圆形人工边界上高阶方位导数的热方程局部人工边界条件

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参考文献一

首先引入一个圆形的人工边界,将无界定义域分为一个有界的计算域和一个无界的外部区域.在外部域上，利用时间上的拉普拉斯变换和空间上的傅里叶级数来实现特殊函数的关系.然后用有理函数逼近特殊函数之间的关系.将拉普拉斯逆变换应用于一系列简单有理函数，最终得到了相应的高阶人工边界条件，其中利用一系列辅助变量避免了高阶导数在时间和空间上的影响.

考虑

$$
\begin{aligned}
&u_t(\mathbf{x},t)=\alpha\Delta u(\mathbf{x},t)+f(\mathbf{x},t), \quad (\mathbf{x},t)\in\Omega\times(0,T]\\
&u(\mathbf{x},0)=u_0(\mathbf{x}), \quad \mathbf{x}\in\Omega\\
&u(\mathbf{x},t)\to 0, \quad \text{as } |\mathbf{x}|\to+\infty
\end{aligned} \tag{1}
$$

其中$\Omega=\{(r,\theta)\mid 0\le r<+\infty, 0\le\theta<2\pi\}$.

在无界区域上考虑一个人工圆形边界:$\Gamma_R=\{(r,\theta)\mid r=R, 0\le\theta<2\pi\}$,将无界区域$\Omega$分成两个部分,用$\Omega_{in}$表示计算区域,则外部区域为$\Omega_e=\{(r,\theta)\mid R<r<+\infty, 0\le\theta<2\pi\}$.

为了设计高阶人工边界条件,我们将外区域上的热方程方程在极坐标系中改写为:

$$
\begin{aligned}
&\frac{1}{\alpha}\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}, in\Omega_e\times(0,T]\\
&u\mid_{t=0}=0, in\Omega_e\\
&u\to 0, \quad \text{as } r\to+\infty
\end{aligned} \tag{2}
$$

对上式利用Laplace变换,得到

$$
\frac{s\hat{u}}{\alpha}=\frac{\partial^2\hat{u}}{\partial r^2}+\frac{1}{r}\frac{\partial\hat{u}}{\partial r}+\frac{1}{r^2}\frac{\partial^2\hat{u}}{\partial\theta^2}. \tag{3}
$$

设

$$
\hat{u}(r,\theta,s)=\sum_{n=-\infty}^{\infty}C_n(r,s)e^{in\theta}. \tag{4}
$$

将(4)代入到(3)中,得到

$$
\frac{s}{\alpha}\sum_{n=-\infty}^{\infty}C_n(r,s)e^{in\theta}=\sum_{n=-\infty}^{\infty}\left[\frac{\partial^2 C_n}{\partial r^2}+\frac{1}{r}\frac{\partial C_n}{\partial r}-\frac{n^2}{r^2}C_n\right]e^{in\theta}.
$$

化简得到

$$
r^2\frac{\partial^2 C_n}{\partial r^2}+r\frac{\partial C_n}{\partial r}-(n^2+r^2\frac{s}{\alpha})C_n=0 \tag{5}
$$

设$r=\bar{r}/\sqrt{s/\alpha}, C_n(r,s)=\bar{C}_n(\bar{r},s)$,得到

$$
\bar{r}^2\frac{\partial^2\bar{C}_n}{\partial\bar{r}^2}+\bar{r}\frac{\partial\bar{C}_n}{\partial\bar{r}}-(n^2+\bar{r}^2)\bar{C}_n=0 \tag{6}
$$

等式(6)是n阶修正贝塞尔方程，有两个线性无关的解$K_n(\sqrt{s}r)$和$I_n(\sqrt{s}r)$,通解为

$$\bar{C}_n(\bar{r},s)=\alpha_n(s)K_n(\sqrt{s}r)+\beta_n(s)I_n(\sqrt{s}r).$$

根据条件(1)可知,$\beta_n(s)$=0,则$C_n=\bar{C}_n(\bar{r},s)=\alpha_n(s)K_n(\sqrt{s}r)$,结合(4)式,得到

$$\frac{\hat{u}(r,\theta,s)}{K_0(\sqrt{s}r)}=\alpha_0(s)+\sum_{n=\pm1}^{\infty}\alpha_n(s)\frac{K_n(\sqrt{s}r)}{K_0(\sqrt{s}r)}e^{in\theta}.$$

将上述方程关于r和θ微分,得到

$$\frac{\partial}{\partial r}\left[\frac{\hat{u}(r,\theta,s)}{K_0(\sqrt{s}r)}\right]=\sum_{n=\pm1}^{\infty}\alpha_n(s)\frac{\partial}{\partial r}\left[\frac{K_n(\sqrt{s}r)}{K_0(\sqrt{s}r)}\right]e^{in\theta},\tag{7}$$

$$\frac{\partial^{2k}}{\partial\theta^{2k}}\left[\frac{\hat{u}(r,\theta,s)}{K_0(\sqrt{s}r)}\right]=\sum_{n=\pm1}^{\infty}(in)^{2k}\alpha_n(s)\frac{K_n(\sqrt{s}r)}{K_0(\sqrt{s}r)}e^{in\theta},\tag{8}$$

我们假设式(7)相当于

$$\frac{\partial}{\partial r}\left[\frac{\hat{u}(r,\theta,s)}{K_0(\sqrt{s}r)}\right]=\sum_{k=1}^{\infty}d_k\frac{\partial^{2k}}{\partial\theta^{2k}}\left[\frac{\hat{u}(r,\theta,s)}{K_0(\sqrt{s}r)}\right],\tag{9}$$

将(8)式代入到(9)式中

$$\frac{\partial}{\partial r}\left[\frac{\hat{u}(r,\theta,s)}{K_0(\sqrt{s}r)}\right]=\sum_{n=\pm1}^{\infty}\left[\alpha_n(s)\frac{K_n(\sqrt{s}r)}{K_0(\sqrt{s}r)}\sum_{k=1}^{\infty}(in)^{2k}d_k\right]e^{in\theta},\tag{10}$$

结合式(7)和(10),得到

$$\sum_{k=1}^{\infty}(in)^{2k}d_k=\frac{K_0(\sqrt{s}r)}{K_n(\sqrt{s}r)}\frac{\partial}{\partial r}\left[\frac{K_n(\sqrt{s}r)}{K_0(\sqrt{s}r)}\right]=\frac{\sqrt{s}{K_n}'(\sqrt{s}r)}{K_n(\sqrt{s}r)}-\frac{\sqrt{s}{K_0}'(\sqrt{s}r)}{K_0(\sqrt{s}r)},\tag{11}$$

在实际计算中,我们用有限项代替无穷项,如下所示

$$\sum_{k=1}^{N}(in)^{2k}d_k=\frac{K_0(\sqrt{s}r)}{K_n(\sqrt{s}r)}\frac{\partial}{\partial r}\left[\frac{K_n(\sqrt{s}r)}{K_0(\sqrt{s}r)}\right]=\frac{\sqrt{s}{K_n}'(\sqrt{s}r)}{K_n(\sqrt{s}r)}-\frac{\sqrt{s}{K_0}'(\sqrt{s}r)}{K_0(\sqrt{s}r)},n=1,2,\cdots,N\tag{12}$$

其中N是给定的正整数,并且系数$\{d_k\}\,(k=1,2,\cdots,N)$由矩阵形式确定

$$\begin{pmatrix} i^2 & i^4 & \cdots & i^{2N}\\ (2i)^2 & (2i)^4 & \cdots & (2i)^{2N}\\ \vdots & \vdots & & \vdots\\ (Ni)^2 & (Ni)^4 & \cdots & (Ni)^{2N}\end{pmatrix}\begin{pmatrix} d_1\\ d_2\\ \vdots\\ d_N\end{pmatrix}=\begin{pmatrix}\frac{\sqrt{s}{K_1}'(\sqrt{s}r)}{K_1(\sqrt{s}r)}-\frac{\sqrt{s}{K_0}'(\sqrt{s}r)}{K_0(\sqrt{s}r)}\\ \frac{\sqrt{s}{K_2}'(\sqrt{s}r)}{K_2(\sqrt{s}r)}-\frac{\sqrt{s}{K_0}'(\sqrt{s}r)}{K_0(\sqrt{s}r)}\\ \vdots\\ \frac{\sqrt{s}{K_N}'(\sqrt{s}r)}{K_N(\sqrt{s}r)}-\frac{\sqrt{s}{K_0}'(\sqrt{s}r)}{K_0(\sqrt{s}r)}\end{pmatrix}\tag{13}$$

重写(9)式

$$\frac{\partial\hat{u}(r,\theta,s)}{\partial r}=\sum_{k=0}^{N}d_k\frac{\partial^{2k}\hat{u}(r,\theta,s)}{\partial\theta^{2k}},\tag{14}$$

其中$d_0=\frac{\sqrt{s}{K_0}'(\sqrt{s}r)}{K_0(\sqrt{s}r)}$,$\{d_k\}\,(k=1,2,\cdots,N)$由式(13)确定.

当逆拉普拉斯变换应用于等式(14)时,可以得到整体近似的人工边界条件.不幸的是,很难实现d_k的逆拉普拉斯变换.为了得到可处理的边界条件,一个有效的替代方法是在有限区间$s\in[s_E,s_W]$上用有理函数逼近系数d_k.假设存在区间$[s_E,s_W]$的分块$s_E=s_0\le s_1\le\cdots\le s_{2L}=s_W$,则对于给定的$r=R$,并使用有理近似,我们有

$$d_k(R,s)\approx\frac{P_L(s)}{Q_L(s)}=\frac{a_0+a_1s+\cdots+a_Ls^L}{1+b_1s+\cdots+b_Ls^L},s\in\{s_0,\cdots,s_{2L}\}\tag{15}$$

式(15)等价于

$$d_k(R,s) \approx \frac{P_L(s)}{Q_L(s)} = c_{k,0} + \sum_{l=1}^{L} \frac{f_{k,l}s}{s-s_{k,l}}, s \in \{s_0,\cdots,s_{2L}\} \tag{16}$$

$s_{k,L}$是多项式$1+b_1s+\cdots+b_Ls^L$的根,$c_{k,0}, f_{k,l}s$通过求解$P_L(s)=Q_L(s)(c_{k,0}+\sum_{l=1}^{L}\frac{f_{k,l}s}{s-s_{k,l}})$获得.

将(16)式代入到(14)式中

$$\frac{\partial \hat{u}(r,\theta,s)}{\partial r} = \sum_{k=0}^{N}\left(c_{k,0} + \sum_{l=1}^{L}\frac{f_{k,l}s}{s-s_{k,l}}\right)\frac{\partial^{2k}\hat{u}(r,\theta,s)}{\partial \theta^{2k}}, \tag{17}$$

将逆拉普拉斯变换直接应用于(17)式将导致高阶导数的出现.为了通过消除所有高阶导数,引入了一个辅助变量.设

$$\hat{\varphi}_k = \frac{\partial^{2k}\hat{u}(r,\theta,s)}{\partial\theta^{2k}}, -\hat{w}_{k,l} = \frac{s}{s-s_{k,l}}\hat{\varphi}_k, k=0,\cdots,N, l=1,\cdots,L.$$

(17)式被写成

$$\begin{cases} \dfrac{\partial \hat{u}}{\partial r} = \sum\limits_{k=0}^{N} c_{k,0}\hat{\varphi}_k - \sum\limits_{k=0}^{N}\sum\limits_{l=1}^{L} f_{k,l}\hat{\omega}_{k,l}, \\ \hat{\varphi}_0 = u, \\ \hat{\varphi}_k = \dfrac{\partial^2\hat{\varphi}_{k-1}}{\partial\theta^2}, \quad (k=1,\ldots,N), \\ -\hat{w}_{k,l} = \dfrac{s}{s-s_{k,l}}\hat{\varphi}_k, (k=0,\cdots,N, l=1,\cdots,L). \end{cases} \tag{18}$$

将逆拉普拉斯变换应用于式(18)并将它们与热方程耦合,我们得到了有界区域上的初边值问题

$$\begin{cases} u_t = \Delta u + f, \quad \text{in } \Omega_{in}\times(0,T], \\ u|_{t=0} = u_0, \quad \text{in } \Omega_{in}, \\ \dfrac{\partial u}{\partial r} = \sum\limits_{k=0}^{N} c_{k,0}\varphi_k - \sum\limits_{k=0}^{N}\sum\limits_{l=1}^{L} f_{k,l}\omega_{k,l}, \quad \text{on } \Gamma_R, \\ \varphi_0 = u, \quad \text{on } \Gamma_R, \\ \varphi_k = \dfrac{\partial^2\varphi_{k-1}}{\partial\theta^2}, \quad (k=1,\ldots,N), \text{ on } \Gamma_R, \\ \partial_t\varphi_k = -\partial_t\omega_{k,l} + s_{k,l}\omega_{k,l}, \quad (k=0,\ldots,N, l=1,\ldots,L), \text{ on } \Gamma_R. \end{cases} \tag{19}$$

参考文献二

我们考虑极坐标系下的经典热方程

$$\begin{aligned} &\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + Q, in\Omega \times (0,T] \\ &u \mid_{t=0} = 0 \\ &u \to 0, \quad \text{as } r \to +\infty \end{aligned} \tag{20}$$

其中$\Omega = \{(r,\theta) \mid 0 \le r < +\infty, 0 \le \theta < 2\pi\}$.

在无界区域上考虑一个人工圆形边界:$\Gamma_R = \{(r,\theta) \mid r = R, 0 \le \theta < 2\pi\}$,将无界区域$\Omega$分成两个部分,用$\Omega_{in}$表示计算区域,则外部区域为$\Omega_e = \{(r,\theta) \mid R < r < +\infty, 0 \le \theta < 2\pi\}$.

设

$$\mathcal{L}_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}, \mathcal{L}_2 = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

先引入一个简化的问题

$$\begin{aligned} &\frac{1}{\alpha}\frac{\partial u}{\partial t} = \mathcal{L}_1 u, R < r < \infty \\ &u \mid_{r=R} = u(R,t) \\ &u \mid_{t=0} = 0, R < r < \infty \\ &u \to 0, \quad \text{as } r \to +\infty \end{aligned} \tag{21}$$

Laplace变换由下式定义

$$\hat{u}(r,s) = \int_0^\infty e^{-st} u(r,t) \mathrm{d}t \tag{22}$$

通过式(21),变换函数$\hat{u}(r,s)$满足

$$\frac{s\hat{u}}{\alpha} = \frac{\partial^2 \hat{u}}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{u}}{\partial r} \tag{23}$$

线性微分方程(23)有两个线性无关的解$K_0(\sqrt{s}r)$和$I_0(\sqrt{s}r)$,其中$K_0(x)$和$I_0(x)$是零阶修正贝塞尔函数.由于(21)的条件,得到

$$\hat{u}(r,s) = CK_0(\sqrt{s}r) \tag{24}$$

对(24)关于r求微分并代入上面的微分方程,我们得到边界Γ_R上所需的单向方程

$$\frac{\partial \hat{u}}{\partial r}(R,s) = \frac{\sqrt{s}K_0'(\sqrt{s}R)}{K_0(\sqrt{s}R)}\hat{u}(R,s) := w(s)\hat{u} \tag{25}$$

其中

$$w(s) = \frac{\sqrt{s}K_0'(\sqrt{s}R)}{K_0(\sqrt{s}R)} = -\frac{\sqrt{s}K_1(\sqrt{s}R)}{K_0(\sqrt{s}R)}$$

精确的ABC现在可以通过应用于式(25)的拉普拉斯逆变换求出,在实际的数值实现中,计算函数$w(s)$的逆拉普拉斯变换是昂贵的.相反,我们将使用由最简单的Padé近似给出的$w(s)$的近似

$$w(s) \approx \frac{\varepsilon s + \beta}{\gamma s + \delta}, |s - s_0| \le l \tag{26}$$

对于给定的参数值s_0,系数$(\varepsilon, \beta, \gamma, \delta)$被唯一地确定.

将(26)式代入到(25)中,得到

$$(\gamma s + \delta)\frac{\partial \hat{u}}{\partial r} = (\varepsilon s + \beta)\hat{u} \tag{27}$$

对(27)式采用逆Laplace变换

$$\frac{\partial}{\partial t}\left(\gamma \frac{\partial u}{\partial r} - \varepsilon u\right) = -\delta \frac{\partial u}{\partial r} + \beta u \tag{28}$$

使用式(28)来获得算子$\mathcal{L}_1$的近似$\mathcal{L}_1^{(3)}$

$$\mathcal{L}_1 \approx \mathcal{L}_1^{(3)} = \left(\gamma \frac{\partial}{\partial r} - \varepsilon\right)^{-1}\left(-\delta \frac{\partial}{\partial r} + \beta\right) \tag{29}$$

则有

$$\frac{u_t}{\alpha} = \mathcal{L}_1^{(3)} u + \mathcal{L}_2 u \tag{30}$$

将算子$\left(\gamma \frac{\partial}{\partial r} - \varepsilon\right)$乘以式(30),我们就得到了线性扩散方程在圆形人工边界Γ_R上的局部吸收边界条件

$$\frac{\gamma}{\alpha} u_{tr} - \frac{\varepsilon}{\alpha} u_t = -\delta u + \beta u - \frac{2\gamma}{R^3} u_{\theta\theta} + \frac{\gamma}{R^2} u_{\theta\theta r} - \frac{\varepsilon}{R^2} u_{\theta\theta} \tag{31}$$

无界区域上的热方程可以归结为有界区域上的初边值问题

$$\begin{aligned}
&\frac{1}{\alpha}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + Q, in \Omega_i \times (0, T] \\
&\frac{\gamma}{\alpha} u_{tr} - \frac{\varepsilon}{\alpha} u_t = -\delta u + \beta u - \frac{2\gamma}{R^3} u_{\theta\theta} + \frac{\gamma}{R^2} u_{\theta\theta r} - \frac{\varepsilon}{R^2} u_{\theta\theta}, on \Gamma_R \\
&u \mid_{t=0} = 0
\end{aligned} \tag{32}$$