Low-Rank Approach for Image Non-Blind Deconvolution with Variance Estimation

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1 Introduction

Image deconvolution is a fundamental problem in image processing that aims to recover a sharp image from its blurred and noisy observation. The degradation process is typically modeled as:

$$G = H * X + N \tag{1}$$

where G is the observed blurry and noisy image, H is the known blur kernel (Point Spread Function), X is the latent sharp image to be estimated, and N is additive white Gaussian noise with variance σ_n^2.

In this work, we present a low-rank approach for image non-blind deconvolution that leverages the non-local self-similarity property of natural images and incorporates variance estimation for improved performance.

2 Mathematical Formulation

2.1 MAP Estimation Framework

Under the Maximum A Posteriori (MAP) estimation framework, we have:

$$\hat{X} = \arg\max_X \log P(X|G) = \arg\max_X [\log P(G|X) + \log P(X)] \tag{2}$$

Assuming Gaussian noise, the likelihood term is:

$$P(G|X) = \frac{1}{(2\pi\sigma_n^2)^{n/2}} \exp\left(-\frac{|H * X - G|^2}{2\sigma_n^2}\right) \tag{3}$$

Taking the negative logarithm and ignoring constants, we get the data fidelity term:

$$-\log P(G|X) \propto \frac{1}{2\sigma_n^2}|H * X - G|^2 \tag{4}$$

The prior term is modeled using a low-rank regularization:

$$-\log P(X) \propto \lambda\Phi(X) \tag{5}$$

where $\Phi(X)$ is a low-rank promoting regularizer. Thus, the optimization problem becomes:

$$\hat{X} = \arg\min_X \left(\frac{1}{2\sigma_n^2} |H * X - G|^2 + \lambda\Phi(X) \right) \tag{6}$$

2.2 ADMM Formulation

To decouple the deblurring and denoising steps, we introduce an auxiliary variable Y with the constraint $Y = X$:

$$\min_{X,Y} \left\{ \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \lambda\Phi(X) \right\} \quad \text{subject to} \quad Y = X \tag{7}$$

The augmented Lagrangian function is:

$$\mathcal{L}(Y, X, M) = \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \lambda\Phi(X) + \langle M, Y - X \rangle + \frac{\rho}{2} |Y - X|^2 \tag{8}$$

where M is the Lagrange multiplier and $\rho > 0$ is the penalty parameter.

3 ADMM Algorithm Derivation

The ADMM algorithm alternates between solving three subproblems:

3.1 Y-subproblem (Deblurring Step)

Fixing X and M, we solve:

$$\begin{aligned} Y^{k+1} &= \arg\min_Y \left\{ \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \langle M^k, Y - X^k \rangle + \frac{\rho}{2} |Y - X^k|^2 \right\} \\ &= \arg\min_Y \left\{ \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \frac{\rho}{2} |Y - X^k + \frac{M^k}{\rho}|^2 \right\} \end{aligned} \tag{9}$$

This is a quadratic optimization problem. Taking the derivative and setting it to zero:

$$\frac{1}{\sigma_n^2} H^T (H * Y - G) + \rho(Y - U^k) = 0 \tag{10}$$

where $U^k = X^k - \frac{M^k}{\rho}$. Transforming to the frequency domain (assuming circular convolution):

$$\frac{1}{\sigma_n^2}\mathcal{F}(H)^* \cdot (\mathcal{F}(H) \cdot \mathcal{F}(Y) - \mathcal{F}(G)) + \rho(\mathcal{F}(Y) - \mathcal{F}(U^k)) = 0 \tag{11}$$

Rearranging terms:

$$\left[\frac{1}{\sigma_n^2}|\mathcal{F}(H)|^2 + \rho\right] \cdot \mathcal{F}(Y) = \frac{1}{\sigma_n^2}\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \rho\mathcal{F}(U^k) \tag{12}$$

Thus, the solution in the frequency domain is:

$$\mathcal{F}(Y^{k+1}) = \frac{\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \rho\sigma_n^2\mathcal{F}(U^k)}{|\mathcal{F}(H)|^2 + \rho\sigma_n^2} \tag{13}$$

Letting $\lambda = \rho\sigma_n^2$, we get the final expression:

$$\mathcal{F}(Y^{k+1}) = \frac{\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \lambda\mathcal{F}(U^k)}{|\mathcal{F}(H)|^2 + \lambda} \tag{14}$$

3.2 X-subproblem (Denoising Step)

Fixing Y and M, we solve:

$$\begin{aligned} X^{k+1} &= \arg\min_X \left\{\lambda\Phi(X) + \langle M^k, Y^{k+1} - X\rangle + \frac{\rho}{2}|Y^{k+1} - X|^2\right\} \\ &= \arg\min_X \left\{\lambda\Phi(X) + \frac{\rho}{2}|X - (Y^{k+1} + \frac{M^k}{\rho})|^2\right\} \end{aligned} \tag{15}$$

Letting $V^{k+1} = Y^{k+1} + \frac{M^k}{\rho} = Y^{k+1} - Z^k$ where $Z^k = -\frac{M^k}{\rho}$ is the scaled Lagrange multiplier, we get:

$$X^{k+1} = \arg\min_X \left\{\lambda\Phi(X) + \frac{\rho}{2}|X - V^{k+1}|^2\right\} \tag{16}$$

This corresponds to denoising V^{k+1} with noise level $\sigma^2 = \frac{1}{\rho}$ and regularization parameter λ.

3.3 Multiplier Update Step

Updating the Lagrange multiplier:

$$M^{k+1} = M^k + \rho(Y^{k+1} - X^{k+1}) \tag{17}$$

Or using the scaled multiplier $Z^k = -\frac{M^k}{\rho}$:

$$Z^{k+1} = Z^k - (Y^{k+1} - X^{k+1}) \tag{18}$$

4 Low-Rank Denoising

The denoising step utilizes the non-local self-similarity property of natural images. For each reference patch, we find similar patches and form a matrix:

$$P_i = [\mathrm{vec}(p_{i_1}), \mathrm{vec}(p_{i_2}), \ldots, \mathrm{vec}(p_{i_M})] \in \mathbb{R}^{d \times M} \tag{19}$$

We then solve the low-rank matrix recovery problem:

$$\hat{X}i = \arg\min_X |X| * + \frac{1}{2\mu} |P_i - X|_F^2 \tag{20}$$

where $| \cdot |_*$ is the nuclear norm. The solution is obtained via singular value thresholding:

$$\hat{X}i = U \mathcal{S} \mu(\Sigma) V^T \tag{21}$$

where $(U, \Sigma, V) = \mathrm{SVD}(P_i)$ and $\mathcal{S}_\mu(\sigma) = \mathrm{sign}(\sigma) \max(|\sigma| - \mu, 0)$ is the soft-thresholding operator.

5 Variance Estimation

We estimate the noise variance at each iteration using:

$$\sigma_k^2 = c_1 \cdot \left(\eta^2 \cdot \left| \frac{\mathcal{F}(H)^*}{|\mathcal{F}(H)|^2 + \rho_k} \right| 2^2 + \sigma k - 1^2 \cdot \left| \frac{\rho_k}{|\mathcal{F}(H)|^2 + \rho_k} \right|_2^2 \right) \tag{22}$$

where c_1 is an empirical constant that compensates for noise correlations.

6 Algorithm Summary

The complete algorithm is summarized as follows:

1. Initialize X^0, $Z^0 = 0$, ρ_0, $\lambda_0 = \rho_0 \sigma_n^2$
2. For $k = 0, 1, 2, \ldots$ until convergence:
   (a) Deblurring: $\mathcal{F}(Y^{k+1}) = \frac{\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \lambda \mathcal{F}(X^k + Z^k)}{|\mathcal{F}(H)|^2 + \lambda}$
   (b) Denoising: $X^{k+1} = \mathrm{denoise}(Y^{k+1} - Z^k, \sigma = \frac{1}{\sqrt{\rho}}, \lambda)$
   (c) Multiplier update: $Z^{k+1} = Z^k - (Y^{k+1} - X^{k+1})$
   (d) Parameter update: $\rho_{k+1} = \beta \rho_k$, $\lambda_{k+1} = \rho_{k+1} \sigma_n^2$
   (e) Variance estimation: Update σ_k^2 using the above formula

7 Conclusion

We have presented a detailed mathematical derivation of a low-rank approach for image non-blind deconvolution with variance estimation. The algorithm combines the efficiency of frequency-domain deblurring with the effectiveness of spatial-domain low-rank denoising, while adaptively estimating noise variance for improved performance.