Birla Institute of Technology and Science, Pilani-K K Birla Goa Campus

Department of Mathematics

Instructor: Dr. P. Dhanumjaya

Ordinary Differential Equations

(Initial Value Problems)

Lab Sheet-1

---

1. The height of the fluid in a tank ($h(t)$) whose outlet flow is dependent on the pressure head (height of fluid) inside the tank and whose inlet flow is a function of time may be modeled via the equation

$$\frac{dh}{dt} = \alpha(t) - \beta\sqrt{h}, \quad h(0) = h_0.$$

Use MATLAB ODE solver function to find the solution $h(t)$ for $0 < t < 30$ if the following values for the parameters are given. Input flow: $\alpha(t) = 10+4\sin(t)$, $\beta = 2$, $h_0 = 1$.

Also find the approximation solution using

(a) Forward Euler

(b) Backward Euler

(c) Modified Euler

(d) 2^{nd} order Runge-Kutta method

(e) 4^{th} order Runge-Kutta method

2. Consider the Predator-Prey equation

$$\begin{aligned} u'(t) &= (2 - v(t))\ u(t), \\ v'(t) &= (u(t) - 1)\ v(t) \end{aligned}$$

with the initial conditions $u(0) = 3, v(0) = 2$.

(a) Use MATLAB ODE solver function to solve the problem within $0 \le t \le 10$, using $n = 100$ steps.

(b) Give two graphs showing the curves $u(t)$ and $v(t)$ against t and a plot of $u(t)$ against $v(t)$.

3. A mass-spring system can be modeled via the following second-order ODE

$$y'' + cy' + \omega^2 y = g(t), \quad y(0) = 1,\ y'(0) = 0.$$

Find the numerical solution in the time interval $0 \le t \le 10$ for the particular set of conditions $c = 5,\ \omega = 2$ and $g(t) = \sin(t)$ using MATLAB.

4. Consider the simple motion of pendulum. Let $\theta(t)$ be the angle made with the vertical at time t. Initially $\theta(0) = \pi/4$ and $\theta'(0) = 0$. The subsequent position is described by the differential equation

$$\theta''(t) - \frac{g}{l}\sin(\theta(t)) = 0,$$

where $g = 9.8m/sec^2$ is the acceleration due to gravity and $l = 0.5$m is the length of the pendulum.

   (a) Solve for $0 \leq t \leq 5$ by taking 40 steps using MATLAB ODE solver function. Plot $\theta(t)$. From this computation, what can you conclude about the pendulum's amplitude as time increases.

   (b) Repeat using 400 steps. What is your estimate of the period of the pendulum's movement.

5. (Vander Pol's equation) The following equation describes the voltage across the triode circuit:

$$\begin{aligned} v''(t) + \epsilon\left(v(t)^2 - 1\right)v'(t) + v(t) &= 0, \\ v(2) &= 1, v'(2) = 0. \end{aligned}$$

Take $\epsilon = 0.897$ and compute the approximate solution at $t = 10$ using MATLAB ODE solver function. Plot the numerical solution.

6. Consider a nonlinear oscillator with a cubic stiffness term to describe the hardening spring effect observed in many mechanical systems

$$\frac{d^2y}{dt^2} + \epsilon\frac{dy}{dt} + y^3 = \gamma\cos(\omega t).$$

   (a) Set the parameter values to $\epsilon = 0.15$, $\gamma = 0.3$ and $\omega = 1$.

   (b) Use the initial conditions: $y(0) = -1$, $y'(0) = 1$.

   (c) Use MATLAB ODE solver function to solve on the t interval $[0, 100]$.

   (d) Plot the solution $y(t)$.

7. Consider the Lorenz system of equations

$$\begin{aligned} \frac{dx_1}{dt} &= \sigma(x_2 - x_1), \\ \frac{dx_2}{dt} &= ((1 + r) - x_3)x_1 - x_2, \\ \frac{dx_3}{dt} &= x_1x_2 - bx_3 \end{aligned}$$

with the initial conditions

$$x_1(0) = -10, \quad x_2(0) = 10, \quad x_3(0) = 25.$$

   (a) Use $\sigma = 10$, $r = 28$, $b = 8/3$.

   (b) Use MATLAB ODE solver function to solve on the t interval $[0, 50]$.

   (c) Plot the Lorenz attractor.

Formula Sheet

1.

$$
\begin{aligned}
u'(x) &\approx \frac{\Delta u(x)}{h} = \frac{u(x+h)-u(x)}{h}, \quad \text{(forward difference formula)} \\
u'(x) &\approx \frac{\nabla u(x)}{h} = \frac{u(x)-u(x-h)}{h}, \quad \text{(backward difference formula)} \\
u'(x) &\approx \frac{\delta u(x)}{2h} = \frac{u(x+h)-u(x-h)}{2h}, \quad \text{(central difference formula)}
\end{aligned}
$$

2. Central difference formula:

$$
u''(x) = \frac{u(x+h)-2u(x)+u(x-h)}{h^2} + O(h^2), \quad h > 0.
$$

3. Single step methods to solve the initial value problem (IVP):

$$
\begin{aligned}
\frac{dy}{dx} &= f(x, y(x)), \\
y(x_0) &= y_0.
\end{aligned}
$$

(a) Forward Euler's method:

$$
y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, 2, \cdots
$$

(b) Backward Euler's method:

$$
y_{n+1} = y_n + h\, f(x_{n+1}, y_{n+1}), \quad n = 0, 1, 2, \cdots
$$

(c) Modified Euler's method:

$$
y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right], \quad n = 0, 1, 2, \cdots,
$$

here $y^*_{n+1} = y_n + h\, f(x_n, y_n)$.

(d) A second-order Runge-Kutta Method:

$$
y_{n+1} = y_n + \frac{h}{2}\left(k_1 + k_2\right), \quad n = 0, 1, 2, \cdots
$$

where $k_1 = f(x_n, y_n)$ and $k_2 = f(x_n + h, y_n + h\, k_1)$.

(e) A fourth-order Runge-Kutta method:

$$y_{n+1} = y_n + \frac{h}{6}\left(k_1 + 2\,k_2 + 2\,k_3 + k_4\right), \quad n = 0, 1, 2, \cdots$$

where

$$\begin{aligned} k_1 &= f(x_n, y_n), & k_2 = f(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}), \\ k_3 &= f(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}), & k_4 = f(x_n + h, y_n + k_3). \end{aligned}$$

4. Multi step methods to solve the initial value problem (IVP):

$$\begin{aligned} \frac{dy}{dx} &= f(x, y(x)), \\ y(x_0) &= y_0. \end{aligned}$$

(a) Two step Adams-Bashforth method:

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right], \quad n = 1, 2, 3, \cdots$$

(b) Four step Adams-Bashforth method:

$$y_{n+1} = y_n + \frac{h}{24}\left[55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})\right],$$

where $n = 3, 4, 5, \cdots$

(c) Three step Adams-Moulton method:

$$y_{n+1} = y_n + \frac{h}{24}\left[9f(x_{n+1}, y_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right],$$

where $n = 2, 3, 4, \cdots$

(d) Adams-Moulton Predictor-Corrector formula:

$$\begin{aligned} y^*_{n+1} &= y_n + \frac{h}{24}\left[55f(x_n, y_n) - 59f(x_{n-1}, y_{n-1}) + 37f(x_{n-2}, y_{n-2}) - 9f(x_{n-3}, y_{n-3})\right], \\ y_{n+1} &= y_n + \frac{h}{24}\left[9f(x_{n+1}, y^*_{n+1}) + 19f(x_n, y_n) - 5f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})\right]. \end{aligned}$$