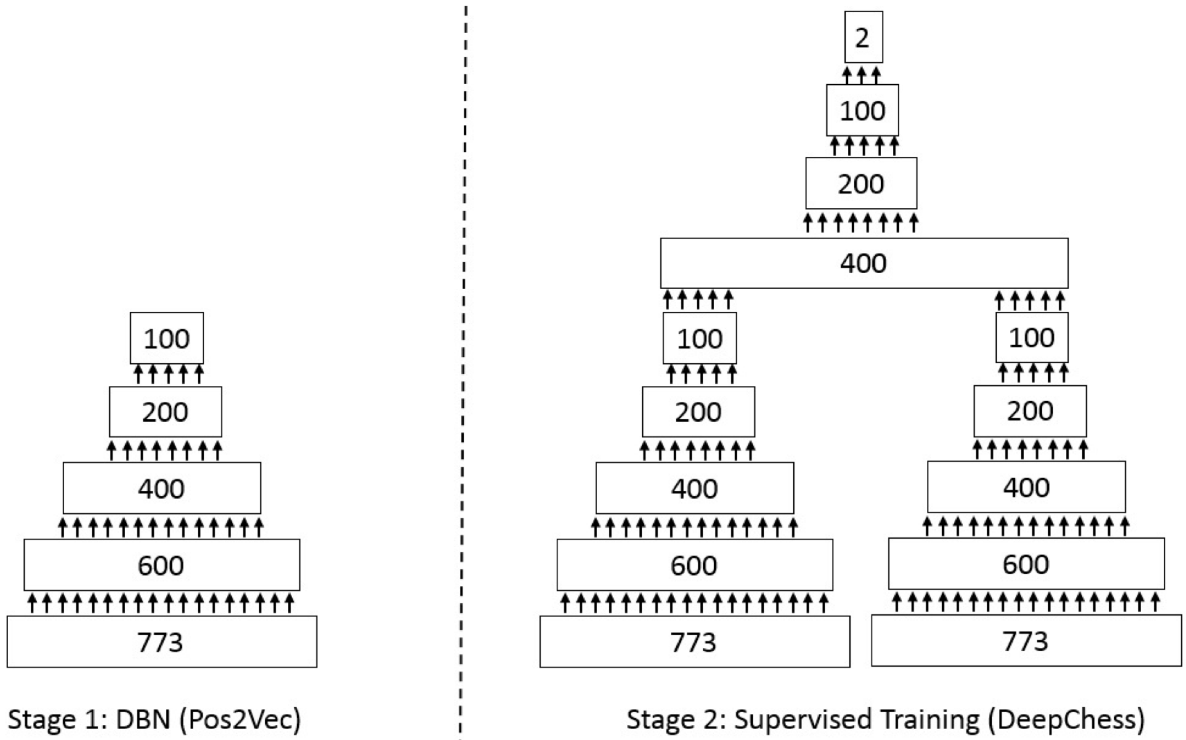
Methodlogy

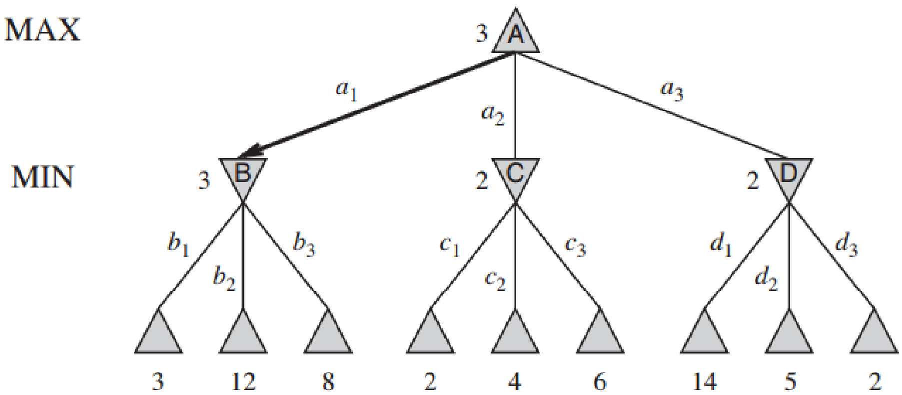
1. Autoencoder + Siamese Network



DeepChess Score Estimator

- Autoencoder: This embedding the state board into a lower-dimensional space.
  - Encoder: maps the input to a lower-dimensional latent space
  - Decoder: maps the latent space back to the original input space
- Siamese Network: This learned to compare two input mapping it to end score
  - The network takes two inputs: the current state and the next state
  - The network outputs a score

Minimax Search




```
function minimax(position, depth, maximizingPlayer)
    if depth == 0 or game over in position
        return static evaluation of position

    if maximizingPlayer
        maxEval = -infinity
```

```
        for each child of position
            eval = minimax(child, depth − 1, false)
            maxEval = max(maxEval, eval)
        return maxEval

    else
        minEval = +infinity
        for each child of position
            eval = minimax(child, depth − 1, true)
            minEval = min(minEval, eval)
        return minEval

// initial call
minimax(currentPosition, 3, true)
```

2. Discrete Generative Model

2.1 Discrete Diffusion Model + ASA

Problem Settings

- We describe Chess game as a Markov Process (Which is a type of Markov Decision Process - Given the current state, the next state is independent of the previous states)
  - State Space ($\mathbb{S}$): The state of the board
  - Action Space ($\mathbb{A}$): The set of all possible moves
  - State transition function ($f(s, a)$): The function that takes a state and an action and returns the next state
  - Action Probability Distribution ($p(a|s)$): The probability of taking action a in state s
  - $o_i = 1 || -1$: terminal reward at time i (win/loss)
  - Value function ($v^p(s)$): The expected return when both player play with the same strategy p

$$v^p(s) = \mathrm{E}\left[o_i | s_i = s, a_{i...I} \sim p\right]$$

$\implies$ Our target is maximize the value function $v^p(s)$ for my engine. Then to do that we need to find the best action a^* in state s.
In other words, we need to find the best policy π^* that maximizes the value function $v^p(s)$. $\implies$ Learning the best probability distribution $\pi^*(a|s)$ for each state s.

Discrete Diffusion Model

- Diffusion Models are a class of generative model which consist 2 processes: forward and reverse processes. Both processes are Markov chains.
  - The forward process is a Markov chain that gradually adds noise to the data,
  - The reverse process is a Markov chain that gradually removes noise from the data.
- Suppose we want to learn a target sample $x_0 \sim q(x_0)$, x_0 is a discrete random variable with K possible values. Defining the forward and backward processes as follows:
  - **Forward process**: gradually adds noise to the data

$$q(x_{1:T} | x_0) = \prod_{t=1}^{T} q(x_t | x_{t-1})$$

  - **Reverse process**: gradually removes noise from the data

$$p_\theta(x_{0:T}) = p(x_T) \prod_{t=1}^{T} p_\theta(x_{t-1} | x_t)$$

- We propose the target function for conditioning the forward process on the action a:

$$K(q_\psi(\cdot|x), p_\theta(\cdot, x)) = \int_{-\infty}^{\infty} q_\psi(z|x)\log(\frac{q_\psi(z|x)}{p_\theta(z|x)})$$

$$\mathrm{L} = \mathrm{E}_{q_\psi(\cdot|x)}[\log(\frac{p_\theta(x,Z)}{q_\psi(Z|x)})]$$

$$\Longrightarrow \begin{aligned} \log p_\theta(x) &= \int_{-\infty}^{\infty} q(\cdot|x)\log(p_\theta(x))dz \\ &= \int_{-\infty}^{\infty} q_\psi(\cdot|x)\log(\frac{p_\theta(x,Z)}{q_\psi(z|x)})dz + \int_{-\infty}^{\infty} q_\psi(z|x)\log(\frac{q_\psi(z|x)}{p_\theta(z|x)})dz \\ &= \mathrm{L} + K(q_\psi(\cdot|x), p_\theta(\cdot, x)) \\ &\geq \mathrm{L} \end{aligned}$$

$$\Longrightarrow \mathrm{L} \geq -\log p_\theta(x)$$

$\Longrightarrow$ We call L as ELBO (Evidence Lower Bound)

When $q(\cdot|x)$ close $p_\theta(\cdot|x)$, then ELBO simiar to \log p_\theta(x) $\Longrightarrow$ L is an upper bound on the negative log-likelihood

$\Longrightarrow$ Minimizing this upper bound L brings us closer to minimizing the true negative log-likelihood.

$\Longrightarrow$ We got the general type of loss function

$$\mathrm{L}_{vb} = -\log p_\theta(x) + K(q_\psi(\cdot|x), p_\theta(\cdot, x))$$

$\Longrightarrow$ We can use above loss for muiltiple steps:

$$\mathrm{L} = \mathrm{E}_{q(x_0)}[\underbrace{D_{\mathrm{KL}}[q(x_T|x_0)\,\|\,p(x_T)]}_{L_T} + \underbrace{\sum_{t=2}^{T}\mathrm{E}_{q(x_t|x_0)}[D_{\mathrm{KL}}[q(x_{t-1}|x_t, x_0)\,\|\,p_\theta(x_{t-1}|x_t)]]}_{L_{t-1}} - \underbrace{\mathrm{E}_{q(x_1|x_0)}[\log p_\theta(x_0|x_1)]}_{L_0}]$$

- We could simplified the above loss function by using the following equation:

$$K(q(x_{t-1}|x_t, x_0)\|p_\theta(x_{t-1}|x_t)) = -\lambda_t 1_{x_t = x_o} x_0^T \log f(x_t, \theta)$$

$$\Longrightarrow L_{vb} = -\mathrm{E}_{q(x_0)}[\sum_{t=1}^{T}\lambda_t \mathrm{E}_{q(x_t|x_0)} 1_{x_t = x_o} x_0^T \log f(x_t, \theta)]$$

- $\lambda_t = \frac{\alpha_{t-1} - \alpha_t}{1 - \alpha_t}$: time-dependent reweighting term that assigns lower weight for noisier x_t
- $\alpha_t \in [0, 1]$: predefined noise scheduler

ASA - Action/State/Action

- Our probability distribution is a function of the state and action.

$$p_\theta(a_i, s_{i+1}, a_{i+1}, \ldots, s_{i+h-1}.a_{i+h-1}|s_i)$$

Training

- Data: use the output of the stockfish engine as the training data.

$$\mathrm{D} = \{(s_i, (a_i^{SF}, s_{i+1}, \ldots, a_{i+h-1}^{SF}))\}$$

Inference

$$\arg\max p_\theta(a_i, s_{i+1}, a_{i+1}, \ldots, s_{i+h-1}, a_{i+h-1}|s_i)$$

Algorithm 1 DIFFUSEARCH Training

Input: dataset $\mathcal{D} = \{(s,(a,z))\}$, neural network $f(\cdot;\boldsymbol{\theta})$, timesteps T.
Output: model parameters $\boldsymbol{\theta}$.
Denote state length $l = |s|$;
repeat
  Draw $(s,(a,z)) \sim \mathcal{D}$ and obtain $\mathbf{x}_{0,1:N} = s \mid\mid a \mid\mid z$ ($\mid\mid$: concat);
  Draw $t \in \text{Uniform}(\{1,\ldots,T\})$;
  Draw $\mathbf{x}_{t,n} \sim q(\mathbf{x}_{t,n}|\mathbf{x}_{0,n})$ for $n \in \{l+1,\ldots,N\}$;
  $L(\boldsymbol{\theta}) = -\lambda_t \sum_{n=l+1}^{N} 1_{\mathbf{x}_{t,n} \neq \mathbf{x}_{0,n}} \mathbf{x}_{0,n}^\top \log f(\mathbf{x}_{t,n};\boldsymbol{\theta})$;
  Minimize $L(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$;
until converged

Algorithm 2 DIFFUSEARCH Inference

Input: board state s, trained network $f(\cdot;\boldsymbol{\theta})$, timesteps T.
Output: next action a.
Denote state length $l = |s|$;
Initialize $\mathbf{x}_{T,1:l} = s$ and $\mathbf{x}_{T,l+1:N} \sim q_{\text{noise}}$;
for $t = T,\ldots,1$ **do**
  for $n = l+1,\ldots,N$ **do**
    Draw $\widetilde{\mathbf{x}}_{0,n} \sim \text{Cat}(f(\mathbf{x}_{t,n};\boldsymbol{\theta}))$;
    Draw $\mathbf{x}_{t-1,n} \sim q(\mathbf{x}_{t-1,n} \mid \mathbf{x}_{t,n}, \widetilde{\mathbf{x}}_{0,n})$;
  end for
end for
Return $a = \mathbf{x}_{0,l+1}$.

2.2 Discrete Flow Matching Model + ASA

3. Heuristic Search