Entropy and optimality in river deltas

Alejandro Tejedor[a], Anthony Longjas[a], Douglas A. Edmonds[b,c], Ilya Zaliapin[d], Tryphon T. Georgiou[e], Andrea Rinaldo[f,g,1], and Efi Foufoula-Georgiou[a,1]

[a]Department of Civil and Environmental Engineering, University of California, Irvine, CA 92697; [b]Department of Earth and Atmospheric Sciences, Indiana University, Bloomington, IN 47405; [c]Center for Geospatial Data Analysis, Indiana University, Bloomington, IN 47405; [d]Department of Mathematics and Statistics, University of Nevada, Reno, NV 89557; [e]Department of Mechanical and Aerospace Engineering, University of California, Irvine, CA 92697; [f]Laboratory of Ecohydrology, Ecole Polytechnique Federale Lausanne, 1015 Lausanne, Switzerland; and [g]Dipartimento di Ingegneria Civile, Edile ed Ambientale, Università di Padova, 35131 Padova, Italy

Contributed by Andrea Rinaldo, August 29, 2017 (sent for review May 22, 2017; reviewed by Axel Kleidon and John B. Shaw)

The form and function of river deltas is intricately linked to the evolving structure of their channel networks, which controls how effectively deltas are nourished with sediments and nutrients. Understanding the coevolution of deltaic channels and their flux organization is crucial for guiding maintenance strategies of these highly stressed systems from a range of anthropogenic activities. To date, however, a unified theory explaining how deltas self-organize to distribute water and sediment up to the shoreline remains elusive. Here, we provide evidence for an optimality principle underlying the self-organized partition of fluxes in delta channel networks. By introducing a suitable nonlocal entropy rate (*nER*) and by analyzing field and simulated deltas, we suggest that delta networks achieve configurations that maximize the diversity of water and sediment flux delivery to the shoreline. We thus suggest that prograding deltas attain dynamically accessible optima of flux distributions on their channel network topologies, thus effectively decoupling evolutionary time scales of geomorphology and hydrology. When interpreted in terms of delta resilience, high *nER* configurations reflect an increased ability to withstand perturbations. However, the distributive mechanism responsible for both diversifying flux delivery to the shoreline and dampening possible perturbations might lead to catastrophic events when those perturbations exceed certain intensity thresholds.

spectral graph theory | information theory | self-organization | resilient deltas

River deltas are depositional landforms forming downstream of major rivers, often home to large populations and important natural resources (1–10). In the last decades, many deltas of the world have been under threat from a range of stressors, including sea-level rise, upstream dam development, and local exploration (2, 4, 5, 11–17). Deltas are nourished by channel networks whose connectivity constrains, if not drives, the evolution, functionality, and resilience of these systems. Remarkably, the properties of delta channel networks differ substantially from the tree-like topology of the rivers that feed them (18). Tree-like networks, defined by the absence of loops, are characteristic of tributary river networks and are found abundantly in nature across different systems and scales (e.g., botanical trees, veins of leaves, blood vessels, lightning, and river networks). The propensity of nature in choosing tree-like configurations has been grounded as an optimality principle. Specifically, tributary river channel networks achieve minimal total energy dissipation, that is, minimal loss of potential energy as water and sediment flow downstream, albeit often manifesting as feasible optimality, that is, a dynamically accessible local minimum due to initial conditions and other constraints (19–22). Similar to river networks, vascular networks in biological systems (e.g., animals, plants, insects, etc.), which transport materials through space filling fractal networks of branching tubes, achieve states of minimal energy dissipation (23, 24). Analogous optimality principles have also been suggested to constrain processes as diverse as root water uptake in plants (minimization of internal dissipation) (25) and land surface energy and water balance (maximization of power) (26–28).

Despite the importance of deltaic systems and the recent advances in quantifying their connectivity properties (18, 29–34), an optimality principle for the organization of their distributary channel networks, akin to that existing for the tributary networks, remains elusive. A recent framework based on spectral graph theory (29, 30, 32) sheds light on the topologic and steady-state flux partitioning characteristics of delta channel networks and their relationship to underlying morphodynamic controls (e.g., sediment composition and tidal and wave energy), paving the way for quantitative delta classification and inference of process from form. However, the diversity of topologic structures of channel networks across a broad spectrum of deltas, and the fact that deltas are highly dynamic systems within which topology and flux partition coevolve, make it challenging to find a universal first-order optimality principle (e.g., minimization of energy, maximization of entropy, or minimization of free energy) governing their formation.

We are physically motivated by the foundational principle that deltas build land by spreading their fluxes on their delta top, as opposed to creating single pathways to the ocean which would diminish the formation of islands that retain sediment and nutrients and reduce land-building potential. This notion resonates with previous results applied to tidal deltas showing that tidal channels self-organize to uniformly distribute the tidal prism across the delta (35). Under this premise, we postulate the

Significance

River deltas are critically important Earthscapes at the land–water interface, supporting dense populations and diverse ecosystems while also providing disproportionately large food and energy resources. Deltas exhibit complex channel networks that dictate how water, sediment, and nutrients are spread over the delta surface. By adapting concepts from information theory, we show that a range of field and numerically generated deltas obey an optimality principle that suggests that deltas self-organize to increase the diversity of sediment transport pathways across the delta channels to the shoreline. We suggest that optimal delta configurations are also more resilient because the same mechanism that diversifies the delivery of fluxes to the shoreline also enhances the dampening of possible perturbations.

Author contributions: A.T., A.L., D.A.E., I.Z., T.T.G., A.R., and E.F.-G. designed research, performed research, analyzed data, and wrote the paper.

Reviewers: A.K., Max Planck Institute for Biogeochemistry; and J.B.S., University of Arkansas.

The authors declare no conflict of interest.

[1]To whom correspondence may be addressed. Email: andrea.rinaldo@epfl.ch or efi@uci.edu.

This article contains supporting information online at www.pnas.org/lookup/suppl/doi:10.1073/pnas.1708404114/-/DCSupplemental.

existence of an optimality principle in delta channel networks in terms of achieving configurations that maximize the diversity of flux delivery from any point of the network to the shoreline. The time scale of topologic reorganization in deltas, which is mainly through major channel avulsions, is of the order of decades to millennia, depending on the deltaic system (36, 37), whereas the flux adaptation on a given channel topology has a characteristic time scale which is orders of magnitude smaller. This separation of scales allows us to study the system on a fixed topology and focus on whether an optimality principle governs the distribution of fluxes on that topology.

Given the nature of our hypothesis, that is, maximization of the diversity (or uncertainty) of flux delivery from any point of the network to the shoreline, we adopt an information theoretic approach to quantify uncertainty based on Shannon entropy (38). Note that Jaynes (39) introduced a formalism demonstrating the equivalence (under thermodynamic equilibrium) of the statistical mechanics and information theory approach to entropy. Furthermore, it has been argued (not exempt from certain controversy) that the mathematical framework formulated by Jaynes (39) serves as a generalization of the statistical mechanics framework for both equilibrium and nonequilibrium systems (40, 41). In this work, we propose the notion of nonlocal entropy rate (nER) and suggest by comparative analysis of field and numerical deltas that indeed deltas self-adjust their flux partition so that they maximize their flux diversity. Note that by “maximization” we do not refer to a global optimization, that is, deltas generally do not achieve flux configurations that correspond to the absolute maximum value of nER but exhibit configurations corresponding to dynamically accessible (local) maximum (feasible optimality). We also discuss the possible implications of the proposed optimality principle with respect to delta resilience in response to perturbations, arguing that flux distributions characterized by extreme values of nER are more resilient, in that a local perturbation (e.g., flux reduction in a channel) will affect the least the distribution of fluxes at the shoreline outlets.

Nonlocal Entropy Rate (*nER*)

Entropy quantifies the uncertainty in the occurrence of events (38), that is, the amount of information needed to describe the outcome of an experiment. Uncertainty intuitively emerges from the notions of probability and surprise. For instance, given a discrete stochastic process $\{X_i\}$, such as rolling a six-sided die, with specified probability distribution of outcomes, for example $\{p_1, p_2, ..., p_6\}$, the occurrence of rolling a 3 when all sides are numbered 3 produces zero surprise. Conversely, on the same die rolling a number other than 3 would produce infinite surprise. Mathematically, this surprise is defined as $-\log(p_i)$. Therefore, the uncertainty of an event, h_i, is the product of $-\log(p_i)$ times its probability of occurrence p_i. Thus, either the occurrence of a completely certain event ($p_i = 1$) or an impossible one ($p_i = 0$) introduces zero uncertainty ($h_i = 0$, by convention $0 \log 0 = 0$). The uncertainty is maximal when all of the N possible outcomes have the same probability of occurrence $1/N$ (e.g., fair die where the number on each face has a probability of occurrence 1/6). The total uncertainty or entropy, H, of a set of N discrete outcomes with probabilities $\{p_1, p_2, ...p_i.., p_N\}$ is equal to the sum of the uncertainties corresponding to each outcome i (42):

$$H = \sum_i^N h_i = -\sum_i^N p_i \log p_i. \quad [1]$$

We aim to develop an entropic metric for delta channel networks that quantifies the diversity of flux delivery to the shoreline. Thus, a delta channel network with low entropy, that is, with low uncertainty in flux pathways, would be one with a dominant channel that carries most of the flux. This configuration is inherently unstable, assuming no redistribution of sediment by marine processes, because that channel would prograde until its water slope reduces and the channel avulses down a steeper path (43, 44). However, a delta channel network that achieves a configuration where all of the possible pathways that drain water and

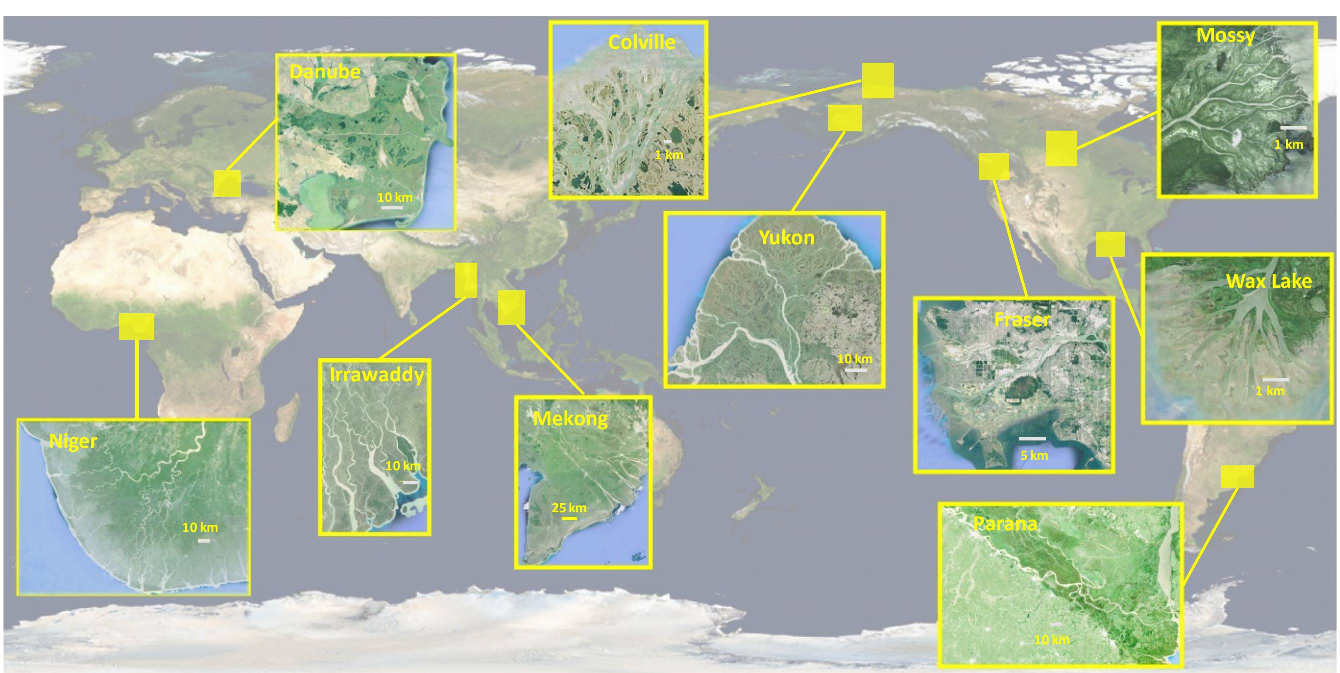

Fig. 1. Field deltas and their corresponding location. Ten deltas with diverse morphodynamic environments and of various degrees of channel complexity were analyzed in terms of their *nER* (see *SI Appendix* for further details about the deltas). Satellite images provided by Landsat/Copernicus, NASA, Digital Globe, and CNES/Airbus were extracted from Google Earth. We acknowledge their respective copyrights.

sediment from any point of the network to the shoreline are equally probable would have the highest entropy. This uncertainty in delivery would have the stabilizing effect of spreading sediment evenly across the shoreline. Notice that maximizing the uncertainty of water and sediment flux delivery to the shoreline does not necessarily imply networks that proportion fluxes equally at every single bifurcation. In fact, this would be inconsistent with morphodynamic theories (45, 46) which showed that asymmetric flux partition at a junction is a requirement for stability.

Conceptualizing delta channel networks as graphs, where nodes correspond to junctions or bifurcations, and links represent channels (*Materials and Methods*), we define a metric of uncertainty of water and sediment flux pathways from the node i to the shoreline as

$$h_i^{NL} = -\sum_k p_{ik} \log p_{ik}, \qquad [2]$$

where p_{ik} is the transition probability from node i to outlet node k. Alternatively, p_{ik} represents the fraction of water and sediment flux from node i that eventually drains to outlet k (see *Materials and Methods* for details on the computation of p_{ik}). We refer to h_i^{NL} as the nonlocal entropy for node i, to emphasize that the transition probabilities are between nodes i and the shoreline nodes k as opposed to among neighboring nodes (local). This notion acknowledges important nonlocal effects on delta dynamics, such as the hydrodynamic backwater where the water surface slope is dependent on the water depth at the shoreline and the local slope between subsequent bifurcations (36, 44).

By weighing h_i^{NL} with the normalized steady-state flux at each node π_i ($\sum_i \pi_i = 1$), we define the nER of the complete delta channel network:

$$nER = \sum_i \pi_i \frac{h_i^{NL}}{h_{i,max}^{NL}} = \sum_i \pi_i \frac{\sum_k p_{ik} \log p_{ik}}{\log \frac{1}{N_i}}, \qquad [3]$$

where $h_{i,max}^{NL} = -\log \frac{1}{N_i}$ is a normalization factor computed for each node i and N_i represents the number of outlet nodes that can be reached from node i. The normalized ***nER*** admits values in the interval [0,1].

Results and Discussion

We hypothesize that deltas distribute the flux at each bifurcation to maximize the uncertainty in the delivery of fluxes from any point of the delta to the shoreline, that is, to achieve a dynamically accessible maximum of nER. We computed the nER for 10 deltas with diverse morphodynamic environments and channel complexity (Fig. 1; see *SI Appendix* for the extracted channel networks and physical information about the deltas). We compared the computed nER for each field delta (based on the actual flow partitions) against 10^5 randomizations of the flux partition at each bifurcation [sampled from a uniform distribution in the interval $(0, 1)$] holding the network structure constant. Despite the broad range of climate, discharge, and sediment influencing these deltas, all but one (the Niger delta) have flux configurations that exhibit extreme values of nER, that is, 9 of 10 field deltas have nER above the 90th percentile of the random distribution (probability of exceedance $P_E < 0.1$) (Fig. 2). The fact that the computed value of nER does not correspond to the absolute maximum of the distribution is not surprising. Natural systems have been argued to achieve stationary configurations that do not correspond to the absolute optimum of the functional describing their organizational principle but to local optima that are accessible given the initial conditions, constraints, and the system dynamics, known as the feasible optimality principle (19–22). Beyond field deltas, we also applied the nER analysis to numerically simulated deltas that formed under varying incoming sediment grain sizes (32, 47, 48) using the physically based hydromorphodynamic model Delft3D (see *SI Appendix* for further details). The results show that five of the six numerical deltas exhibit extreme values of nER with probability of exceedance $P_E < 0.1$ (Fig. 3), further supporting our optimality hypothesis. Note that the simulated delta ($D_{50} = 0.01$ mm) that does not satisfy the optimality of nER is an extreme case, in terms of cohesiveness, for field deltas. Very cohesive banks are harder to erode and form levee breaches infrequently, delaying the triggering mechanism of avulsions. As a result, the system maintains itself at states at which the fluxes are not at equilibrium with its underlying channel network topology. This is reflected in suboptimal states of flux distribution and thus nER.

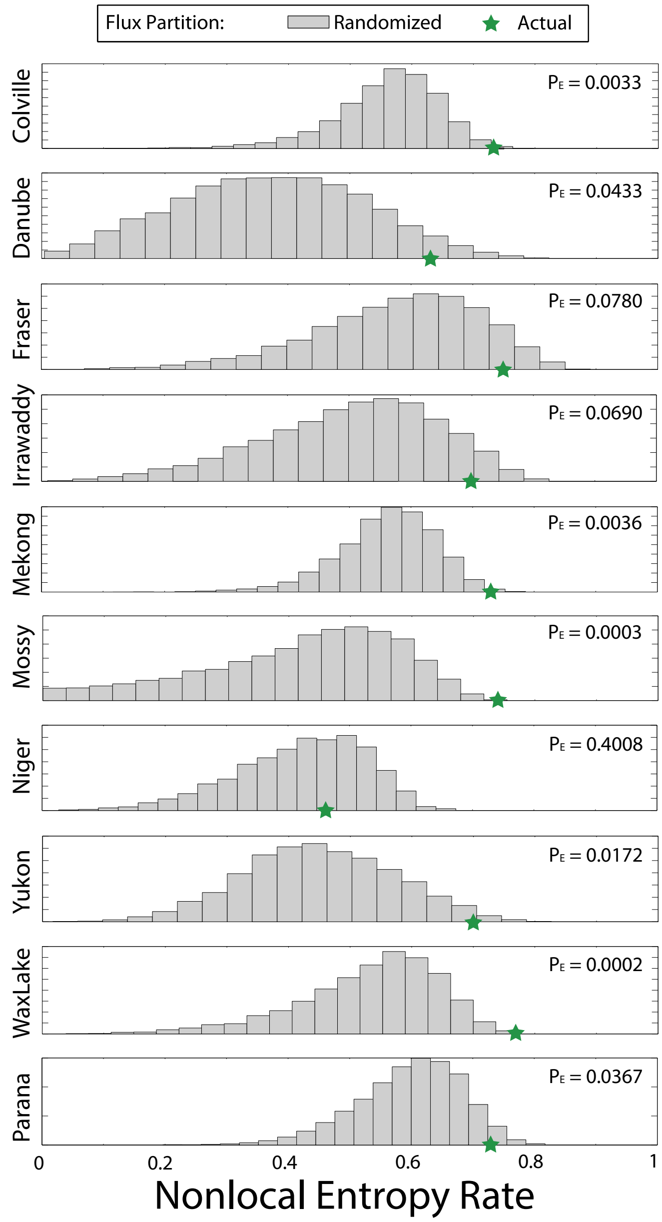

Fig. 2. *nER* for 10 field deltas. Green stars represent the values of *nER* computed for each field delta, using channel width (extracted from Landsat) as proxy for flux partition in bifurcations. We compared the values of *nER* for each delta with 10^5 randomizations of flux partitions (histograms). Nine out of the 10 deltas analyzed exhibit a maximal value of *nER*, defining maximal as a value where the probability of exceedance, P_E, by a random realization is less than 0.1.

EARTH, ATMOSPHERIC, AND PLANETARY SCIENCES

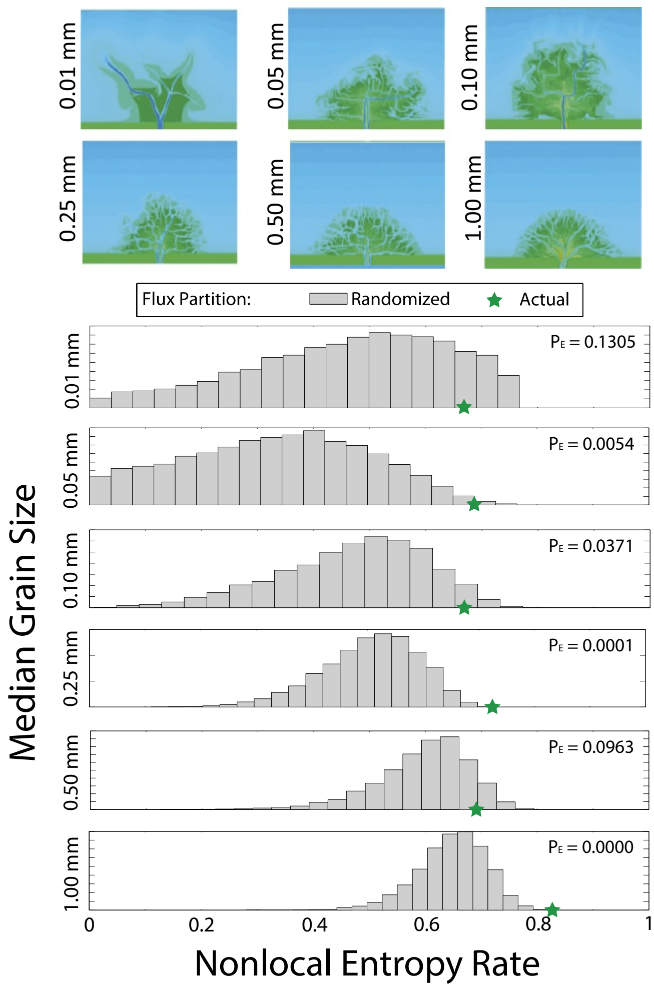

Fig. 3. *nER* for simulated deltas. We examine the *nER* of numerically simulated deltas obtained by the Delft3D model. The simulated deltas are river-dominated, with no vegetation, and with a lognormal distribution of incoming sediment size with median grain size D_{50} varying from 0.01 to 1.0 mm and the same variance in the log space (for more details see *SI Appendix*). These deltas exhibit a wide variety of channel network topologies as shown in ref. 32. Similar to the analysis conducted for the field deltas, green stars represent the values of *nER* using channel width (extracted from simulations) as proxy for flux partition in each bifurcation. Compared with 10^5 randomizations of flux partition (histogram), five out of six deltas analyzed exhibit a maximal value of *nER*, defining maximal as a value where the probability of exceedance by a random realization is less than 0.1.

Our results suggest that the flux partitions at each bifurcation, which have evolved naturally, are not random but rather follow a rule that optimizes the delta system as a whole. In fact, an interesting paradox arises from our analysis. Although the entropy introduced locally by each bifurcation, considered as an insulated unit, is suboptimal [the maximum would correspond to a symmetric bifurcation which is not consistent with stability theory of delta bifurcations that requires asymmetric local flux partition (45)], the specific assemblage of those bifurcations forming the delta network as a system is optimal (in terms of nER) and consistent with maximization of the diversity of fluxes delivered to the shoreline.

Turning attention to delta dynamics, we further hypothesize that during an avulsion the delta nER would decrease (see schematic in Fig. 4*A*). This is because during this phase of topologic reorganization the flux distribution inherited from the previous channel network structure is in general suboptimal with respect to the incipient channel network reworked by the avulsion (i.e., the new channel structure created during the avulsion received a disproportionally small share of the flux, creating an asymmetry in the flux delivery to the shoreline and thus reducing the value of nER). Testing this hypothesis in field deltas is challenging because avulsions occur infrequently. However, using numerical models we can observe that during an avulsion cycle the nER drops significantly at the onset of a new flow path and following the abandonment of the old channel (Fig. 4*B*). Since the time scale of the avulsion itself is negligible in comparison with the lifespan of the topologies before and following the avulsion, it is observed that the flux partition is able to self-organize to achieve a configuration that maximizes nER. This supports our assumption that the time scale of the flux reorganization is several orders of magnitude smaller than the characteristic time scale of topologic reorganization which is set by the time lapse between avulsion cycles.

An important implication of this optimality principle can be interpreted in terms of the resilience of deltas to withstand perturbations. Intuitively, if a perturbation (e.g., flux reduction) is applied to a delta during its high-nER state, the perturbation will be damped as it will spread through the diverse pathways connecting the delta top to the shoreline. However, if the same perturbation is applied to a delta in a low-nER state, the perturbation will be more confined to a localized part of the delta but will exert a more severe disturbance. As revealed by our analysis, river deltas operate in configurations characterized by high values of nER, supporting the idea that deltas self-organize to achieve resilient morphologies priming self-maintenance. As a word of caution, especially relevant in the current scenario where deltas are subjected to increasing anthropogenic stresses, this distributive mechanism that dampens the intensity of perturbations can also lead to delta-wide catastrophic disturbances and tipping points when those perturbations exceed certain thresholds.

Conclusions

Deltas are highly productive regions supporting extensive agriculture and aquaculture and diverse ecosystems and containing natural resources such as hydrocarbon deposits. Climate change and human actions, both in the upstream basins and locally, act as stressors on these landscapes, calling for a thorough understanding of these complex systems and their response to perturbations. We examined the existence of an optimality principle that governs the self-organization of water and sediment fluxes on delta channel networks. Specifically, (*i*) we put forth the hypothesis that maximizing *nER*, which quantifies the diversity in flux delivery to the shoreline, is a selective criterion in the evolutionary dynamics of delta networks; (*ii*) we tested this hypothesis by analyzing 10 field deltas of diverse complexity, age, and environmental settings and showed that all but one, the Niger delta, exhibited maximum *nER*; (*iii*) we further supported the existence of an optimality principle by analysis of Delft3D simulated deltas; (*iv*) we showed that during major reorganization, such as avulsions, *nER* exhibits suboptimal values and increases back to high rates (maximum values) when the flux distribution self-adjusts to the new delta channel network topology; and, finally, (*v*) we discussed the relation between entropy and resilience, arguing that delta flux configurations characterized by maximal *nER* are more resilient in the face of random perturbations. In the anthropocene where human activities have become a major agent of geomorphic change, understanding delta self-organization within an optimality perspective offers new ways of thinking about delta dynamics and disturbances that might hinder self-maintenance.

Materials and Methods

Deltas as Directed Graphs. Tejedor et al. (29, 30) presented a rigorous framework based on graph theory within which a delta channel network is represented by a directed graph, that is, a collection of vertices (bifurcations and junctions) and directed edges (channels in-between vertices, where the direction is given by the flow). All information about network connectivity and directionality of the flow can be stored in a sparse matrix called the adjacency matrix, A. Specifically, A is an $N \times N$ matrix, where N is the number of vertices, and whose entry a_{ij} is unity if vertex i receives fluxes directly from vertex j (i.e., vertices i and j are connected by a link directed from j to i) and zero otherwise. From A we can derive an important matrix called Laplacian, which is equivalent to a diffusivity operator in a graph. To construct the Laplacian we need first to introduce the degree matrices for directed graphs. The in-degree (out-degree) matrix D^{in} (D^{out}) is an $N \times N$ diagonal matrix whose entries d_{ii} depict the number of links entering (exiting) vertex i and are computed as the sum of the entries in the i-th row (column) of A. The Laplacian matrix, L^{in} (L^{out}), is defined as $D^{in} - A$ ($D^{out} - A$). Tejedor et al. (29, 30) showed that certain eigenvectors of the Laplacian operator contain important topologic information of the deltaic network. Furthermore, information about flux propagation can be obtained if A is replaced with the weighted adjacency matrix W, where the weights w_{ij} correspond to the fraction of flux in link (ji) with respect to the flux in vertex j. Similar to the at-station hydraulic geometry relationship (49)—width to landscape-forming discharge—reported for tributary rivers (50) and tidal channels (51), we assume the flux partition at the bifurcation to be proportional to the width of the downstream channels (18). Note that even though we do not consider explicitly in the computation of steady-state fluxes relevant processes such as water–sediment interchange between channel and islands (31, 52, 53), vegetation (54), tides (35), and so on, all of these processes set the hydrogeomorphic attributes of the channel network. Therefore, the computation of steady-state fluxes in the channel network based on physical attributes such as channel widths can be interpreted to a certain degree as the result of the integrative effect of all of the main processes acting on a delta. For the purposes of this paper, there are two probability distributions that can be computed by simple algebraic manipulation in the above-mentioned operators, namely, the steady-state flux distribution and the node-to-outlet transition probability distribution.

Steady-State Flux Distribution. Having a delta represented as a directed acyclic graph (DAG) allows us to compute the steady-state flux by assuming conservation of mass. For instance, Tejedor et al. (29) showed how the steady-state flux can be formulated as an eigenvalue–eigenvector problem of the Laplacian matrix of the graph. Here, we present the input/output model as a more intuitive way to compute the delta steady-state flux. Let us consider a DAG fed from the most upstream node (apex) with a constant unit flux. Without loss of generality, and for simplicity in the subsequent derivations, the apex is assumed to be labeled as node 1. Then, we can define the stationary distribution, F, as

$$F = \begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} + W\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} + W^2\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix} + \cdots = (I - W)^{-1}\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}, \qquad [4]$$

where the column vector $e_1 = (1\ 0\ 0\ \cdots\ 0)^T$ corresponds to the initial state, and W corresponds to the weighted adjacency matrix of the graph. Alternatively, e_1 represents a constant inflow at the apex normalized to 1, and $W^k e_1$ the resulting downstream response at time/distance specified by k. For a DAG, there exists at least one indexing of the graph such that each offspring vertex has a higher index than its parent vertex. In this indexing, by construction, the matrix W is strictly upper triangular, and therefore the matrix W is nilpotent, guaranteeing the convergence of the sum as expressed in Eq. **4**. In *SI Appendix* we prove that the stationary flux distribution F can also be obtained as the stationary probability of a Markov process π.

Node–Outlet Transition Probability Distribution. The node-to-outlet transition probability, p_{ik}, is defined as the probability that a package of flux at node i drains to outlet k. Thus, the transport from each node i, to the different outlets M, can be understood as a discrete stochastic process with probability distribution $\{p_{ik}\}$, with $k = 1, .., M$. Tejedor et al. (29) showed that when a delta channel network is represented by a directed acyclic graph G with a weighted adjacency matrix W,

i) The null space of the weighted in-degree Laplacian $L_W^{in}(G^R)$ for the reverse graph G^R has dimension (multiplicity of the eigenvalue zero) equal to the number of outlets M;

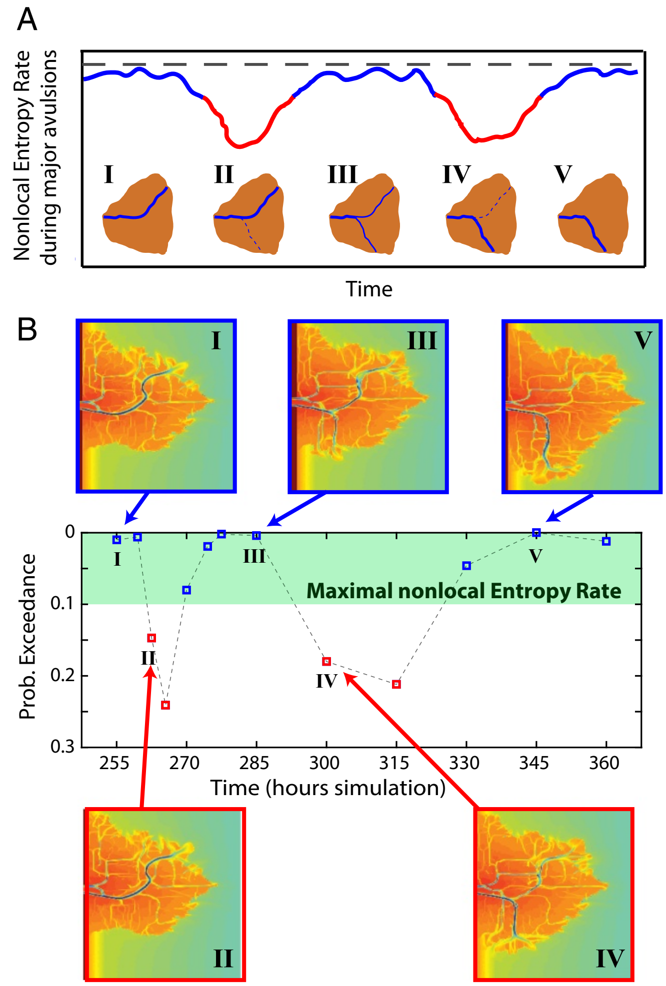

Fig. 4. *nER* during an avulsion cycle. (*A*) The results presented in Figs. 2 and 3 support the hypothesis that river deltas self-organize to maximize *nER*. However, deltas in dynamic equilibrium experience avulsions, periodically causing major reorganization of their channel networks. Our hypothesis is that during these periods of major reorganization delta *nER* drops significantly until the delta fluxes are self-organized back to a new structure as shown in the schematic of *A*. (*Insets*, from left to right) Schematics of the five stages within a single avulsion cycle (notice that for illustration purposes, only the main channel was drawn to better depict the avulsion cycle): (*I*) starting from a given channel network topology, (*II*) a new channel structure is created downstream of the avulsion node draining initially a small part of the flux, but progressively receiving larger proportions of fluxes until (*III*) the new and the old channel structures receive similar amounts of flux, and eventually transition to (*IV*) a configuration wherein the new channel structure has more flux, to finally lead to (*V*) the abandonment of the former channel structure downstream of the avulsion node. (*B*) We have tested this hypothesis using Delft3D simulations and analyzing subsequent instances of a delta evolution, wherein an avulsion cycle is observed. The probability (Prob.) of exceedance of the delta *nER* by a randomization of the flux configuration is displayed. The results agree with the posed hypothesis, showing that *nER* drops at the beginning and end of the avulsion cycle, keeping a maximal value otherwise.

ii) There exists a unique basis γ_k, $k = 1, \cdots, M$, of this null space in $\mathbb{R}^N$ (i.e., the basis consists of M vectors each having N entries) with the property

$$\gamma_k(i) = \begin{cases}1, & i = k\\0, & i \neq k\end{cases} \quad \text{for} \quad k = 1, \cdots, M. \qquad [5]$$

That is, the entry of the vector γ_k corresponding to outlet k is one, and zero at all other outlets.

iii) The value $\gamma_k(i)$ represents the portion of flux at the vertex i that drains to the outlet k, that is, p_{ik}.

Thus, if we define a matrix T, whose columns form the basis of the null space of $L_W^{in}(G^R)$, $\{\gamma\}$, then the i-th row corresponds to the probability distribution $\{p_{ik}\}$.

ACKNOWLEDGMENTS. This work is part of the International BF-DELTAS project on "Catalyzing action toward sustainability of deltaic systems" funded by the Belmont Forum (NSF Grant EAR-1342944) and the Linked Institutions for Future Earth (NSF Grant EAR-1242458). This work was partially supported by the FESD Delta Dynamics Collaboratory (NSF Grant EAR-1135427), the Water Sustainability and Climate Program (NSF Grant EAR-1209402), and NSF Grant ECCS-1509387. A.T. acknowledges financial support from a National Center for Earth-surface Dynamics 2 postdoctoral fellowship (NSF Grant EAR-1246761).

1. Cohen J, Small C, Mellinger A, Gallup J, Sachs J (1997) Estimates of coastal populations. *Science* 278:1209–1213.
2. Ericson J, Vörösmarty C, Dingman S, Ward L, Meybeck M (2006) Effective sea-level rise and deltas: Causes of change and human dimension implications. *Glob Planet Change* 50:63–82.
3. Woodroffe C, Nicholls R, Saito Y, Chen Z, Goodbred S (2006) *Landscape Variability and the Response of Asian Megadeltas to Environmental Change,* ed Harvey N (Springer, Dordrecht, The Netherlands), pp 277–314.
4. Syvitski J, et al. (2009) Sinking deltas due to human activities. *Nat Geosci* 2:681–686.
5. Vörösmarty C, et al. (2009) Battling to save the world's river deltas. *Bull At Sci* 65: 31–43.
6. Hoanh C, Szuster B, Suan-Pheng K, Ismail A, Noble A, eds (2010) *Tropical Deltas and Coastal Zones: Food Production, Communities and Environment at the Land-Water Interface* (CAB, Wallingford, UK), p 496.
7. Barbier E, et al. (2011) The value of estuarine and coastal ecosystem services. *Ecol Monogr* 81:169–193.
8. Foufoula-Georgiou E, et al. (2013) *A Vision for a Coordinated International Effort on Delta Sustainability*, eds Young G, Perillo G (IAHS, Gothenburg, Sweden), pp 3–11.
9. Brondizio E, et al. (2016) Catalyzing action towards the sustainability of deltas. *Curr Opin Environ Sustain* 19:182–194.
10. Szabo S, et al. (2016) Population dynamics, delta vulnerability and environmental change: Comparison of the Mekong, Ganges-Brahmaputra and Amazon delta regions. *Sustain Sci* 11:539–554.
11. Syvitski J, Vörösmarty C, Kettner A, Green P (2005) Impact of humans on the flux of terrestrial sediment to the global coastal ocean. *Science* 308:376–380.
12. Blum M, Roberts H (2009) Drowning of the Mississippi delta due to insufficient sediment supply and global sea-level rise. *Nat Geosci* 2:488–491.
13. Kirwan M, Megonigal J (2013) Tidal wetland stability in the face of human impacts and sea-level rise. *Nature* 504:53–60.
14. Giosan L, Syvitski J, Constantinescu S, Day J (2014) Climate change: Protect the world's deltas. *Nature* 516:31–33.
15. Tessler Z, et al. (2015) Profiling risk and sustainability in coastal deltas of the world. *Science* 349:638–643.
16. Wang S, et al. (2015) Reduced sediment transport in the yellow river due to anthropogenic changes. *Nat Geosci* 9:38–41.
17. Sebesvari Z, et al. (2016) A review of vulnerability indicators for deltaic social-ecological systems. *Sustain Sci* 11:575–590.
18. Edmonds D, Paola C, Hoyal D, Sheets B (2011) Quantitative metrics that describe river deltas and their channel networks. *J Geophys Res* 116:2156–2202.
19. Rodriguez-Iturbe I, Rinaldo A (2001) *Fractal River Basins: Chance and Self-Organization* (Cambridge Univ Press, New York), 2nd Ed, p 547.
20. Banavar J, Maritan A, Rinaldo A (2000) Topology of the fittest network. *Phys Rev Lett* 84:4745–4748.
21. Banavar J, Colaiori F, Flammini A, Maritan A, Rinaldo A (2001) Scaling, optimality, and landscape evolution. *J Stat Phys* 104:1–48.
22. Rinaldo A, Rigon R, Banavar J, Maritan A, Rodriguez-Iturbe I (2014) Evolution and selection of river networks: Statics, dynamics, and complexity. *Proc Natl Acad Sci USA* 111:2417–2424.
23. West G, Brown J, Enquist B (1997) A general model for the origin of allometric scaling laws in biology. *Science* 276:122–126.
24. West G, Brown J, Enquist B (1999) A general model for the structure and allometry of plant vascular systems. *Nature* 400:664–667.
25. Hildebrandt A, Kleidon A, Bechmann M (2016) A thermodynamic formulation of root water uptake. *Hydrol Earth Syst Sci Discuss* 20:3441–3454.
26. Kleidon A, Renner M, Porada P (2014) Estimates of the climatological land surface energy and water balance derived from maximum convective power. *Hydrol Earth Syst Sci Discuss* 18:2201–2218.
27. Kleidon A, Kravitz B, Renner M (2015) The hydrological sensitivity to global warming and solar geoengineering derived from thermodynamic constraints. *Geophys Res Lett* 42:138–144.
28. Dhara C, Renner M, Kleidon A (2016) Broad climatological variation of surface energy balance partitioning across land and ocean predicted from the maximum power limit. *Geophys Res Lett* 43:7686–7693.
29. Tejedor A, Longjas A, Zaliapin I, Foufoula-Georgiou E (2015) Delta channel networks: 1. A graph-theoretic approach for studying connectivity and steady state transport on deltaic surfaces. *Water Resour Res* 51:3998–4018.
30. Tejedor A, Longjas A, Zaliapin I, Foufoula-Georgiou E (2015) Delta channel networks: 2. Metrics of topologic and dynamic complexity for delta comparison, physical inference, and vulnerability assessment. *Water Resour Res* 51:4019–4045.
31. Hiatt M, Passalacqua P (2015) Hydrological connectivity in river deltas: The first-order importance of channel-island exchange. *Water Resour Res* 51:2264–2282.
32. Tejedor A, et al. (2016) Quantifying the signature of sediment composition on the topologic and dynamic complexity of river deltas. *Geophys Res Lett* 43:3280–3287.
33. Liang M, Van Dyk C, Passalacqua P (2016) Quantifying the patterns and dynamics of river deltas under conditions of steady forcing and relative sea level rise. *J Geophys Res Earth Surf* 121:465–496.
34. Passalacqua P (2017) The delta connectome: A network-based framework for studying connectivity in river deltas. *Geomorphology* 277:50–62.
35. Fagherazzi S (2008) Self-organization of tidal deltas. *Proc Natl Acad Sci USA* 105:18692–18695.
36. Chatanantavet P, Lamb M, Nittrouer J (2012) Backwater controls of avulsion location on deltas. *Geophys Res Lett* 39:L01402.
37. Ganti V, Chu Z, Lamb M, Nittrouer J, Parker G (2014) Testing morphodynamic controls on the location and frequency of river avulsions on fans versus deltas: Huanghe (Yellow river), China. *Geophys Res Lett* 41:7882–7890.
38. Shannon C (1948) A mathematical theory of communication. *Bell Syst Tech J* 27:379–423, 623–656.
39. Jaynes E (1957) Information theory and statistical mechanics. *Phys Rev* 106:620–630.
40. Dewar R (2003) Information theory explanation of the fluctuation theorem, maximum entropy production and self-organized criticality in non-equilibrium stationary states. *J Phys A Math Gen* 36:631–641.
41. Martyushev L, Seleznev V (2006) Maximum entropy production principle in physics, chemistry and biology. *Phys Rep* 426:1–45.
42. Cover T, Thomas J (2006) *Elements of Information Theory* (Wiley, Hoboken, NJ), 2nd Ed, p 748.
43. Slingerland R, Smith N (2004) River avulsions and their deposits. *Annu Rev Earth Pl Sc* 32:257–285.
44. Ganti V, Chadwick A, Hassenruck-Gudipati H, Lamb M (2016) Avulsion cycles and their stratigraphic signature on an experimental back-water-controlled delta. *J Geophys Res Earth Surf* 121:1651–1675.
45. Edmonds D, Slingerland R (2008) Stability of delta distributary networks and their bifurcations. *Water Resour Res* 44:W09426.
46. Bolla Pittaluga M, Coco G, Kleinhans M (2015) A unified framework for stability of channel bifurcations in gravel and sand fluvial systems. *Geophys Res Lett* 42:7521–7536.
47. Edmonds D, Slingerland R (2010) Significant effect of sediment cohesion on delta morphology. *Nat Geosci* 3:105–109.
48. Caldwell R, Edmonds D (2014) The effects of sediment properties on deltaic processes and morphologies: A numerical modeling study. *J Geophys Res Earth Surf* 119:961–982.
49. Leopold L, Maddock T, Jr. (1953) The hydraulic geometry of stream channels and some physiographic implications. *US Geol Surv Prof Pap* 252:1–52.
50. Dodov B, Foufoula-Georgiou E (2004) Generalized hydraulic geometry: Derivation based on a multiscaling formalism. *Water Resour Res* 40:W06302.
51. Rinaldo A, Fagherazzi S, Lanzoni S, Marani M, Dietrich W (1999) Tidal networks: 2. Watershed delineation and comparative network morphology. *Water Resour Res* 35:3905–3917.
52. Shaw J, Mohrig D, Wagner R (2016) Flow patterns and morphology of a prograding river delta. *J Geophys Res Earth Surf* 121:372–391.
53. Wagner W, et al. (2017) Elevation change and stability on a prograding delta. *Geophys Res Lett* 44:1786–1794.
54. Nardin W, Edmonds DA (2014) Optimum vegetation height and density for inorganic sedimentation in deltaic marshes. *Nat Geosci* 7:722–726.