

# Low-Rank Approach for Image Non-Blind Deconvolution with Variance Estimation

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## 1 Introduction

Image deconvolution is a fundamental problem in image processing that aims to recover a sharp image from its blurred and noisy observation. The degradation process is typically modeled as:

$$G = H * X + N \quad (1)$$

where  $G$  is the observed blurry and noisy image,  $H$  is the known blur kernel (Point Spread Function),  $X$  is the latent sharp image to be estimated, and  $N$  is additive white Gaussian noise with variance  $\sigma_n^2$ .

In this work, we present a low-rank approach for image non-blind deconvolution that leverages the non-local self-similarity property of natural images and incorporates variance estimation for improved performance.

## 2 Mathematical Formulation

### 2.1 MAP Estimation Framework

Under the Maximum A Posteriori (MAP) estimation framework, we have:

$$\hat{X} = \arg \max_X \log P(X|G) = \arg \max_X [\log P(G|X) + \log P(X)] \quad (2)$$

Assuming Gaussian noise, the likelihood term is:

$$P(G|X) = \frac{1}{(2\pi\sigma_n^2)^{n/2}} \exp\left(-\frac{|H * X - G|^2}{2\sigma_n^2}\right) \quad (3)$$

Taking the negative logarithm and ignoring constants, we get the data fidelity term:

$$-\log P(G|X) \propto \frac{1}{2\sigma_n^2} |H * X - G|^2 \quad (4)$$

The prior term is modeled using a low-rank regularization:

$$-\log P(X) \propto \lambda \Phi(X) \quad (5)$$

where  $\Phi(X)$  is a low-rank promoting regularizer. Thus, the optimization problem becomes:

$$\hat{X} = \arg \min_X \left( \frac{1}{2\sigma_n^2} |H * X - G|^2 + \lambda \Phi(X) \right) \quad (6)$$

## 2.2 ADMM Formulation

To decouple the deblurring and denoising steps, we introduce an auxiliary variable  $Y$  with the constraint  $Y = X$ :

$$\min_{X,Y} \left\{ \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \lambda \Phi(X) \right\} \quad \text{subject to} \quad Y = X \quad (7)$$

The augmented Lagrangian function is:

$$\mathcal{L}(Y, X, M) = \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \lambda \Phi(X) + \langle M, Y - X \rangle + \frac{\rho}{2} |Y - X|^2 \quad (8)$$

where  $M$  is the Lagrange multiplier and  $\rho > 0$  is the penalty parameter.

## 3 ADMM Algorithm Derivation

The ADMM algorithm alternates between solving three subproblems:

### 3.1 Y-subproblem (Deblurring Step)

Fixing  $X$  and  $M$ , we solve:

$$\begin{aligned} Y^{k+1} &= \arg \min_Y \left\{ \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \langle M^k, Y - X^k \rangle + \frac{\rho}{2} |Y - X^k|^2 \right\} \\ &= \arg \min_Y \left\{ \frac{1}{2\sigma_n^2} |H * Y - G|^2 + \frac{\rho}{2} \left| Y - X^k + \frac{M^k}{\rho} \right|^2 \right\} \end{aligned} \quad (9)$$

This is a quadratic optimization problem. Taking the derivative and setting it to zero:

$$\frac{1}{\sigma_n^2} H^T (H * Y - G) + \rho(Y - U^k) = 0 \quad (10)$$

where  $U^k = X^k - \frac{M^k}{\rho}$ . Transforming to the frequency domain (assuming circular convolution):

$$\frac{1}{\sigma_n^2} \mathcal{F}(H)^* \cdot (\mathcal{F}(H) \cdot \mathcal{F}(Y) - \mathcal{F}(G)) + \rho(\mathcal{F}(Y) - \mathcal{F}(U^k)) = 0 \quad (11)$$

Rearranging terms:

$$\left[ \frac{1}{\sigma_n^2} |\mathcal{F}(H)|^2 + \rho \right] \cdot \mathcal{F}(Y) = \frac{1}{\sigma_n^2} \mathcal{F}(H)^* \cdot \mathcal{F}(G) + \rho \mathcal{F}(U^k) \quad (12)$$

Thus, the solution in the frequency domain is:

$$\mathcal{F}(Y^{k+1}) = \frac{\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \rho \sigma_n^2 \mathcal{F}(U^k)}{|\mathcal{F}(H)|^2 + \rho \sigma_n^2} \quad (13)$$

Letting  $\lambda = \rho \sigma_n^2$ , we get the final expression:

$$\mathcal{F}(Y^{k+1}) = \frac{\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \lambda \mathcal{F}(U^k)}{|\mathcal{F}(H)|^2 + \lambda} \quad (14)$$

### 3.2 X-subproblem (Denoising Step)

Fixing  $Y$  and  $M$ , we solve:

$$\begin{aligned} X^{k+1} &= \arg \min_X \left\{ \lambda \Phi(X) + \langle M^k, Y^{k+1} - X \rangle + \frac{\rho}{2} |Y^{k+1} - X|^2 \right\} \\ &= \arg \min_X \left\{ \lambda \Phi(X) + \frac{\rho}{2} |X - (Y^{k+1} + \frac{M^k}{\rho})|^2 \right\} \end{aligned} \quad (15)$$

Letting  $V^{k+1} = Y^{k+1} + \frac{M^k}{\rho} = Y^{k+1} - Z^k$  where  $Z^k = -\frac{M^k}{\rho}$  is the scaled Lagrange multiplier, we get:

$$X^{k+1} = \arg \min_X \left\{ \lambda \Phi(X) + \frac{\rho}{2} |X - V^{k+1}|^2 \right\} \quad (16)$$

This corresponds to denoising  $V^{k+1}$  with noise level  $\sigma^2 = \frac{1}{\rho}$  and regularization parameter  $\lambda$ .

### 3.3 Multiplier Update Step

Updating the Lagrange multiplier:

$$M^{k+1} = M^k + \rho(Y^{k+1} - X^{k+1}) \quad (17)$$

Or using the scaled multiplier  $Z^k = -\frac{M^k}{\rho}$ :

$$Z^{k+1} = Z^k - (Y^{k+1} - X^{k+1}) \quad (18)$$

## 4 Low-Rank Denoising

The denoising step utilizes the non-local self-similarity property of natural images. For each reference patch, we find similar patches and form a matrix:

$$P_i = [\text{vec}(p_{i_1}), \text{vec}(p_{i_2}), \dots, \text{vec}(p_{i_M})] \in \mathbb{R}^{d \times M} \quad (19)$$

We then solve the low-rank matrix recovery problem:

$$\hat{X}i = \arg \min_X |X|_* + \frac{1}{2\mu} \|P_i - X\|_F^2 \quad (20)$$

where  $|\cdot|_*$  is the nuclear norm. The solution is obtained via singular value thresholding:

$$\hat{X}i = U \mathcal{S}_\mu(\Sigma) V^T \quad (21)$$

where  $(U, \Sigma, V) = \text{SVD}(P_i)$  and  $\mathcal{S}_\mu(\sigma) = \text{sign}(\sigma) \max(|\sigma| - \mu, 0)$  is the soft-thresholding operator.

## 5 Variance Estimation

We estimate the noise variance at each iteration using:

$$\sigma_k^2 = c_1 \cdot \left( \eta^2 \cdot \left| \frac{\mathcal{F}(H)^*}{|\mathcal{F}(H)|^2 + \rho_k} \right|^2 + \sigma_{k-1}^2 \cdot \left| \frac{\rho_k}{|\mathcal{F}(H)|^2 + \rho_k} \right|^2 \right) \quad (22)$$

where  $c_1$  is an empirical constant that compensates for noise correlations.

## 6 Algorithm Summary

The complete algorithm is summarized as follows:

1. Initialize  $X^0, Z^0 = 0, \rho_0, \lambda_0 = \rho_0 \sigma_n^2$
2. For  $k = 0, 1, 2, \dots$  until convergence:
  - (a) Deblurring:  $\mathcal{F}(Y^{k+1}) = \frac{\mathcal{F}(H)^* \cdot \mathcal{F}(G) + \lambda \mathcal{F}(X^k + Z^k)}{|\mathcal{F}(H)|^2 + \lambda}$
  - (b) Denoising:  $X^{k+1} = \text{denoise}(Y^{k+1} - Z^k, \sigma = \frac{1}{\sqrt{\rho}}, \lambda)$
  - (c) Multiplier update:  $Z^{k+1} = Z^k - (Y^{k+1} - X^{k+1})$
  - (d) Parameter update:  $\rho_{k+1} = \beta \rho_k, \lambda_{k+1} = \rho_{k+1} \sigma_n^2$
  - (e) Variance estimation: Update  $\sigma_k^2$  using the above formula

## 7 Conclusion

We have presented a detailed mathematical derivation of a low-rank approach for image non-blind deconvolution with variance estimation. The algorithm combines the efficiency of frequency-domain deblurring with the effectiveness of spatial-domain low-rank denoising, while adaptively estimating noise variance for improved performance.