Yaw Moment Diagram (MMM) Equations

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1 Nomenclature

Symbol	Name
O	World Frame
I	Intermediate Frame
B	Body Frame
A	Aero Frame
T	Tire Frame
W	Wheel Frame

Table 1: Frame Nomenclature
(See SAE-J670, Z-Up Convention)

Symbol	Name	Frame
δ_W	Steering Wheel Angle	
$\eta_{T/B}$	Pedal Requests	
$z_{r,j}$	Road Displacement	O

Table 2: Input Nomenclature

Symbol	Name	Frame
$\dot{x}, \ddot{x}$	Longitudinal Velocity, Acceleration	I
$\dot{y}, \ddot{y}$	Lateral Velocity, Acceleration	I
$z, \dot{z}, \ddot{z}$	Vertical Displacement, Rate, Acceleration	I
$\theta, \dot{\theta}, \ddot{\theta}$	Pitch Angle, Rate, Acceleration	I
$\phi, \dot{\phi}, \ddot{\phi}$	Roll Angle, Rate, Acceleration	I
$\psi, \dot{\psi}, \ddot{\psi}$	Yaw Heading, Rate, Acceleration	I
$z_{u,j}, \dot{z}_{u,j}, \ddot{z}_{u,j}$	Wheel Displacement, Rate, Acceleration	I

Table 3: State Nomenclature

Symbol	Name	Frame
$\vec{F}_{T_j}$	Tire Loads	T
F_{T,x_j}	Longitudinal Tire Loads	T
F_{T,y_j}	Lateral Tire Loads	T
F_{T,z_j}	Normal Tire Loads	T
M_{T,x_j}	Overturning Tire Moments	T
M_{T,y_j}	Tire Rolling Resistances	T
M_{T,z_j}	Aligning Tire Moments	T
$\vec{r}_{T_j}$	Tire Positions (w.r.t. C.G.)	I
α_j	Slip Angle	T
κ_j	Slip Ratio	T
P_{i_j}	Inflation Pressure	N/A
$z_{c,j}$	Sprung Corner Displacement	I
$\vec{F}_{A_j}$	Aerodynamic Loads	A
$F_{A,x}$	Drag Force	A
$F_{A,y}$	Side Force	A
$F_{A,z}$	Downforce	A
$M_{A,x}$	Roll Moment	A
$M_{A,y}$	Pitch Moment	A
$M_{A,z}$	Yaw Moment	A

Table 4: Main Evaluations Nomenclature

Symbol	Name	Frame
m_s	Sprung Mass	B
I_s	Sprung Inertia	B
m_{u_j}	Unsprung Mass	W
I_{u_j}	Unsprung Inertia	W
δ_j	Tire Steer Angle	I
γ_j	Inclination Angle	T
φ_j	Caster	B
Γ_j	Kingpin Inclination (KPI)	W
???		B

Table 5: Parameter Nomenclature

2 Chassis Dynamics

Rigid Body State Vectors:

$$\chi = [x \quad y \quad z]^T \tag{2.1}$$
$$\Omega = [\theta \quad \phi \quad \psi]^T \tag{2.2}$$

2.1 10-DOF Full Equations

10-DOF Transient Rigid Body Motion Equations (@C.G. aligned w/ Intermediate Frame):

$$m_s\left(\ddot{\chi} + \dot{\Omega} \times \dot{\chi}\right) + (m_{uf} + m_{ur})\left(\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{bmatrix} \tag{2.3}$$

$$(R(\phi,\theta,0)\, I_s)\, \ddot{\Omega} + \dot{\Omega} \times (R(\phi,\theta,0)\, I_s)\, \dot{\Omega} + \begin{bmatrix} 0 \\ 0 \\ I_{u,zz}\ddot{\psi} \end{bmatrix} = \begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} \tag{2.4}$$

$$m_{u,j}\ddot{z}_{u,j} = k_{t,j}\,(z_{r,i} - z_{u,i}) - k_{r,i}\,(z_{u,j} - z_{c,i}) - b_{r,i}\,(\dot{z}_{u,i} - \dot{z}_{c,i}) \tag{2.5}$$

Additional Definitions

$$R_x(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix} \qquad R_y(\theta_y) = \begin{bmatrix} \cos\theta_y & 0 & -\sin\theta_y \\ 0 & 1 & 0 \\ \sin\theta_y & 0 & \cos\theta_y \end{bmatrix}$$

$$R_z(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x, \theta_y, \theta_z) = R_x(\theta_x)R_y(\theta_y)R_z(\theta_z)$$

$$z_c = z + \frac{r_{T,x}sin(\theta) + r_{T,y}cos(\theta)sin(\phi)}{cos(\theta)cos(\phi)}$$

2.2 Quasi-Transient Reduction

Quasi-Transient Conditions:

$$\begin{aligned} \dot{z} &= 0, & \ddot{z} &= 0 \\ \dot{\theta} &= 0, & \ddot{\theta} &= 0 \\ \dot{\phi} &= 0, & \ddot{\phi} &= 0 \\ \dot{z}_{u,j} &= 0, & \ddot{z}_{u,j} &= 0 \end{aligned}$$

10-DOF Quasi-Transient Rigid Body Motion Equations:

$$(m_s + m_{uf} + m_{ur}) \left(\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{bmatrix} \tag{2.6}$$

$$R(\phi, \theta, 0)\, I_s \begin{bmatrix} 0 \\ 0 \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \times R(\phi, \theta, 0)\, I_s \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I_{u,zz} \ddot{\psi} \end{bmatrix} = \begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} \tag{2.7}$$

$$k_{t,j} \left(z_{r,i} - z_{u,i} \right) = k_{r,i} \left(z_{u,j} - z_{c,i} \right) \tag{2.8}$$

2.3 No Body Pitch/Roll

No Body Pitch / Roll Conditions:

$$\theta = 0, \qquad \phi = 0$$

10-DOF Quasi-Transient, Planar Rigid Body Motion Equations:

$$m\left(\begin{bmatrix} \ddot{x} \\ \ddot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \times \begin{bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \sum F_x \\ \sum F_y \\ \sum F_z \end{bmatrix} \tag{2.9}$$

$$I_s \begin{bmatrix} 0 \\ 0 \\ \ddot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \times I_s \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I_{u,zz}\ddot{\psi} \end{bmatrix} = \begin{bmatrix} \sum M_x \\ \sum M_y \\ \sum M_z \end{bmatrix} \tag{2.10}$$

$$k_{t,j}\left(z_{r,i} - z_{u,i}\right) = k_{r,i}\left(z_{u,j} - z\right) \tag{2.11}$$

10-DOF QT, Planar Rigid Body Motion Residual Equations:

$$m\left(\ddot{x} - \dot{y}\dot{\psi}\right) - \sum F_x = 0 \tag{2.12}$$

$$m\left(\ddot{y} + \dot{x}\dot{\psi}\right) - \sum F_y = 0 \tag{2.13}$$

$$\sum F_z = 0 \tag{2.14}$$

$$I_{s,xz}\ddot{\psi} - I_{s,yz}\dot{\psi}^2 - \sum M_x = 0 \tag{2.15}$$

$$I_{s,yz}\ddot{\psi} + I_{s,xz}\dot{\psi}^2 - \sum M_y = 0 \tag{2.16}$$

$$\left(I_{s,zz} + I_{u,zz}\right)\ddot{\psi} - \sum M_z = 0 \tag{2.17}$$