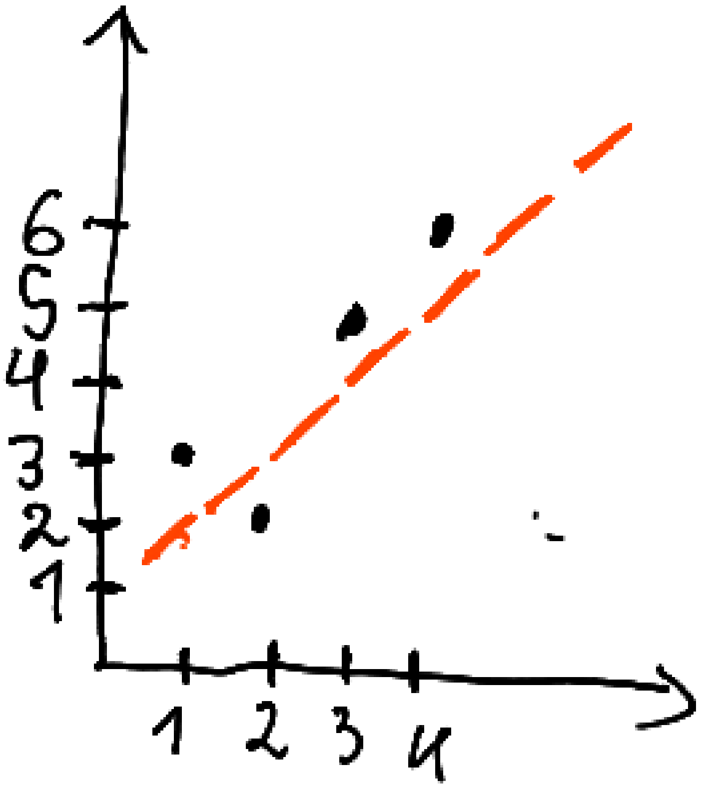
X	Y
1	3
2	2
3	5
4	6



$$\underbrace{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}_{\vec{x}_0} w_0 + \underbrace{\begin{bmatrix}1\\2\\3\\4\end{bmatrix}}_{\vec{x}_1} w_1 = \underbrace{\begin{bmatrix}3\\2\\5\\6\end{bmatrix}}_{\vec{y}}$$

$$\Leftrightarrow \begin{bmatrix}1 & 1\\1 & 2\\1 & 3\\1 & 4\end{bmatrix}\begin{bmatrix}w_0\\w_1\end{bmatrix} = \begin{bmatrix}3\\2\\5\\6\end{bmatrix}$$

$$\Leftrightarrow X \cdot \vec{w} = \vec{y}$$

$$\Longleftrightarrow \quad \vec{w} = X^{-1} \cdot \vec{y}$$

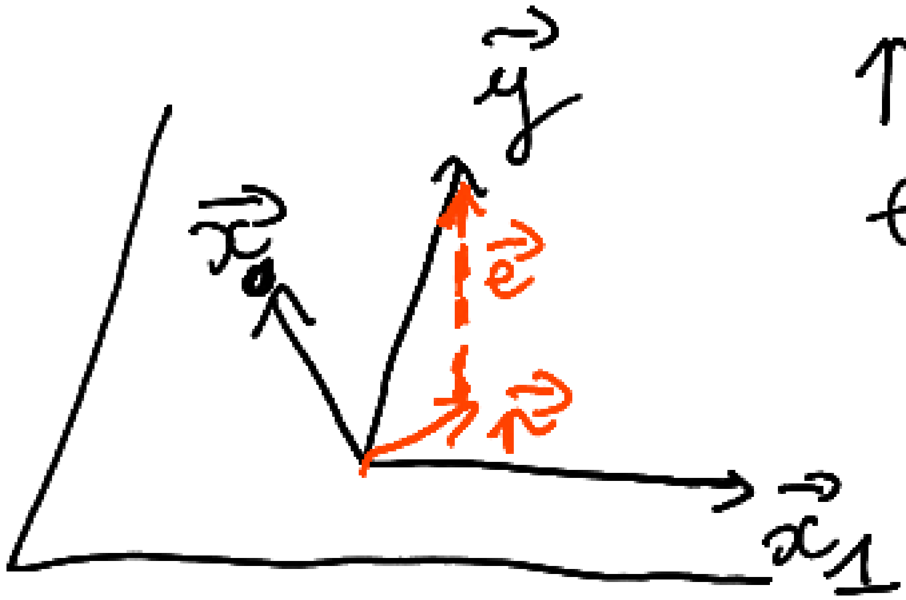
But X dimension is (4×2), meaning X is not a square matrix, so X is not invertible

$\Rightarrow$ cannot solve the equation

$\Rightarrow$ $\vec{y}$ and $\vec{x_0}, \vec{x_1}$ cannot establish a linear combination

$\Rightarrow$ $\vec{y}$ and $\vec{x_0}, \vec{x_1}$ are not in the same plane

$\Rightarrow$ find the projection $\vec{p}$ of $\vec{y}$ instead.



p: projection

e: error (ε)

1) $$\vec{p} = \vec{x_0} \cdot w_0 + \vec{x_1} \cdot w_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot w_0 + \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot w_1$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

$$= X \cdot \vec{w}$$

2) $$\vec{y} = \vec{p} + \vec{e}$$

$$\Rightarrow \vec{e} = \vec{y} - \vec{p}$$

$$\Rightarrow \vec{e} = \vec{y} - X \cdot \vec{w}$$

3) Since $\begin{cases} (\vec{x}_0 \cdot \vec{e}) = 90° \\ (\vec{x}_1 ; \vec{e}) = 90° \end{cases}$

$$\Leftrightarrow \begin{cases} \vec{x}_0 \cdot \vec{e} = 0 \\ \vec{x}_1 \cdot \vec{e} = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = 0 \\ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = 0 \end{cases}$$

$$\xrightarrow{\text{transpose}} \begin{cases} [1\ 1\ 1\ 1] \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = 0 \\ [1\ 2\ 3\ 4] \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = 0 \end{cases}$$

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Leftrightarrow X^T \cdot \vec{e} = 0$$

$$\Leftrightarrow X^T \cdot (\vec{y} - X \cdot \vec{w}) = 0$$

$$\Leftrightarrow X^T \cdot X \cdot \vec{w} = X^T \cdot \vec{y}$$

matrix $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$ has two vectors, and they are linearly independent.

$\Leftrightarrow$ the vectors of X^T are also linearly independent.

$\Leftrightarrow$ $(X^T \cdot X)$ vectors have the same property.

But $(X^T \cdot X)$ is a square matrix

$\Leftrightarrow$ $(X^T \cdot X)$ is invertibe

$$\Leftrightarrow \vec{w} = (X^T \cdot X)^{-1} \cdot X^T \cdot \vec{y}$$

$$\Leftrightarrow \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \right)^{-1} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix}$$

(2×1)

$$\underbrace{\underbrace{\underbrace{(2\times4)\cdot(4\times2)}_{(2\times2)} \cdot (2\times4)}_{(2\times4)} \cdot (4\times1)}_{(2\times1)}$$

$$\Leftrightarrow \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.2 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} w_0 = 1 \\ w_1 = 1.2 \end{cases} \Leftrightarrow \hat{y} = 1 + x^* 1.2$$

$$\Leftrightarrow \hat{y} = \text{projection} = \vec{p} = X \cdot \vec{w}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} = \begin{bmatrix} 2.2 \\ 3.4 \\ 4.6 \\ 5.8 \end{bmatrix}$$

$$\Leftrightarrow \vec{e} = \vec{y} - \vec{p} = \vec{y} - \hat{y}$$

$$= \begin{bmatrix} 3 \\ 2 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 2.2 \\ 3.4 \\ 4.6 \\ 5.8 \end{bmatrix}$$

$$= \begin{bmatrix} 0.8 \\ -1.4 \\ 0.4 \\ 0.2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

Summary Way:

$$X \cdot \vec{w} = \vec{y}$$

$$\Leftrightarrow X^T \cdot X \cdot \vec{w} = X^T \cdot \vec{y}$$

$$\Leftrightarrow \vec{w} = (X^T X)^{-1} \cdot X^T \cdot \vec{y}$$