Paragraph A.1 Exercises

A.1-1

From linearity property of the summations, we know that,

$$\sum_{k=1}^{n} (c_1 a_k + c_2 b_k) = c_1 \sum_{k=1}^{n} a_k + c_2 \sum_{k=1}^{n} b_k$$

So considering $a_k = k$, $c_1 = 2$, $c_2 = -1$ and $b_k = 1$ we have:

$$\sum_{k=1}^{n} (2k+1) = 2 \sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \tag{1}$$

We know from theory that $\sum_{k=1}^{n} k = \frac{1}{2} n(n+1)$ and that $\sum_{k=1}^{n} 1 = n$.
So in total we have:

$$(1) = 2(\frac{1}{2} n(n+1)) - n = n^2$$

So the final result is n^2.

A.1-2

We know that the harmonic series is

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$

We want to calculate

$$\sum_{k=1}^{n} \frac{1}{2k-1},$$

which means that we calculate for all odd k from 1 to 2n, which can be written as

$$\sum_{\text{k is odd}}^{2n} \frac{1}{k}$$

It is obvious that:

$$\sum_{1}^{2n} \frac{1}{k} = \sum_{\text{k is odd}}^{2n} \frac{1}{k} + \sum_{\text{k is even}}^{2n} \frac{1}{k} \tag{2}$$

We can see that:

$$\sum_{\text{k is even}}^{2n} \frac{1}{k} = \sum_{1}^{n} \frac{1}{2k} = \frac{1}{2}\sum_{1}^{n} \frac{1}{k} = \frac{1}{2}(\ln n + O(1)) \tag{3}$$

For $\sum_{1}^{2n} \frac{1}{k}$ we also have:

$$\sum_{1}^{2n} \frac{1}{k} = \ln 2n + O(1) = \ln 2 + \ln n + O(1) = \ln n + O(1) \tag{4}$$

From 2, 3 and 4 we get:

$$\sum_{\text{k is odd}}^{2n} \frac{1}{k} = \sum_{1}^{2n} \frac{1}{k} - \sum_{\text{k is even}}^{2n} \frac{1}{k} = \ln n - \frac{1}{2}\ln n + O(1) = \frac{1}{2}\ln n + O(1) =$$
$$\ln \sqrt{n} + O(1)$$

A.1-3

We know that the geometric series is $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $0 < |x| < 1$.

We can differentiate both parts of this series, so we get:

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, 0 < |x| < 1 \tag{5}$$

Differantiating again, we have:

$$\sum_{k=0}^{\infty} k(k-1)x^{k-2} = \frac{-2(-1)}{(1-x)^3} \Rightarrow$$
$$\sum_{k=0}^{\infty} (k^2-k)x^{k-2} = \frac{2}{(1-x)^3}$$

By multiplying with x^2 and using the linearity property, we have:

$$\sum_{k=0}^{\infty}(k^2-k)x^k = \sum_{k=0}^{\infty}k^2x^k - \sum_{k=0}^{\infty}kx^k = \frac{2x^2}{(1-x)^3} \tag{6}$$

By multiplying equation (5) with x, we get:

$$\sum_{k=0}^{\infty}kx^k = \frac{x}{(1-x)^2} \tag{7}$$

So from equations (6) and (7), we get:

$$\sum_{k=0}^{\infty}k^2x^k - \frac{x}{(1-x)^2} = \frac{2x^2}{(1-x)^3} \Rightarrow$$

$$\sum_{k=0}^{\infty}k^2x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{2x^2+x(1-x)}{(1-x)^3} =$$

$$\frac{x^2+x}{(1-x)^3} \Rightarrow$$

$$\sum_{k=0}^{\infty}k^2x^k = \frac{x(1+x)}{(1-x)^3}, 0 < |x| < 1$$

A.1-4

We want to show that

$$\sum_{k=0}^{\infty}\frac{(k-1)}{2^k} = 0$$

.

From linearity property we know that

$$\sum_{k=0}^{\infty}\frac{(k-1)}{2^k} = \sum_{k=0}^{\infty}\frac{k}{2^k} - \sum_{k=0}^{\infty}\frac{1}{2^k} \tag{8}$$

We know that the geometric series is $\sum_{k=0}^{\infty}x^k = \frac{1}{1-x}$, for $0 < |x| < 1$.

By replacing x with $\frac{1}{2}$, we get

$$\sum_{k=0}^{\infty}(\frac{1}{2})^k = \frac{1}{1-\frac{1}{2}} \Rightarrow$$
$$\sum_{k=0}^{\infty}(\frac{1}{2^k}) = 2$$

That way we have found the value of the second sum of Equation 8.

Moreover, we know that by differentiating the geometric series and multiplying by x, we get $\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$. Again, by replacing x with $\frac{1}{2}$, we get:

$$\sum_{k=0}^{\infty} k(\frac{1}{2})^k = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} \Rightarrow$$
$$\sum_{k=0}^{\infty} \frac{k}{2^k} = \frac{\frac{1}{2}}{(\frac{1}{4})} = 2$$

So no we have found the value of the first sum of Equation 8.

Now it is easy to see that the full sum of Equation 8 equals with $2-2=0$.

A.1-5

We need to evaluate the sum $\sum_{k=1}^{\infty}(2k+1)x^{2k}$.

Because of the linearity property, we have:

$$\sum_{k=1}^{\infty}(2k+1)x^{2k} = 2 * \sum_{k=1}^{\infty} kx^{2k} - \sum_{k=1}^{\infty} x^{2k} \tag{9}$$

We know that the geometric series is $\sum_{k=0}^{\infty} y^k = \frac{1}{1-y}$, for $0 < |y| < 1$.

By replacing y with x^2, we have:

$$\sum_{k=0}^{\infty}(x^2)^k = \frac{1}{1-x^2} \Rightarrow$$
$$\sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$$

We want our sum to start from 1, and the value of the sum for $k = 0$ is 1:

$$\sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2} \Rightarrow$$
$$\sum_{k=1}^{\infty} x^{2k} + 1 = \frac{1}{1-x^2} \Rightarrow$$
$$\sum_{k=1}^{\infty} x^{2k} = \frac{1}{1-x^2} - \frac{1-x^2}{1-x^2} \Rightarrow$$
$$\sum_{k=1}^{\infty} x^{2k} = \frac{x^2}{1-x^2}$$

So, we have found the value of the second sum of Equation 9.

Similarly we know that if we differentiate the geometric series and multiply by y, we get $\sum_{k=0}^{\infty} ky^k = \frac{y}{(1-y)^2}$.

By replacing y with x^2, we have:

$$\sum_{k=0}^{\infty} k(x^2)^k = \frac{x^2}{(1-x^2)^2} \Rightarrow$$
$$\sum_{k=0}^{\infty} kx^{2k} = \frac{x^2}{(1-x^2)^2}$$

Again, we want the sum to start from 1, but here the value of the sum for $k = 0$ is 0, so we have:

$$\sum_{k=0}^{\infty} kx^{2k} = \sum_{k=1}^{\infty} kx^{2k} = \frac{x^2}{(1-x^2)^2}$$

So, we have found the value of the first sum of Equation 9.

In total we have:

$$\sum_{k=1}^{\infty} (2k+1)x^{2k} = \frac{2x^2}{(1-x^2)^2} + \frac{x^2}{1-x^2} = \frac{2x^2}{(1-x^2)^2} + \frac{x^2-x^4}{(1-x^2)^2} = \frac{3x^2-x^4}{(1-x^2)^2}$$

A.1-6

We want to prove that: $\sum_{k=1}^{n} O(g_k(i)) = O(\sum_{k=1}^{n} g_k(i))$ by using the linearity property.

Let's assume that there are functions $g_k(i)$, so that for each i, the following holds true $|g_k(i)| \leq M_k f_k(i)$ for a postiive real number M_k and for all $i \geq i_0$. The we say that $g_k(i) = O(f_k(i))$.

We select $c = \max_{1 \leq k \leq n} (M_k)$.

The inequality $|g_k(i)| \leq c f_k(i)$ still fits under the definition of big-O, since we picked the largest of the M_k.

So if we sum all the g_k from 0 until n, we have,

$$\sum_{k=1}^{n} g_k(i) \leq \sum_{k=1}^{n} M_k f_k(i) \Rightarrow \tag{10}$$

$$\sum_{k=1}^{n} g_k(i) \leq \sum_{k=1}^{n} c f_k(i) \Rightarrow \tag{11}$$

$$\sum_{k=1}^{n} g_k(i) = \sum_{k=1}^{n} O(f_k(i)) \tag{12}$$

Because of linearity property, we get:

$$\sum_{k=1}^{n} c f_k(i) = c \sum_{k=1}^{n} f_k(i) \tag{13}$$

So from Equation 12 and Equation 13, we have:

$$\sum_{k=1}^{n} g_k(i) \leq \sum_{k=1}^{n} c f_k(i) \Rightarrow \tag{14}$$

$$\sum_{k=1}^{n} g_k(i) \leq c \sum_{k=1}^{n} f_k(i) \Rightarrow \tag{15}$$

$$\sum_{k=1}^{n} g_k(i) = O(\sum_{k=1}^{n} f_k(i)) \tag{16}$$

So now we see that Equation 12 and Equation 16 are equal, so we proved our original equality, that $\sum_{k=1}^{n} O(f_k(i)) = O(\sum_{k=1}^{n} f_k(i))$, for any kind of function f.

A.1-7

We will use the fact that $\lg \left(\prod_{k=1}^{n} a_k\right) = \sum_{k=1}^{n} \lg a_k$.

We have :

$$\lg\left(\prod_{k=1}^{n} 2*4^k\right) = \sum_{k=1}^{n} \lg(2*4^k) =$$
$$\sum_{k=1}^{n} (\lg 2 + \lg 4^k) =$$
$$\sum_{k=1}^{n} (\lg 2 + k \lg 4) = \sum_{k=1}^{n} (1 + 2k) =$$
$$\sum_{k=1}^{n} 1 + 2\sum_{k=1}^{n} k = n + 2\frac{1}{2}n(n+1) =$$
$$n^2 + 2n$$

By using the fact that $2^{\lg(\prod_{k=1}^{n} a_k)} = \prod_{k=1}^{n} a_k$, we have:

$$2^{\lg(\prod_{k=1}^{n} 2*4^k)} = 2^(n^2 + 2n) =$$
$$2^{n^2} + 4^n$$

A.1-8

We will expand the product:

$$\prod_{k=2}^{n} (1 - \frac{1}{k^2}) = \prod_{k=2}^{n} (\frac{k^2-1}{k^2}) =$$
$$\prod_{k=2}^{n} (\frac{(k-1)(k+1)}{k^2}) =$$
$$\frac{1*3}{2*2} * \frac{2*4}{3*3} * \frac{3*5}{4*4} ... \frac{(n-1)(n+1)}{n*n} =$$
$$\frac{1*(n+1)}{2*n} = \frac{(n+1)}{2n}$$