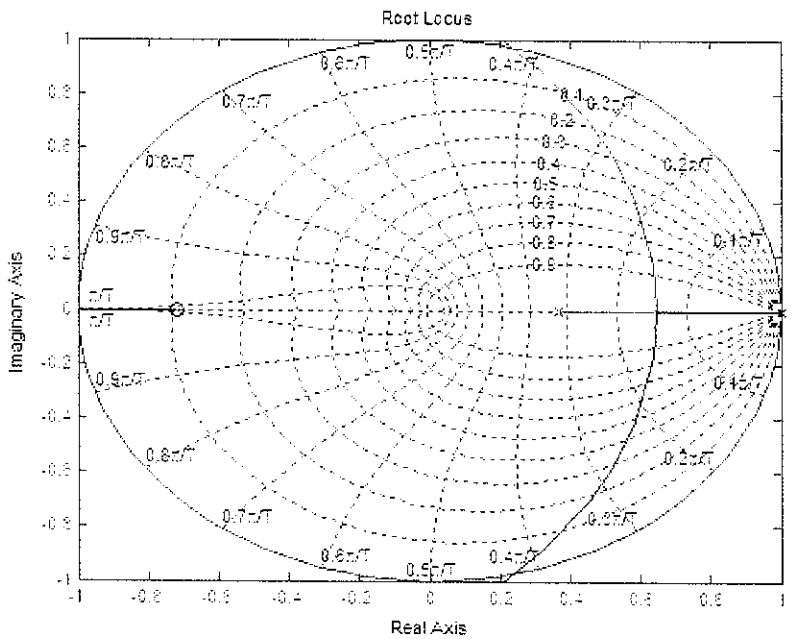

Figure 8.5 Root locus with lines of constant damping factor and natural frequency

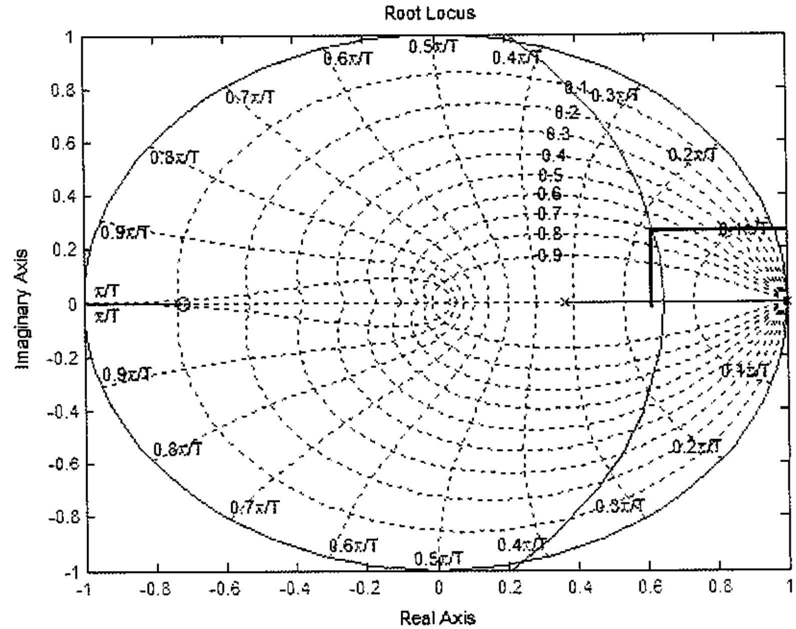

Figure 8.6 Reading the roots when $\zeta = 0.7$

Solution

The above equation is in the form $1 + kF(z) = 0$, where

$$F(z) = \frac{z-0.2}{z^2 - 1.5z + 0.5}.$$

The system has two poles at $z = 1$ and at $z = 0.5$. There are two zeros, one at $z = -0.2$ and the other at infinity. The locus will start from the two poles and terminate at the two zeros.

1. The section on the real axis between $z = 0.5$ and $z = 1$ is on the locus. Similarly, the section on the real axis between $z = -\infty$ and $z = 0.2$ is on the locus.
2. Since $n_p - n_z = 1$, there is one asymptote and the angle of this asymptote is

$$\theta = \frac{180r}{n_p - n_z} = \pm 180^\circ \quad \text{for } r = \pm 1$$

   Note that since the angle of the asymptotes are $\pm 180^\circ$ it meaningless to find the real axis intersection point of the asymptotes.
3. The breakaway points can be found from

$$\frac{dF(z)}{dz} = 0$$

or

$$(z^2 - 1.5z + 0.5) - (z - 0.2)(2z - 1.5) = 0,$$

which gives

$$z^2 - 0.4z - 0.2 = 0$$

and the roots are at

$$z = -0.290 \quad \text{and} \quad z = 0.689.$$

4. The value of k at the breakaway points can be calculated from

$$k = -\frac{1}{F(z)}\bigg|_{z=-0.290,0.689}$$

which gives $k = 0.12$ and $k = 2.08$. The root locus of the system is shown in Figure 8.7. It is clear from this plot that the system is always stable since all poles are inside the unit circle for all values of k.

Lines of constant damping factor and constant angular frequency are plotted on the same axis in Figure 8.8.

Assuming that $T = 1$ s, $\omega_n > 0.6$ if the roots are on the left-hand side of the constant angular frequency line $\omega_n = 0.2\pi/T$. The damping factor will be greater than 0.6 if the roots are below the constant damping ratio line $\zeta = 0.6$. A point satisfying these properties has been chosen

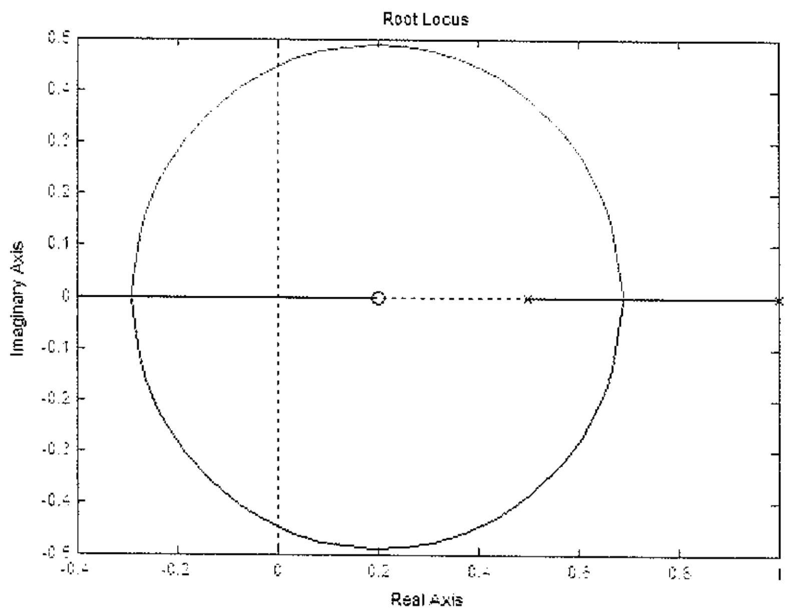

Figure 8.7 Root locus for Example 8.10

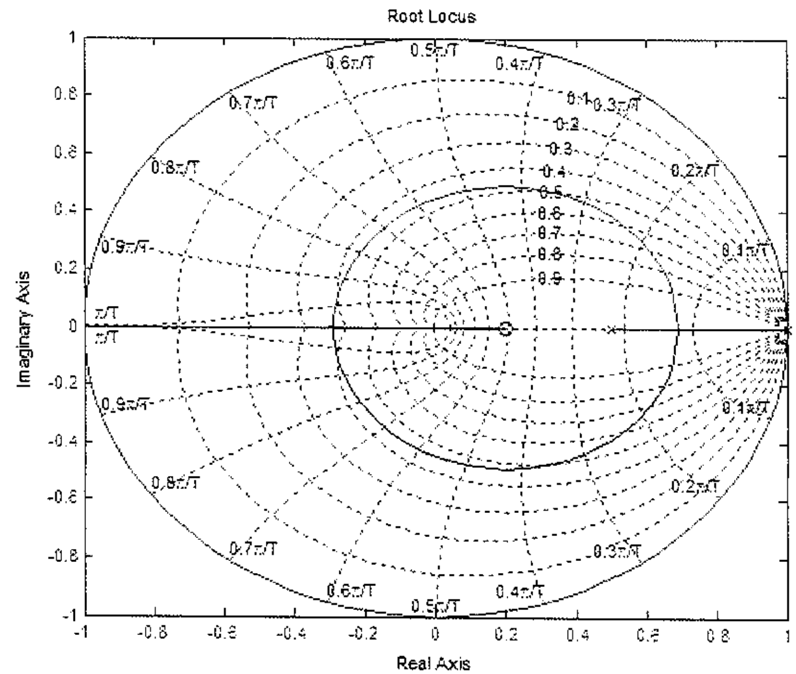

Figure 8.8 Root locus with lines of constant damping factor and natural frequency

and shown in Figure 8.9. The roots at this point are given as $s_{1,2} = 0.55 \pm j0.32$. The value of k can now be calculated as

$$k = -\frac{1}{F(z)}\bigg|_{z=0.55\pm j0.32}$$

which gives $k = 0.377$.

8.5 NYQUIST CRITERION

The Nyquist criterion is one of the widely used stability analysis techniques in the s-plane, based on the frequency response of the system. To determine the frequency response of a continuous system transfer function $G(s)$, we replace s by $j\omega$ and use the transfer function $G(j\omega)$. In the s-plane, the Nyquist criterion is based on the plot of the magnitude $|GH(j\omega)|$ against the angle $\angle GH(j\omega)$ as ω is varied.

In a similar manner, the frequency response of a transfer function $G(z)$ in the z-plane can be obtained by making the substitution $z = e^{j\omega T}$. The Nyquist plot in the z-plane can then be obtained by plotting the magnitude of $|GH(z)|_{z=e^{j\omega T}}$ against the angle $\angle GH(z)|_{z=e^{j\omega T}}$ as ω is varied. The criterion is then

$$Z = N + P,$$

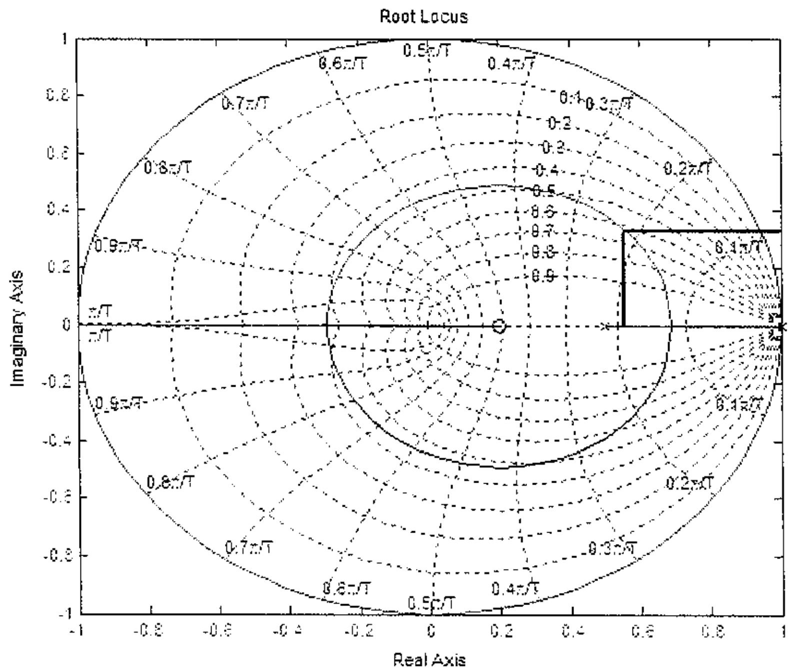

Figure 8.9 Point for $\zeta > 0.6$ and $\omega_n > 0.6$

where N is the number of clockwise circles around the point -1, P the number of poles of $GH(z)$ that are outside the unit circle, and Z the number of zeros of $GH(z)$ that are outside the unit circle.

For a stable system, Z must be equal to zero, and hence the number of anticlockwise circles around the point -1 must be equal to the number of poles of $GH(z)$.

If $GH(z)$ has no poles outside the unit circle then the criterion becomes simple and for stability the Nyquist plot must not encircle the point -1.

An example is given below.

Example 8.11

The transfer function of a closed-loop sampled data system is given by

$$\frac{G(z)}{1 + GH(z)},$$

where

$$GH(z) = \frac{0.4}{(z - 0.5)(z - 0.2)}.$$

Determine the stability of this system using the Nyquist criterion. Assume that $T = 1$ s.

Solution

Setting $z = e^{j\omega T} = \cos\omega T + j\sin\omega T = \cos\omega + j\sin\omega$,

$$G(z)|_{z=e^{j\omega T}} = \frac{0.4}{(\cos\omega + j\sin\omega - 0.5)(\cos\omega + j\sin\omega - 0.2)}$$

or

$$G(z)|_{z=e^{j\omega T}} = \frac{0.4}{(\cos^2\omega - \sin^2\omega - 0.7\cos\omega + 0.1) + j(2\sin\omega\cos\omega - 0.7\sin\omega)}.$$

This has magnitude

$$|G(z)| = \frac{0.4}{\sqrt{(\cos^2\omega - \sin^2\omega - 0.7\cos\omega + 0.1)^2 + (2\sin\omega\cos\omega - 0.7\sin\omega)^2}}$$

and phase

$$\angle G(z) = \tan^{-1}\frac{2\sin\omega\cos\omega - 0.7\sin\omega}{\cos^2\omega - \sin^2\omega - 0.7\cos\omega + 0.1}.$$

Table 8.2 lists the variation of the magnitude of $G(z)$ with the phase angle. The Nyquist plot for this example is shown in Figure 8.10. Since $N = 0$ and $P = 0$, the closed-loop system has no poles outside the unit circle in the x-plane and the system is stable.

The Nyquist diagram can also be plotted by transforming the system into the w-plane and then using the standard s-plane Nyquist criterion. An example is given below.

Example 8.12

The open-loop transfer function of a unity feedback sampled data system is given by

$$G(z) = \frac{z}{(z - 1)(z - 0.4)}.$$

Derive expressions for the magnitude and the phase of $|G(z)|$ by transforming the system into the w-plane.

Solution

The w transformation is defined as

$$z = \frac{1 + w}{1 - w}$$

which gives

$$G(w) = \frac{(1 + w)/(1 - w)}{((1 + w/1 - w) - 1)((1 + w/1 - w) - 0.4)} = \frac{1 + w}{2w(0.6 + 1.4w)},$$

or

$$G(w) = \frac{1 + w}{1.2w + 2.8w^2}.$$

Table 8.2 Magnitude and phase of $G(z)$

w	$\lvert G(z)\rvert$	$\angle G(z)$
0	1.0000E+000	0
1.0472E−001	9.8753E−001	−1.9430E+001
2.0944E−001	9.5248E−001	−3.8460E+001
3.1416E−001	9.0081E−001	−5.6779E+001
4.1888E−001	8.3961E−001	−7.4209E+001
5.2360E−001	7.7510E−001	−9.0690E+001
6.2832E−001	7.1166E−001	−1.0625E+002
7.3304E−001	6.5193E−001	−1.2096E+002
8.3776E−001	5.9719E−001	−1.3492E+002
9.4248E−001	5.4789E−001	−1.4820E+002
1.0472E+000	5.0395E−001	−1.6089E+002
1.1519E+000	4.6505E−001	−1.7308E+002
1.2566E+000	4.3075E−001	1.7518E+002
1.3614E+000	4.0058E−001	1.6384E+002
1.4661E+000	3.7408E−001	1.5283E+002
1.5708E+000	3.5082E−001	1.4213E+002
1.6755E+000	3.3044E−001	1.3168E+002
1.7802E+000	3.1259E−001	1.2147E+002
1.8850E+000	2.9698E−001	1.1146E+002
1.9897E+000	2.8337E−001	1.0162E+002
2.0944E+000	2.7154E−001	9.1945E+001
2.1991E+000	2.6130E−001	8.2401E+001
2.3038E+000	2.5250E−001	7.2974E+001
2.4086E+000	2.4501E−001	6.3646E+001
2.5133E+000	2.3872E−001	5.4404E+001
2.6180E+000	2.3354E−001	4.5232E+001
2.7227E+000	2.2939E−001	3.6118E+001
2.8274E+000	2.2622E−001	2.7050E+001
2.9322E+000	2.2399E−001	1.8015E+001
3.0369E+000	2.2266E−001	9.0018E+000

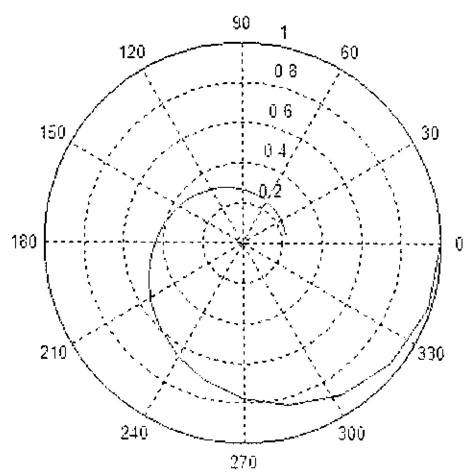

Figure 8.10 Nyquist plot for Example 8.11

But since the w-plane can be regarded an analogue of the s-plane, i.e. $s = \sigma + jw$ in the s-plane and $w = \sigma_w + jw_w$ in the z-plane, for the frequency response we can set

$$w = jw_w$$

which yields

$$G(jw_w) = \frac{1 + w_w}{1.2jw_w - 2.8w^2{}_w}.$$

The magnitude and the phase are then given by

$$|G(jw_w)| = \frac{1 + w_w}{\sqrt{(1.2w_w)^2 + (2.8w^2{}_w)^2}}$$

and

$$\angle G(jw_w) = \tan^{-1}\frac{1.2}{2.8w_w},$$

where w_w is related to w by the expression

$$w_w = \tan\left(\frac{wT}{2}\right).$$

8.6 BODE DIAGRAMS

The Bode diagrams used in the analysis of continuous-time systems are not very practical when used directly in the z-plane. This is because of the e^{jwT} term present in the sampled data system transfer functions when the frequency response is to be obtained. However, it is possible to draw the Bode diagrams of sampled data systems by transforming the system into the w-plane by making the substitution

$$z = \frac{1 + w}{1 - w}, \tag{8.7}$$

where the frequency in the w-plane (w_w) is related to the frequency in the s-plane (w) by the expression

$$w_w = \tan\left(\frac{wT}{2}\right). \tag{8.8}$$

It is common in practice to use a similar transformation to the one given above, known as the w'-plane transformation, which gives a closer analogy between the frequency in the s-plane and the w'-*plane*. The w'-*plane* transformation defined as

$$w' = \frac{2}{T}\frac{z - 1}{z + 1}, \tag{8.9}$$

or

$$z = \frac{1 + (T/2)w'}{1 - (T/2)w'}, \tag{8.10}$$

and the frequencies in the two planes are related by the expression

$$w' = \frac{2}{T}\tan\frac{wT}{2}. \tag{8.11}$$

Note that for small values of the real frequency (s-plane frequency) such that wT is small, (8.11) reduces to

$$w' = \frac{2}{T}\tan\frac{wT}{2} \approx \frac{2}{T}\left(\frac{wT}{2}\right) = w \tag{8.12}$$

Thus, the w'-plane frequency is approximately equal to the s-plane frequency. This approximation is only valid for small values of wT such that $\tan(wT/2) \approx wT$, i.e.

$$\frac{wT}{2} \leq \frac{\pi}{10},$$

which can also be written as

$$w \leq \frac{2\pi}{10T}$$

or

$$w \leq \frac{w_s}{10}, \tag{8.13}$$

where w_s is the sampling frequency in radians per second. The interpretation of this is that the w'-plane and the s-plane frequencies will be approximately equal when the frequency is less than one-tenth of the sampling frequency.

We can use the transformations given in (8.10) and (8.11) to transform a sampled data system into the w'-plane and then use the standard continuous system Bode diagram analysis.

Some example Bode plots for sampled data systems are given below.

Example 8.13

Consider the closed-loop sampled data system given in Figure 8.11. Draw the Bode diagram and determine the stability of this system. Assume that $T = 0.1$ s.

Solution

From Figure 8.11,

$$G(z) = Z\left\{\frac{1-e^{-sT}}{s}\frac{5}{s+5}\right\} = \frac{1-e^{-0.5}}{z-e^{-0.5}},$$

or

$$G(z) = \frac{0.393}{z-0.606}.$$

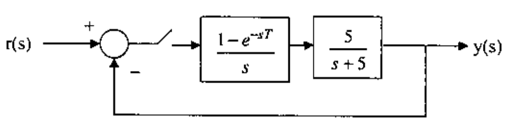

Figure 8.11 Closed-loop system

Transforming the system into the w'-plane gives

$$G(w') = \frac{0.393 - 0.0196w}{0.08w - 0.393},$$

or

$$G(w') = \frac{4.9(1-0.05w')}{w'+4.9}.$$

The magnitude of the frequency response and the phase can now be calculated if we set $w' = jv$ where v is the analogue of true frequency ω. Thus,

$$G(jv) = \frac{4.9(1-0.05jv)}{jv+4.9}.$$

The magnitude is

$$|G(jv)| = \frac{4.9\sqrt{1+(0.25v)^2}}{\sqrt{v^2+4.9^2}}$$

and

$$\angle G(jv) = -\tan^{-1}(0.05) - \tan^{-1}\left(\frac{v}{4.9}\right).$$

The Bode diagram of the system is shown in Figure 8.12. The system is stable.

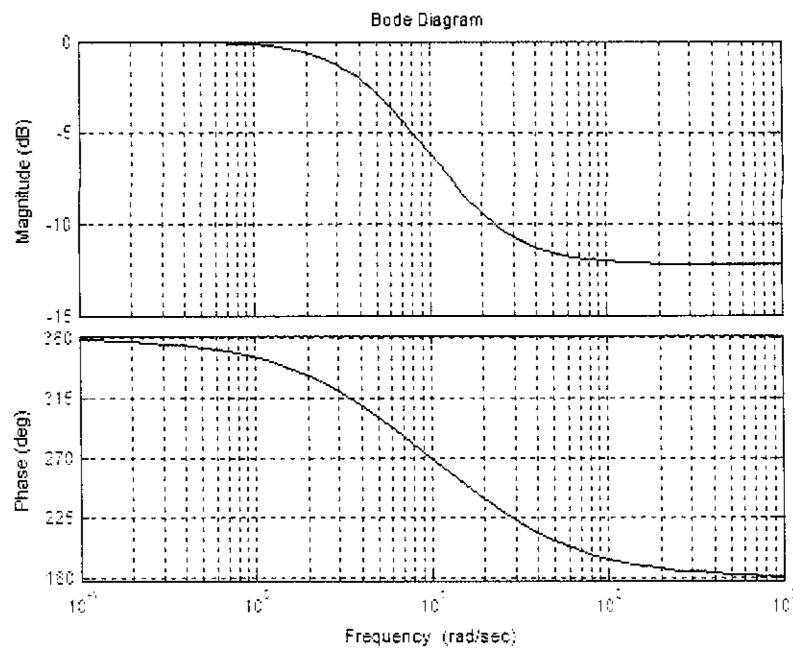

Figure 8.12 Bode diagram of the system

Example 8.14

The loop transfer function of a unity feedback sampled data system is given by

$$G(z) = \frac{0.368z + 0.264}{z^2 - 1.368z + 0.368}.$$

Draw the Bode diagram and analyse the stability of the system. Assume that $T = 1$ s.

Solution

Using the transformation

$$z = \frac{1 + (T/2)w}{1 - (T/2)w} = \frac{1 + 0.5w}{1 - 0.5w},$$

we get

$$G(w) = \frac{0.368(1 + 0.5w/1 - 0.5w) + 0.264}{(1 + 0.5w/1 - 0.5w)^2 - 1.368(1 + 0.5w/1 - 0.5w) + 0.368}$$

or

$$G(w) = -\frac{0.0381(w - 2)(w + 12.14)}{w(w + 0.924)}.$$

To obtain the frequency response, we can replace w with jv, giving

$$G(jv) = -\frac{0.0381(jv - 2)(jv + 12.14)}{jv(jv + 0.924)}.$$

The magnitude is then

$$|G(jv)| = \frac{0.0381\sqrt{v^2 + 2^2}\sqrt{v^2 + 12.14^2}}{v\sqrt{v^2 + 0.924^2}}$$

and

$$\angle G(jv) = \tan^{-1}\frac{v}{2} + \tan^{-1}\frac{v}{12.14} - 90 - \tan^{-1}\frac{v}{0.924}.$$

The Bode diagram is shown in Figure 8.13. The system is stable with a gain margin of 5 dB and a phase margin of 26°.

8.7 EXERCISES

1. Given below are the characteristic equations of some sampled data systems. Using Jury's test, determine if the systems are stable.
   (a) $z^2 - 1.8z + 0.72 = 0$
   (b) $z^2 - 0.5z + 1.2 = 0$
   (c) $z^3 - 2.1z^2 + 2.0z - 0.5 = 0$
   (d) $z^3 - 2.3z^2 + 1.61z - 0.32 = 0$
2. The characteristic equation of a sampled data system is given by

   $$(z - 0.5)(z^2 - 0.5z + 1.2) = 0.$$

   Determine the stability of the system.

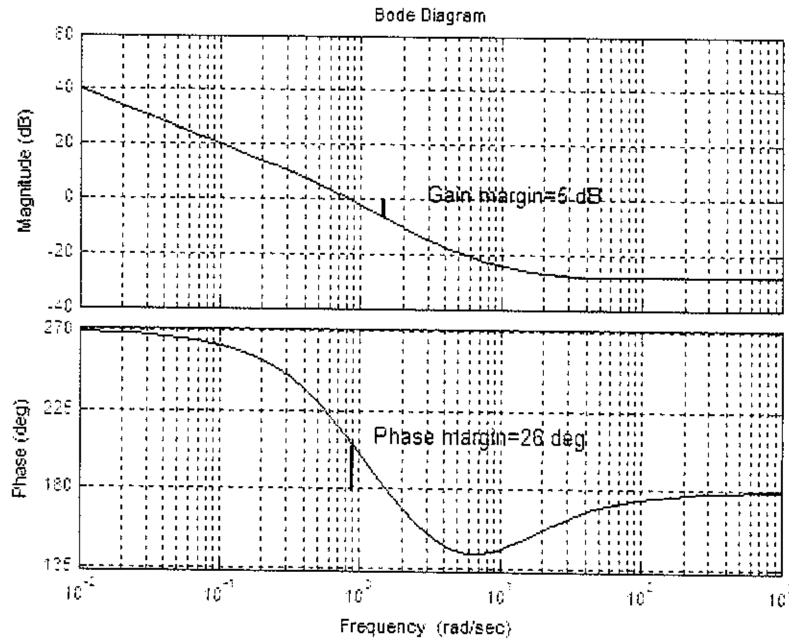

Figure 8.13 Bode diagram of the system

3. For the system shown in Figure 8.14, determine the range of K for stability using Jury's test
4. Repeat Exercise 3 using the Routh–Hurwitz criterion.
5. Repeat Exercise 3 using the root locus.
6. The forward gain of a unity feedback sampled data system is given by

   $$G(z) = \frac{K(z - 0.2)}{(z - 0.8)(z - 0.6)}.$$

   (a) Write an expression for the closed-loop transfer function of the system.
   (b) Draw the root locus of the system and hence determine the stability.

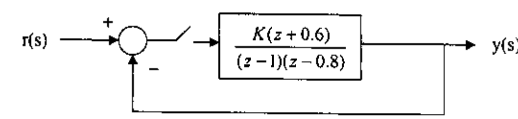

Figure 8.14 System for Exercise 3

9

Discrete Controller Design

The design of a digital control system begins with an accurate model of the process to be controlled. Then a control algorithm is developed that will give the required system response. The loop is closed by using a digital computer as the controller. The computer implements the control algorithm in order to achieve the required response.

Several methods can be used for the design of a digital controller:

- A system transfer function is modelled and obtained in the s-plane. The transfer function is then transformed into the z-plane and the controller is designed in the z-plane.
- System transfer function is modelled as a digital system and the controller is directly designed in the z-plane.
- The continuous system transfer function is transformed into the w-plane. A suitable controller is then designed in the w-plane using the well-established time response (e.g. root locus) or frequency response (e.g. Bode diagram) techniques. The final design is transformed into the z-plane and the algorithm is implemented on the digital computer.

In this chapter we are mainly interested in the design of a digital controller using the first method, i.e. the controller is designed directly in the z-plane.

The procedure for designing the controller in the z-plane can be outlined as follows:

- Derive the transfer function of the system either by using a mathematical approach or by performing a frequency or a time response analysis.
- Transform the system transfer function into the z-plane.
- Design a suitable digital controller in the z-plane.
- Implement the controller algorithm on a digital computer.

A discrete-time system can be in many different forms, depending on the type of input and the type of sensor used. Figure 9.1 shows a discrete-time system where the reference input is an analog signal, and the process output is also an analog signal. Analog-to-digital converters are then used to convert these signals into digital form so that they can be processed by a digital computer. A zero-order hold at the output of the digital controller approximates a D/A converter which produces an analog signal to drive the plant.

In Figure 9.2 the reference input is a digital signal, which is usually set using a keyboard or can be hard-coded into the controller algorithm. The feedback signal is also digital and the

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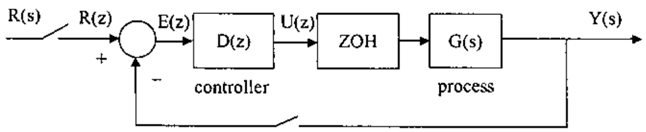

Figure 9.1 Discrete-time system with analog reference input

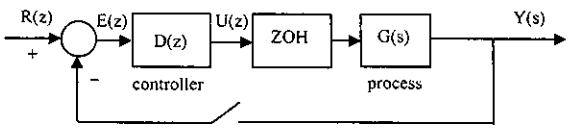

Figure 9.2 Discrete-time system with digital reference input

error signal is formed by the computer after subtracting the feedback signal from the reference input. The digital controller then implements the control algorithm and derives the plant.

9.1 DIGITAL CONTROLLERS

In general, we can make use of the block diagram shown in Figure 9.3 when designing a digital controller. In this figure, $R(z)$ is the reference input, $E(z)$ is the error signal, $U(z)$ is the output of the controller, and $Y(z)$ is the output of the system. $HG(z)$ represents the digitized plant transfer function together with the zero-order hold.

The closed-loop transfer function of the system in Figure 9.3 can be written as

$$\frac{Y(z)}{R(z)} = \frac{D(z)HG(z)}{1 + D(z)HG(z)}. \tag{9.1}$$

Now, suppose that we wish the closed-loop transfer function to be $T(z)$, i.e.

$$T(z) = \frac{Y(z)}{R(z)}. \tag{9.2}$$

Then the required controller that will give this closed-loop response can be found by using (9.1) and (9.2):

$$D(z) = \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)}. \tag{9.3}$$

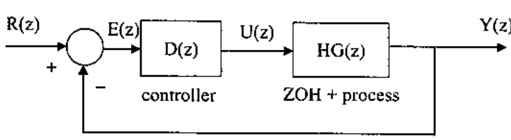

Figure 9.3 Discrete-time system

Equation (9.3) states that the required controller $D(z)$ can be designed if we know the model of the process. The controller $D(z)$ must be chosen so that it is stable and can be realized. One of the restrictions affecting realizability is that $D(z)$ must not have a numerator whose order exceeds that of the numerator. Some common controllers based on (9.3) are described below.

9.1.1 Dead-Beat Controller

The dead-beat controller is one in which a step input is followed by the system but delayed by one or more sampling periods, i.e. the system response is required to be equal to unity at every sampling instant after the application of a unit step input.

The required closed-loop transfer function is then

$$T(z) = z^{-k}, \quad \text{where } k \geq 1. \tag{9.4}$$

From (9.3), the required digital controller transfer function is

$$D(z) = \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)} = \frac{1}{HG(z)} \left(\frac{z^{-k}}{1 - z^{-k}} \right). \tag{9.5}$$

An example design of a controller using the dead-beat algorithm is given below.

Example 9.1

The open-loop transfer function of a plant is given by

$$G(s) = \frac{e^{-2s}}{1 + 10s}.$$

Design a dead-beat digital controller for the system. Assume that $T = 1$ s.

Solution

The transfer function of the system with a zero-order hold is given by

$$HG(z) = Z \left\{ \frac{1 - e^{-sT}}{s} G(s) \right\} = (1 - z^{-1}) Z \left\{ \frac{e^{-2s}}{s(1 + 10s)} \right\}$$

or

$$HG(z) = (1 - z^{-1}) z^{-2} Z \left\{ \frac{1}{s(1 + 10s)} \right\} = (1 - z^{-1}) z^{-2} Z \left\{ \frac{1/10}{s(s + 1/10)} \right\}.$$

From z-transform tables we obtain

$$HG(z) = (1 - z^{-1}) z^{-2} \frac{z(1 - e^{-0.1})}{(z - 1)(z - e^{-0.1})} = z^{-3} \frac{(1 - e^{-0.1})}{1 - e^{-0.1} z^{-1}}$$

or

$$HG(z) = \frac{0.095 z^{-3}}{1 - 0.904 z^{-1}}.$$

From Equations (9.3) and (9.5),

$$D(z) = \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)} = \frac{1 - 0.904 z^{-1}}{0.095 z^{-3}} \frac{z^{-k}}{1 - z^{-k}}.$$

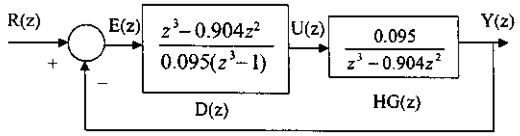

Figure 9.4 Block diagram of the system of Example 9.1

For realizability, we can choose $k \geq 3$. Choosing $k = 3$, we obtain

$$D(z) = \frac{1 - 0.904z^{-1}}{0.095z^{-3}} \frac{z^{-3}}{1 - z^{-3}}$$

or

$$D(z) = \frac{z^3 - 0.904z^2}{0.095(z^3 - 1)}.$$

Figure 9.4 shows the system block diagram with the controller, while Figure 9.5 shows the step response of the system. The output response is unity after 3 s (third sample) and stays at this value. It is important to realize that the response is correct only at the sampling instants and the response can have an oscillatory behaviour between the sampling instants.

The control signal applied to the plant is shown in Figure 9.6. Although the dead-beat controller has provided an excellent response, the magnitude of the control signal may not be acceptable, and it may even saturate in practice.

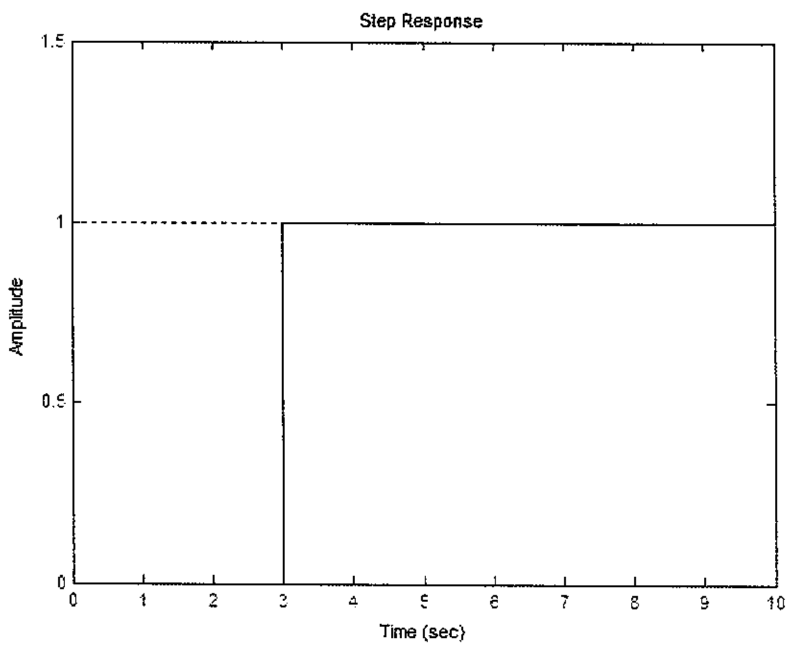

Figure 9.5 Step response of the system

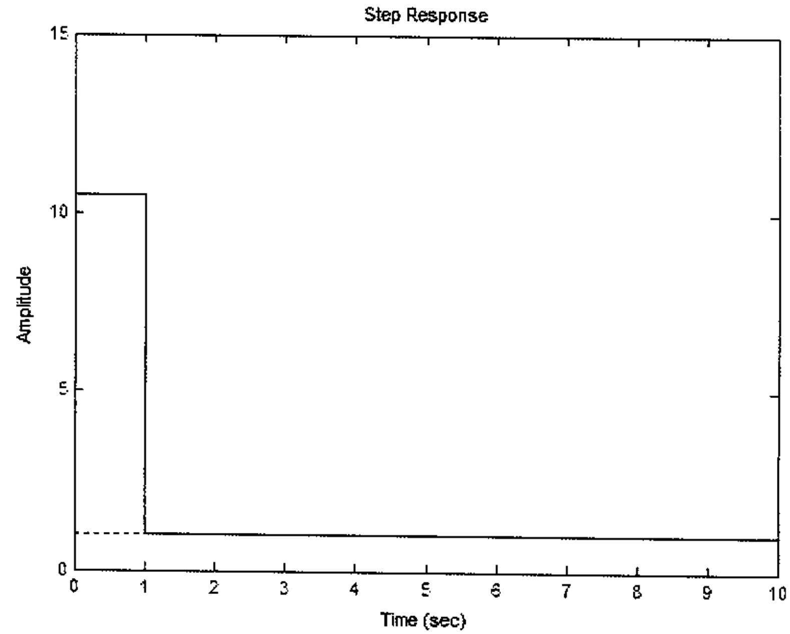

Figure 9.6 Control signal

The dead-beat controller is very sensitive to plant characteristics and a small change in the plant may lead to ringing or oscillatory response.

9.1.2 Dahlin Controller

The Dahlin controller is a modification of the dead-beat controller and produces an exponential response which is smoother than that of the dead-beat controller.

The required response of the system in the s-plane can be shown to be

$$Y(s) = \frac{1}{s} \frac{e^{-as}}{1 + sq},$$

where a and q are chosen to give the required response (see Figure 9.7). If we let $a = kT$, then the z-transform of the output is

$$Y(z) = \frac{z^{-k-1}(1 - e^{-T/q})}{(1 - z^{-1})(1 - e^{-T/q}z^{-1})}$$

and the required transfer function is

$$T(z) = \frac{Y(z)}{R(z)} = \frac{z^{-k-1}(1 - e^{-T/q})}{(1 - z^{-1})(1 - e^{-T/q}z^{-1})} \frac{(1 - z^{-1})}{1}$$

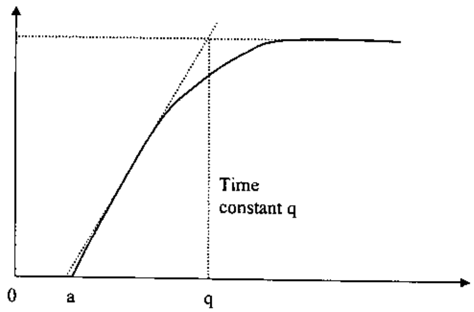

Figure 9.7 Dahlin controller response

or

$$T(z) = \frac{z^{-k-1}(1 - e^{-T/q})}{1 - e^{-T/q}z^{-1}}.$$

Using (9.3), we can find the transfer function of the required controller:

$$D(z) = \frac{1}{HG(z)}\frac{T(z)}{1 - T(z)} = \frac{1}{HG(Z)}\frac{z^{-k-1}(1 - e^{-T/q})}{1 - e^{-T/q}z^{-1} - (1 - e^{-T/q})z^{-k-1}}.$$

An example is given below to illustrate the use of the Dahlin controller.

Example 9.2

The open-loop transfer function of a plant is given by

$$G(s) = \frac{e^{-2s}}{1 + 10s}.$$

Design a Dahlin digital controller for the system. Assume that $T = 1$ s.

Solution

The transfer function of the system with a zero-order hold is given by

$$HG(z) = Z\left\{\frac{1 - e^{-sT}}{s}G(s)\right\} = (1 - z^{-1})Z\left\{\frac{e^{-2s}}{s(1 + 10s)}\right\}$$

or

$$HG(z) = (1 - z^{-1})z^{-2}Z\left\{\frac{1}{s(1 + 10s)}\right\} = (1 - z^{-1})z^{-2}Z\left\{\frac{1/10}{s(s + 1/10)}\right\}.$$

From z-transform tables we obtain

$$HG(z) = (1 - z^{-1})z^{-2}\frac{z(1 - e^{-0.1})}{(z - 1)(z - e^{-0.1})} = z^{-3}\frac{(1 - e^{-0.1})}{1 - e^{-0.1}z^{-1}}$$

or

$$HG(z) = \frac{0.095z^{-3}}{1 - 0.904z^{-1}}.$$

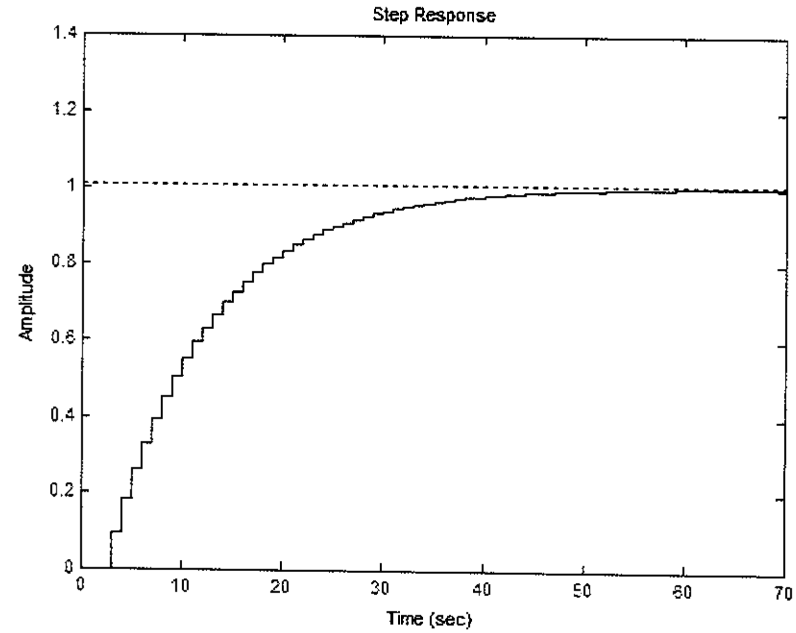

Figure 9.8 System response with Dahlin controller

For the controller, if we choose $q = 10$, then

$$D(z) = \frac{1}{HG(z)}\frac{T(z)}{1 - T(z)} = \frac{1 - 0.904z^{-1}}{0.095z^{-3}}\frac{z^{-k-1}(1 - e^{-0.1})}{1 - e^{-0.1}z^{-1} - (1 - e^{-0.1})z^{-k-1}}$$

or

$$D(z) = \frac{1 - 0.904z^{-1}}{0.095z^{-3}}\frac{0.095z^{-k-1}}{1 - 0.904z^{-1} - 0.095z^{-k-1}}$$

For realizability, if we choose $k = 2$, we obtain

$$D(z) = \frac{0.095z^3 - 0.0858z^2}{0.095z^3 - 0.0858z^2 - 0.0090}.$$

Figure 9.8 shows the step response of the system. It is clear that the response is exponential as expected.

The response of the controller is shown in Figure 9.9. Although the system response is slower, the controller signal is more acceptable.

9.1.3 Pole-Placement Control – Analytical

The response of a system is determined by the positions of its closed-loop poles. Thus, by placing the poles at the required points we should be able to control the response of a system.

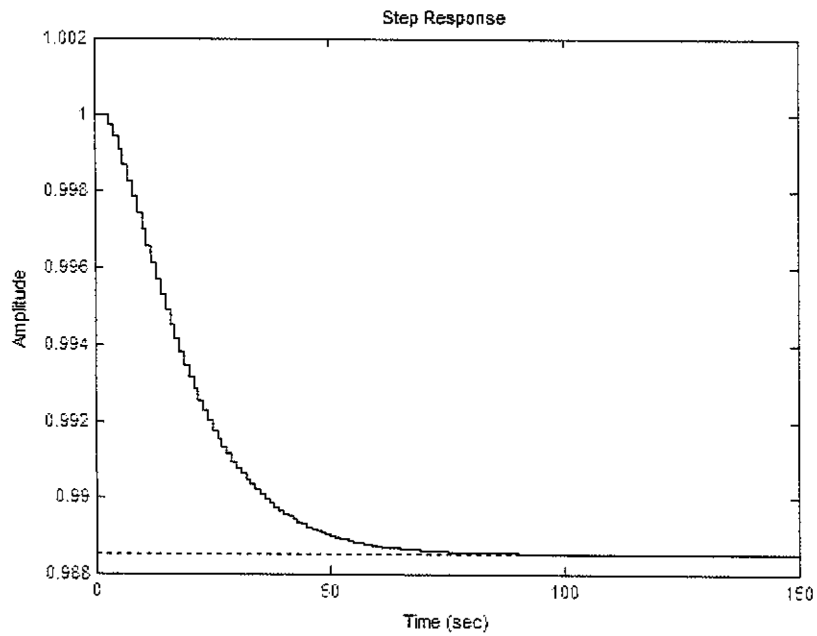

Figure 9.9 Controller response

Given the pole positions of a system, (9.3) gives the required transfer function of the controller as

$$D(z) = \frac{1}{HG(z)} \frac{T(z)}{1 - T(z)}.$$

$T(z)$ is the required transfer function, which is normally in the form of a polynomial. The denominator of $T(z)$ is constructed from the positions of the required roots. The numerator polynomial can then be selected to satisfy certain criteria in the system. An example is given below.

Example 9.3

The open-loop transfer function of a system together with a zero-order hold is given by

$$HG(z) = \frac{0.03(z + 0.75)}{z^2 - 1.5z + 0.5}.$$

Design a digital controller so that the closed-loop system will have $\zeta = 0.6$ and $w_d = 3$ rad/s. The steady-state error to a step input should be zero. Also, the steady-state error to a ramp input should be 0.2. Assume that $T = 0.2$ s.

Solution

The roots of a second-order system are given by

$$z_{1,2} = e^{-\zeta\omega_n T \pm j\omega_n T\sqrt{1-\zeta^2}} = e^{-\zeta w_n T}(\cos\omega_n T\sqrt{1-\zeta^2} \pm j\sin\omega_n T\sqrt{1-\zeta^2}).$$

Thus, the required pole positions are

$$z_{1,2} = e^{-0.6\times 3.75\times 0.2}(\cos(0.2\times 3) \pm j\sin(0.2\times 3)) = 0.526 \pm j0.360.$$

The required controller then has the transfer function

$$T(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots}{(z - 0.526 + j0.360)(z - 0.526 - j0.360)}$$

which gives

$$T(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + \ldots}{1 - 1.052 z^{-1} + 0.405 z^{-2}}. \tag{9.6}$$

We now have to determine the parameters of the numerator polynomial. To ensure realizability, $b_0 = 0$ and the numerator must only have the b_1 and $b_2$terms. Equation (9.6) then becomes

$$T(z) = \frac{b_1 z^{-1} + b_2 z^{-2}}{1 - 1.052 z^{-1} + 0.405 z^{-2}}. \tag{9.7}$$

The other parameters can be determined from the steady-state requirements.

The steady-state error is given by

$$E(z) = R(z)[1 - T(z)].$$

For a unit step input, the steady-state error can be determined from the final value theorem, i.e.

$$E_{ss} = \lim_{z\to 1} \frac{z-1}{z} \frac{z}{z-1}[v]$$

or

$$E_{ss} = 1 - T(1). \tag{9.8}$$

From (9.8), for a zero steady-state error to a step input,

$$T(1) = 1$$

From (9.7), we have

$$T(1) = \frac{b_1 + b_2}{0.353} = 1$$

or

$$b_1 + b_2 = 0.353, \tag{9.9}$$

and

$$T(z) = \frac{b_1 z + b_2}{z^2 - 1.052z + 0.405}. \tag{9.10}$$

If K_v is the system velocity constant, for a steady-state error to a ramp input we can write

$$E_{ss} = \lim_{z\to 1} \frac{(z-1)}{z} \frac{Tz}{(z-1)^2}[1 - T(z)] = \frac{1}{K_v}$$

or, using L'Hospital's rule,

$$\left.\frac{dT}{dz}\right|_{z=1} = -\frac{1}{K_v T}.$$

Thus from (9.10),

$$\left.\frac{dT}{dz}\right|_{z=1} = \frac{b_1(z^2 - 1.052z + 0.405) - (b_1 z + b_2)(2z - 1.052)}{(z^2 - 1.052z + 0.405)^2} = -\frac{1}{K_v T} = -\frac{0.2}{0.2} = -1,$$

giving

$$\frac{0.353b_1 - (b_1 + b_2)0.948}{0.353^2} = -1$$

or

$$0.595b_1 + 0.948b_2 = 0.124, \tag{9.11}$$

From (9.9) and (9.11) we obtain,

$$b_1 = 0.596 \text{ and } b_2 = -0.243.$$

Equation (9.10) then becomes

$$T(z) = \frac{0.596z - 0.243}{z^2 - 1.052z + 0.405}. \tag{9.12}$$

Equation (9.12) is the required transfer function. We can substitute in Equation (9.3) to find the transfer function of the controller:

$$D(z) = \frac{1}{HG(z)}\frac{T(z)}{1 - T(z)} = \frac{z^2 - 1.5z + 0.5}{0.03(z + 0.75)}\frac{T(z)}{1 - T(z)}$$

or,

$$D(z) = \frac{z^2 - 1.5z + 0.5}{0.03(z + 0.75)}\frac{0.596z - 0.243}{z^2 - 1.648z + 0.648}$$

which can be written as

$$D(z) = \frac{0.596z^3 - 1.137z^2 + 0.662z - 0.121}{0.03z^3 - 0.027z^2 - 0.018z + 0.015} \tag{9.13}$$

The step response of the system with the controller is shown in Figure 9.10.

9.1.4 Pole-Placement Control – Graphical

In the previous subsection we saw how the response of a closed-loop system can be shaped by placing its poles at required points in the z-plane. In this subsection we will be looking at some examples of pole placement using the root-locus graphical approach.

When it is required to place the poles of a system at required points in the z-plane we can either modify the gain of the system or use a dynamic compensator (such as a phase lead or a phase lag). Given a first-order system, we can modify only the d.c. gain to achieve the required time constant. For a second-order system we can generally modify the d.c. gain to achieve a constant damping ratio greater than or less than a required value, and, depending on the system, we may also be able to design for a required natural frequency by simply varying the d.c. gain. For more complex requirements, such as placing the system poles at specific points in the z-plane, we will need to use dynamic compensators, and a simple gain adjustment alone

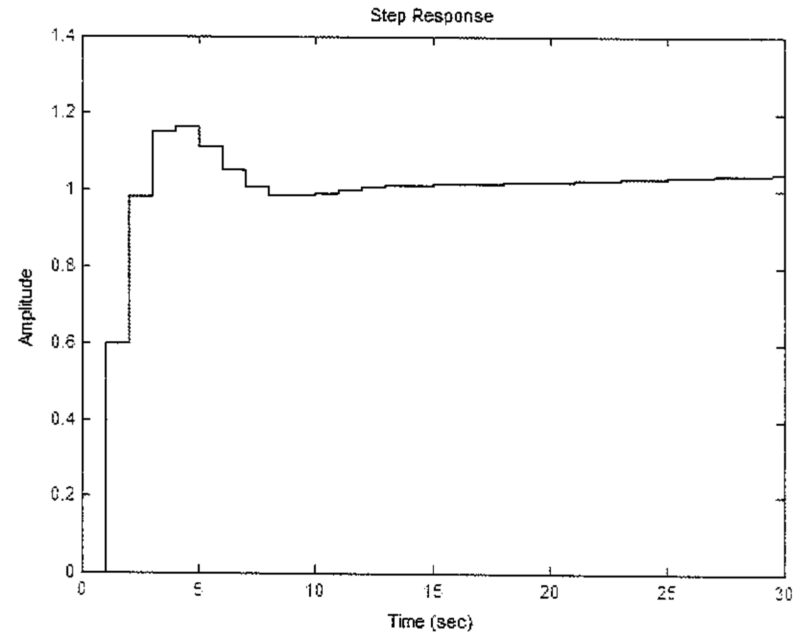

Figure 9.10 Step response of the system

will not be adequate. Some example pole-placement techniques are given below using the root locus approach.

Example 9.4

The block diagram of a sampled data control system is shown in Figure 9.11. Find the value of d.c. gain K which yields a damping ratio of $\zeta = 0.7$.

Solution

In this example, we will draw the root locus of the system as the gain K is varied, and then we will superimpose the lines of constant damping ratio on the locus. The value of K for the required damping ratio can then be read from the locus.

The root locus of the system is shown in Figure 9.12. The locus has been expanded for clarity between the real axis points $(-1, 1)$ and the imaginary axis points $(-1, 1)$, and the lines of constant damping ratio are shown in Figure 9.13. A vertical and a horizontal line are

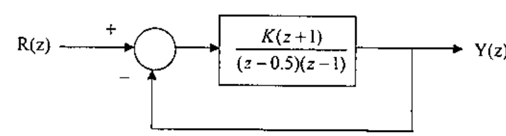

Figure 9.11 Block diagram for Example 9.4

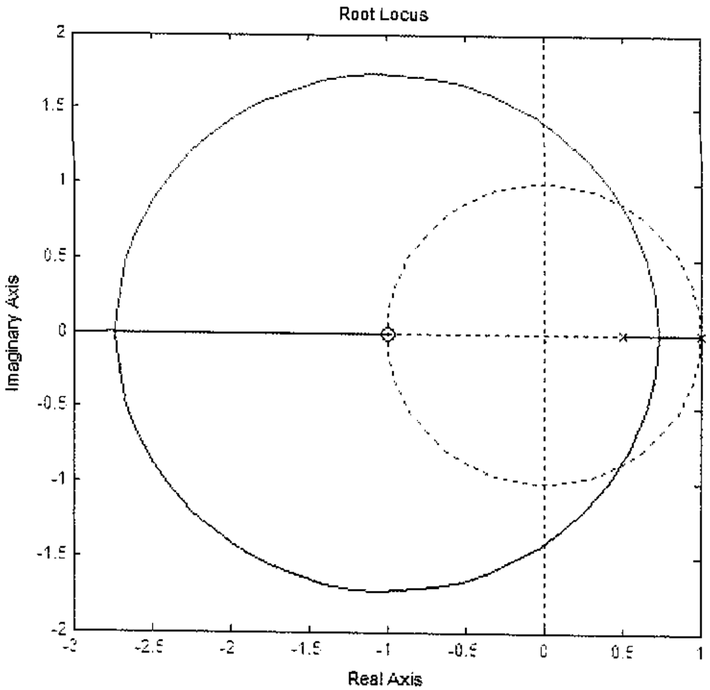

Figure 9.12 Root locus of the system

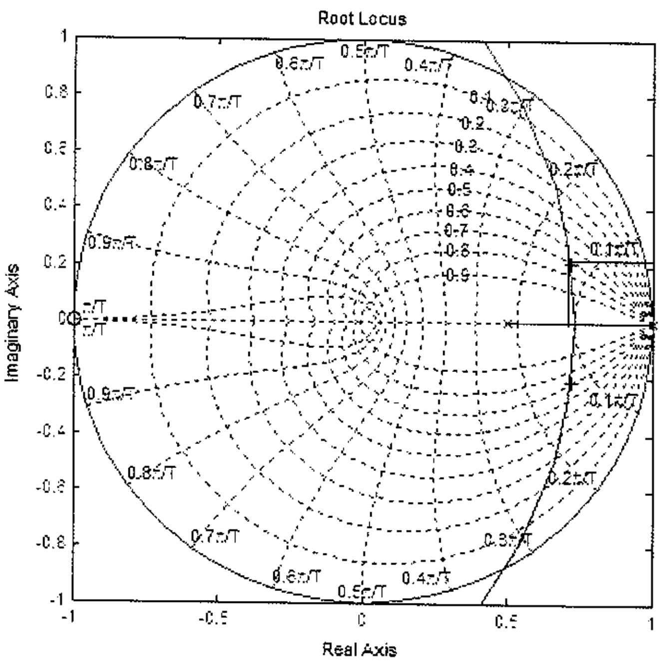

Figure 9.13 Root locus with the lines of constant damping ratio

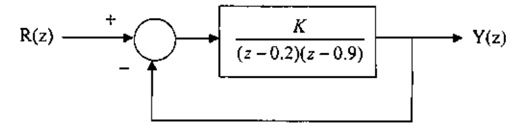

Figure 9.14 Block diagram for Example 9.5

drawn from the point where the damping factor is 0.7. At the required point the roots are $z_{1,2} = 0.7191 \pm j0.2114$. The value of K at this point is calculated to be $K = 0.0618$.

In this example, the required specification was obtained by simply modifying the d.c. gain of the system. A more complex example is given below where it is required to place the poles at specific points in the z-plane.

Example 9.5

The block diagram of a digital control system is given in Figure 9.14. It is required to design a controller for this system such that the system poles are at the points $z_{1,2} = 0.3 \pm j0.3$.

Solution

In this example, we will draw the root locus of the system and then use a dynamic compensator to modify the shape of the locus so that it passes through the required points in the z-plane.

The root locus of the system without the compensator is shown in Figure 9.15. The point where we want the roots to be is marked with a $\times$ and clearly the locus will not pass through this point by simply modifying the d.c. gain K of the system.

The angle of $G(z)$ at the required point is

$$\angle G(z) = -\angle(0.3 + j0.3 - 0.2) - \angle(0.3 + j0.3 - 0.9)$$

or

$$\angle G(z) = -\tan^{-1}\frac{0.3}{0.1} - \tan^{-1}\frac{0.3}{-0.6} = -45^\circ.$$

Since the sum of the angles at a point in the root locus must be a multiple of -180°, the compensator must introduce an angle of $-180 - (-45) = -135^\circ$. The required angle can be obtained using a compensator with a transfer function

$$D(Z) = \frac{z-n}{z-p}.$$

The angle introduced by the compensator is

$$\angle D(Z) = \angle(0.3 + j0.3 - n) - \angle(0.3 + j0.3 - p) = -135^\circ$$

or

$$\tan^{-1}\frac{0.3}{0.3-n} - \tan^{-1}\frac{0.3}{0.3-p} = -135^\circ.$$

There are many combinations of p and n which will give the required angle. For example, if we choose $n = 0.5$, then,

$$124^\circ - \tan^{-1}\frac{0.3}{0.3-p} = -135^\circ$$

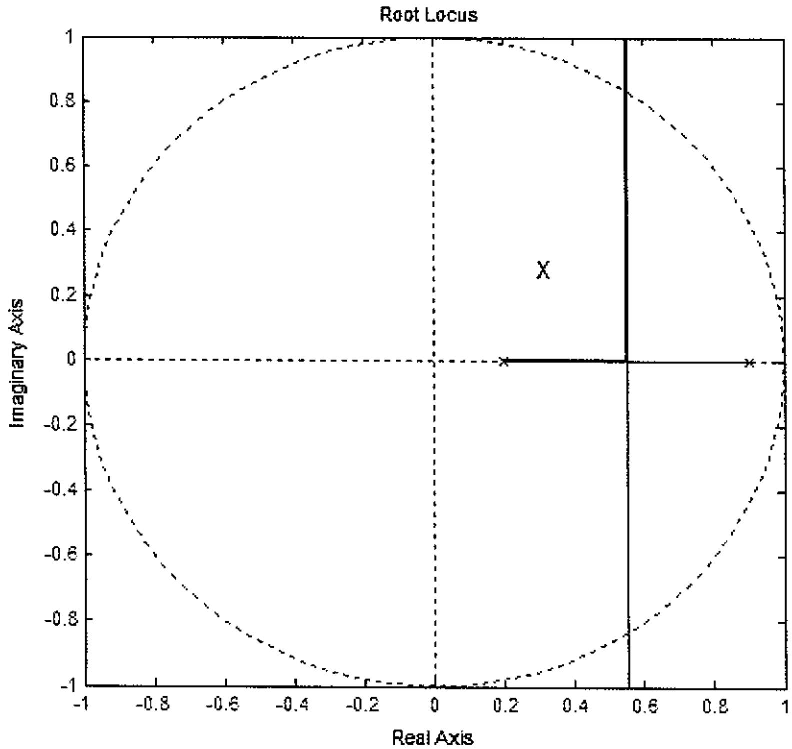

Figure 9.15 Root locus of the system without compensator

or

$$p = 0.242.$$

The required controller transfer function is then

$$D(z) = \frac{z - 0.5}{z - 0.242}.$$

The compensator introduces a zero at $z = 0.5$ and a pole at $z = 0.242$. The root locus of the compensated system is shown in Figure 9.16. Clearly the new locus passes through the required points $z_{1,2} = 0.3 \pm j0.3$, and it will be at these points that the d.c. gain is $K = 0.185$. The step response of the system with the compensator is shown in Figure 9.17. It is clear from this diagram that the system has a steady-state error.

The block diagram of the controller and the system is given in Figure 9.18.

Example 9.6

The block diagram of a system is as shown in Figure 9.19. It is required to design a controller for this system with percent overshoot (PO) less than 17 % and settling time $t_s \le 10$ s. Assume that $T = 0.1$ s.

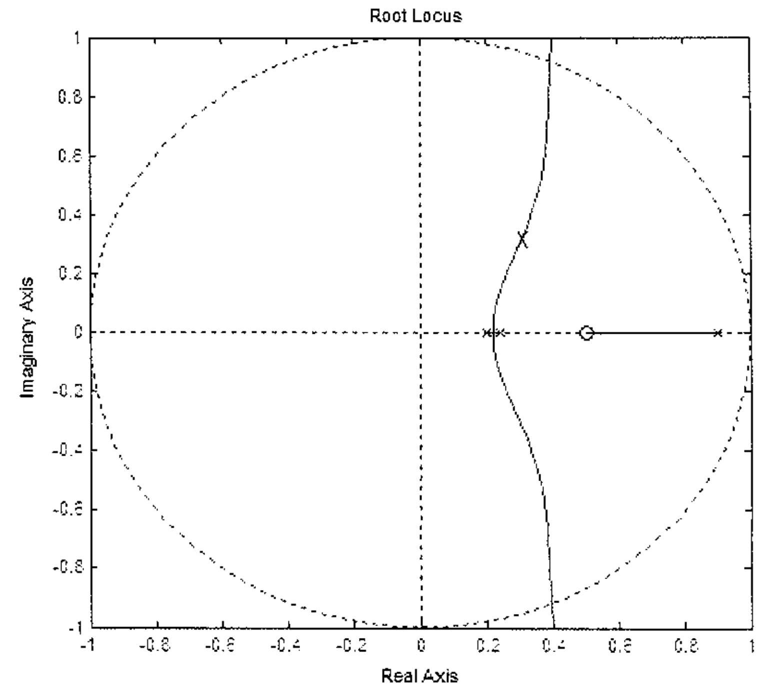

Figure 9.16 Root locus of the compensated system

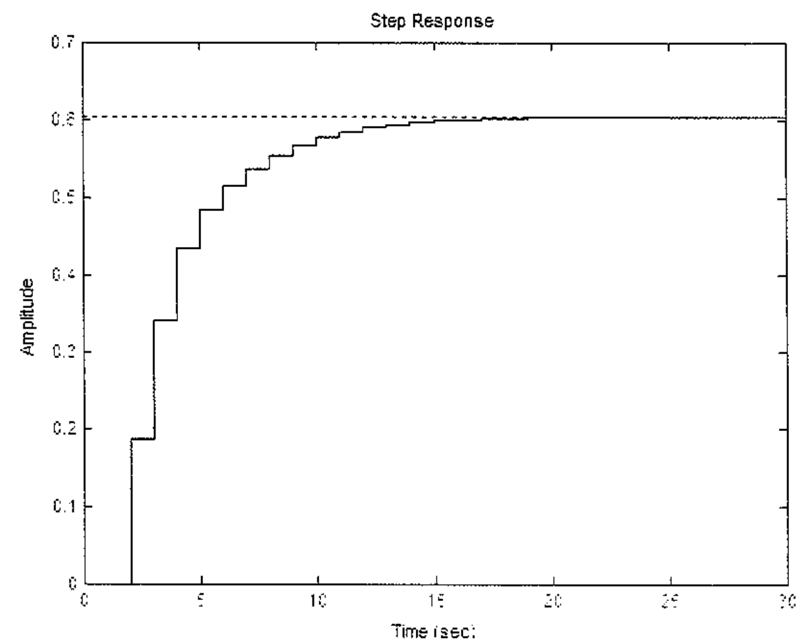

Figure 9.17 Step response of the system

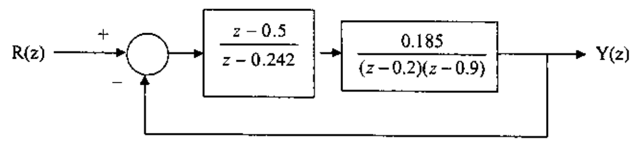

Figure 9.18 Block diagram of the controller and the system

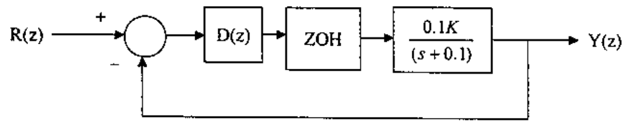

Figure 9.19 Block diagram for Example 9.6

Solution

The damping ratio, natural frequency and hence the required root positions can be determined as follows:

$$\text{For } PO < 17\%, \quad \zeta \geq 0.5.$$

$$\text{For } t_s \leq 10, \quad \zeta\omega_n \geq \frac{4.6}{t_s} \quad \text{or} \quad \omega_n \geq 0.92 \text{ rad/s}.$$

Hence, the required pole positions are found to be

$$z_{1,2} = e^{-\zeta\omega_n T}\left(\cos \omega_n T\sqrt{1-\zeta^2} + j \sin \omega_n T\sqrt{1-\zeta^2}\right)$$

or

$$z_{1,2} = 0.441 \pm j0.451.$$

The z-transform of the plant, together with the zero-order hold, is given by

$$G(z) = \frac{z-1}{z} Z\left[\frac{0.1K}{s^2(s+0.1)}\right] = \frac{0.00484K(z+0.9672)}{(z-1)(z-0.9048)}.$$

The root locus of the uncompensated system and the required root position is shown in Figure 9.20.

It is clear from the figure that the root locus will not pass through the marked point by simply changing the d.c. gain. We can design a compensator as in Example 9.5 such that the locus passes through the required point, i.e.

$$D(Z) = \frac{z-n}{z-p}.$$

The angle of $G(z)$ at the required point is

$$\angle G(z) = \angle 0.441 + j0.451 + 0.9672 - \angle(0.441 + j0.451 - 1) - \angle(0.441 + j0.451 - 0.9048)$$

or

$$\angle G(z) = \tan^{-1}\frac{0.451}{1.4082} - \tan^{-1}\frac{0.451}{-0.559} - \tan^{-1}\frac{0.451}{-0.4638} = -259°.$$

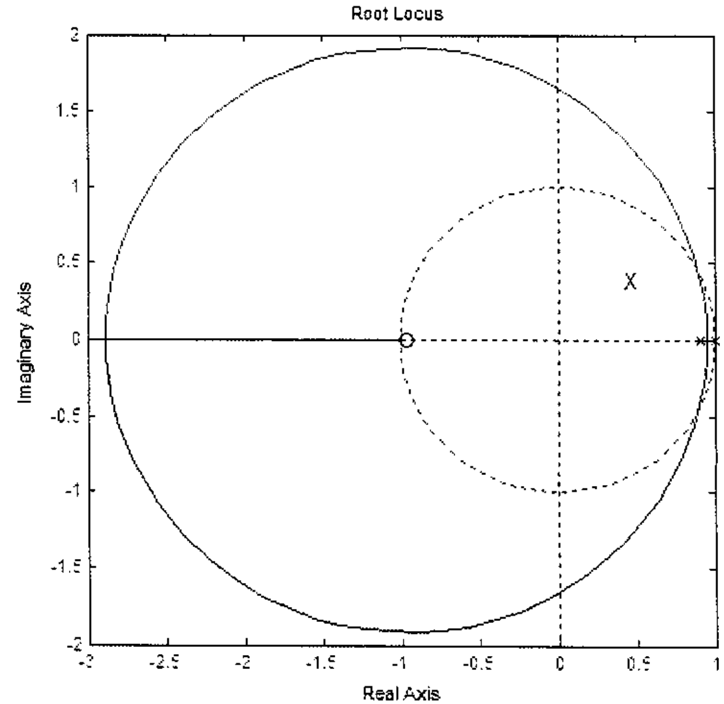

Figure 9.20 Root locus of uncompensated system

Since the sum of the angles at a point in root locus must be a multiple of $-180°$, the compensator must introduce an angle of $-180 - (-259) = 79°$. The required angle can be obtained using a compensator with a transfer function, and the angle introduced by the compensator is

$$\angle D(Z) = \angle(0.441 + j0.451 - n) - \angle(0.441 + j0.451 - p) = 79°$$

or

$$\tan^{-1}\frac{0.451}{0.441-n} - \tan^{-1}\frac{0.451}{0.441-p} = 79°.$$

If we choose $n = 0.6$, then

$$109° - \tan^{-1}\frac{0.451}{0.441-p} = 79°$$

or

$$p = -0.340.$$

The transfer function of the compensator is thus

$$D(z) = \frac{z-0.6}{z+0.340}.$$

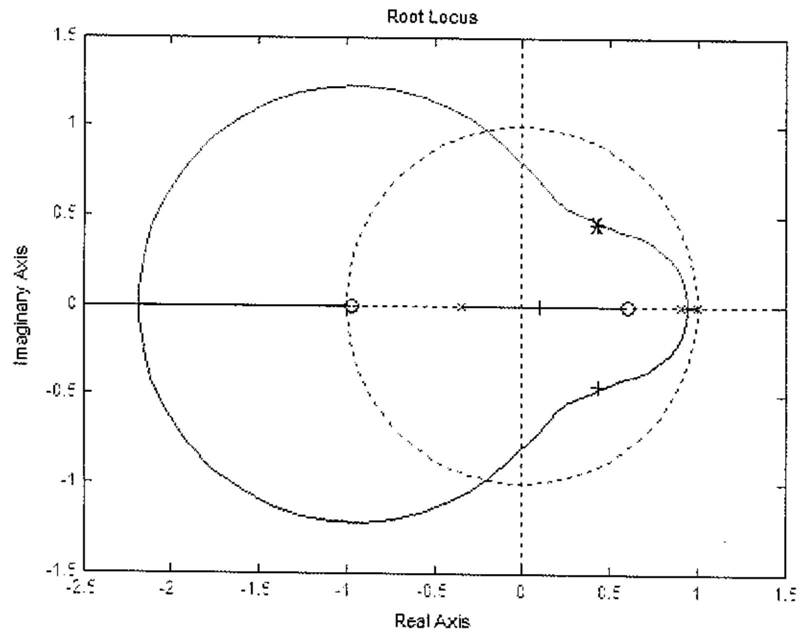

Figure 9.21 Root locus of the compensated system

Figure 9.21 shows the root locus of the compensated system. Clearly the locus passes through the required point. The d.c. gain at this point is $K = 123.9$.

The time response of the compensated system is shown in Figure 9.22.

9.2 PID CONTROLLER

The proportional–integral–derivative (PID) controller is often referred to as a 'three-term' controller. It is currently one of the most frequently used controllers in the process industry. In a PID controller the control variable is generated from a term proportional to the error, a term which is the integral of the error, and a term which is the derivative of the error.

- *Proportional*: the error is multiplied by a gain K_p. A very high gain may cause instability, and a very low gain may cause the system to drift away.
- *Integral*: the integral of the error is taken and multiplied by a gain K_i. The gain can be adjusted to drive the error to zero in the required time. A too high gain may cause oscillations and a too low gain may result in a sluggish response.
- *Derivative*: The derivative of the error is multiplied by a gain K_d. Again, if the gain is too high the system may oscillate and if the gain is too low the response may be sluggish.

Figure 9.23 shows the block diagram of the classical continuous-time PID controller. Tuning the controller involves adjusting the parameters K_p, K_d and K_i in order to obtain a satisfactory

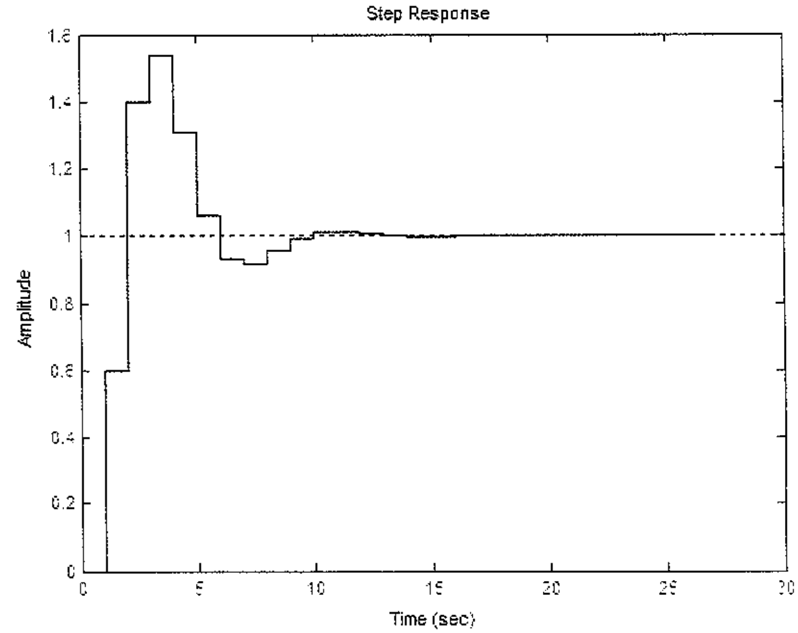

Figure 9.22 Time response of the system

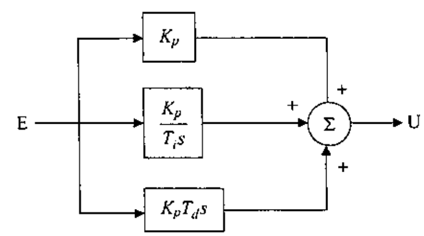

Figure 9.23 Continuous-time system PID controller

response. The characteristics of PID controllers are well known and well established, and most modern controllers are based on some form of PID.

The input–output relationship of a PID controller can be expressed as

$$u(t) = K_p \left[e(t) + \frac{1}{T_i} \int_0^t e(t)dt + T_d \frac{de(t)}{dt} \right], \tag{9.14}$$

where $u(t)$ is the output from the controller and $e(t) = r(t) - y(t)$, in which $r(t)$ is the desired set-point (reference input) and $y(t)$ is the plant output. T_i and T_d are known as the integral and derivative action time, respectively. Notice that (9.14) is sometimes written as

$$u(t) = K_p e(t) + K_i \int_0^t e(t)dt + K_d \frac{de(t)}{dt} + u_0, \tag{9.15}$$

where

$$K_i = \frac{K_p}{T_i} \quad \text{and} \quad K_d = K_p T_d. \tag{9.16}$$

Taking the Laplace transform of (9.14), we can write the transfer function of a continuous-time PID as

$$\frac{U(s)}{E(s)} = K_p + \frac{K_p}{T_i s} + K_p T_d s. \tag{9.17}$$

To implement the PID controller using a digital computer we have to convert (9.14) from a continuous to a discrete representation. There are several methods for doing this and the simplest is to use the trapezoidal approximation for the integral and the backward difference approximation for the derivative:

$$\frac{de(t)}{dt} \approx \frac{e(kT) - e(kT - T)}{T} \quad \text{and} \quad \int_0^t e(t)dt \approx \sum_{k=1}^{n} T\, e(kT).$$

Equation (9.14) thus becomes

$$u(kT) = K_p \left[e(kT) + T_d \frac{e(kT) - e(kT - T)}{T} + \frac{T}{T_i} \sum_{k=1}^{n} e(kT) \right] + u_0. \tag{9.18}$$

The PID given by (9.18) is now in a suitable form which can be implemented on a digital computer. This form of the PID controller is also known as the *positional* PID controller. Notice that a new control action is implemented at every sample time.

The discrete form of the PID controller can also be derived by finding the z-transform of (9.17):

$$\frac{U(z)}{E(z)} = K_p \left[1 + \frac{T}{T_i(1 - z^{-1})} + T_d \frac{(1 - z^{-1})}{T} \right]. \tag{9.19}$$

Expanding (9.19) gives

$$u(kT) = u(kT - T) + K_p[e(kT) - e(kT - T)] + \frac{K_p T}{T_i} e(kT)$$
$$+ \frac{K_p T_d}{T}[e(kT) - 2e(kT - T) - e(kT - 2T)]. \tag{9.20}$$

This form of the PID controller is known as the *velocity* PID controller. Here the current control action uses the previous control value as a reference. Because only a change in the control action is used, this form of the PID controller provides a smoother bumpless control when the error is small. If a large error exists, the response of the velocity PID controller may be slow, especially if the integral action time T_i is large.

The two forms of the PID algorithm, (9.18) and (9.20), may look quite different, but they are in fact similar to each other. Consider the positional controller (9.18). Shifting back one

sampling interval, we obtain

$$u(kT - T) = K_p \left[e(kT - T) + T_d \frac{e(kT - T) - e(kT - 2T)}{T} + \frac{T}{T_i} \sum_{k=1}^{n-1} e(kT) \right] + u_0.$$

Subtracting from (9.18), we obtain the velocity form of the controller, as given by (9.20).

9.2.1 Saturation and Integral Wind-Up

In practical applications the output value of a control action is limited by physical constraints. For example, the maximum voltage output from a device is limited. Similarly, the maximum flow rate that a pump can supply is limited by the physical capacity of the pump. As a result of this physical limitation, the error signal does not return to zero and the integral term keeps adding up continuously. This effect is called integral wind-up (or integral saturation), and as a result of it long periods of overshoot can occur in the plant response. A simple example of what happens is the following. Suppose we wish to control the position of a motor and a large set-point change occurs, resulting in a large error signal. The controller will then try to reduce the error between the set-point and the output. The integral term will grow by summing the error signals at each sample and a large control action will be applied to the motor. But because of the physical limitation of the motor electronics the motor will not be able to respond linearly to the applied control signal. If the set-point now changes in the other direction, then the integral term is still large and will not respond immediately to the set-point request. Consequently, the system will have a poor response when it comes out of this condition.

The integral wind-up problem affects positional PID controllers. With velocity PID controllers, the error signals are not summed up and as a result integral wind-up will not occur, even though the control signal is physically constrained.

Many techniques have been developed to eliminate integral wind-up from the PID controllers, and some of the popular ones are as follows:

- Stop the integral summation when saturation occurs. This is also called conditional integration. The idea is to set the integrator input to zero if the controller output is saturated and the input and output are of the same sign.
- Fix the limits of the integral term between a minimum and a maximum.
- Reduce the integrator input by some constant if the controller output is saturated. Usually the integral value is decreased by an amount proportional to the difference between the unsaturated and saturated (i.e maximum) controller output.
- Use the velocity form of the PID controller.

9.2.2 Derivative Kick

Another possible problem when using PID controllers is caused by the derivative action of the controller. This may happen when the set-point changes sharply, causing the error signal to change suddenly. Under such a condition, the derivative term can give the output a *kick*, known as a *derivative kick*. This is usually avoided in practice by moving the derivative term to the feedback loop. The proportional term may also cause a sudden kick in the output and it is common to move the proportional term to the feedback loop.

9.2.3 PID Tuning

When a PID controller is used in a system it is important to tune the controller to give the required response. Tuning a PID controller involves selecting values for the controller parameters K_p, T_i and T_d. There are many techniques for tuning a controller, ranging from the first techniques described by J.G. Ziegler and N.B. Nichols (known as the Ziegler–Nichols tuning algorithm) in 1942 and 1943, to recent auto-tuning controllers. In this section we shall look at the tuning of PID controllers using the Ziegler–Nichols tuning algorithm.

Ziegler and Nichols suggested values for the PID parameters of a plant based on open-loop or closed-loop tests of the plant. According to Ziegler and Nichols, the open-loop transfer function of a system can be approximated with a time delay and a single-order system, i.e.

$$G(s) = \frac{Ke^{-sT_D}}{1+sT_1}, \tag{9.21}$$

where T_D is the system time delay (i.e. transportation delay), and T_1 is the time constant of the system.

9.2.3.1 Open-Loop Tuning

For open-loop tuning, we first find the plant parameters by applying a step input to the open-loop system. The plant parameters K, T_D and T_1 are then found from the result of the step test as shown in Figure 9.24.

Ziegler and Nichols then suggest using the PID controller settings given in Table 9.1 when the loop is closed. These parameters are based on the concept of minimizing the integral of the absolute error after applying a step change to the set-point.

An example is given below to illustrate the method used.

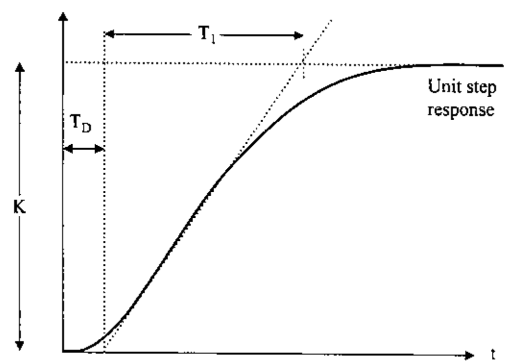

Figure 9.24 Finding plant parameters K, T_D and T_1

Table 9.1 Open-loop Ziegler–Nichols settings

Controller	K_p	T_i	T_d
Proportional	$\frac{T_1}{KT_D}$		
PI	$\frac{0.9T_1}{KT_D}$	$3.3T_D$	
PID	$\frac{1.2T_1}{KT_D}$	$2T_D$	$0.5T_D$

Example 9.7

The open-loop unit step response of a thermal system is shown in Figure 9.25. Obtain the transfer function of this system and use the Ziegler–Nichols tuning algorithm to design (a) a proportional controller, (b) to design a proportional plus integral (PI) controller, and (c) to design a PID controller. Draw the block diagram of the system in each case.

Solution

From Figure 9.25, the system parameters are obtained as $K = 40°\text{C}$, $T_D = 5\,\text{s}$ and $T_1 = 20\,\text{s}$, and the transfer function of the plant is

$$G(s) = \frac{40e^{-5s}}{1+20s}.$$

Proportional controller. According to Table 9.1, the Ziegler–Nichols settings for a proportional controller are:

$$K_p = \frac{T_1}{KT_D}.$$

Thus,

$$K_p = \frac{20}{40 \times 5} = 0.1,$$

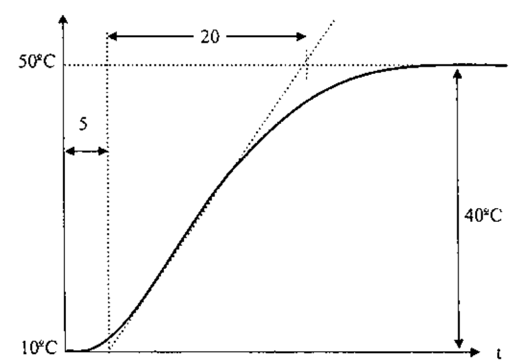

Figure 9.25 Unit step response of the system for Example 9.7

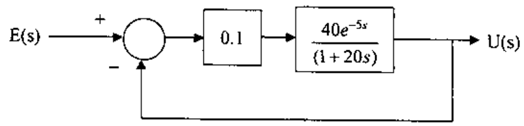

Figure 9.26 Block diagram of the system with proportional controller

The transfer function of the controller is then

$$\frac{U(s)}{E(s)} = 0.1,$$

and the block diagram of the closed-loop system with the controller is shown in Figure 9.26.

PI controller. According to Table 9.1, the Ziegler–Nichols settings for a PI controller are

$$K_p = \frac{0.9T_1}{KT_D} \quad \text{and} \quad T_i = 3.3T_D.$$

Thus,

$$K_p = \frac{0.9 \times 20}{40 \times 5} = 0.09 \quad \text{and} \quad T_i = 3.3 \times 5 = 16.5.$$

The transfer function of the controller is then

$$\frac{U(s)}{E(s)} = 0.09\left[1 + \frac{1}{16.5s}\right] = \frac{0.09(16.5s+1)}{16.5s}$$

and the block diagram of the closed-loop system with the controller is shown in Figure 9.27.

PID controller. According to Table 9.1, the Ziegler–Nichols settings for a PID controller are

$$K_p = \frac{1.2T_1}{KT_D}, \quad T_i = 2T_D, \quad T_d = 0.5T_D.$$

Thus,

$$K_p = \frac{1.2 \times 20}{40 \times 5} = 0.12, \quad T_i = 2 \times 5 = 10, \quad T_d = 0.5 \times 5 = 2.5.$$

The transfer function of the required PID controller is

$$\frac{U(s)}{E(s)} = K_p\left[1 + \frac{1}{T_i s} + Tds\right] = 0.12\left[1 + \frac{1}{10s} + 2.5s\right]$$

or

$$\frac{U(s)}{E(s)} = \frac{3s^2 + 1.2s + 0.12}{10s}.$$

The block diagram of the system, together with the controller, is shown in Figure 9.28.

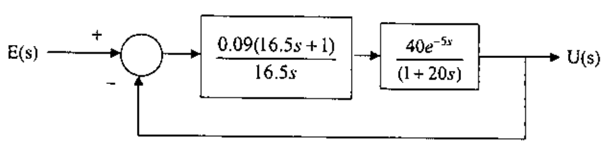

Figure 9.27 Block diagram of the system with PI controller

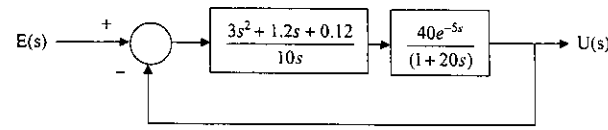

Figure 9.28 Block diagram of the system with PID controller

9.2.3.2 Closed-Loop Tuning

The Ziegler–Nichols closed-loop tuning algorithm is based on plant closed-loop tests. The procedure is as follows:

- Disable any derivative and integral action in the controller and leave only the proportional action.
- Carry out a set-point step test and observe the system response.
- Repeat the set-point test with increased (or decreased) controller gain until a stable oscillation is achieved (see Figure 9.29). This gain is called the *ultimate gain*, K_u.
- Read the period of the steady oscillation and let this be P_u.
- Calculate the controller parameters according to the following formulae: $K_p = 0.45K_u$, $T_i = P_u/1.2$ in the case of the PI controller; and $K_p = 0.6K_u$, $T_i = P_u/2$, $T_d = P_u/8$ in the case of the PID controller.

9.3 EXERCISES

1. The open-loop transfer function of a plant is given by:

$$G(s) = \frac{e^{-4s}}{1+2s}.$$

(a) Design a dead-beat digital controller for the system. Assume that $T = 1$ s.
(b) Draw the block diagram of the system together with the controller.
(c) Plot the time response of the system.

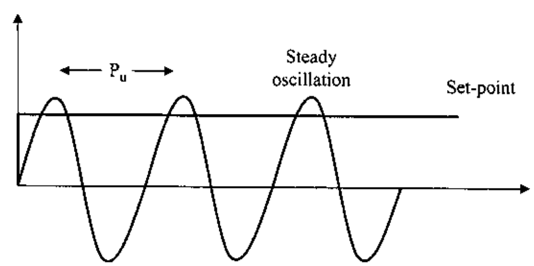

Figure 9.29 Ziegler–Nichols closed-loop test