(Selected) Solution for Matrix Groups for Undergraduates by Kristopher Tapp

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Selected questions. Note: I stoppped at page 40 of the book, 47 on .pdf

Abstract

This is from the Student Mathematical Library Vol 29 edition. I just want to do some problem sets along my way when I read it.

Chapter 1 Exercises

Ex. 1.2. Prove Equation 1.3.

Equation 1.3.

$$(A \cdot B)^T = B^T \cdot A^T$$

Say we have $A \in M_{m,n}(\mathbb{K})$ and $B \in M_{n,p}(\mathbb{K})$,

$$\begin{aligned}
(A \cdot B)^T &= \left(\begin{bmatrix} A_{1,1} & \dots & A_{1,n} \\ \vdots & \ddots & \\ A_{m,1} & & A_{m,n} \end{bmatrix} \begin{bmatrix} B_{1,1} & \dots & B_{1,p} \\ \vdots & \ddots & \\ B_{n,1} & & B_{n,p} \end{bmatrix} \right)^T \\
&= \begin{bmatrix} (AB)_{1,1} & (AB)_{1,2} & \dots & (AB)_{1,p} \\ \vdots & \ddots & & \\ (AB)_{m,1} & \dots & \dots & (AB)_{m,p} \end{bmatrix}^T \\
&= \begin{bmatrix} (AB)_{1,1} & (AB)_{2,1} & \dots & (AB)_{m,1} \\ \vdots & \ddots & & \\ (AB)_{1,p} & \dots & \dots & (AB)_{m,p} \end{bmatrix} \\
&= \begin{bmatrix} B_{1,1} & \dots & B_{n,1} \\ \vdots & \ddots & \\ B_{1,p} & & B_{n,p} \end{bmatrix} \begin{bmatrix} A_{1,1} & \dots & A_{m,1} \\ \vdots & \ddots & \\ A_{1,n} & & A_{m,n} \end{bmatrix} \\
&= B^T \cdot A^T
\end{aligned}$$

Here we use the definion of (1.2) of product mulitplication here: $(AB)_{ij} =$ row i of $A\cdot$ column of B.

Ahhh I fucked up the indexing, but just write the entries out and compare would work.

Ex. 1.3. Prove Equation 1.4.

Equation 1.4.

$$\text{trace}(AB) = \text{trace}(BA)$$

For all $A, B \in M_n(\mathbb{K})$ when $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$

$$
\begin{aligned}
\operatorname{trace}(AB) &= (AB)_{11} + \cdots + (AB)_{nn} \\
&= \sum_{i=1}^{n} \sum_{s=1}^{n} A_{i,s} \cdot B_{s,i} \\
&= \operatorname{trace}(BA) \square
\end{aligned}
$$

Ex 1.4. Let $A, B \in M_n(\mathbb{K})$. Prove that if $AB = I$, then $BA = I$.

From $AB = I$, we take the determinant

$$\det(AB) = \det(A)\det(B) = \det(I) = 1$$

we then use the "the determinant is nonzero if and only if the matrix is invertible" relationship to concluse that both A and B is invertible.

$$
\begin{aligned}
I &= BB^{-1} \\
&= BIB^{-1} \\
&= B(AB)B^{-1} \\
&= BA\left(BB^{-1}\right) \\
&= BA\square
\end{aligned}
$$

Ex. 1.11. Show by example that for $A \in M_n(\mathbb{H})$, $L_A : \mathbb{H}^n \to \mathbb{H}^n$ is not necessarily $\mathbb{H}$-linear.

By Definion 1.9., A function $f : V_1 \to V_2$is called $\mathbb{K}$-linear (or simply linear) if for all $a, b \in \mathbb{K}$ and all $X, Y \in V_1$,

$$f(a \cdot X + b \cdot Y) = a \cdot f(X) + b \cdot f(Y)$$

By Definion 1.10., we have $L_A(X) \coloneqq \left(A \cdot X^T\right)^T$;

So we want some

$$f(a \cdot X + b \cdot Y) \neq a \cdot f(X) + b \cdot f(Y)$$

Example setup: $n = 2$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then we want some $X^T = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ where $f(X) = L_A(X) = \begin{bmatrix} a_{11}X_1 + a_{12}X_2 \\ a_{21}X_1 + a_{22}X_2 \end{bmatrix}^T$

Value	quaternion(real)	quaternion(i)	quaternion(j)	quaternion(k)
a_{11}	1			
a_{12}		1		
a_{21}			1	
a_{22}				1
X_1	1	1	1	
X_2		1	1	1
Y_1	1	1		
Y_2	1			1
a	1		1	
b	1	1		

Given these values (zeros for the empty boxes), our left hand side is $L_A(aX + bY) = (-3 + 3i + j - k, -3 - i + 3j - 3k)$ and right hand side $aL_A(X) + bL_A(Y) = (-1 + 5i - j - k, -5 - 3i + j + k)$.