Discrete Dubins Paths

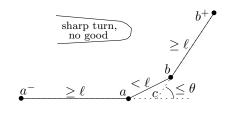Sylvester Eriksson-Bique[*]   David Kirkpatrick[†]   Valentin Polishchuk[‡]

Abstract

A *Dubins path* is a shortest path with bounded curvature. The seminal result in non-holonomic motion planning is that (in the absence of obstacles) a Dubins path consists either from a circular arc followed by a segment followed by another arc, or from three circular arcs [Dubins, 1957]. Dubins original proof uses advanced calculus; later, Dubins result was reproved using control theory techniques [Reeds and Shepp, 1990], [Sussmann and Tang, 1991], [Boissonnat, Cérézo, and Leblond, 1994].

We introduce and study a discrete analogue of curvature-constrained motion. We show that shortest "bounded-curvature" polygonal paths have the same structure as Dubins paths. The properties of Dubins paths follow from our results as a limiting case—this gives a new, "discrete" proof of Dubins result.

1   Introduction

Curvature-constrained paths are a fundamental tool in planning motion with bounded turning radius. Because the paths are *smooth*, they may look more appealing and realistic than *polygonal* paths. Still, polygonal paths are a much more common model in geometry, exactly because of their discrete nature: polygonal paths are more natural to plan, describe and follow. For instance, in air traffic management—one of our motivating applications—an aircraft flight plan ("waypoints list") is represented on the strategic level by a polygonal path whose vertices are the GPS waypoints; see Fig. 14 in Appendix A for examples. The actual smoothly turning trajectory at a waypoint is decided by the pilot on the tactical level when executing the turn (see [20] for more on curvature-constrained route planning in air transport). We are thus motivated to work out a *discretized* model of curvature-constrained motion.

Angles and Lengths   The first and foremost constraint put on a "bounded-curvature" polygonal path P is that it never turns "too sharply". Formally, let $0 \le \theta \le \pi/2$ be a given angle (assume that θ perfectly divides 2π); we require that at its every vertex P turns by at most θ. Such a restriction alone, however, does not yet guarantee that P will serve as a bona fide discrete analogue of a bounded-curvature path: many short successive edges of P, each turning only slightly, can simulate a sharp turn. Thus we also introduce a length constraint.

Specifically, let $\ell > 0$ be a given length; call a path edge *short* if it is shorter than ℓ. We require that in a discrete curvature-constrained path no two short edges are adjacent, and that

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the turn from one edge adjacent to a short edge to the other edge adjacent to the short edge is at most θ. More specifically, let ab be a short non-terminal edge; let a^-a, bb^+ be the (non-short) edges adjacent to ab. Suppose that ab is a *non-inflection* edge, i.e., that a^-a and bb^+ lie on the same side of the supporting line of ab. Let c be the point of intersection of supporting lines of a^-a and bb^+. We require that the supplementary angle of the angle acb is at most θ.

Summarizing, we arrive at the following definition:

Definition 1.1. A *discrete curvature-constrained path* is a polygonal path with the following properties:

 Turn constraints: the turn at every vertex is at most θ;

 Length constraints: there are no adjacent short edges;

 Turn-over-length constraints: for any short non-inflection edge e the turn from one non-short edge adjacent to e to the other non-short edge adjacent to e is at most θ.

Configurations A *configuration* $\mathcal{U} = (u, U)$ is a pair (point, direction), where the direction is a length-ℓ vector. If a robot follows a curvature-constrained path from $\mathcal{U} = (u, U)$, then the robot starts at u having velocity U. In our discrete setting we require that the robot starts at u and "does not turn too sharply" there, which we formalize as follows: imagine that a "pre-edge" $u'u$ is attached to P at u, where $u' = u - U$ (Fig. 1); we require that this longer path is also a discrete curvature-constrained path, i.e., that the turn at u is at most θ, and if the first edge of P is short, the turn from the pre-edge to the second edge of P is at most θ. (The pre-edge is not an edge of P; we attach it only conceptually, to formalize the restriction of P's turn at u.) We say that P *starts at* $\mathcal{U}$. Similarly, let $\mathcal{V} = (v, V)$ be another configuration. We say that P *ends at* $\mathcal{V}$ if v is the last vertex of P and the path obtained by appending to P the segment vv', where $v' = v + V$, is still a curvature-constrained path.

Definition 1.2. A *discrete Dubins path,* or *discrete curvature-constrained geodesic (DC^2G)*, is a shortest discrete curvature-constrained path that starts at $\mathcal{U}$ and ends at $\mathcal{V}$.

1.1 Related work

The study of curvature-constrained path planning has a rich history that long predates and goes well beyond robot motion planning, for example the work of Markov [24] on the construction
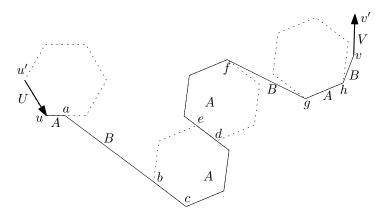

Figure 1: A $\mathcal{U}$-$\mathcal{V}$ path of type $ABAABAB$ (see Section 4 for definition of the type and classification of different parts of the path.)

of railway segments. A fundamental result in curvature-constrained motion planning, due to Dubins [12] (hence the name—*Dubins paths*—for shortest bounded-curvature paths), states that, in the absence of obstacles, shortest paths in the plane, whose mean curvature is bounded by 1, are one of two types: a (unit) circular arc followed by a line segment followed by another arc, or a sequence of three circular arcs. This result has been rederived and extended using techniques from optimal control theory in [4, 27, 38]. Variations and generalizations of the problem were studied in [3, 5–7, 9–11, 15, 23, 26, 28, 30, 32, 35–37].

The books [21, 22] are general references; for some very recent work on bounded-curvature paths see [1, 2, 8, 13, 14, 16, 17, 19]. A discretization of curvature-constrained motion was studied by Wilfong [40, 41]; his setting is different from ours since he considered discretized *environment*, and not discrete paths. A practical way of producing paths with length and turn constraints is presented in [39].

1.2 Our approach

We study properties of discrete Dubins paths, as well as their continuous counterparts. In Section 2 we examine different types of edges of a discrete Dubins path, and conclude that *at least* half of the edges of the path must have length *exactly* ℓ. We go on to proving that the path has *at most one* edge with length *strictly greater* than ℓ, and *at most two* inflection edges; moreover, if the path does contain a longer-than-ℓ edge, then it does not have inflection edges. Overall we obtain that a discrete Dubins path "mostly" consists of non-inflection length-ℓ edges, which prompts us to define "discrete circular arcs" as chains of length-ℓ edges with turns θ from one edge to the next (Section 3). We use the arcs to define the "type" of a discrete Dubins path similarly to how Dubins defined the type of a smooth path (Section 5): we split the path into (discrete) circular arcs and "bridges". Using the freedom of bridges, formalized in Section 5, together with the results from Section 2, we eliminate certain types as infeasible for a discrete Dubins path, and arrive at the characterization of discrete Dubins paths in terms of the type (Theorem 5.1).

In Section 6 we obtain a new proof of Dubins' result for the type of smooth curvature-constrained shortest paths as the limiting case of our results for discrete Dubins paths (Theorem 6.5). In comparison to the earlier proofs of Dubins' characterization (which used calculus, Pontryagin's principle, Arzelá–Ascoli theorem, etc.) [4, 12, 27, 38], our argument is more "discrete" and thus is arguably simpler than the previous ones. Similar methods to prove "continuous" results using discretization were already used by Shur in his paper of 1921; interestingly, exactly these problems, considered by Schur [34] and Schmidt [33], led Dubins to his result.

2 Edges of discrete Dubins paths

We now start our investigation of discrete Dubins paths edges. Recall that an internal edge e of a polygonal path is an *inflection* edge if the edges adjacent to e lie on the opposite sides of (the supporting line of) e. Inflection edges receive special treatment from us (which shows up already in the definition of discrete curvature-constrained paths – we ignored turn-over-length constraints for short inflection edges) because, intuitively, a path does not "gain" curvature while traversing an inflection edge. This is due to an equivalent definition of an inflection edge: e is an inflection edge if the path makes opposite turns at the vertices of e.

Recall also that an edge is *short* if it is shorter than ℓ. We say that an edge is *normal* (resp. *long*) if its length is exactly (resp. larger than) ℓ.

In this section we present a series of results showing that, with only a small constant number of exceptions, a discrete Dubins path consists entirely of non-inflection normal edges.

2.1 Normal edges are "frequent"

Our first two observations are that in a discrete Dubins path long, short and inflection edges are separated by normal edges. The proofs of the next two lemmas are elementary applications of shortcutting arguments; we nevertheless spell out the full proofs since they serve as examples of the kind of arguments we use throughout.
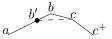
Lemma 2.1. *A long or a short edge can be adjacent only to a normal edge.*

Proof. Suppose that ab, bc are two adjacent long edges. Choose points $a' \in ab, b' \in c'c$ close to b so that aa' and $c'c$ are long, and $a'c'$ is short. Replacing the subpath abc with $aa'c'c$ results in a shorter path, which is a feasible discrete curvature-constrained path. Indeed, the turns at a' and at c' are smaller than the original turn at b' – hence the turn constraints are satisfied (the other turns of the path do not change because of the shortcut). The length constraint is satisfied because aa' and $c'c$ are both long. Finally, the turn-over-length constraint is satisfied for $a'c'$ because the turn from aa' to $c'c$ is the same as it was from ab to bc.

Suppose now that a long edge ab is adjacent to a short edge bc. Select a point $b' \in ab$ close to b so that ab' is still long and $b'c$ is still short. The path going through b' instead of b is shorter; let us check that it is still feasible: The turn at b' is smaller than the turn at b was, so the turn constraints are satisfied. The length constraints are obviously satisfied. Finally, the turn from ab' to the edge cc^+ following $b'c$ is the same as the turn from ab to cc^+, so the turn-over-length constraints are satisfied too. $\square$

Lemma 2.2. *An inflection edge can be adjacent only to a non-inflection normal edge.*

Proof. Let bc, cd be the edges that follow an inflection edge ab. If bc is not normal, rotate ab around a while sliding b along bc (Fig. 2). Clearly, this shortens the path. To see that the path remains feasible, let us check that all constraints in the definition of a discrete curvature-constrained path are satisfied:

Turn constraints The inflection edge endpoints are the only vertices the turns at which change due to the modification of the path; however, both turns only decrease.

Length constraints The only edge whose length decreases due to the rotation is bc. If it was short before the rotation, then it was adjacent to non-short edges, i.e., both ab and cd were non-short. The length of ab only increases with the modification, and the length of cd stays the same; thus bc is still adjacent to non-short edges. On the other hand, if bc was long, we can move b by small enough distance to ensure that bc stays long.

Turn-over-length constraints Again, the only edge to worry about is bc. If it is an inflection edge, the constraint is irrelevant for it. If bc is non-inflection, the turn from ab to cd only decreases with the rotation.

Thus, bc must be normal. Now, if bc is a terminal edge ($c = v$), then it is not an inflection edge, and we are done. Otherwise bc is an internal edge. However, if bc is an inflection edge, then the vertex b could be bypassed by connecting a to c directly, shortening the path. $\square$

2.2 "Freedom" of inflection and long edges

The proof of Lemma 2.2 used the possibility to move b locally along bc while rotating the edge $e = ab$ about a. In the next lemma, we show that b has actually much more freedom than just moving along the adjacent edge: there is a whole halfplane of "admissible" directions.
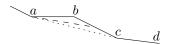
Figure 2: If bc is long (resp. short), then b can be slid towards c slightly while keeping bc long (resp. short) – rotating ab around a decreases the turns at a and b, and shortens the path (the dashed segment is the shortcut). If bc is normal, then ac (dotted) is long; since also $\angle acd > \angle bcd$, we have that ac is a feasible shortcut.
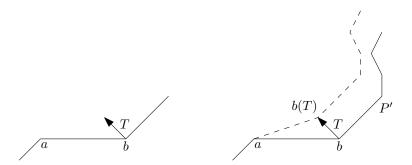

Figure 3: Left: If e is horizontal and b is a left-turn vertex, then admissible vectors point up. Right: The modification.

Definition 2.3. A vector T is *admissible* for e if, when applied to b, the vector points to the side of (the supporting line of) e on which bc lies (Fig. 3, left).

Let P' be the part of P starting from b, and let $P(T)$ be the path obtained from P by rigidly translating P' by T, and rotating e around a so as to keep the path connected (i.e., by connecting a to $b(T) = b + T$, the new position b); Fig. 3, right.

Lemma 2.4. *For any admissible vector T there exists $\varepsilon > 0$ such that $P(\varepsilon T)$ is a feasible discrete curvature-constrained path.*

Proof. The inflection edge endpoints are the only vertices the turns at which may change due to the modification of the path. It is easy to see that if T is admissible, the turns at both endpoints of the edge decrease. Thus, the turn and the turn-over-length constraints are satisfied. As for the length constraints, the inflection edge is the only edge whose length may decrease due to the modification; for that to possibly break the length constraint, the edge must be adjacent to a short edge (and also must itself become short in $P(\varepsilon T)$ while being normal in P – otherwise, if the edge was long in P, it will remain long also in $P(\varepsilon T)$, for a sufficiently small ε). But having an inflection edge adjacent to a short edge contradicts Lemma 2.2. $\square$

Note that even though P was assumed to be a *shortest* path (optimality of P was used in proving that the length and the turn-over-length constraints hold for neighbors of the bridge), we do not claim that $P(\varepsilon T)$ will be the shortest path between its start and destination; we only claim that $P(\varepsilon T)$ is *feasible*.

We now argue that long non-inflection edges have even more freedom than inflection edges: no matter where a vertex of such an edge is moved, the edge can stay connected to the vertex while maintaining path feasibility. There is a price to pay for the connectivity though: the long edge does not stay intact, but gets "broken" into a normal edge and the remainder. That is, an additional vertex of a path appears at a point at distance ℓ from one of the edge endpoints (such addition of a vertex in the middle of a long edge will be central to the notion of "bridges" in Section 4).

Specifically, let ab be a long non-inflection edge; let P'' be the part of P up to a, and let P' be the part of P starting from b. For a vector T let $P'(T)$ be the subpath P' translated by T.
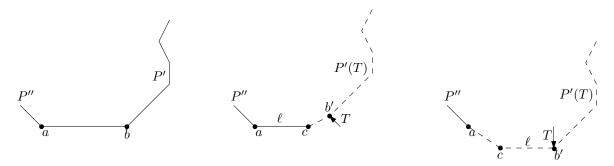
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Figure 4: Left: The original path; ab is horizontal. Middle: The modification if T points up. Right: The modification if T points down.

Lemma 2.5. *For any T there exists $\varepsilon > 0$ and a point c such that the path P''-a-c-b'-$P'(\varepsilon T)$ (i.e., the concatenation of P'', the 2-link path acb', and $P'(\varepsilon T)$) is a feasible discrete curvature-constrained path.*

Proof. Assume that ab is horizontal and that the path turns left at both a and b (so the edges adjacent to ab are above it; Fig. 4, left). If T points up, we choose $c \in ab$ so that $|ac| = \ell$ (Fig. 4, middle). The turn constraints and turn-over-length constraints are not broken because the sum of the turns at c and b' equals to the turn that the original path P made at b. The length constraints are satisfied because even if cb' is short it cannot be adjacent to a short edge (otherwise in the original path P the long edge ab was adjacent to a short edge – a contradiction to Lemma 2.1).

The proof for the case when T points down is analogous: ab is broken at the point c at distance ℓ from b, the normal part is translated by T, and the remainder is rotated to keep the connectivity (Fig. 4, right). $\qquad\square$

As immediate corollaries of Lemmas 2.4 and 2.5 we obtain that inflection and long edges have "length freedom":

Lemma 2.6. *Let ab be an inflection edge, and let P' be the part of P after b; for $\varepsilon \in [0,1]$ let $P(\varepsilon)$ be the path obtained from P by rigidly translating P' towards a along ab by distance $\varepsilon|ab|$. There exists $\varepsilon > 0$ such that $P(\varepsilon)$ is a feasible discrete curvature-constrained path.*

Proof. The vector $\vec{ba}$ is admissible for ab. $\qquad\square$

Lemma 2.7. *Let ab be a long edge, and let P' be the part of P starting from b; for $\varepsilon \in [0,1]$ let $P(\varepsilon)$ be the path obtained from P by rigidly translating P' towards a along sb by distance $\varepsilon|ab|$. There exists $\varepsilon > 0$ such that $P(\varepsilon)$ is a feasible discrete curvature-constrained path.*

Proof. Choose ε small enough so that $|ab| - \varepsilon|ab| > \ell$. $\qquad\square$

2.3 Inflection and long edges are but a few

Using the local freedom of inflection and long edges, formalized above, we now show that in a discrete Dubins path there are only a couple of inflection+long edges. Say that inflection edges ab and cd have *similar turns* w.r.t. P if the turn at a is the same as the turn at c — both right of both left (and hence the turns at b and d are also the same — both left or both right).

Lemma 2.8. *A discrete Dubins path cannot have*

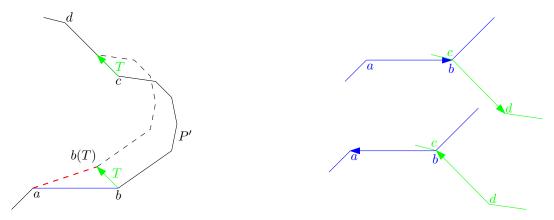1. *two inflection edges with similar turns, unless the edges are parallel*

Figure 5: Left: P' is moved along one edge while rotating the other; blue+green is replaced by red. Right: If $\vec{cd}$ is not admissible for ab, then $\vec{ba}$ is admissible for dc.
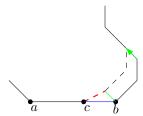

Figure 6: The modification replaces blue+green by red.

2. *a non-inflection long edge and an inflection edge*

3. *two long non-inflection edges*

Proof of 1. Let P' be the part of P between b and c; since inflection edges with similar turns cannot be adjacent, P' is non-empty. By the triangle inequality, each of the following two local modifications shortens P (Fig. 5): (1) slide P' along cd moving c towards d and rotating ab around a so as to keep it connected to b; (2) change the roles of ab and cd, i.e., slide P' along ba moving b towards a and rotating dc around d so as to keep it connected to c. Moreover, one of these modification keeps the path feasible because either $\vec{cd}$ is admissible for ab, or $\vec{ba}$ is admissible for dc (this is because of any two non-parallel vectors, one is to the left of the other).

Proof of 2. By Lemma 2.2, the edges are not adjacent. Rigidly shift the subpath between the edges along the inflection edge (shortening the edge); a small shift is feasible due to Lemma 2.6. Keep the path connected by breaking the long edge as appropriate (Lemma 2.5). By the triangle inequality, this modification shortens the path (Fig. 6).

Proof of 3. Analogous to 2, using Lemma 2.7 in place of Lemma 2.6.

□

3   Discrete arcs

The results from the previous section imply that, loosely speaking, pretty much all that is left for a discrete Dubins path is to have long chains of normal non-inflection edges, connected by few inflection and/or long and/or short edges. In this section we introduce "discrete circular arcs", which are essentially chains of normal edges with turns θ.

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Definition 3.1. A *discrete circle* is a regular $\frac{2\pi}{\theta}$-gon with side ℓ.

In what follows, we often omit the modifier "discrete" and call discrete circles just circles, discrete arcs just arcs, etc.

We define a discrete arc as a maximal subpath of P that is also a subpath of a discrete circle; the definition is somewhat involved because formally, any length-at-most-ℓ edge of P is a subpath of a discrete circle – and we want to avoid calling each and every short edge an arc:

Definition 3.2. A *discrete circular arc* is a subpath $\pi = (p_1, \ldots, p_m)$ of a curvature-constrained path P, with the following properties:

1. π is a subpath of some discrete circle;

2. π is maximal in the sense that it is not contained in another subpath of P which is also a subpath of some discrete circle;

3. if $m = 2$, then at least one of p_1, p_2 is a vertex of P, and in addition

   (a) if p_1 is a vertex of P, but p_2 is not, then $p_1 = u$ and the turn from the pre-edge to $p_1 p_2$ is exactly θ (i.e., $\angle u' u p_2 = \theta$);

   (b) if p_2 is a vertex of P, but p_1 is not, then $p_2 = v$ and the turn from $p_1 p_2$ to the post-edge is exactly θ (i.e., $\angle p_1 v v' = \theta$);

   (c) if both p_1, p_2 are internal vertices of P (i.e., $p_1 \neq u, p_2 \neq v$), then $|p_1 p_2| = \ell$;

   (d) if both p_1, p_2 are vertices of P and $p_1 = u$, then the turn from the pre-edge to $p_1 p_2$ is exactly θ or $|p_1 p_2| = \ell$;

   (e) if both p_1, p_2 are vertices of P and $p_2 = v$, then the turn from $p_1 p_2$ to the post-edge is exactly θ or $|p_1 p_2| = \ell$.

Few remarks are appropriate here (refer to Fig. 1):

- If $m = 2$ and $p_1 p_2$ is short, then one of p_1, p_2 is a terminal vertex of P (e.g., in Fig. 1 the short edge ua is an arc since the turn onto it from the pre-edge is θ). If $m = 2$ and p_1, p_2 are both internal vertices of P (e.g., g and h in Fig. 1), then (by the maximality of the arc) the turn of P at each of p_1, p_2 is less than θ.

- If $m = 3$, then either $|p_1 p_2| = \ell$ or $|p_2 p_3| = \ell$ (since the arc is a subpath of a discrete circle, it has no long edges; no two short edges are adjacent).

- If $m > 3$, then the $m - 3$ internal edges of an arc are normal, and the turn of P at each internal vertex of the arc is exactly θ.

- Every vertex at which P turns by θ, is part of an arc, and every edge incident to a turn-θ vertex, will share a portion with an arc. If a short inflection edge makes angles θ with both adjacent edges, the short edge is a part of two arcs; that is, consecutive arcs along P may overlap.

Note that only the internal vertices of an arc must be vertices of P; the terminal vertices of an arc may lie in the middle of P's edges (in Fig. 1, such are both terminal vertices b, e of the arc bc-de, as well as the terminal vertex d of the arc de-f). We use such vertices below when defining the type of a path, as well as in the local modifications by which we eliminate forbidden pairs of edges (we also used such a vertex earlier when identifying the possibility to break a long edge in Lemma 2.5).

3.1 Arcs and inflection

One important aspect in which discrete curvature-constrained paths differ form their smooth counterparts is that several consecutive discrete arcs may be similarly oriented (all going clockwise or all counterclockwise); in a smooth path adjacent arcs necessarily have a common inflection tangent. We show that, nevertheless, having two adjacent similarly-oriented arcs in a sequence of arcs is quite restrictive for a discrete Dubins path – essentially, *all* arcs must then be similarly oriented (Lemma 3.3). In addition, a path of more than 3 similarly oriented arcs may be either shortened or transformed to an equal-length path with only 2 arcs (Lemma 3.5).

Specifically, for adjacent arcs X, Y let X-Y denote the subpath going from the start of X to the end of Y. The (long and technical) proof of the following lemma can be found in Appendix B:

Lemma 3.3. *Let X, Y, Z be consecutive arcs. If the subpath X-Y has no inflection edge, then neither does Y-Z.*

Recall (Section 2.2) that a long edge has more freedom than an inflection edge (compare Lemmas 2.4 and Lemmas 2.7); in a sense, a long edge (which is capable of being broken) has "twice as much freedom" as an inflection edge – admissible vectors for the latter lie in a halfplane while for the former *any* vector is "admissible". In particular, the arguments in the proof of the above lemma can be carried out for long edges in place of inflection, leading to

Lemma 3.4. *Let X, Y, Z be consecutive arcs. If the subpath X-Y has no inflection edge, then the subpath Y-Z does not have a long edge.*

We now show that a path consisting of 3 similarly oriented arcs may be transformed to an equal-length path "with more θ-turns". Say that a path is *flush* if the turn from the pre-edge onto the first edge of the path is exactly θ. The (technical) proof of the next lemma can be found in Appendix C:

Lemma 3.5. *A non-flush discrete Dubins path consisting of 3 consecutive arcs and having no inflection edge can be transformed to an equal-length path that is either flush, or consists of at most 2 arcs.*

4 Path typification and canonization

Smooth Dubins paths can be classified as either arc-segment-arc or arc-arc-arc type. In this section, we develop a notion of type for a discrete curvature-constrained path, similar to the type of a smooth Dubins path, by labeling subpaths with A's (discrete circular arcs) and B's (bridges – segments connecting consecutive arcs). Specifically, the type of a path is defined as follows:

1. Identify all arcs, and label each arc by an A.

2. The unlabeled subpaths are now a set of straight segments disjoint other than possibly at endpoints; each such segment is called a *bridge* and labeled with a B.

The *length* of a type is the number of letters in it.
   Remarks:

- We emphasize again that a bridge/arc may start or end in the middle of an edge of the path; i.e., the turn of P at a bridge endpoint may be 0 (such is, e.g., the endpoint b of the bridge ab in Fig. 1). However, no edge of the path can support more than one bridge.

- (As mentioned above,) consecutive arcs may overlap (e.g., arcs *bc-de* and *de-f* in Fig. 1 have a common edge *de*). Bridges, on the contrary, have disjoint interiors, which are also disjoint from arcs.

- The main feature of the bridge is the "freedom to rotate locally" (formalized in Lemma 4.3).

- A type is uniquely defined.

The type is defined for arbitrary discrete curvature-constrained paths, not only for shortest such paths (discrete Dubins paths). Since our ultimate goal is the characterization of the shortest paths, from now on we will consider only discrete Dubins paths.

4.1 Canonical form

In our development so far, we tacitly assumed that a path makes a non-zero turn at every vertex, i.e., that the path has no vertices in the middle of its edges. This is a usual and natural assumption in the treatment of polygonal paths, and we lose no generality by making it: any 0-turn vertex could be removed without violating the feasibility of the path – removing such a vertex does not influence the path turns and does not decrease the edge lengths). On the contrary, *adding* vertices in the middle of edges may, in general, render a path infeasible (e.g., if a vertex is added on a normal edge, the edge splits into two adjacent short edges). Still, when proving that a path may have only one bridge, we would like to have vertices at certain points internal to edges. Specifically, we postulate that the endpoints of the bridges are *always* path vertices (by default, this may not be the case – like, e.g., for point *b* in Fig. 1); we need this to be able to "break" the path there (this is similar to how we broke long edges in Lemma 2.5). That is, even if the endpoint of a bridge is in the middle of an edge, we add a vertex of the path there. The next lemma shows that such addition of vertices is feasible; the (straightforward) proof of the lemma appears in Appendix D:

Lemma 4.1. *Let P be a discrete Dubins path. The path P' obtained by adding to P as vertices all bridge endpoints is a discrete Dubins path.*

We say that the path P' is in *canonical* form. Where needed, we will assume that a discrete Dubins path is in canonical form, i.e., we will often jump between the original path P and its canonical form (the above lemma allows us to do this). In particular, when necessary, we will assume that bridges are edges of the path, and can be rotated around their endpoints. Note that the during the canonization, vertices are appended only in the middle of long edges, which is similar to Lemma 2.5.

We emphasize that we first do path typification (splitting the path into As and Bs), and only then – canonization (adding A's endpoints to vertices of the path). This is natural, since the type is defined for arbitrary paths, while the canonical form – only for shortest paths.

One useful property of a path in canonical form is that it can be split at vertices into subpaths, each of which is discrete bounded-curvature path. If in addition, the pre-edge and post-edges are long or normal, then it is also a discrete Dubins paths itself; we state this as a lemma without proof:

Lemma 4.2. *Let π the subpath between vertices b, c of a discrete Dubins path in canonical form; let a be the vertex preceding b, and d be the vertex following c (if $b = u$, then $a = u'$; if $c = v$, then $d = v'$). Then π is a discrete bounded curvature path. If in addition $\vec{ab}, \vec{cd}$ are long or normal, then π is in fact a discrete Dubins path starting at $(b, \vec{ab}/|\vec{ab}|)$ and ending at $(c, \vec{cd}/|\vec{cd}|)$.*

Lemma 4.2 allows one in some cases to speak about a particular "subtype" – a type subsequence, as a path in it's self. This is helpful when eliminating forbidden subsequence $AAAA$ (Section 5.2).
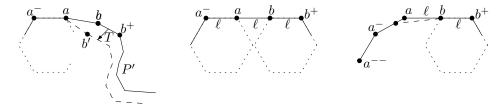
Figure 7: Left: The modification. The turn constraints are satisfied by definition of a bridge. The length and turn-over-length constraints can break only if ab is normal, ab' is short, and one of the edges adjacent to the bridge is short. Middle: Both a and b were added during canonization. Right: Only b was added during the canonization: if a^-a is short, a shortcut (dashed) is possible.

4.2 "Freedom" of bridges

We defined the bridges so that their vertices have certain freedom similar to (or, actually, exceeding) the freedom of inflection and long edges (Section 2.2). Specifically, let ab be a bridge in a discrete Dubins path P, and let P' be the part of P starting from b. For a vector T let $P(T)$ be the path obtained from P by rigidly translating P' by T, and connecting a to $b' = b + T$ (the new position b) (Fig. 7, left). The next lemma is proved in Appendix E:

Lemma 4.3. *For any T there exists $\varepsilon > 0$ such that $P(\varepsilon T)$ is a feasible discrete curvature-constrained path.*

4.3 Bridges are but a few

We append bridges to the list of rare occurrences in a discrete Dubins path (the start of the list is in Lemma 2.8). Say that two bridges ab, cd have *similar direction* w.r.t. P if they are parallel and the vectors $\vec{ab}, \vec{cd}$ point in the same direction. In Appendix F we prove the following:

Lemma 4.4. *A discrete Dubins path may not have*

1. *a bridge and an inflection edge*

2. *a long edge and a bridge*

3. *two bridges, unless they have similar direction.*

5 Types of discrete Dubins paths

We define a set of "true types" and a set of "forbidden subtypes" (Section 5.1); using the facts that inflection edges, long edges and bridges are rare (Lemmas 2.8 and 4.4) and that consecutive arcs must inflect (Lemmas 3.3 and 3.5), we show that forbidden subtypes can never appear in a discrete Dubins path (Section 5.2). It will follow that for any start and end configurations there always exists a discrete Dubins path of a true type.

5.1 True types and Forbidden subtypes

We prove that every discrete Dubins path with *minimal-length* type is of the types $\mathcal{T} = \{B, A, AB, BA, AA, ABA, AAA\}$ – the *true* types. This will be shown by considering the set $\mathcal{F} = \{BB, BAB, AAB, BAA, AAAA\}$ of *forbidden subsequences* and demonstrating that if the type has a member of $\mathcal{F}$ as a subsequence, then either the path can be strictly shortened or modified to a same-length path with a shorter type. More specifically, we show that the forbidden subtypes $\mathcal{F} \setminus AAAA$ can *never* appear in a discrete Dubins path, while for the subtype $AAAA$
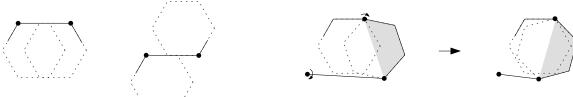
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Figure 8: Edge-edge: the original path had a long edge.

Figure 9: Vertex-vertex: rotate the second arc as a rigid body; rotate the bridge to keep the path connected.

there always exists a true-type path with the same or smaller length. By symmetry we only need to prove that the subsequences $BB, BAB, AAB, AAAA$ can be excluded.

5.2 Eliminating forbidden subsequences

BB

Two adjacent bridges must belong to the same edge of P (otherwise the bridges are not parallel – a contradiction to Lemma 4.4(3)), which is impossible by definition of a bridge.

BAB

By Lemma 4.4(3), the two bridges must have similar direction; this is possible only if there is an inflection edge between them, contradicting Lemma 4.4(1).

AAB

Let $\mathcal{C}_1, \mathcal{C}_2$ be the supporting circles of the two adjacent arcs. There are 3 ways of how the two arcs may connect: (1) $\mathcal{C}_1, \mathcal{C}_2$ are flush, sharing a whole edge or part of an edge (Fig. 8); (2) they share a single vertex (Fig. 9); (3) an edge of one circle pivots on a vertex of the other (Fig. 11). We consider the 3 cases separately.

Edge-edge Let $e = \mathcal{C}_1 \cap \mathcal{C}_2$ be the common part of the two arcs. If e is normal (i.e., $\mathcal{C}_1, \mathcal{C}_2$ share a whole edge), then e is an inflection edge (for otherwise $\mathcal{C}_1 = \mathcal{C}_2$) – a contradiction to Lemma 4.4(1). If e is short, it is either a proper subset of a long edge of the original path or is an inflection edge (Fig. 8). However, having a long edge and a bridge contradicts Lemma 4.4(2), and having an inflection edge a bridge contradicts Lemma 4.4(1).

Vertex-vertex It is easy to see that the second arc has the local freedom to rotate, in either direction, around the common vertex of the arcs without violating the feasibility of the path (Fig. 9). Due to the rotation, the other endpoint of the arc may move slightly, which is no problem because it is bridge endpoint (Lemma 4.3). Rotating the arc in one of the directions decreases the length of the bridge (and hence the length of the path), unless the bridge belongs to the line through the arc's endpoints, in which case either the last edge of the arc is an inflection edge (contradiction to Lemma 4.4(1)) or the bridge is inside $\mathcal{C}_2$ (contradiction to the turn constraint); refer to Fig. 10.

Vertex-edge Suppose that an edge b^-b of $\mathcal{C}_2$ rocks on a vertex a of $\mathcal{C}_1$ (Fig. 11). Then ab is short (for otherwise, we are in the vertex-vertex or edge-edge case), and hence the second arc has more then one edge (b cannot be a terminal vertex – there is a bridge following the arc). But then ab is an inflection edge (for otherwise, the turn-over-length constraint is violated
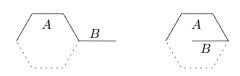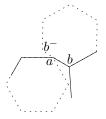
Figure 10: Bridge aligned with the arc endpoints.

Figure 11: Vertex-edge: if the turn at b is to the right, the turn over ab is too sharp; if the turn at b is to the left, then ab is an inflection edge.

for it – the turn at b is already θ). Having an inflection edge and a bridge contradicts Lemma 4.4(1).

AAAA

We show that a subpath consisting of 4 arcs $A_1A_2A_3A_4$ can either be shortened or transformed to an equal-length path with fewer arcs. Incidentally, *all* existing proofs of the structure of smooth Dubins paths go through proving non-optimality of $AAAA$ paths [12, Lemma 2], [4, Lemma 11], [38, Lemma 26], [27, Lemma 7]; we are no exception. We do case analysis based on the number of inflection edges in an $AAAA$ path.

More than 2 inflection edges In this case two inflection edges have similar turns – a contradiction to Lemma 2.8(1).

Two inflection edges Clearly, for any arcs $A_i, A_{i+1}, i = 1, 2, 3$, the subpath A_i-A_{i+1} can contain at most 1 inflection edge. Now, no matter which subpaths contain the two inflection edges, we can always apply Lemma 3.3 to reach a contradiction.

One inflection edge If the subpath A_1-A_2 has no inflection edge, then, by Lemma 3.3, neither do A_2-A_3 and A_3-A_4, so there is no inflection edge at all. On the other hand, having the inflection edge in the subpath A_1-A_2 (or, equivalently, in A_2-A_1) contradicts Lemma 3.3 applied to the arcs A_3, A_2, A_1.

There are no inflection edges Similarly to the AAB case above, there are 3 ways of how two consecutive arcs may connect: sharing (part of) an edge, sharing a single vertex, or with an edge of one arc pivoting on a vertex of the other (see Figs. 8, 9 and 11). Again similarly to AAB, in the first case the original path (i.e., the path before the canonization) had a long edge, which constradicts Lemma 3.4. Still similarly to AAB, in the third case the turn from one arc to the next is too sharp. Thus, all three connections between the consectuive arcs in AAAA are vertex-vertex types, and the edges incident to the connection vertices are normal (Fig. 12).

Consider now the subpath $\pi = v_1 - \cdots - v_4$ consisting of 3 arcs. The edge of A_1, incident to v_1 is normal; thus, by Lemma 4.2, π is a discrete Dubins path. The turn at v_1 is not θ (for otherwise A_1 and A_2 would have been the same arc); thus, π is not flush. By Lemma 3.5, either π can be replaced by a same-length 2-arc path, or π can be made flush (without changing length); in the latter case A_1, A_2 merge into a single arc. In both cases, the number of arcs in π decreases to 2, and the $AAAA$ subpath becomes AAA.
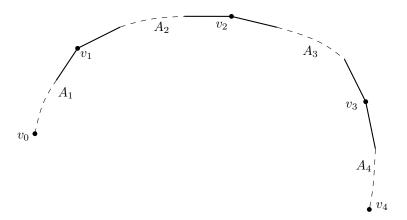
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Figure 12: The arc $A_i, i = 1\ldots 4$ goes from v_{i-1} to v_i. The edges incident to v_1, v_2, v_3 are normal.

5.3 Structure of a discrete Dubins path

We are now ready to show that discrete Dubins paths have the same structure as Dubins paths; in the next section we use this to obtain properties of the (smooth) Dubins paths as a limiting case.

Theorem 5.1. *For any configurations $\mathcal{U}$, $\mathcal{V}$ the collection of discrete curvature-constrained $\mathcal{U}$-$\mathcal{V}$ paths contains a path of minimum length that consists of a discrete circular arc followed by a segment followed by a discrete circular arc, or is a sequence of at most 3 discrete circular arcs.*

Proof. We only have to prove that the shortest path exists; by forbidden subsequence elimination (Section 5.2), the path must be of true type. (Note that the local shortenings that we exhibited above, do not prove existence of a shortest path; potentially it could happen that our local modifications continue ad infinitum, never reaching the minimum.)

For $n \in \mathbb{N}$ let l_n be the infimum of the lengths of $\mathcal{U}$-$\mathcal{V}$ paths with at most n vertices (if no such path exists, $l_n = \infty$). A $\mathcal{U}$-$\mathcal{V}$ path cannot be shorter than $l^* = \inf_{n \in \mathbb{N}} l_n$. We will first prove that there exists $n_0 \in \mathbb{N}$ such that any path with more than n_0 vertices is strictly longer than l^*; this implies that $l^* = \inf_{n \leq n_0} l_n = l_{n_0}$. We next prove that for any n, l_n attains its minimum, i.e., that there exists a path with at most n vertices whose length is exactly l_n. It will follow that there exists a path with length $l^* = l_{n_0}$.

Clearly, there exists at least one turn-constrained $\mathcal{U}$-$\mathcal{V}$ path with finite length; let the length be L. Any path with n vertices has at least $\lfloor \frac{n-3}{2} \rfloor$ long or normal edges (there are $n-1$ edges, at most 2 of them are terminal; the rest have at most one short per long or normal by Lemma 2.1). Thus the length of the path is at least $\lfloor (n-3)/2 \rfloor \ell$. In particular, a path with more than $n_0 = \lceil 2L/\ell \rceil + 3$ vertices will be strictly longer than L.

Any path with at most n vertices may be specified by a point in $(x_i, y_i)_{i=1}^n \in \mathbb{R}^{2n}$. The set of all paths of length at most L is a compact subset of $\mathbb{R}^{2n}$. Let $\mathcal{P}$ be the set of all discrete curvature-constrained $\mathcal{U}$-$\mathcal{V}$ paths; since the length is a continuous function on $\mathcal{P}$, the existence of the shortest path will follow as soon as we can show that $\mathcal{P}$ is closed. We do it by examining the constraints one-by-one:

Turn constraints Each constraint can be written as a *non-strict* inequality; hence turn-constrained paths form a closed set.

Length constraints The condition that ith edge of the path is short defines an open set $E_i \subset \mathbb{R}^{2n}$; thus the constraint that the ith and the $(i+1)$st edges are not both short, excludes an open set $E_i \cap E_{i+1}$. Since there are finitely many (pairs of consecutive) edges, the length constraints for all of them exclude an open set, leaving the feasible set closed.

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Turn-over-length constraints Let $T_i \subset E_i$ be the set of points for which the turn-over-length constraint is *not* satisfied for the ith edge; since T_i is open, the set of infeasible points (w.r.t. the turn-over-length constraints) is open, meaning that its complement—the feasible set—is closed.

$\square$

6   Smooth curvature-constrained paths

From now on, we turn our attention to the smooth ("usual", non-discrete) bounded-curvature paths. Following Dubins [12] seminal work, we will not require that paths have curvature defined at every point; instead, we consider paths whose *mean* curvature is bounded everywhere (see the first paragraph in Dubins paper for a discussion of this technicality). Specifically, let γ be a smooth (continuously differentiable) path, parameterized by its arclength. For any t in the domain of γ, let $\gamma'(t)$ denote the derivative of γ at t – the unit vector tangent to γ at t. We require that the average curvature of γ is at most 1, which is formalized as follows: for any $t < s < t + \pi$ let $\angle st$ denote the angle between the directions of $\gamma'(s)$ and $\gamma'(t)$; the bound on the average curvature means that $\angle st \leq s - t$. (In other words, γ', viewed as mapping from the domain of γ to the unit circle, is 1-Lipschitz.) We call such paths *admissible*.

In this section we show how to discretize an admissible path into a discrete curvature-constrained path; the finer the discretization, the closer the discrete path is to the smooth one. This allows us to obtain the properties of shortest smooth paths as a limit of the properties of their discrete counterparts, which we established above (Theorem 5.1). We discretize a smooth path by splitting it into 3 parts (with the first and the last part possibly empty). The middle part is discretized by choosing regularly spaced points, while the first and the last part are whatever remains from the regular discretization; the first and the last part are made equal-length. We prove that the resulting polygonal path has at most 2 short edges (the first and the last edge), and that it is a feasible discrete curvature-constrained path. The details follow.

6.1   Preliminary lemmas

The correctness of our discretization hinges on the next two lemmas.

Lemma 6.1. *For any t and $t < s < t + \pi$ in the domain of γ, $|\gamma(s) - \gamma(t)| \geq 2\sin\frac{s-t}{2}$.*

Proof. Assume w.l.o.g. that $t = 0$ and that $\gamma(t)$ is at the origin ($\gamma(0) = (0,0)$); the lemma is then equivalent to $|\gamma(s)| \geq 2\sin\frac{s}{2}$ (Fig. 13, left). We prove this by lower-bounding the derivative of $|\gamma(s)|^2$:

$$(|\gamma(s)|^2)' = 2\gamma(s) \cdot \gamma'(s) = 2\int_0^s \gamma'(\tau) \cdot \gamma'(s)\, d\tau = 2\int_0^s \cos\angle s\tau\, d\tau \geq 2\int_0^s \cos(s - \tau)\, d\tau = 2\sin s$$

Hence $|\gamma(s)|^2 \geq 2(1 - \cos s) = 4\sin^2\frac{s}{2}$. $\square$

Lemma 6.2. *The angle between $\gamma'(t)$ and the ray $\gamma(t)\gamma(s)$ is at most $\frac{s-t}{2}$.*

Proof. Assume again that $t = 0$ and that $\gamma(0) = O$; also assume w.l.o.g. that $\gamma'(0)$ is horizontal ($\gamma'(0) = (1,0)$). Let $\gamma(s) = (x(s), y(s))$, and let $k(s) = y(s)/x(s)$ be the slope of the ray $O\gamma(s)$ (Fig. 13, left). Then the lemma is equivalent to $k(s) \leq \tan\frac{s}{2}$, which we will prove by showing that $k' \leq \frac{1-\cos s}{\sin^2 s} = \frac{1}{2\cos^2(s/2)} = (\tan\frac{s}{2})'$.
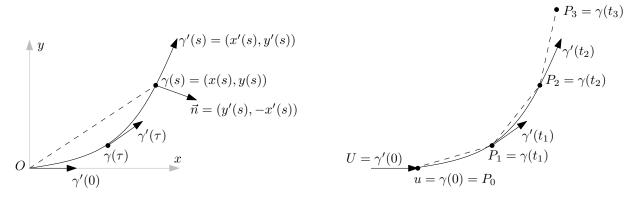
Figure 13: Left: $|O\gamma(s)| \geq 2\sin\frac{s}{2}$, the slope of $O\gamma(s)$ is at most $\tan\frac{s}{2}$. Right: P_1P_2 and P_2P_3 are non-short. The turn from P_1P_2 onto $\gamma'(t_2)$ is at most $\theta/2$; such is also the turn from $\gamma'(t_2)$ onto P_2P_3. If $\delta > 0$, then $t_1 = \delta/2 < \theta/2$, and P_0P_1 is short; in this case, the angles between P_0P_1 and the tangents to γ at P_0, P_1 are at most $\theta/4$ each.

First of all, for any s we have $x'(s) = (x'(s), y'(s)) \cdot (1,0) = \gamma'(s) \cdot \gamma'(0) = \cos\angle s0 \geq \cos s$; thus, $x(s) \geq \sin s$. Next, consider the unit vector $\vec{n} = (y'(s), -x'(s))$, orthogonal to $\gamma'(s)$ (Fig. 13, left). By definition, for any $\tau < s$ the angle between $\gamma'(\tau)$ and $\gamma'(s)$ is at most $s - \tau$; hence, the angle between $\gamma'(\tau)$ and $\vec{n}$ is at least $\pi/2 - (s - \tau)$, from whence $(x'(\tau), y'(\tau)) \cdot \vec{n} \leq \cos(\pi/2 - (s - \tau))$, or $x'(\tau)y'(s) - y'(\tau)x'(s) \leq \sin(s - \tau)$. Integrating over τ from 0 to s, we get $x(s)y'(s) - y(s)x'(s) \leq 1 - \cos s$. Combining this with $x(s) \geq \sin s$, we obtain what we need: $k' = (y/x)' = \frac{y'x - x'y}{x^2} \leq \frac{1 - \cos s}{\sin^2 s}$. $\qquad\square$

6.2 Discretization

Let $\theta < |\gamma|$ be a number, and suppose that $|\gamma| = m\theta + \delta$ for some $m \in \mathbb{N}, \delta < \theta$ (i.e., $m = \lfloor|\gamma|/\theta\rfloor$ and δ is the remainder).

Definition 6.3. The θ-discretization of γ is a polygonal path P with vertices $P_i = \gamma(t_i), i = 0\ldots k$ such that

- $t_0 = 0$, $t_k = |\gamma|$.

- If $\delta = 0$, then $k = m = |\gamma|/\theta$, and $t_i - t_{i-1} = \theta$ for all $i = 1\ldots m$.

- If $\delta > 0$, then $k = m + 2$, $t_i - t_{i-1} = \theta$ for $i = 1\ldots m+1$, and $t_1 - t_0 = t_{m+2} - t_{m+1} = \delta/2$.

Let $\ell = 2\sin\frac{\theta}{2}$. By Lemma 6.1, whenever $t_i - t_{i-1} = \theta$, we have that $P_iP_{i-1} \geq \ell$. Thus, all edges of P, except possibly for the first and the last one, are non-short. If the first and the last edges are short, the path has at least 3 edges, so the two short edges are non-adjacent. Thus, length constraints hold for P.

We now examine the turns of P. Instead of directly looking at the angles between edges of P, we look at the angle that an edge makes at its endpoints with tangents to γ; we bound these angles using Lemma 6.2. Specifically, let $P_{i-1}P_i, P_iP_{i+1}$ be two non-short edges. By Lemma 6.2, the angle between $P_{i-1}P_i$ and the tangent $\gamma'(t_i)$ to γ at $P(i)$ is at most $\theta/2$ (Fig. 13, right); symmetrically, the angle between the tangent and P_iP_{i+1} is also at most $\theta/2$. Thus, the turn of P at P_i is at most θ, and the angle constraint is satisfied. Next, if P_0P_1 is a short edge, then $t_1 - t_0 < \delta/2 < \theta/2$, and by Lemma 6.1, the turn from $\gamma'(0)$ onto the edge is at most $\theta/4$. Similarly, the turn from P_0P_1 to $\gamma'(t_1)$ is at most $\theta/4$; finally the turn from $\gamma'(t_1)$ onto P_1P_2 is at most $\theta/2$. Thus, overall, the turn of P from $\gamma'(0)$ onto the first non-short edge, P_1P_2, is at most $\theta/4 + \theta/4 + \theta/2 = \theta$.

Let $\mathcal{U} = (\gamma(0), \gamma(0)'), \mathcal{V} = (\gamma(|\gamma|), \gamma(|\gamma|)')$. It follows from the above that P is a $\mathcal{U}$-$\mathcal{V}$ discrete curvature-constrained path with parameters θ and ℓ, i.e., a path whose turns are constrained by θ and lengths – by ℓ (recall Definition 1.1 for the exact meaning of the parameters in the constraints).

Finally, for $n \in \mathbb{N}$ let $l_n = 2 \sin \frac{\pi}{n}$ be the side length of the regular n-gon inscribed in unit circle; let $\theta_n = 2\pi/n$. Suppose that n is large enough so that $\theta_n < |\gamma|$, and let P_n be the θ_n-discretization of γ. The preceding discussion implies the following:

Lemma 6.4. *P_n is a $\mathcal{U}$-$\mathcal{V}$ discrete curvature-constrained path with parameters θ_n and l_n.*

6.3 Dubins paths properties

We are now ready to give new proofs of Dubins result. Let $\mathcal{U}$, $\mathcal{V}$ be two arbitrary configurations, and let $\mathbb{C}$ be the collection of $\mathcal{U}$-$\mathcal{V}$ paths with bounded mean curvature. Say that a path $\gamma \in \mathbb{C}$ is *Dubins-type* if it consists of a circular arc followed by a segment followed by a circular arc, or is a sequence of at most 3 circular arcs.

Theorem 6.5 ([12])**.** *$\mathbb{C}$ contains a Dubins-type path of minimum length.*

Proof. Let τ_n be a $\mathcal{U}$-$\mathcal{V}$ discrete Dubins path with parameters θ_n and l_n. By Theorem 5.1 there are only finitely many types of discrete Dubins paths; thus, we may assume, possibly changing to a subsequence, that all paths τ_n have the same type. Suppose, for instance that τ_n are of type AAA (the other cases are similar). As $n \to \infty$ each of the discrete arcs converges to a circular arc. Denote the limit curve, consisting of the 3 circular arcs, by τ^*.

Let $\gamma \neq \tau^*$ be another $\mathcal{U}$-$\mathcal{V}$ path with bounded mean curvature. For the sequence P_n of θ_n-discretizations of γ we have $|\tau_n| \leq |P_n| \leq |\gamma|$. Taking the limit, we obtain $|\tau^*| \leq |\gamma|$, which proves that τ^* is a curve of minimum length. $\qquad\square$

7 Conclusion

We studied a discrete model of curvature-constrained motion. We chose one particular way of defining the discrete motion; many other versions are possible. Also, even within our framework, the definition of discrete Dubins path may be modified in many ways (the pre- and post-edges uu', vv' may be mandated to be edges of P, all edges may be required to have length at least ℓ, the turn constraints may be imposed differently, etc.); some ways might lead to more natural (and possibly, shorter) definitions than ours. Our choice of the definition was prompted merely by technical details – we found it easier to allow short edges etc.; since short edges disappear in the limit $\ell \to 0$, the length and turn constraints at terminal edges are not a significant factor.

In any case, we admit that there may exist other definitions of discrete curvature-constrained motion; perhaps, the only objective requirements for the model are that smooth paths can be discretized, and that the model contains smooth paths as a limiting case of the discretization. This allows one to obtain results concerning curvature-constrained motion via the limiting argument similar to ours. As one example of such a result we mention the theorem that a curvature-constrained path must have length at least π in order to be able to enter the interior of the unit circles tangent to the path's initial configuration. The theorem was proved using "continuous" methods in several prior works ([12, Proposition 6], [18, 25, 29], [31, Lemma 2]); the discretize-and-take-the-limit technique may give yet another proof (of course, we do not claim to have invented the technique in this paper – it has been used for centuries, e.g., for isoperimetric problems.)

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Appendix

A   Flight paths

Figure 14: Images from avsim.com, flightsim.com, spyzone.com

B   Proof of Lemma 3.3

If X and Y are "flush", i.e., if the last edge of Y has the same supporting line as the first edge of Y, then the connection between the arcs is a long edge (Fig. 15, left). In this case, Lemma 2.8(2) implies that there can be no inflection edge anywhere in the path, in particular – in Y-Z, so we are done. Thus we will assume that X and Y intersect in a vertex (Fig. 15, middle).

Let $b = X \cap Y$ be the common vertex of X and Y, and suppose that Y-Z has an inflection edge cd. Since b is the first vertex of Y-Z, and an inflection edge in a subpath cannot be its first edge (an inflection edge is an internal edge), we have that $b \neq c$. Let P_{bc} denote the subpath between b and c. Since arcs do not have inflection edges, P_{bc} is a subpath of Y (possibly, $P_{bc} = Y$ — if cd is not an edge of Y; in that case, however, cd must be an edge of Z, like in Fig. 15, middle).

Let a be the vertex (of X) preceding b, and let P_{ac} denote the subpath between a and c. Suppose that at b (and hence also at all internal vertices of X-Y) the path turns to the right. We would like to use an angular freedom at a to rotate P_{ac} counterclockwise around a. This is always possible, except when $a = u$ and the turn from the pre-edge onto ab is to the left and is
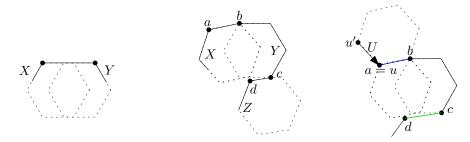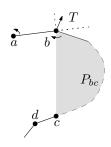
Figure 15: Left: If X, Y are flush, there is a long edge. Middle: $b = X \cap Y$. $b \neq c$. c is a vertex of Y, d is a vertex of Z. cd is aligned either with an edge of Y or with an edge of Z (or both — Y and Z are flush). Right: If a has no angular freedom to rotate counterclockwise, then ab is the second inflection edge in the path $u'u$-P; apply the argument of Lemma 2.8(1) to the path.

exactly θ (Fig. 15, right). In the latter case, however, we can shorten P by sliding P_{bc} either along ba or along cd as in the proof of Lemma 2.8(1) – this is, essentially, an application of the lemma to the path obtained by appending the pre-edge to P (the appended path has 2 inflection edges – ab and cd). Thus in what follows we will assume that P_{ac} can be rotated around a.

Assume that c is at the origin and that cb is vertical and points up. We will do different local modifications depending on the direction of cd. All modifications will move P_{bc} rigidly – either rotating it around b or translating. The rotation is no problem at b – there is a rotational freedom in any direction. In translating P_{bc} we will ensure that the translation vector T, when applied to b, is to the left of the directed line ab and makes an obtuse angle with ab – this way ab will be rotated counterclockwise and also will increase length (so that the length and turn-over-length constraints continue hold). Some of our modifications rotate ac counterclockwise about a, and one rotates P_{ac} counterclockwise about a.
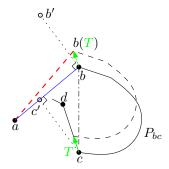

First of all, since X-Y is a right-turning path, both ab and cd are to the right of cb; thus, cd cannot live in the first or fourth quadrant. If cd is in the third quadrant, we simply rotate P_{bc} clockwise. This will locally move c in the $-x$ direction, which is admissible for dc (Definition 2.3) and also shortens the edge. So from now on we assume that cd is in the second quadrant. We separately consider the cases of abc being non-obtuse and obtuse.

If $\angle abc$ is non-obtuse (Fig. 16), drop the perpendicular cc' from c to (the supporting line of) ab. There are two possibilities of where cd lies w.r.t. cc':

cd belongs to the (closure of the) wedge bcc' (Fig. 16, left) Slide P_{bc} along cd towards d by a small vector T. Since cd is to the right of cc', we have that $bb(T)$ is to the right of bb', where $b(T) = b + T$ and bb' is parallel to cc' (and hence perpendicular to ab); thus, the modification does not shorten ab, and the length and turn-over-length constraints are satisfied. Since cd is to the left of cb, $bb(T)$ is to the left of the vertical ray starting at b, and hence $abb(T)$ has a left turn at b, which means that the turn at a decreases, so the turn constraints are also satisfied for the modified path.

cd is strictly outside the wedge bcc' (Fig. 16, right) Rotate ab about a counterclockwise, translating P_{bc} (parallel to itself, without rotation) – locally this means that P_{bc} is translated in the direction cc', which is admissible for cd by the assumption that cc' is to the right of cd. To see that the length of cd decreases note that the angle $\angle dcc'$ is acute – this is because both cd and cc' live in the same (the third) quadrant.

If $\angle abc$ is obtuse (Fig. 17), draw the ray cc^* parallel to ab. Again, there are two possibilities of where cd lies w.r.t. cc^*:
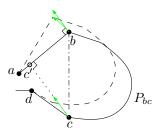
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Figure 16: $cc' \perp ab$. (The figure is schematic: e.g., ab is too long.) If $\angle bcd \leq \angle bcc'$ (left), take $T = \varepsilon \vec{cd}$; the modification replaces blue+green by red. If $\angle bcd > \angle bcc'$ (right), rotate ba around a; this locally moves P_{bc} in the direction cc' – admissible for cd and shortening cd.

cd belongs to the (closure of the) wedge bcc^* (Fig. 17, left) Rotate ab about a counterclockwise, translating P_{bc} (parallel to itself, without rotation) – locally this means that P_{bc} is translated in the direction cc' normal to ab, which is admissible for cd because cd is to the left of cb (since cd is in the third quadrant) while cc' is to the right of bc (since $\angle abc$ is obtuse). To see that the length of cd decreases note that the angle $\angle dcc'$ is acute – this is because cd is to the right of cc^* (since $\angle bcd \leq \angle bcc^*$).

cd is strictly outside the wedge bcc^* (Fig. 16, right) This is the hardest case as no single transformation from the usual suspects—rotation around a, b, shift along cd—works alone (either the path becomes infeasible or lengthens). We thus carefully apply two transformations in succession.

Specifically, let l be the supporting line of cd; we fix l and *do not* move it as cd changes during our transformations (in fact, after *both* our transformations are applied, cd gets back aligned with l). Our first transformation is a rotation of P_{bc} clockwise around b by a small angle φ_b; since we rotate clockwise, the turn from any edge of P_{bc} to l decreases by φ_b, and since cd is in the second quadrant, the rotation moves c below l. (Note the difference: the path's turn angle at c *increases* by some unspecified amount – the turn direction is not admissible for cd; still, the turn onto the line l, which stays put and is not moved with the modification, *decreases* by exactly φ_b.) Our second transformation is a rotation of P_{ac} counterclockwise around a; we rotate until c comes up to l. Let φ_a be the angle of the rotation.

Let us see what happened with the path after the two transformations. The turns at all vertices except a, b, c stayed the same. At a and b we had the freedom, so the turn constraints there are not violated; at c the turn decreased by $\varphi_b - \varphi_a$. Claim B.1 below assures that $\varphi_b > \varphi_a$, and so the turn constraints are satisfied. Now, the only edge that changed the length is cd, but it is an inflection edge, so the turn-over-length constrains are irrelevant for it, and the length constraints are satisfied because an inflection edge is adjacent to normal edges (Lemma 2.2). Thus, the transformed path stays feasible. In addition, Claim B.1 shows that our mission is complete – the transformations make cd shorter.

This finishes proof Lemma 3.3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Claim B.1. $|c_2 d| < |cd|$, where c_2 is the final position of c. Also, $\varphi_b > \varphi_a$.

Proof. Refer to Fig. 18. Let a' be the foot of the perpendicular dropped onto cd form a. We claim that a' is to the left of c. Indeed, let b' be the foot of the perpendicular dropped onto
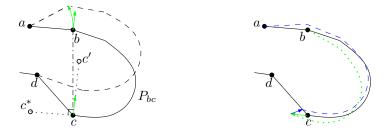
Figure 17: $cc^*|||ab$. If $\angle bcd \leq \angle bcc^*$ (left), rotate ba around a; this locally moves P_{bc} in the direction cc' – admissible for cd and shortening cd. If $\angle bcd > \angle bcc'$ (right), first rotate P_{bc} around b and then rotate P_{ac} counterclockwise around a restoring the direction of cd.
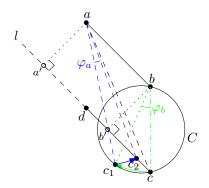

Figure 18: a' is to the left of b' to the left of c. $\angle abc > \angle abc_1 \implies |ac| > |ac_1| = |ac_2| \implies |c_2d| < |cd|$. $\varphi_a = \angle c_1ac_2 < \angle c_1ac < \angle c_1bc = \varphi_b$.

cd form b. Since a is to the left of b, a' is to the left of b'. But b' is to the left of c because $\angle abc > \frac{\pi}{2}$.

Let c_1, c_2 be the images of c after the first and the second transformation resp. For the triangles abc, abc_1 we have $|bc| = |bc_1|$, $\angle abc_1 = \angle abc - \varphi_b$, so by the cosine theorem $|ac_1| < |ac|$. Since $|ac_2| = |ac_1|$, we have $|ac_2| < |ac|$. Since the foot of the perpendicular from a onto cd lies to the left of c, we have that c_2 is closer to d than c (proving the first statement of the claim) and also that $\varphi_a = \angle c_1ac_2 < \angle c_1ac$.

Let C be the circle through b, c, c_1. By continuity, for small enough φ_b, a is outside C — for $\varphi_b = 0$, the circle is the diametrical circle of bc, and a is strictly outside it because $\angle bcd > \angle bcc^*$ (where cc^* is the line parallel to ab — recall that we are still in this case). Thus, $\angle c_1ac < \angle c_1bc = \varphi_b$. $\square$

C   Proof of Lemma 3.5

Let P be the path, and X, Y, Z be the arcs of P (Fig. 19, left). Suppose that P turns to the right at its internal vertices. If the turn from the last edge of Z onto the post-edge is to the left, then P augmented with the post-edge has an inflection edge (the last edge of Z), and we can shorten P as in the proof of Lemma 3.3 — this is, essentially, an application of the lemma to the augmented path consisting of Y, Z, and the post-edge treated as a separate arc. Thus, we will assume that the turn onto the post-edge is to the right; similarly, the turn from the pre-edge is to the right. By assumption the turn at u is strictly less than θ.

By Lemma 3.4, P does not have long edges; thus the consecutive arcs intersect at vertices (cf. Fig. 19, left). Let $b = X \cap Y, c = Y \cap Z$. We modify P by a sequence of 3 rotations using the angular freedom at u, b, c, v. Specifically, for a real number ε let $c(\varepsilon)$ denote the image of c
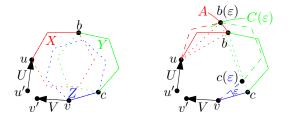
Figure 19: Left: The whole u'-v' path is right-turning. Right: $|c(\varepsilon)b(\varepsilon)| = |cb|, |ub(\varepsilon)| = |ub|$.

after rotation by ε around v, and let $C(\varepsilon)$ be the circle of radius cb the centered at $c(\varepsilon)$ (Fig. 19, right). Let A be the circle of radius $|ub|$ centered at u, and let $b(\varepsilon) = A \cap C(\varepsilon)$. Since u, b, c are not collinear, the circles A and $C(0)$ intersect "properly" (formally, there is another point of intersection in addition to $b = b(0)$; we set $b(\varepsilon)$ to be the point of the intersection closer to b). This means that there is some positive-length interval $\mathcal{E}$ containing 0 in the interior, such that for any $\varepsilon \in \mathcal{E}$ we have $A \cap C(\varepsilon) \neq \emptyset$.

For $\varepsilon \in \mathcal{E}$ let $P(\varepsilon) = X(\varepsilon)Y(\varepsilon)Z(\varepsilon)$ be the path consisting of 3 arcs: u-$b(\varepsilon)$ arc $X(\varepsilon)$, $b(\varepsilon)$-$c(\varepsilon)$ arc $Y(\varepsilon)$, and $c(\varepsilon)$-v arc $X(\varepsilon)$; the arcs are perturbed versions of X, Y, Z and the length of $P(\varepsilon)$ equals that of P. Because of the non-zero freedom at each of u, b, c, v, there exists a positive-length interval $\mathcal{E}' \subset \mathcal{E}$ such that for any ε in $\mathcal{E}'$ the path $P(\varepsilon)$ is a feasible path. Indeed, since our perturbation does not change lengths, the only reason why P may become infeasible is because of a turn constraint. But P had non-zero freedom at u, b, c *both* in clockwise and counterclockwise directions – by our assumption P was not flush. As far as v is concerned, if the turn of P at v onto the post-edge is not θ, then P has a similar (both clockwise and counterclockwise) freedom at v; if the turn at v is exactly θ, then the freedom at v is one-sided, i.e., $\mathcal{E}'$ has 0 as an endpoint. Still, even in the latter case, c can be rotated clockwise around v by a non-zero ε while keeping the path feasible.

Let ε^* be an endpoint of $\mathcal{E}'$, such that the turn at v onto the post-edge is not θ and is to the right. (If the turn of P at v was θ, then ε^* is the non-zero endpoint of $\mathcal{E}'$; otherwise, ε^* can be either of $\mathcal{E}'$'s endpoints.)

If ε^* is an endpoint of $\mathcal{E}$, then we can feasibly perturb P so that u, b and c become collinear. If any of the arcs X, Y has more than 1 edge, then collinearity of the arcs endpoints implies existence of an inflection edge in $P(\varepsilon^*)$, which means that $P(\varepsilon^*)$ can be shortened, and hence (since $P(\varepsilon^*)$ and P are equal-length) that P was not a shortest path. On the other hand, if both $X = ub$ and $Y = bc$, then the collinearity of u, b, c implies that uc is a long edge of $P(\varepsilon^*)$, which also means, by Lemma 3.4, that $P(\varepsilon^*)$ is not shortest.

If ε^* is not an endpoint of $\mathcal{E}$, then an angular constraint must become tight at ε^* — either the turn at v becomes θ and to the left (and then proceed as in the first paragraph), or the turn at one of $u(\varepsilon^*), b(\varepsilon^*), c(\varepsilon^*)$ becomes θ. In the latter case, either the path is flush at u, or the number of arcs in $P(\varepsilon^*)$ becomes less than 3.

D   Proof of Lemma 4.1

Since we add turns of degree 0, the turn constraints and the turn-over-length constraints are not violated. To check the length constraints assume for the sake of contradiction that there exist adjacent short edges ab, bc in P'.

If the turn at b is not 0, at least one of the vertices a, c must have been appended to P (for otherwise already the original path P had adjacent short edges ab, bc); suppose it is a (Fig. 20, left). Let a^- be the vertex of P preceding a. By definition, no edge may contain two bridges; thus ab is the only bridge on the edge a^-b, and the part a^-a is a normal edge of some arc.
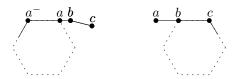
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Figure 20: Left: If the turn at b is not zero, then a^- is a vertex of P and a^-a is a normal edge – part of an arc, so P had a long edge a^-b adjacent to short edge bc. Right: If the turn at b is zero, then bc is a normal edge – part of an arc.

Hence the edge a^-b is long. By similar argument, if c is appended, then bc^+ is long. If c is not appended, then bc is a short edge of the original path. In any case P had a long edge a^-b adjacent to a short or long edge bc – contradiction to Lemma 2.1.

Assume now that the turn at b is 0 (Fig. 20, right). Then b has been appended, and thus is a bridge endpoint. An edge can contain at most two appended vertices, so one of a, c is an original vertex of P; suppose it is c. But then, since b is an endpoint of an arc, bc is a normal edge – contradiction to our assumption that ab, bc are short.

E   Proof of Lemma 4.3

Let us check that all constraints in the definition of a discrete curvature-constrained path are satisfied for $P(\varepsilon T)$ for small enough ε.

Turn constraints The bridge endpoints are the only vertices the turns at which may change due to the modification of the path. From definition of arc, a bridge never makes an angle θ with an adjacent edge (or else part of the bridge would have belonged to an arc). Thus, the turns at both a and b can change without violating the turn constraints, provided the changes are small.

Length constraints The bridge is the only edge whose length may change due to the modification; for that to possibly break the length constraints, the bridge must be adjacent to a short edge, and must itself become short in $P(\varepsilon T)$ while being normal in P (otherwise, if the bridge was long in P, it will remain long also in $P(\varepsilon T)$, for a sufficiently small ε). However, having a normal bridge adjacent to a short edge contradicts optimality of P. To see this, let us look more closely when a path can have a normal bridge:

- If both a and b were vertices of P, then ab would not have been a bridge (it would have been part of an arc).
- If none of a, b was a vertex of P (i.e., if both were added during the canonization), then ab is adjacent to normal edges on both sides (the edges, together with the bridge, formed a length-3ℓ edge of P with turns of θ at both vertices); Fig. 7, middle.
- If a was originally a vertex of P but b was added during the canonization (Fig. 7, right) then the edge bb^+ incident to b is normal, so the only remaining problematic case is when the edge a^-a, incident to a is short. But in this case the original path had a short edge adjacent to a long edge bb^+ contradictory to Lemma 2.1.
- The situation when b was a vertex of P but a was added during the canonization is symmetric to the above.

Turn-over-length constraints Again, the only problematic case is when the bridge was normal in P, became short in $P(\varepsilon T)$, and the turn from a^-a to bb^+ is larger then θ. Similarly to the above, if the bridge was normal, then its at least one incident turn was 0 in P; hence for small ε the turn-over-length constraints will not be violated.

Figure 21: Left: $b = c$. It must be that b is a vertex of P and ab is normal, implying that a was added during canonization, and that P had a long edge a^-b adjacent to an inflection edge bd – contradiction to Lemma 2.2. Right: The modification replaces blue+green by red.
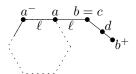

Figure 22: One of the adjacent bridges is normal, and hence is a proper subset of a long edge; the long edge is adjacent to a normal edge, and the second bridge has "no space" to fit on the normal edge.

F Proof of Lemma 4.4

Proof of 1. Let ab be a bridge, and cd and inflection edge. First suppose that ab and cd are adjacent, say $b = c$ (Fig. 21, left). If b were not a vertex of P before the canonization (i.e., if b was added in the middle of an edge of P), then bd is not an inflection edge (the turn at b is 0); so b must be a vertex of P. If ab is not normal, then it cannot be adjacent to an inflection edge by Lemma 2.2; so ab must be normal. Then a is not a vertex of P, since otherwise ab, being a normal edge, would not have been a bridge (it would have been part of an arc). Thus, a was added during the canonization in the middle of an edge a^-b of P. Since $|a^-a| = \ell$, we have that ab is long, and is adjacent to an inflection edge cd – a contradiction to Lemma 2.2.

Suppose now that ab and cd are not adjacent, and let P' be the subpath of P between b and c (Fig. 21, right). We rigidly translate P' so that c slides towards d and ab rotates around a keeping connectivity to b. By Lemmas 2.6 and 4.3, the path remains feasible. By the triangle inequality, the path shortens.

Proof of 2. Similar to proof of 1: If the bridge and the long edge are adjacent, the bridge must be normal (by Lemma 2.1), but then it belongs to a long edge of the original (non-canonized) path, which implies that the original path had adjacent long edges – a contradiction to Lemma 2.1. If the bridge and the long edge are not adjacent, the part between them can be moved along the long edge just as it was moved along the inflection edge in the proof of 1; Lemma 2.7 is used in place of Lemma 2.6 – to ensure feasibility of the motion.

Proof of 3. Similar to proofs of 1 and 2: First suppose that the two bridges ab, cd are adjacent, i.e., $b = c$ (Fig. 22). Since no edge can have two bridges, b must be a vertex of the original path, which implies (Lemma 2.1) that at least one of the bridges, say ab, is normal. The normal bridge must be a proper subset of a long edge a^-b of the original path. Let bb^+ be the edge of the original path adjacent to a^-b; by Lemma 2.1, bb^+ is normal. Now, if $d \neq b^+$, then bd would be a short edge adjacent to a long edge a^-b (contradicting Lemma 2.1); on the other hand, if $d = b^+$ then bd cannot be a bridge – it should be (part of) and arc.

If the two bridges are not adjacent, the part between them can be moved along one of them just as it was moved along the inflection edge in the proof of 1; Lemma 4.3 is used to ensure feasibility of the motion.

$\square$