12卷2期　　微　波　学　报　　Vol. 12 No. 2
1996年6月　　JOURNAL OF MICROWAVES　　Jun. 1996

完全匹配层(PML)吸收边界条件的理论分析*

Theoretical Analysis on the Perfectly Matched Layer (PML) Absorbing Boundary Condition

陈　彬　　方大纲
(南京理工大学,南京 210094)
陈晓明
(南京电子器件研究所,南京 210016)

Chen Bin, Fang Dagang
(Nanjing University of Science and Technology, Nanjing 210094, *China)*
Chen Xiaoming
(Nanjing Electronic Devices Institute, Nanjing 210016, *China)*

【摘要】本文以屏蔽微带线为例,应用谱域方法推导了时域差分(FDTD)法中所用的完全匹配层(PML)吸收边界条件的公式,并且严格证明了有关参数的选择原则。计算表明,在屏蔽微带线中,这种吸收边界条件的反射系数可以小于－80dB。本文所给出的推导方法对于平面分层介质结构是普遍适用的,因为谱域中的每一个波谱实际上都代表一个平面波。文中所给出的推导揭示了这种吸收边界条件的机理,它对于正确使用以及进一步改进这种吸收边界条件都是有意义的。

关键词:FDTD法,完全匹配层(PML),吸收边界条件

Abstract: Taking the shielded microstrip line as an example, this paper gives the derivation of formulas of perfectly matched layer (PML) boundary condition by using the spectral domain method. The principle on the determination of the related parameters is also illustrated theoretically. The computational results show that the reflection coefficient caused by PML boundary condition may be lower than －80dB. In spectral domain, each spectrum represents a plane wave, therefore, the derivation given in this paper is general for an arbitrary microstrip structure and may give the insight on the absorbing mechanism which is significant to the application and improvement of this kind of absorbing boundary conditions.

Key words: FDTD method, Perfectly matched layer (PML), Absorbing boundary condition

* 初稿收到日期:1995年9月8日,定稿收到日期:1995年11月26日。国家自然科学基金资助项目

一、引 言

时域有限差分(FDTD)方法已逐渐成为解决电磁散射和电磁波传输等问题的有力工具。研究 FDTD 方法的核心问题是寻求一种理想的吸收边界,使截断面反射最小。1994 年 J. P. Berenger 提出了"完全匹配层(PML)"这种新边界。在二维自由空间,他得出结论:PML 吸收边界的反射系数可低于-70dB,比其它各种边界改善约 40dB。A. Taflove 等把他的结果推广到了三维空间[1,2],但没有证明有关计算公式[3~5]。

我们对 PML 吸收边界也进行了深入的研究。本文用谱域法严格证明了 PML 边界有关参数的选择原则及计算公式,并以微带线为例,计算出 PML 吸收边界的反射系数。结果表明,反射系数小于-80dB,PML 吸收边界是 FDTD 方法中边界处理上重要的成果之一。

二、公式推导

参考图 1。$z > 0$ 的区域即为 PML 吸收边界,此区域最外层也为完善电壁(PEC)。

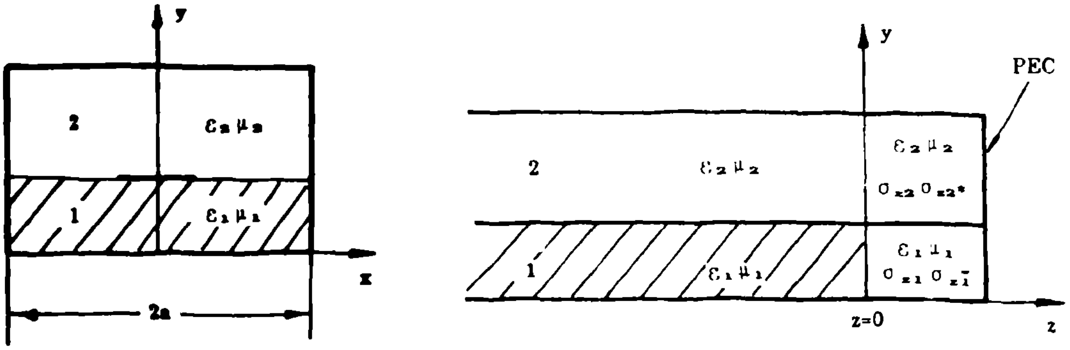

图 1 屏蔽微带线及 PML 的设置

完全匹配层是一种非物理性的电磁波吸收层,用电导率 σ 和磁导率 σ^* 表征,σ 和 σ^* 可分解为 σ_x、σ_y、σ_z 和 σ_x^*、σ_y^*、σ_z^*。当 PML 满足以下两条件时,它对电磁波不反射:

(1) 为吸收某一方向的电磁波(如 z 向),则 σ 和 σ^* 在其它两个方向上的分量 σ_x、σ_y、σ_x^*、σ_y^* 均为零。

(2) σ 和 σ^* 与电介质常数 ε 和磁导率 μ 满足下列方程:

$$\frac{\sigma_{zi}}{\varepsilon_i} = \frac{\sigma_{zi}^*}{\mu_i} \qquad i = 1,2$$

下面用谱域法证明上述两个结论。

麦氏方程一般形式为:

$$\nabla \times \vec{E} = \mu \frac{\partial \vec{H}}{\partial t} + \sigma^* \vec{H} \tag{1.1}$$

$$\nabla \times \vec{H} = \varepsilon \frac{\partial \vec{E}}{\partial t} + \sigma \vec{E} \tag{1.2}$$

屏蔽微带线中传输混合模,当上两式电导率和磁导率均取零时,满足(1)式。对于TM模,$H_{zi}=0$,可从(1)式得出:

$$\frac{\partial^2 E_{zi}}{\partial x^2}+\frac{\partial^2 E_{zi}}{\partial y^2}+k_{ci}^2E_{zi}=0 \tag{2.1}$$

$$k_{ci}^2=k_i^2-k_z^2 \tag{2.2}$$

$$k_i^2=\omega^2\mu_i\varepsilon_i \qquad i=1,2 \tag{2.3}$$

以及其它场分量与 E_{zi} 的关系表达式。用谱域法,将场分量作如下傅里叶变换:

$$\widetilde{F}(x,y,z)=\int_{-a}^{a}F(x,y,z)e^{j\alpha x}dx$$

则(2)式变为:

$$\frac{\partial \widetilde{E}_{zi}}{\partial y^2}-\gamma_i^2\widetilde{E}_{zi}=0 \tag{3.1}$$

$$\gamma_i^2=\alpha^2-k_{ci}^2 \qquad i=1,2 \tag{3.2}$$

其它场分量与 $\widetilde{E}_z$ 的关系为:

$$\widetilde{H}_{xi}=\frac{j\omega\varepsilon_i}{k_{ci}^2}\frac{\partial\widetilde{E}_{zi}}{\partial y} \tag{4.1}$$

$$\widetilde{H}_{yi}=-\frac{\alpha\omega\varepsilon_i}{k_{ci}^2}\widetilde{E}_{zi} \tag{4.2}$$

$$\widetilde{E}_{xi}=-\frac{\alpha k_z}{k_{ci}^2}\widetilde{E}_{zi} \tag{4.3}$$

$$\widetilde{E}_{yi}=-\frac{jk_z}{k_{ci}^2}\frac{\partial\widetilde{E}_{zi}}{\partial y} \tag{4.4}$$

根据屏蔽微带线的边界,$\widetilde{E}_{zi}$ 可取下列形式:

$$\widetilde{E}_{z1}(x,y,z)=A_\pm^e\,\mathrm{sh}\gamma_1 y e^{\mp jk_z z} \tag{5.1}$$

$$\widetilde{E}_{z2}(x,y,z)=B_\pm^e\,\mathrm{sh}\gamma_2(h-y)e^{\mp jk_z z} \tag{5.2}$$

$A_\pm^e$ 和 $B_\pm^e$ 中,上标"e"表示TM模,下标"+"对应正 z 方向的波,下标"-"对应负 z 方向的波。将(5.1)和(5.2)式代入(4.1)～(4.4)式,可得:

$$\widetilde{E}_{x1}^\pm=\mp\frac{\alpha k_z}{k_{c1}^2}A_\pm^e\,\mathrm{sh}\gamma_1 y e^{\mp jk_z z} \tag{6.1}$$

$$\widetilde{E}_{y1}^\pm=\mp\frac{jk_z\gamma_1A_\pm^e}{k_{c1}^2}\mathrm{ch}\gamma_1 y e^{\mp jk_z z} \tag{6.2}$$

$$\widetilde{H}_{x1}^\pm=\frac{j\omega\varepsilon_1\gamma_1A_\pm^e}{k_{c1}^2}\mathrm{ch}\gamma_1 y e^{\mp jk_z z} \tag{6.3}$$

$$\widetilde{H}_{y1}^\pm=\frac{-\alpha\omega\varepsilon_1A_\pm^e}{k_{c1}^2}\mathrm{sh}\gamma_1 y e^{\mp jk_z z} \tag{6.4}$$

$$\widetilde{E}_{x2}^\pm=\mp\frac{\alpha k_zB_\pm^e}{k_{c2}^2}\mathrm{sh}\gamma_2(h-y)e^{\mp jk_z z} \tag{6.5}$$

$$\widetilde{E}_{y2}^\pm=\pm\frac{jk_z\gamma_2B_\pm^e}{k_{c2}^2}\mathrm{ch}\gamma_2(h-y)e^{\mp jk_z z} \tag{6.6}$$

$$\widetilde{H}_{x2}^{\pm}=-\frac{j\omega\epsilon_2\gamma_2B_{\pm}^{e}}{k_{c2}^2}\mathrm{ch}\gamma_2(h-y)e^{\mp jk_z z} \tag{6.7}$$

$$\widetilde{H}_{y2}^{\pm}=-\frac{\alpha\omega\epsilon_2B_{\pm}^{e}}{k_{c2}^2}\mathrm{sh}\gamma_2(h-y)e^{\mp jk_z z} \tag{6.8}$$

(6.1)～(6.8)式即为屏蔽微带线中TM波电磁场各分量的表达式。

在PML吸收边界中，麦氏方程(1)式分解成12个场方程，其特点是在某一方向的场分量分解成在另外两方向的两个分量，σ 和 σ^* 各分解成三个分量。具体分解形式参考文献[1]。对于 z 向传输波，取 $\sigma_x=\sigma_y=\sigma_x^*=\sigma_y^*=0$。再根据文献[1]中所给方程，可得PML边界中电磁场的表达式：

$$H_{xi}=\frac{1}{j\omega\mu_i+\sigma_{zi}^*}\frac{\partial E_{yi}}{\partial z}-\frac{1}{j\omega\mu_i}\frac{\partial E_{zi}}{\partial y} \tag{7.1}$$

$$H_{yi}=\frac{1}{j\omega\mu_i}\frac{\partial E_{zi}}{\partial x}-\frac{1}{j\omega\mu_i+\sigma_{zi}^*}\frac{\partial E_{xi}}{\partial z} \tag{7.2}$$

$$H_{zi}=\frac{1}{j\omega\mu_i}\left[\frac{\partial E_{xi}}{\partial y}-\frac{\partial E_{yi}}{\partial x}\right] \tag{7.3}$$

$$E_{xi}=\frac{1}{j\omega\epsilon_i}\frac{\partial H_{zi}}{\partial y}-\frac{1}{j\omega\epsilon_i+\sigma_{zi}}\frac{\partial H_{yi}}{\partial z} \tag{7.4}$$

$$E_{yi}=\frac{1}{j\omega\epsilon_i+\sigma_{zi}}\frac{\partial H_{xi}}{\partial z}-\frac{1}{j\omega\epsilon_i}\frac{\partial H_{zi}}{\partial x} \tag{7.5}$$

$$E_{zi}=\frac{1}{j\omega\epsilon_i}\left[\frac{\partial H_{yi}}{\partial x}-\frac{\partial H_{xi}}{\partial y}\right] \tag{7.6}$$

式中各场分量还满足下列关系：

$$E_{xi}=E_{xyi}+E_{xzi} \qquad H_{xi}=H_{xyi}+H_{xzi}$$
$$E_{yi}=E_{yxi}+E_{yzi} \qquad H_{yi}=H_{yxi}+H_{yzi}$$
$$E_{zi}=E_{zxi}+E_{zyi} \qquad H_{zi}=H_{zxi}+H_{zyi}$$
$$i=1,2$$

对于TM模 $H_{zi}=0$，由(7.1)～(7.6)各式可导出 E_{zi} 满足下列波动方程：

$$\frac{\partial^2E'_{zi}}{\partial x^2}+\frac{\partial^2E'_{zi}}{\partial y^2}+k_{ci}'^2E'_{zi}=0 \tag{8.1}$$

$$k_{ci}'^2=k_i^2+\frac{k_i^2k_{zi}'^2}{(j\omega\epsilon_i+\sigma_{zi})(j\omega\mu_i+\sigma_{zi}^*)} \tag{8.2}$$

同样采用谱域法，通过傅里叶变换，(7)、(8)式中有关项成为：

$$\frac{\partial^2\widetilde{E}'_{zi}}{\partial y^2}-\gamma_i'^2\widetilde{E}'_{zi}=0 \tag{9.1}$$

$$\gamma_i'^2=\alpha^2-k_{ci}'^2 \tag{9.2}$$

$$\widetilde{H}'_{xi}=\frac{j\omega\epsilon_i}{k_{ci}'^2}\frac{\partial\widetilde{E}'_{zi}}{\partial y} \tag{10.1}$$

$$\widetilde{H}'_{yi}=-\frac{\alpha\omega\epsilon_i}{k_{ci}'^2}\widetilde{E}'_{zi} \tag{10.2}$$

$$\widetilde{E}'_{xi} = -j\frac{\alpha k'_{zi}\omega\epsilon_i}{k'^2_{ci}(j\omega\epsilon_i + \sigma_{zi})}\widetilde{E}'_{zi} \tag{10.3}$$

$$\widetilde{E}'_{yi} = \frac{jk'_{zi}\omega\epsilon_i}{k^2_{ci}(j\omega\epsilon_i + \sigma_{zi})}\frac{\partial\widetilde{E}'_{zi}}{\partial y} \tag{10.4}$$

根据边界条件和方程(9)式,$\widetilde{E}'_{zi}$ 可取下列形式:

$$\widetilde{E}'_{z1}(\alpha,y,z) = A^{e'}\mathrm{sh}\gamma'_1 y e^{-jk'_{z1}z} \tag{11.1}$$

$$\widetilde{E}'_{z2}(\alpha,y,z) = B^{e'}\mathrm{sh}\gamma'_2(h-y)e^{-jk'_{z2}z} \tag{11.2}$$

再由(10)式可得其余场分量:

$$\widetilde{E}'_{x1} = -\frac{j\alpha k'_{z1}\omega\epsilon_1}{k'^2_{c1}(j\omega\epsilon_1 + \sigma_{z1})}A^{e'}\mathrm{sh}\gamma'_1 y e^{-jk'_{z1}z} \tag{12.1}$$

$$\widetilde{E}'_{y1} = \frac{k'_{z1}\omega\epsilon_1\gamma'_1}{k'^2_{c1}(j\omega\epsilon_1 + \sigma_{z1})}A^{e'}\mathrm{ch}\gamma'_1 y e^{-jk'_{z1}z} \tag{12.2}$$

$$\widetilde{H}'_{x1} = \frac{j\omega\epsilon_1\gamma'_1}{k'^2_{c1}}A^{e'}\mathrm{ch}\gamma'_1 y e^{-jk'_{z1}z} \tag{12.3}$$

$$\widetilde{H}'_{y1} = -\frac{\alpha\omega\epsilon_1}{k'^2_{c1}}A^{e'}\mathrm{sh}\gamma'_1 y e^{-jk'_{z1}z} \tag{12.4}$$

$$\widetilde{E}'_{x2} = -\frac{j\alpha k'_{z2}\omega\epsilon_2}{k'^2_{c2}(j\omega\epsilon_2 + \sigma_{z2})}B^{e'}\mathrm{sh}\gamma'_2(h-y)e^{-jk'_{z2}z} \tag{12.5}$$

$$\widetilde{E}'_{y2} = -\frac{k'_{z2}\omega\epsilon_2\gamma'_2}{k'^2_{c2}(j\omega\epsilon_2 + \sigma_{z2})}B^{e'}\mathrm{ch}\gamma'_2(h-y)e^{-jk'_{z2}z} \tag{12.6}$$

$$\widetilde{H}'_{x2} = -\frac{j\omega\epsilon_2\gamma'_2}{k'^2_{c2}}B^{e'}\mathrm{ch}\gamma'_2(h-y)e^{-jk'_{z2}z} \tag{12.7}$$

$$\widetilde{H}'_{y2} = -\frac{\alpha\omega\epsilon_2}{k'^2_{c2}}B^{e'}\mathrm{sh}\gamma'_2(h-y)e^{-jk'_{z2}z} \tag{12.8}$$

0 界面上,切向场分量连续,则由(6)和(12)式可得:

$$-\frac{\alpha k_z}{k^2_{c1}}(A^e_+ - A^e_-)\mathrm{sh}\gamma_1 y = -\frac{j\alpha k'_{z1}\omega\epsilon_1}{k'^2_{c1}(j\omega\epsilon_1 + \sigma_{z1})}A^{e'}\mathrm{sh}\gamma'_1 y \tag{13.1}$$

$$-\frac{jk_z\gamma_1}{k^2_{c1}}(A^e_+ - A^e_-)\mathrm{ch}\gamma_1 y = \frac{k'_{z1}\omega\epsilon_1\gamma'_1}{k'^2_{c1}(j\omega\epsilon_1 + \sigma_{z1})}A^{e'}\mathrm{ch}\gamma'_1 y \tag{13.2}$$

$$\frac{j\omega\epsilon_1\gamma_1}{k^2_{c1}}(A^e_+ + A^e_-)\mathrm{ch}\gamma_1 y = \frac{j\omega\epsilon_1\gamma'_1}{k'^2_{c1}}A^{e'}\mathrm{ch}\gamma'_1 y \tag{13.3}$$

$$-\frac{\alpha\omega\epsilon_1}{k^2_{c1}}(A^e_+ + A^e_-)\mathrm{sh}\gamma_1 y = -\frac{\alpha\omega\epsilon_1}{k'^2_{c1}}A^{e'}\mathrm{sh}\gamma_1 y \tag{13.4}$$

$$-\frac{\alpha k_z}{k^2_{c2}}(B^e_+ - B^e_-)\mathrm{sh}\gamma_2(h-y) = -\frac{j\alpha k'_{z2}\omega\epsilon_2}{k'^2_{c2}(j\omega\epsilon_2 + \sigma_{z2})}B^{e'}\mathrm{sh}\gamma'_2(h-y) \tag{13.5}$$

$$\frac{jk_z\gamma_2}{k^2_{c2}}(B^e_+ - B^e_-)\mathrm{ch}\gamma_2(h-y) = -\frac{k'_{z2}\omega\epsilon_2\gamma_2}{k'_{c2}(j\omega\epsilon_2 + \sigma_{z2})}B^{e'}\mathrm{ch}\gamma'_2(h-y) \tag{13.6}$$

$$-\frac{j\omega\epsilon_2\gamma_2}{k^2_{c2}}(B^e_+ + B^e_-)\mathrm{ch}\gamma_2(h-y) = -\frac{j\omega\epsilon_2\gamma'_2}{k'^2_{c2}}B^{e'}\mathrm{ch}\gamma'_2(h-y) \tag{13.7}$$

$$B^e_-)\mathrm{sh}\gamma_2(h-y) = -\frac{\alpha\omega\epsilon_2}{k'_{c2}}B^{e'}\mathrm{sh}\gamma'_2(h-\ldots \tag{13.8}$$

因为(13.1)～(13.8)各式对于任何 y 值都成立,因此必有:

$$r_i = r_i' \qquad i = 1,2 \tag{14}$$

由(3.2)和(9.2)式又可得:

$$k_{ci}^2 = k_{ci}'^2 \qquad i = 1,2 \tag{15}$$

再由(2.2)式和(8.2)式得到:

$$k_{zi}' = k_z\sqrt{(1+\frac{\sigma_{zi}}{j\omega\epsilon_i})(1+\frac{\sigma_{zi}^*}{j\omega\mu_i})} \tag{16}$$

为使介质1和介质2分界面上场分量连续,应有 $k_{z1}' = k_{z2}'$,即

$$(1+\frac{\sigma_{z1}}{j\omega\epsilon_1})(1+\frac{\sigma_{z1}^*}{j\omega\mu_1}) = (1+\frac{\sigma_{z2}}{j\omega\epsilon_2})(1+\frac{\sigma_{z2}^*}{j\omega\mu_2}) \tag{17}$$

将(14)～(16)式代入(13)式,并且令 $z=0$ 分界面上的反射系数为:$\Gamma_1 = A_-^e / A_+^e$,$\Gamma_2 = B_-^e / B_+^e$,可得:

$$\Gamma_1 = \frac{\sqrt{1+\frac{\sigma_{z1}}{j\omega\epsilon_1}} - \sqrt{1+\frac{\sigma_{z1}^*}{j\omega\mu_1}}}{\sqrt{1+\frac{\sigma_{z1}}{j\omega\epsilon_1}} + \sqrt{1+\frac{\sigma_{z1}^*}{j\omega\mu_1}}} \tag{18.1}$$

$$\Gamma_2 = \frac{\sqrt{1+\frac{\sigma_{z2}}{j\omega\epsilon_2}} - \sqrt{1+\frac{\sigma_{z2}^*}{j\omega\mu_2}}}{\sqrt{1+\frac{\sigma_{z2}}{j\omega\epsilon_2}} + \sqrt{1+\frac{\sigma_{z2}^*}{j\omega\mu_2}}} \tag{18.2}$$

显然,要使 Γ_1 和 Γ_2 均为零,即电磁波无反射地透过 $z=0$ 分界面,PML 介质参数必须满足:

$$\frac{\sigma_{z1}}{\epsilon_1} = \frac{\sigma_{z1}^*}{\mu_1}, \quad \frac{\sigma_{z2}}{\epsilon_2} = \frac{\sigma_{z2}^*}{\mu_2} \tag{19}$$

将(19)式代回(16)式和(17)式,可得:

$$k_{z1}' = k_z(1+\frac{\sigma_{z1}}{j\omega\epsilon_1}) = k_z(1+\frac{\sigma_{z1}^*}{j\omega\mu_1}) = k_z(1+\frac{\sigma_{z2}}{j\omega\epsilon_2}) = k_z(1+\frac{\sigma_{z2}^*}{j\omega\mu_2}) = k_{z2}' \tag{20}$$

$$\frac{\sigma_{z1}}{\epsilon_1} = \frac{\sigma_{z2}}{\epsilon_2} = \frac{\sigma_{z1}^*}{\mu_1} = \frac{\sigma_{z2}^*}{\mu_2} \tag{21}$$

注意,(21)式是在 $\sigma_x = \sigma_y = \sigma_x^* = \sigma_y^* = 0$ 的前提下得出的。至此,我们得到所需证明的结果。

透过 $z=0$ 分界面的电磁波在PML中是逐渐衰减的,经PEC面的反射,再次衰减,到达分界面。假设PML中 σ 和深度 ρ 的关系为:

$$\sigma(\rho) = \sigma_{max}(\frac{\rho}{\rho_0})^2 \tag{22}$$

式中,ρ_0 为PML的总厚度。

由(20)式可知,PML中电磁场衰减因子为$\frac{k_z\sigma_{z1}}{\omega\epsilon_1}$。则通过简单的积分可得到 $z=0$ 分界面上的反射系数为:

$$R_0 = e^{-2\sigma_{max}\rho_0 k_z/(n+1)\omega\epsilon_1}$$

R_0 的选择具有一定的任意性,故取 $k_z = k_0$,可得:

$$R_0 = e^{-2\sigma_{max}\rho_0/(n+1)\varepsilon_1 c} \tag{23}$$

三、计算结果

图2给出了屏蔽微带线中PML吸收边界的反射系数。(23)式中 $R_0 = 10^{-8}$, $n = 2$, 另外 $\varepsilon_{r1} = 9.8$, $\varepsilon_{r2} = 1$, 介质厚度和金属带宽度均为0.8mm。由(21)式可知 σ_{max} 和 σ^*_{max} 分别等于 347.79s/m, $4.71 \times 10^6 \Omega/m$, 35.49s/m, $4.71 \times 10^6 \Omega/m$。图中实线表示PML划分为16层的结果,虚线是10层的结果。

从图2的结果可知,PML的反射系数小于－80dB。

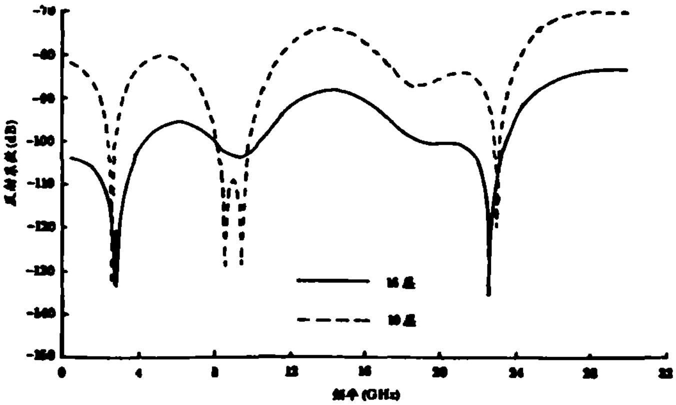

图2　屏蔽微带线中PML边界的反射系数

四、结　　论

本文以屏蔽微带线为例;应用谱域的方法,推导出了PML的有关计算公式。事实上,本文的推导方法对于平面分层介质结构是普遍适用的,因为谱域中的每一个波谱实际上都代表一个平面波。对于无限大平面分层介质,用任意入射角的平面波来推导,也可以得到相同结果。

计算结果表明,PML吸收边界是FDTD方法中理想的边界处理方法,它为FDTD的应用开辟了更为广阔的前景。

参 考 文 献

1 D. S. Katz, E. T. Thiele and A. Taflove, "Validation and Extension to Three Dimensions of the Berenger PML Absorbing Boundary Condition for FDTD Meshes," IEEE Microwave and Guided Wave Letters, Aug. 1994, Vol. 4, No. 8, p 268～270

2 C. E. Reuter, R. M. Joseph, E. T. Thiele, et al, "Ultra-wideband Absorbing Boundary Condition for Termination of Waveguiding Structures in FDTD Simulations," ibid, Oct. 1994, Vol. 4, No. 10, p 344～346

3 B. Chen, D. G. Fang and B. H. Zhou, "Modified Berenger PML Absorbing Boundary Condition for FDTD," ibid, 1995, Vol. 5, No. 11

4 陈彬,方大纲,周壁华,"修正的完全匹配层吸收边界条件数值色散问题",1995年全国天线年会论文集

5 陈晓明,方大纲,"FDTD中的完全匹配层吸收边界",1995年全国微波年会论文集