Paragraph A.1 Excercises

A.1-1

From linearity property of the summations, we know that,

$$\sum_{k=1}^{n} (ca_k + b_k) = c \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

So considering $a_k = k$, $c = 2$ and $b_k = 1$ we have:

$$\sum_{k=1}^{n} (2k + 1) = 2 \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1 \tag{1}$$

We know from theory that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and that $\sum_{k=1}^{n} 1 = n$.
So in total we have:

$$(1) = 2(\frac{1}{2}n(n+1)) + n = n^2 + 2n$$

So the final result is $n^2 + 2n$.

A.1-2

We know that the harmonic series is

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \ln n + O(1)$$

We want to calculate

$$\sum_{k=1}^{n} \frac{1}{2k-1},$$

which means that we calculate for all odd k from 1 to 2n, which can be written as

$$\sum_{\text{k is odd}}^{2n} \frac{1}{k}$$

It is obvious that:

$$\sum_{1}^{2n} \frac{1}{k} = \sum_{\text{k is odd}}^{2n} \frac{1}{k} + \sum_{\text{k is even}}^{2n} \frac{1}{k} \tag{2}$$

We can see that:

$$\sum_{\text{k is even}}^{2n} \frac{1}{k} = \sum_{1}^{n} \frac{1}{2k} = \frac{1}{2} \sum_{1}^{n} \frac{1}{k} = \frac{1}{2}(\ln n + O(1)) \tag{3}$$

For $\sum_{1}^{2n} \frac{1}{k}$ we also have:

$$\sum_{1}^{2n} \frac{1}{k} = \ln 2n + O(1) = \ln 2 + \ln n + O(1) = \ln n + O(1) \tag{4}$$

From 2, 3 and 4 we get:

$$\sum_{\text{k is odd}}^{2n} \frac{1}{k} = \sum_{1}^{2n} \frac{1}{k} - \sum_{\text{k is even}}^{2n} \frac{1}{k} = \ln n - \frac{1}{2} \ln n + O(1) = \frac{1}{2} \ln n + O(1) =$$
$$\ln \sqrt{n} + O(1)$$

A.1-3

We know that the geometric series is $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, for $0 < |x| < 1$.

We can differentiate both parts of this series, so we get:

$$\sum_{k=0}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}, 0 < |x| < 1 \tag{5}$$

Differantiating again, we have:

$$\sum_{k=0}^{\infty} k(k-1)x^{k-2} = \frac{-2(-1)}{(1-x)^3} \Rightarrow$$
$$\sum_{k=0}^{\infty} (k^2 - k)x^{k-2} = \frac{2}{(1-x)^3}$$

By multiplying with x^2 and using the linearity property, we have:

$$\sum_{k=0}^{\infty}(k^2-k)x^k = \sum_{k=0}^{\infty}k^2x^k - \sum_{k=0}^{\infty}kx^k = \frac{2x^2}{(1-x)^3} \tag{6}$$

By multiplying equation (5) with x, we get:

$$\sum_{k=0}^{\infty}kx^k = \frac{x}{(1-x)^2} \tag{7}$$

So from equations (6) and (7), we get:

$$\sum_{k=0}^{\infty}k^2x^k - \frac{x}{(1-x)^2} = \frac{2x^2}{(1-x)^3} \Rightarrow$$

$$\sum_{k=0}^{\infty}k^2x^k = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{2x^2+x(1-x)}{(1-x)^3} =$$

$$\frac{x^2+x}{(1-x)^3} \Rightarrow$$

$$\sum_{k=0}^{\infty}k^2x^k = \frac{x(1+x)}{(1-x)^3}, 0 < x| < 1$$