MA2104 Multivariable Calculus

1. Points and vectors in 2- and 3-dimensional spaces. Lines and planes.

Main topics

- Points and vectors in 2- and 3-D. Graphical representation.
- Vector algebra. Length and distance. Dot product and cross product.
- Lines and planes in 3-D.

$\mathbb{R}^2$ and $\mathbb{R}^3$. Rectangular coordinate systems.

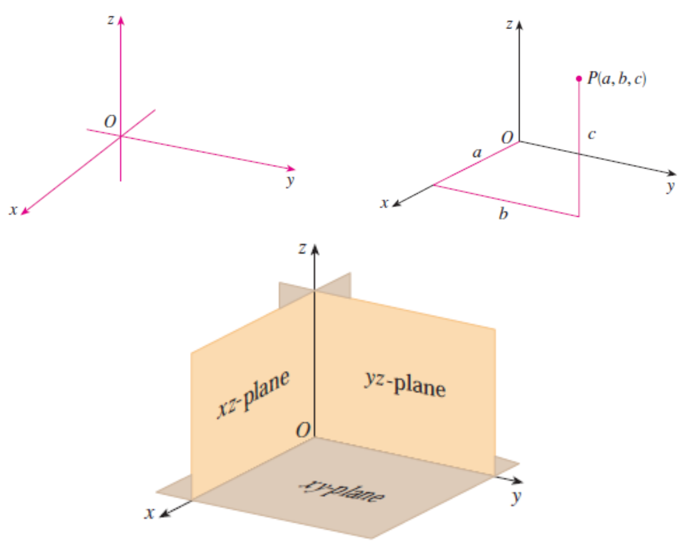
The Euclidean plane and Euclidean 3-dimensional space are denoted by $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Points in $\mathbb{R}^2$ and $\mathbb{R}^3$ are written as ordered pairs/ordered triples: (a, b) and (a, b, c) respectively, where a, b, c are real numbers.

I will assume that you know the usual rectangular (x, y)-coordinate system in $\mathbb{R}^2$.

The figure below shows the retangular (x, y, z)-coordinate system in $\mathbb{R}^3$ and the representation of a general point (a, b, c) in the coordinate system.

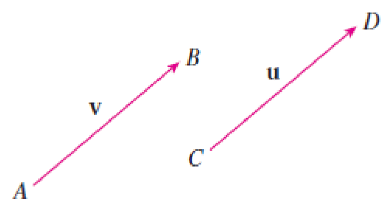
Note that the coordinate system we have chosen to use is the **right-handed coordinate system**. This means that if we look up along the z-direction, then the y-axis is to the right of the x-axis.



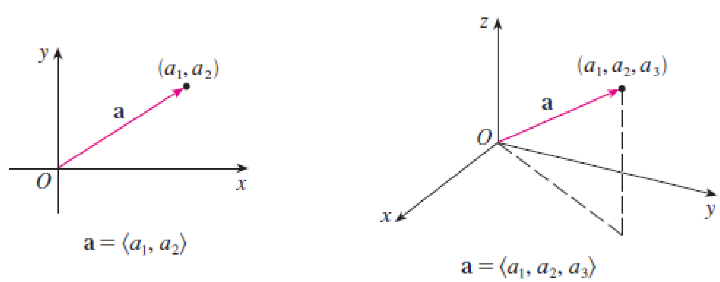
(a) Coordinate planes

Vectors

Let A and B be two points in $\mathbb{R}^2$ or $\mathbb{R}^3$. The arrow that goes from point A to point B is called a *vector*, denoted by $\overrightarrow{AB}$. Note that if the arrow is translated in parallel (and keeping its length) so that it begins at C and ends at D, then $\overrightarrow{AB}=\overrightarrow{CD}$. Vectors are also denoted by a single bold face letter such as $\mathbf{v}$.



In a rectangular coordinate system, a vector can be resolved into its components as indicated in the figure below. To distinguish vectors from points, we write the components of a vector inside angled brackets: $\mathbf{a} = \langle a_1, a_2\rangle$ (in 2-D) or $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ (in 3-D). In the discussion below, I will generally work with 3-D vectors. For 2-D vectors, just suppress the last component.

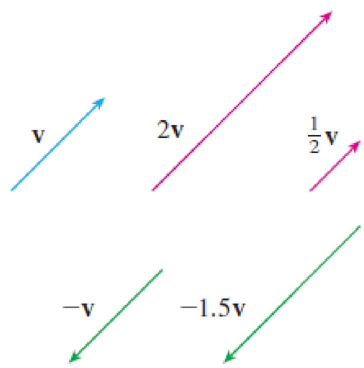


If points A and B have coordinates (a_1, a_2, a_3) and (b_1, b_2, b_3) respectively. Then the components of the vector $\overrightarrow{AB}$ are

$$\overrightarrow{AB} = \langle b_1 - a_1, b_2 - a_2, b_3 - a_3\rangle.$$

Vector algebra.

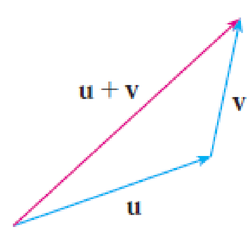
Suppose that $\mathbf{v}$ is a vector and c is a real number (also called a scalar). The *scalar product* $c\mathbf{v}$ is the vector whose length is $|c|$ times that of $\mathbf{v}$ and in the same direction as $\mathbf{v}$ if $c \geq 0$ and in the opposite direction to $\mathbf{v}$ if $c < 0$.



If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$c\mathbf{v} = \langle cv_1, cv_2, cv_3 \rangle.$$

The *sum* of two vectors is obtained pictorially by putting the start of one arrow at the tip of the other and taking the resultant arrow as shown in the following diagram.



If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then

$$\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle.$$

The *difference* of two vectors $\mathbf{u}$ and $\mathbf{v}$ is the vector

$$\mathbf{u} - \mathbf{v} \stackrel{\text{def}}{=} \mathbf{u} + (-1)\mathbf{v} = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle.$$

Vector addition and scalar multiplication obey the following rules. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be vectors and c, d be scalars. Then

1. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
2. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
3. $\mathbf{a} + \mathbf{0} = \mathbf{a}$
4. $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
5. $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
6. $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
7. $(cd)\mathbf{a} = c(d\mathbf{a})$
8. $1\mathbf{a} = \mathbf{a}$

Expression of vectors in terms of the coordinate unit vectors.

Another frequently used way to express vectors is in terms of the coordinate unit vectors defined by

$$\mathbf{i} = \langle 1, 0, 0 \rangle, \quad \mathbf{j} = \langle 0, 1, 0 \rangle, \quad \mathbf{k} = \langle 0, 0, 1 \rangle.$$

Clearly,

$$\langle a_1, a_2, a_3 \rangle = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Length and scalar product

The *length* of a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, then the *scalar or dot product* of $\mathbf{u}$ and $\mathbf{v}$ is

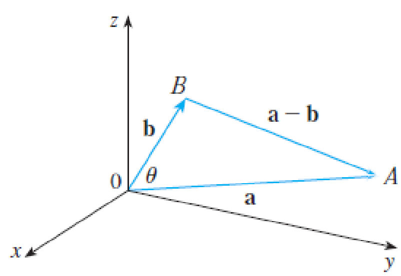
$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

Theorem 1. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be nonzero vectors. If* θ *is the angle between the two vectors, then*

$$|\mathbf{a}|\,|\mathbf{b}| \cos\theta = \mathbf{a} \cdot \mathbf{b}.$$

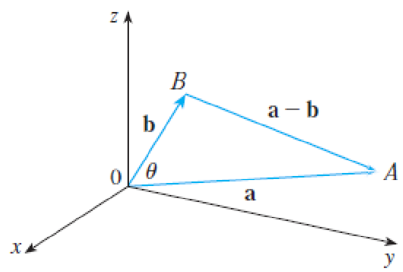


Corollary 2. *Two nonzero vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *are perpendicular if and only if* $\mathbf{a} \cdot \mathbf{b} = 0$.

Corollary 3. *For any vectors* $\mathbf{a}$ *and* $\mathbf{b}$, $|\mathbf{a} \cdot \mathbf{b}| \le |\mathbf{a}|\,|\mathbf{b}|$.

Theorem 1. Let $\mathbf{a}$ and $\mathbf{b}$ be nonzero vectors. If θ is the angle between the two vectors, then

$$|\mathbf{a}|\,|\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}.$$

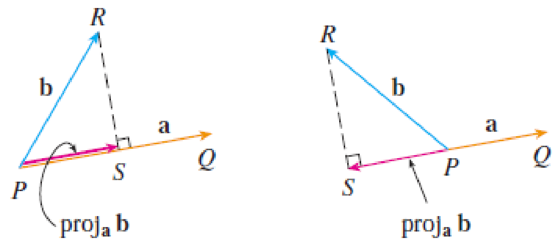


Proof. From the Cosine Rule

$$\begin{aligned}
2|\mathbf{a}|\,|\mathbf{b}|\,\cos\theta &= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2 \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 - (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) \\
&= |\mathbf{a}|^2 + |\mathbf{b}|^2 - |\mathbf{a}|^2 + 2\mathbf{a}\cdot\mathbf{b} - |\mathbf{b}|^2 \\
&= 2\mathbf{a}\cdot\mathbf{b}.
\end{aligned}$$

Hence $|\mathbf{a}|\,|\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}$. □

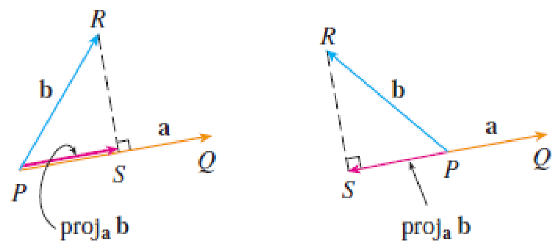
Take two nonzero vectors **a** and **b**, the figure below shows the *projection of* **b** *onto* **a**, denoted by $\text{proj}_{\mathbf{a}}\mathbf{b}$.



Example. Show that

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\,\mathbf{a}.$$

Example. Let $\mathbf{a} = \langle 1, 3, -2\rangle$ and $\mathbf{b} = \langle -1, 2, 5\rangle$. Find $\mathbf{c}, \mathbf{d}$ so that $\mathbf{b} = \mathbf{c} + \mathbf{d}$, $\mathbf{c}$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{d} \perp \mathbf{a}$.



Consider the figure on the left. Since $\text{proj}_{\mathbf{a}}\mathbf{b}$ is in the direction of $\mathbf{a}$,

$$\text{proj}_{\mathbf{a}}\mathbf{b} = c\mathbf{a} \text{ for some } c \geq 0.$$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

$$|\text{proj}_{\mathbf{a}}\mathbf{b}| = |\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}.$$

Hence

$$\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = |\text{proj}_{\mathbf{a}}\mathbf{b}| = |c\mathbf{a}| = c|\mathbf{a}|$$

since $c \geq 0$. Thus

$$\text{proj}_{\mathbf{a}}\mathbf{b} = c\mathbf{a} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}.$$

Example. Let $\mathbf{a} = \langle 1, 3, -2\rangle$ and $\mathbf{b} = \langle -1, 2, 5\rangle$. Find $\mathbf{c}, \mathbf{d}$ so that $\mathbf{b} = \mathbf{c} + \mathbf{d}$, $\mathbf{c}$ is a scalar multiple of $\mathbf{a}$ and $\mathbf{d} \perp \mathbf{a}$.

$$\mathbf{c} = \text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \frac{-5}{14}\langle 1, 3, -2\rangle.$$

$$\mathbf{d} = \mathbf{b} - \mathbf{c} = \langle -1, 2, 5\rangle + \frac{5}{14}\langle 1, 3, -2\rangle = \frac{1}{14}\langle -9, 43, 60\rangle.$$

Cross product

Cross product only applies to vectors in 3-D.

For any vectors $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$ in 3-D, their *cross product* is

$$\mathbf{a}\times\mathbf{b} \overset{\text{def}}{=} \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$$
$$= (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

To compute the determinant, we have applied formal co-factor expansion along the first row.

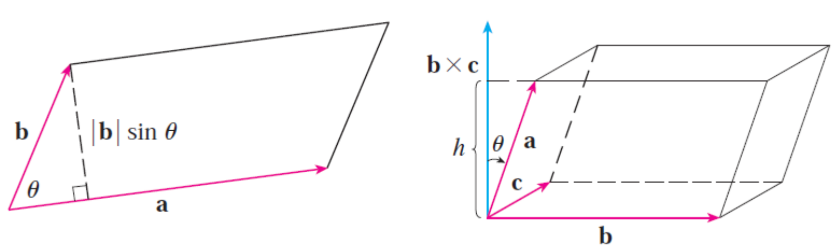
Theorem 4. *Let* $\mathbf{a}$ *and* $\mathbf{b}$ *be 3-D vectors.*

(1) $\mathbf{a}\times\mathbf{a} = \mathbf{0}$ *(zero vector).*
(2) $\mathbf{a}\times\mathbf{b}$ *is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$*.*
(3) *If* θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *and* $0 \le \theta \le \pi$*, then*

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta.$$

Corollary 5. *Let* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$ *be 3-D vectors.*

(1) $\mathbf{a}$ *and* $\mathbf{b}$ *are parallel if and only if* $\mathbf{a}\times\mathbf{b} = \mathbf{0}$*.*
(2) $|\mathbf{a}\times\mathbf{b}|$ *is the area of the parallelogram determined by the vectors* $\mathbf{a}$ *and* $\mathbf{b}$*.*
(3) $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$ *is the volume of the parallelopiped determined by the vectors* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$*.*



$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is called the *scalar triple product* of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Theorem 4. Let $\mathbf{a}$ and $\mathbf{b}$ be 3-D vectors.

(1) $\mathbf{a} \times \mathbf{a} = \mathbf{0}$ (zero vector).

(2) $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(3) If θ is the angle between $\mathbf{a}$ and $\mathbf{b}$ and $0 \leq \theta \leq \pi$, then

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta.$$

(1) Easy.

(2) Check that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ and $\mathbf{b} \cdot (\mathbf{a} \times \mathbf{b}) = 0$.

(3)

$$\begin{aligned}
|\mathbf{a} \times \mathbf{b}|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2 \\
&= a_2^2b_3^2 + a_3^2b_2^2 + a_3^2b_1^2 + a_1^2b_3^2 + a_1^2b_2^2 + a_2^2b_1^2 \\
&\quad - 2(a_1a_2b_1b_2 + a_1a_3b_1b_3 + a_2a_3b_1b_3) \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1^2b_1^2 + a_2^2b_2^2 + a_3^2b_3^2) \\
&\quad - 2(a_1a_2b_1b_2 + a_1a_3b_1b_3 + a_2a_3b_1b_3) \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2 \\
&= |\mathbf{a}|^2|\mathbf{b}|^2 - |\mathbf{a}|^2|\mathbf{b}|^2\cos^2\theta \\
&= |\mathbf{a}|^2|\mathbf{b}|^2\sin^2\theta.
\end{aligned}$$

Since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$. Hence

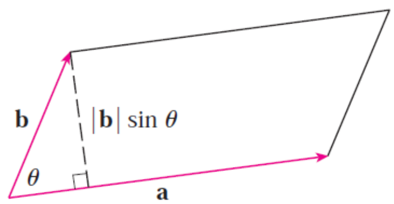
$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta.$$

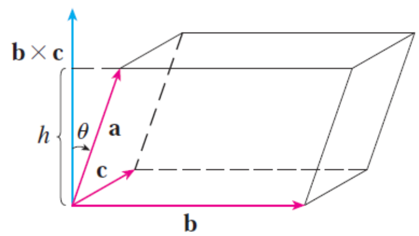
Corollary 5 Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be 3-D vectors.

(1) $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

(2) $|\mathbf{a} \times \mathbf{b}|$ is the area of the parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$.

(3) $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$ is the volume of the parallelopiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.





(1) $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\theta = 0$ or π if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

(2) Area of the parallelogram determined by the vectors $\mathbf{a}$ and $\mathbf{b}$ is given by base $\times$ height

$$= |\mathbf{a}| \cdot |\mathbf{b}| \sin\theta = |\mathbf{a} \times \mathbf{b}|.$$

(3) Volume of the parallelopiped is
(area of parallelogram determined by $\mathbf{b}$ and $\mathbf{c}$)$\times$ height h

$$= h|\mathbf{b} \times \mathbf{c}| = |\mathbf{a}|\,|\cos\theta| \cdot |\mathbf{b} \times \mathbf{c}| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|.$$

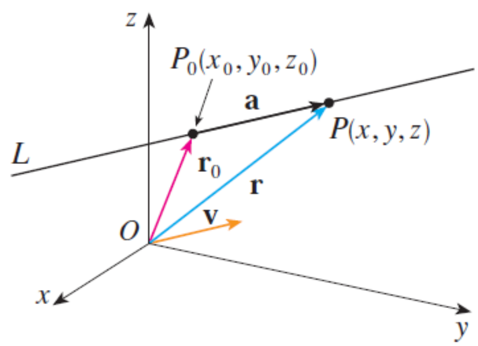
If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
5. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
6. $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Lines.

A straight line in $\mathbb{R}^3$ is determined by

(1) A point and a direction; *or*
(2) Two points.



Refer to the figure above. It is given that the line L passes through the point $P_0(x_0, y_0, z_0)$ and is parallel to the vector $\mathbf{v} = \langle a, b, c \rangle$.
We have

$$\begin{aligned} P \text{ lies on } L &\iff \overrightarrow{P_0P} \parallel \mathbf{v} \\ &\iff \overrightarrow{P_0P} = t\mathbf{v} \text{ for some scalar } t. \end{aligned}$$

Denote the vectors $\overrightarrow{OP_0}$ and $\overrightarrow{OP}$ by $\mathbf{r_0}$ and $\mathbf{r}$ respectively. Then $\overrightarrow{P_0P} = \mathbf{r} - \mathbf{r_0}$. So the equation for L (in vector form) is

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v}, \;\; t \in \mathbb{R}.$$

(t is called a *parameter*.)
If we put in $\mathbf{r} = \langle x, y, z \rangle$ and $\mathbf{r_0} = \langle x_0, y_0, z_0 \rangle$ and write the equations for the 3 components separately, we have the *parametric equations for L*

$$x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc.$$

If a, b, c are nonzero, we can solve for the parameter t in all 3 equations and equate the results to get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

This is the *symmetric equation for L.*

Remark. If, e.g., $a, b \neq 0$ and $c = 0$, then the symmetric equations for L would be

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}, \quad z = z_0.$$

Example. Find the vector equation, parametric equations and symmetric equation for the line L that passes through $(1, -2, 3)$ and is parallel to $\langle 3, -1, 4 \rangle$. Where does this line intersect the xy-plane?

Suppose that a straight line L passes through points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$. Then L passes through P_1 and is parallel to the vector $\overrightarrow{P_1P_2} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$. So the equations for L can be found as before.

Example. Find the vector equation, parametric equations and symmetric equation for the line L that passes through $(2, 5, -3)$ and $(-1, 3, 0)$.

Example. Find the vector equation, parametric equations and symmetric equation for the line L that passes through $(1, -2, 3)$ and is parallel to $\langle 3, -1, 4\rangle$. Where does this line intersect the xy-plane?

The vector equation is

$$\mathbf{r}(t) = \langle 1, -2, 3\rangle + t\langle 3, -1, 4\rangle = \langle 1 + 3t, -2 - t, 3 + 4t\rangle.$$

The parametric equations are

$$x = 1 + 3t, \quad y = -2 - t, \quad z = 3 + 4t.$$

Solving for t in the parametric equations, we get

$$\frac{x-1}{3} = -y - 2 = \frac{z-3}{4},$$

which is the symmetric equation.

The line hits the xy-plane when $z = 0$. From the symmetric equation, we see that when $z = 0$,

$$\frac{x-1}{3} = -y - 2 = \frac{-3}{4}.$$

Hence $x = \frac{-5}{4}$ and $y = \frac{-5}{4}$.

So L intersects the xy-plane at the point $(\frac{-5}{4}, \frac{-5}{4}, 0)$.

Example. Find the vector equation, parametric equations and symmetric equation for the line L that passes through $(2, 5, -3)$ and $(-1, 3, 0)$.

L passes through $(-2, 5, -3)$ and has direction

$$\langle -1, 3, 0\rangle - \langle 2, 5, -3\rangle = \langle -3, -2, 3\rangle.$$

Vector equation:

$$\mathbf{r}(t) = \langle -2, 5, -3\rangle + t\langle -3, -2, 3\rangle.$$

Parametric equations:

$$x = -2 - 3t, \quad y = 5 - 2t, \quad z = -3 + 3t.$$

Symmetric equation:

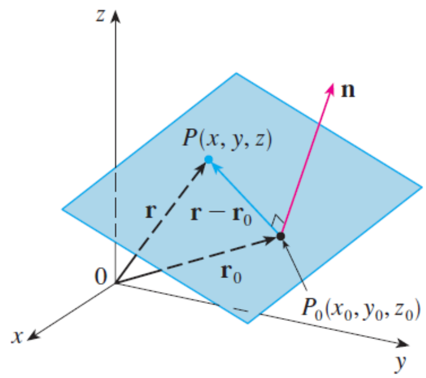
$$\frac{x+2}{-3} = \frac{y-5}{-2} = \frac{z+3}{3}.$$

Planes.

A plane in $\mathbb{R}^3$ is determined by

(1) A point in the plane and a direction perpendicular to the plane (called the *normal direction*), *or*

(2) Three non-collinear points on the plane.



In the figure above, a plane is given so that it passes through $P_0(x_0, y_0, z_0)$ and has normal vector (i.e., perpendicular direction) $\mathbf{n} = \langle a, b, c\rangle$.

$$\begin{aligned} P(x,y,z) \text{ lies on the plane } &\iff \overrightarrow{P_0P} \perp \mathbf{n} \\ &\iff (\mathbf{r} - \mathbf{r_0}) \cdot \mathbf{n} = 0, \end{aligned}$$

where $\mathbf{r} = \overrightarrow{OP} = \langle x, y, z\rangle$ and $\mathbf{r_0} = \overrightarrow{OP_0} = \langle x_0, y_0, z_0\rangle$. So the equation of the plane can be written as

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

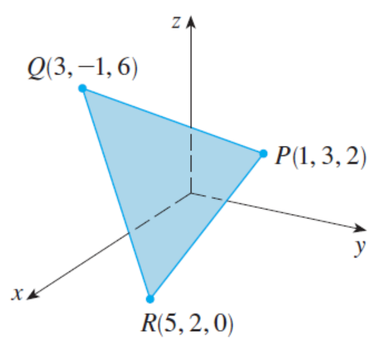
or

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

Example. A plane has equation $3x - 2y + z = 7$. Find a normal vector to the plane and a point on the plane.

Normal vector: $\langle 3, -2, 1 \rangle$.
To find a point on the plane, set $x = y = 0$ and solve for z.
When $x = y = 0$, $z = 7$. So a point on the plane is $(0, 0, 7)$.

Suppose three non-collinear points P_1, P_2 and P_3 are given on a plane.
Then $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ are two non-parallel vectors lying in the plane.
So $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is a nonzero vector perpendicular to both $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$. Thus $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ must be a normal vector to the plane.



Example. Find an equation for the plane $\mathcal{P}$ that passes through the points $P(1,3,2), Q(3,-1,6)$ and $R(5,2,0)$. Does the plane $\mathcal{P}$ contain the line L with symmetric equations

$$\frac{x-2}{7} = \frac{z+4}{6}, \ y = 1?$$

Normal vector:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \langle 2,-4,4\rangle \times \langle 4,-1,-2\rangle = \langle 12,20,14\rangle.$$

Equation of plane:

$$\begin{aligned} &\langle x-1, y-3, z-2\rangle \cdot \langle 12,20,14\rangle = 0 \\ \Longleftrightarrow\ & 12x + 20y + 14z - 100 = 0 \\ \Longleftrightarrow\ & 6x + 10y + 7z = 50. \end{aligned}$$

If (x,y,z) is a point on L, then

$$x = \frac{7}{6}(z+4) + 2 \text{ and } y = 1.$$

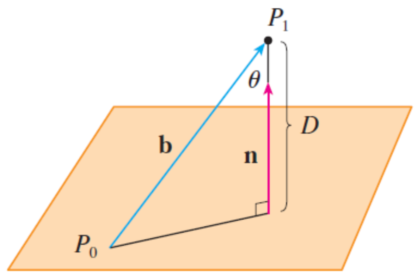
Hence

$$\begin{aligned} 6x + 10y + 7z &= 6[\frac{7}{6}(z+4) + 2] + 10 + 7z \\ &= 7(z+4) + 12 + 10 - 7z = 50. \end{aligned}$$

Hence L lies on $\mathcal{P}$.

Distance from a point to a plane.

Refer to the picture below.



The distance from P_1 to the plane $\mathcal{P}$ that passes through P_0 and has normal $\mathbf{n}$ is (taking $\mathbf{b} = \overrightarrow{P_0P}$)

$$|\text{proj}_{\mathbf{b}}\mathbf{n}| = \Big|\frac{\mathbf{b}\cdot\mathbf{n}}{|\mathbf{n}|^2}\,\mathbf{n}\Big| = \frac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

Example. Find the distance from the point $(3, 7, 5)$ to the plane with equation $2x - 5y - 3z = 15$.

Find a point P_0 on the plane: set $x = y = 0$, then $-3z = 15$. So $z = -5$. A point on the plane is $P_0(0, 0, -5)$.

Take $P = (3, 7, 5)$. Then $\mathbf{b} = \overrightarrow{P_0P} = \langle 3, 7, 10\rangle$.

Normal to the plane: $\mathbf{n} = \langle 2, -5, -3\rangle$.

Distance from P_0 to $\mathcal{P}$:

$$\frac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{n}|} = \frac{|6 - 35 - 30|}{\sqrt{2^2 + 5^2 + 3^2}} = \frac{69}{\sqrt{38}}.$$

Angles between lines/planes.

The angle between two planes is the angle made by their normal vectors.

Example. Find the angle between the planes with equations

$$2x - 5y - 3z = 4 \text{ and } 3x + y + z = 22.$$

The two normal vectors are

$$\mathbf{n}_1 = \langle 2, -5, -3 \rangle \text{ and } \mathbf{n}_2 = \langle 3, 1, 1 \rangle.$$

If θ is the angle between these vectors, then

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \frac{-2}{\sqrt{38}\,\sqrt{11}}.$$

Thus

$$\theta = \cos^{-1}(\frac{-2}{\sqrt{418}}) = 1.669 \text{ radians} = 96.61^\circ.$$

Example. Find the angle made by the line

$$x = 3 + 2t, \quad y = -1 - t, \quad z = 7 + 5t$$

with the plane $2x - 5y - 3z = 4$.

Normal vector for the plane: $\mathbf{n} = \langle 2, -5, -3 \rangle$.

Direction vector for the line: $\mathbf{v} = \langle 2, -1, 5 \rangle$.

Angle between $\mathbf{n}$ and $\mathbf{v}$ is

$\theta = \cos^{-1}(\frac{\mathbf{n} \cdot \mathbf{v}}{|\mathbf{n}|\,|\mathbf{v}|}) = \cos^{-1}(\frac{-5}{\sqrt{38}\sqrt{30}}) = 1.71943\,\text{radians} = 98.52^\circ.$

Angle between plane and line is $|90^\circ - \theta| = 8.52^\circ$.

Example. Show that the lines

$$x = 3 + 2t, \quad y = -1 - t, \quad z = 7 + 5t$$
$$\mathbf{r} = \langle 0, 3, 6 \rangle + t\langle -1, 3, 4 \rangle$$

intersect and find the angle between them.

To find the intersection, change one of the parameters to s and solve the simultaneous equations:

$$x = 3 + 2t, \quad y = -1 - t, \quad z = 7 + 5t$$
$$\mathbf{r} = \langle 0, 3, 6 \rangle + s\langle -1, 3, 4 \rangle$$

Thus

$$3 + 2t = -s, \quad -1 - t = 3 + 3s, \quad 7 + 5t = 6 + 4s.$$

From the first two equations, we have

$$-1 - t = 3 + 3(-3 - 2t) = -6 - 6t.$$

Hence $t = -1$. Then $s = -3 - 2t = -1$.
We must check that the third equation is also satisfied.
If $t = -1$ and $s = -1$, then

$$7 + 5t = 2 \text{ and } 6 + 4s = 2.$$

So the final equation also holds.
Therefore, the two lines intersect. The point of intersection is $(3 + 2(-1), -1 - (-1), 7 + 5(-1)) = (1, 0, 2)$.

The direction vectors of the two lines are $\mathbf{v}_1 = \langle 2, -1, 5 \rangle$ and $\mathbf{v}_2 = \langle -1, 3, 4 \rangle$.
The angle between them is

$$\cos^{-1}\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|\,|\mathbf{v}_2|}\right) = \cos^{-1}\left(\frac{15}{\sqrt{30}\sqrt{26}}\right) = 122.49^\circ.$$

2. Space curves

Main topics

- Curves. Tangent vector. Tangent line. Unit tangent. Unit normal. Binormal. Osculating plane.
- Arc length.
- Curvature.
- Curvilinear motion.
- Planetary motion.

Curves.

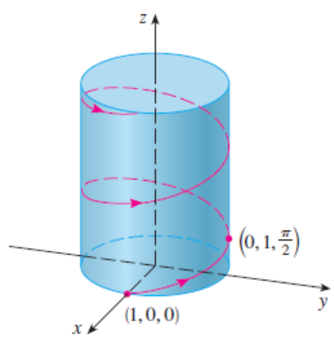
A *curve* in 3-space is a function $\mathbf{r} : I \to \mathbb{R}^3$, where I is a real interval. $\mathbf{r}$ can be written in terms of its components

$$\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}.$$

We can also write the curve in the form of *parametric equations*

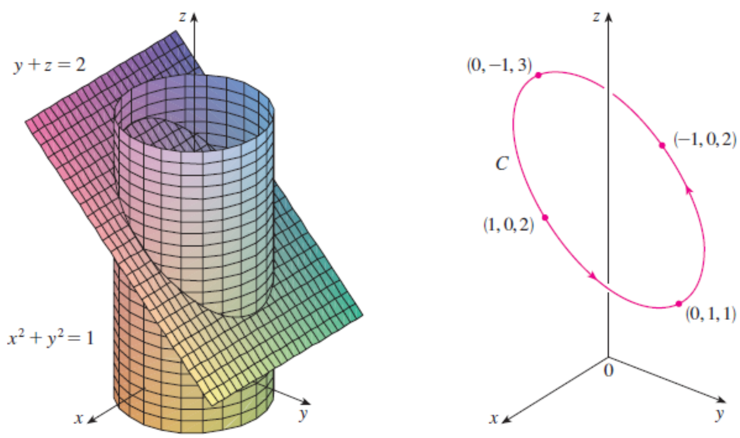
$$x = f(t), \quad y = g(t), \quad z = h(t).$$

Example. Sketch the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$.



A curve is sometimes described in some other way. To express it as a function $\mathbf{r} : I \to \mathbb{R}^3$ is called a *parametrization* of the curve.

Example. Parametrize the curve that is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.



We may take $x = \cos t$, $y = \sin t$ so that $x^2 + y^2 = 1$.
Then $z = 2 - y = 2 - \sin t$.
So a parametrization is

$$\mathbf{r}(t) = \langle \cos t, \sin t, 2 - \sin t \rangle.$$

Define

$$\lim_{t\to a}\mathbf{r}(t)=\lim_{t\to a}f(t)\mathbf{i}+\lim_{t\to a}g(t)\mathbf{j}+\lim_{t\to a}h(t)\mathbf{k}$$

if the limits of the components f, g and h exists at a.
$\mathbf{r}$ is *continuous* at a if $\lim_{t\to a}\mathbf{r}(t)=\mathbf{r}(a)$.

Example. Suppose that

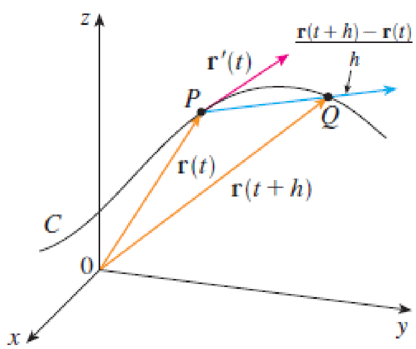
$$\mathbf{r}(t)=\langle t^2-1, e^t, \frac{\sin t}{t}\rangle \quad \text{if } t\neq 0.$$

Find $\lim_{t\to 0}\mathbf{r}(t)$. If $\mathbf{r}(0)=\langle -1,1,0\rangle$, is $\mathbf{r}$ continuous at 0?

A curve $\mathbf{r}$ is *differentiable* at t if

$$\frac{d\mathbf{r}}{dt}=\mathbf{r}'(t)=\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$$

exists. $\mathbf{r}$ is differentiable if it is differentiable at all $t\in I$.



In the figure above, we can see that $\mathbf{r}'(t)$ is the limit of the blue arrow as Q approaches P. So $\mathbf{r}'(t)$ is a *tangent* to the curve $\mathbf{r}$ at the point $P=\mathbf{r}(t)$. If $\mathbf{r}'(t)\neq\mathbf{0}$, the vector

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

is called the *unit tangent*.

Example. Let $\mathbf{r}(t) = \langle \cos t, \sin t, t^2 \rangle$. Find the tangents and unit tangents at $t = \pm 2\pi$. Obtain parametric equations for the tangent line to $\mathbf{r}$ at $t = 2\pi$.

$$\mathbf{r}'(t) = \langle -\sin t, \cos t, 2t \rangle.$$

So tangents are

$\mathbf{r}'(2\pi) = \langle 0, 1, 4\pi \rangle$ at $t = 2\pi$, $\mathbf{r}'(-2\pi) = \langle 0, 1, -4\pi \rangle$ at $t = -2\pi$.

Unit tangents are

$$\mathbf{T}(2\pi) = \frac{1}{\sqrt{1 + 16\pi^2}} \langle 0, 1, 4\pi \rangle \text{ at } t = 2\pi$$

$$\mathbf{T}(-2\pi) = \frac{1}{\sqrt{1 + 16\pi^2}} \langle 0, 1, -4\pi \rangle \text{ at } t = -2\pi.$$

Tangent line at $t = 2\pi$ passes through $\mathbf{r}(2\pi) = (1, 0, 4\pi^2)$ and has direction vector $\mathbf{r}'(2\pi) = \langle 0, 1, 4\pi \rangle$.
Parametric equations for the tangent line:

$$x = 1, \quad y = t, \quad z = 4\pi^2 + 4\pi t.$$

THEOREM Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\dfrac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

2. $\dfrac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

5. $\dfrac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

6. $\dfrac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)

Example. Suppose that $\mathbf{r}$ is differentiable and $|\mathbf{r}(t)| = 1$ for all t. Then $\mathbf{r}(t) \perp \mathbf{r}'(t)$ for all t.

Since $|\mathbf{r}(t)| = 1$,

$$\mathbf{r}(t) \cdot \mathbf{r}(t) = |\mathbf{r}(t)|^2 = 1 \text{ for all } t.$$

Differentiate this equation using property 4 above:

$$\begin{aligned} & \mathbf{r}'(t) \cdot (r)(t) + \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \\ \iff & 2\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \\ \iff & \mathbf{r}(t) \cdot \mathbf{r}'(t) = 0 \\ \iff & \mathbf{r}(t) \perp \mathbf{r}'(t). \end{aligned}$$

A *smooth curve* is a differentiable curve $\mathbf{r}$ so that $\mathbf{r}'(t) \neq \mathbf{0}$ for any t. If $\mathbf{r}$ is a smooth curve, then the unit tangent $\mathbf{T}(t)$ is defined at any t.

Example. Let $\mathbf{r}$ be a smooth curve so that $\mathbf{T}$ is differentiable and $\mathbf{T}'(t) \neq \mathbf{0}$ for any t. Define

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}.$$

Then $\mathbf{N}(t) \perp \mathbf{T}(t)$ for any t. $\mathbf{N}(t)$ is called the *unit normal* to the curve $\mathbf{r}$ at t.

Proof. Since $|\mathbf{T}(t)| = 1$ for all t, by the above, $T(t) \perp T'(t)$ for all t.
Since $\mathbf{N}$ is a multiple of $\mathbf{T}'(t)$, $\mathbf{N}(t) \perp \mathbf{T}(t)$. $\square$

The *binormal vector* to the curve at t is the vector $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$, provided both $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are defined.
The plane passing through $\mathbf{r}(t)$ and perpendicular to $\mathbf{B}(t)$ is called the *osculating plane* of the curve $\mathbf{r}$ at t.
The osculating plane is the plane that comes closes to containing the the curve $\mathbf{r}$ near the point t.

Example. Let $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$. Determine the unit tangent, unit normal, binormal and the osculating plane at $t = 1$.

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle.$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 1, 2t, 3t^2 \rangle}{\sqrt{1 + 4t^2 + 9t^4}} = (1{+}4t^2{+}9t^4)^{-1/2} \langle 1, 2t, 3t^2 \rangle.$$

$$\boxed{\mathbf{T}(1) = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle.}$$

$$\begin{aligned}\mathbf{T}'(t) &= (1 + 4t^2 + 9t^4)^{-1/2} \langle 0, 2, 6t \rangle \\ &\quad + \frac{-1}{2}(1 + 4t^2 + 9t^4)^{-3/2}(8t + 36t^3)\langle 1, 2t, 3^2 t \rangle.\end{aligned}$$

$$\mathbf{T}'(1) = \frac{1}{\sqrt{14}} \langle 0, 2, 6 \rangle - \frac{22}{14^{3/2}} \langle 1, 2, 3 \rangle = \frac{\langle -22, -16, 18 \rangle}{14^{3/2}}.$$

$$\boxed{\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \frac{\langle -22, -16, 18 \rangle}{|\langle -22, -16, 18 \rangle|} = \frac{\langle -11, -8, 9 \rangle}{\sqrt{266}}.}$$

$$\boxed{\mathbf{B}(1) = \frac{\langle 1, 2, 3 \rangle \times \langle -11, -8, 9 \rangle}{\sqrt{14}\sqrt{266}} = \frac{\langle 21, -21, 7 \rangle}{\sqrt{931}}.}$$

$\mathbf{B}(1)$ is a normal to the osculating plane $\mathcal{P}$.
A point on the plane is $\mathbf{r}(1) = (1, 1, 1)$.
Equation for $\mathcal{P}$:

$$\begin{aligned}&21(x - 1) - 21(y - 1) + 7(z - 1) = 0 \\ \iff &3x - 3y + z = 1.\end{aligned}$$

Integral of a vector function.

Let $\mathbf{r} : I \to \mathbb{R}^3$ be a vector function. If $a, b \in I$, $a < b$, a *partition* of the interval $[a, b]$ is a set of points

$$S = \{a = t_0 < t_1 < \cdots < t_n = b\}.$$

Let $|S| = \max\{t_k - t_{k-1} : 1 \leq k \leq n\}$. A *Riemann sum* of $\mathbf{r}$ with respect to the partition S is

$$R(\mathbf{r}, S) = \sum_{k=1}^{n} \mathbf{r}(s_k)(t_k - t_{k-1}), \text{ where } t_{k-1} \leq s_k \leq t_k.$$

The *integral* of $\mathbf{r}$ on $[a, b]$ is defined to be

$$\int_a^b \mathbf{r}(t)\,dt = \lim_{|S| \to 0} R(\mathbf{r}, S)$$

if the limit exists.

Theorem 6. *If $\mathbf{r} : I \to \mathbb{R}^3$ has components $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then $\int_a^b \mathbf{r}(t)\,dt$ exists if and only if all three integrals $\int_a^b f(t)\,dt$, $\int_a^b g(t)\,dt$, $\int_a^b h(t)\,dt$ exist; in which case*

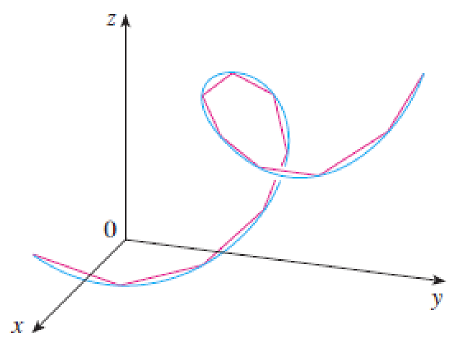
$$\int_a^b \mathbf{r}(t)\,dt = \langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \rangle.$$

Example. Let $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$. Compute $\int_0^\pi \mathbf{r}(t)\,dt$.

$$\begin{aligned}
\int_0^\pi \mathbf{r}(t)\,dt &= \langle \int_0^\pi \cos t\,dt, \int_0^\pi \sin t\,dt, \int_0^\pi t\,dt \rangle \\
&= \langle \sin t\Big|_0^\pi, -\cos t\Big|_0^\pi, \frac{t^2}{2}\Big|_0^\pi \rangle \\
&= \langle 0, 2, \frac{\pi^2}{2} \rangle.
\end{aligned}$$

Arc length.

Let $\mathbf{r} : I \to \mathbb{R}^3$ be a curve so that $\mathbf{r}'(t)$ exists and is continuous for $a \le t \le b$. To find the length of the arc, we fit a polygon to the curve and take limit as the division becomes finer.



Thus, to find the length of the arc described by $\mathbf{r}$ from $t = a$ to $t = b$, take any partition

$$S = \{a = t_0 < t_1 < \cdots < t_n = b\}$$

of $[a, b]$. The length of the polygonal arc is

$$\sum_{k=1}^{n} |\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})| = \sum_{k=1}^{n} \frac{|\mathbf{r}(t_k) - \mathbf{r}(t_{k-1})|}{t_k - t_{k-1}} (t_k - t_{k-1})$$

$$\approx \sum_{k=1}^{n} |\mathbf{r}'(t_k)|(t_k - t_{k-1}).$$

The last term is a Riemann sum of the integral $\int_a^b |\mathbf{r}'(t)|\, dt$. To find the arc length for the curve $\mathbf{r}$, take limit as $|S| \to 0$. The limit of the Riemann sums is exactly the integral $\int_a^b |\mathbf{r}'(t)|\, dt$. Therefore,

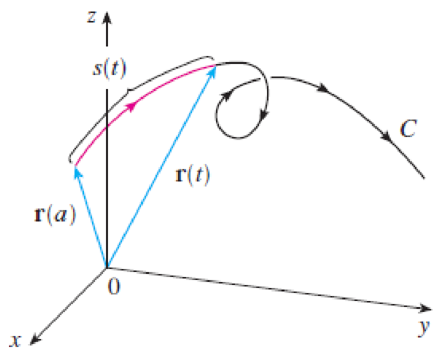
$$\text{arc length of } \mathbf{r} \text{ from } a \text{ to } b = \int_a^b |\mathbf{r}'(t)|\, dt.$$

Example. Find the length of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, 2t \rangle$ from $t = 0$ to $t = 1/2$.

The length is

$$\begin{aligned}
\int_0^1 |\mathbf{r}'(t)|\, dt &= \int_0^1 |\langle -\sin t, \cos t, 2 \rangle|\, dt \\
&= \int_0^1 \sqrt{\sin^2 t + \cos^2 t + 4}\, dt \\
&= \int_0^1 \sqrt{5}\, dt \\
&= \sqrt{5}.
\end{aligned}$$

Parametrization by arc length. Curvature.



The figure above shows the arc length $s(t)$ of the curve $\mathbf{r}$ from a fixed point $\mathbf{r}(a)$ to $\mathbf{r}(t)$. We call $s(t)$ the *arc length function*. Expressing $\mathbf{r}$ as a function of s is called a *parametrization of* $\mathbf{r}$ *by arc length*.

Note that $s'(t) = |\mathbf{r}'(t)|$.

Example. Find the arc length function and the arc length parametrization of $\mathbf{r}(t) = \langle \cos t, \sin t, 2t\rangle$ measured from the point $(1, 0, 0)$ (when $t = 0$) in the direction of increasing t.

$s'(t) = |\mathbf{r}'(t)| = \sqrt{5}$. $s(0) = 0$.
Hence

$$s(t) - s(0) = \int_0^t s'(u)\, du = \sqrt{5}t.$$

Thus $s(t) = \sqrt{5}t$.
Solving for t in terms of s gives $t = \frac{s}{\sqrt{5}}$.
Therefore, parametrization of $\mathbf{r}$ in terms of s is

$$\mathbf{r}(s) = \langle \cos \frac{s}{\sqrt{5}}, \sin \frac{s}{\sqrt{5}}, \frac{2s}{\sqrt{5}}\rangle.$$

Suppose that a curve is parametrized by arc length $\mathbf{r}(s)$. The *curvature* at a point $\mathbf{r}(a)$ is

$$\kappa = |\frac{d\mathbf{T}}{ds}|,$$

where $\mathbf{T} = \frac{\mathbf{r}'(s)}{|\mathbf{r}'(s)|}$ is the unit tangent.
If $\mathbf{r}(t)$ is parametrized by any t (which may not be arc length), then by the Chain Rule,

$$\kappa = |\frac{d\mathbf{T}}{ds}| = |\frac{d\mathbf{T}/dt}{ds/dt}|.$$

But $\frac{ds}{dt} = |\mathbf{r}'(t)|$. Hence

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}.$$

Example. Find the curvature κ of $\mathbf{r}(t) = \langle 1, t, t^2 \rangle$ at any t where κ exists.

$$\mathbf{r}'(t) = \langle 0, 1, 2t \rangle. \quad \mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 0, 1, 2t \rangle}{\sqrt{1+4t^2}}.$$

$$\mathbf{T}'(t) = \frac{\langle 0, -4t, 2 \rangle}{(1+4t^2)^{3/2}}.$$

So

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{\frac{2}{1+4t^2}}{\sqrt{1+4t^2}} = \frac{2}{(1+4t^2)^{3/2}}.$$

Example. Show that the circle of radius r has constant curvature $\kappa = \frac{1}{r}$.

Theorem 7.

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Proof. We have

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = (\mathbf{r}'(t) \cdot \mathbf{r}'(t))^{-1/2}\mathbf{r}'(t).$$

Differentiating gives

$$\begin{aligned}
\mathbf{T}'(t) &= (\mathbf{r}'(t) \cdot \mathbf{r}'(t))^{-1/2}\mathbf{r}''(t) \\
&\quad + \frac{-1}{2}(\mathbf{r}'(t) \cdot \mathbf{r}'(t))^{-3/2}\frac{d}{dt}(\mathbf{r}'(t) \cdot \mathbf{r}'(t))\mathbf{r}'(t) \\
&= (\mathbf{r}'(t) \cdot \mathbf{r}'(t))^{-1/2}\mathbf{r}''(t) - (\mathbf{r}'(t) \cdot \mathbf{r}'(t))^{-3/2}(\mathbf{r}'(t) \cdot \mathbf{r}''(t))\mathbf{r}'(t) \\
&= \frac{|\mathbf{r}'(t)|^2\mathbf{r}''(t) - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))\mathbf{r}'(t)}{|\mathbf{r}'(t)|^3}.
\end{aligned}$$

Recall the formula

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Taking $\mathbf{a} = \mathbf{c} = \mathbf{r}'(t)$ and $\mathbf{b} = \mathbf{r}''(t)$ gives

$$\mathbf{r}'(t) \times (\mathbf{r}''(t) \times \mathbf{r}'(t)) = |\mathbf{r}'(t)|^2\mathbf{r}''(t) - (\mathbf{r}'(t) \cdot \mathbf{r}''(t))\mathbf{r}'(t).$$

Therefore,

$$\begin{aligned}
|\mathbf{T}'(t)| &= \frac{|\mathbf{r}'(t) \times (\mathbf{r}''(t) \times \mathbf{r}'(t))|}{|\mathbf{r}'(t)|^3} \\
&= \frac{|\mathbf{r}'(t)|\,|\mathbf{r}''(t) \times \mathbf{r}'(t))|\sin\theta}{|\mathbf{r}'(t)|^3},
\end{aligned}$$

where θ is the angle between $\mathbf{r}'(t)$ and $\mathbf{r}''(t) \times \mathbf{r}'(t)$.
But $\mathbf{r}'(t) \times \mathbf{r}''(t) \perp \mathbf{r}'(t)$. So $\theta = \pi/2$.
Finally, we obtain

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

$\square$

Example. Redo the last example using this formula.

$\mathbf{r}(t) = \langle 1, t, t^2 \rangle$.
So

$$\mathbf{r}'(t) = \langle 0, 1, 2t \rangle, \quad \mathbf{r}''(t) = \langle 0, 0, 2 \rangle.$$

Therefore,

$$\begin{aligned}
\kappa(t) &= \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} \\
&= \frac{|\langle 0, 1, 2t \rangle \times \langle 0, 0, 2 \rangle|}{|\langle 0, 1, 2t \rangle|^3} \\
&= \frac{|\langle 2, 0, 0 \rangle|}{|\langle 0, 1, 2t \rangle|^3} = \frac{2}{(1 + 4t^2)^{3/2}}.
\end{aligned}$$

Osculating circle.

Let $\mathbf{r}(t)$ be a curve. Suppose that at a point $P = \mathbf{r}(a)$, it has unit tangent $\mathbf{T}$, unit normal $\mathbf{N}$ and curvature κ. The circle that lies on the plane generated by $\mathbf{T}$ and $\mathbf{N}$, with center at $\mathbf{r}(a) + \frac{\mathbf{N}}{\kappa}$ and radius $\frac{1}{\kappa}$ is called the *osculating circle* of the curve at P. It is the circle that most closely approximate the curve at P.

Example. Find an equation of the osculating circle of the curve $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ at the point $P(-1, 0, \pi)$.

Note that P is the point where $t = \pi$.
One can compute

$$\mathbf{T}(\pi) = \frac{\langle 0, -1, 1 \rangle}{\sqrt{2}}, \quad \mathbf{N}(\pi) = \langle 1, 0, 0 \rangle.$$

The osculating circle lies on the plane $\mathcal{P}$ with normal vector

$$\mathbf{n} = \mathbf{T}(\pi) \times \mathbf{N}(\pi) = \frac{\langle 0, 1, 1 \rangle}{\sqrt{2}}.$$

A point on $\mathcal{P}$ is $(-1, 0, \pi)$
Equation for $\mathcal{P}$: $y + z - \pi = 0$.

$$\kappa(\pi) = \frac{|\mathbf{r}'(\pi) \times \mathbf{r}''(\pi)|}{|\mathbf{r}'(\pi)|^3} = \frac{1}{2}.$$

Radius of the osculating circle is $1/\kappa = 2$.
Center of the osculating circle is $(-1, 0, \pi) + \frac{\mathbf{N}}{\kappa} = (1, 0, \pi)$.
If (x, y, z) is a point on the osculating circle, then

$$\begin{cases} (x-1)^2 + y^2 + (z-\pi)^2 = 4 \\ y + z - \pi = 0. \end{cases}$$

From the second equation, $y = -(z - \pi)$.
Putting this in the first equation gives

$$\begin{aligned} & (x-1)^2 + (z-\pi)^2 + (z-\pi)^2 = 4 \\ \Longleftrightarrow\ & (x-1)^2 + 2(z-\pi)^2 = 4 \\ \Longleftrightarrow\ & \left(\frac{x-1}{2}\right)^2 + \left(\frac{z-\pi}{\sqrt{2}}\right)^2 = 1. \end{aligned}$$

Set

$$\cos t = \frac{x-1}{2}, \ \sin t = \frac{z-\pi}{\sqrt{2}}.$$

Then the last equation is satisfied.

Also,

$$y = -(z - \pi) = -\sqrt{2}\sin t.$$

The parametric equations of the osculating circle is

$$x = 2\cos t + 1, \quad y = -\sqrt{2}\sin t, \quad z = \pi + \sqrt{2}\sin t.$$

Motion of a particle.

If $\mathbf{r}(t)$ denotes the position of a particle at time t, then

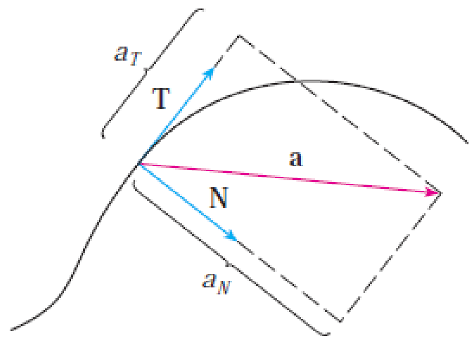
$$\mathbf{r}'(t) = \mathbf{v}(t) \text{ is the } \textit{velocity} \text{ at time } t,$$
$$\mathbf{r}''(t) = \mathbf{a}(t) \text{ is the } \textit{acceleration} \text{ at time } t.$$

Example. Find the velocity and acceleration vectors of a particle traveling at constant angular speed ω on a circle of radius r.

Put the circle in the xy-plane with center at the origin. Suppose that the particle is at the point $(r, 0)$ at time $t = 0$. Then at time t, the particle is at $\mathbf{r}(t) = \langle r\cos(\omega t), r\sin(\omega t)\rangle$. Therefore,

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -\omega r\sin(\omega t), \omega r\cos(\omega t)\rangle,$$
$$\mathbf{a}(t) = \mathbf{r}''(t) = -\langle \omega^2 r\cos(\omega t), \omega^2 r\sin(\omega t)\rangle.$$

We can express the acceleration in terms of its tangential and normal components.



Theorem 8. *Let* $v(t) = |\mathbf{v}(t)|$ *be the* speed. *Then*

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}, \text{ where } a_T = v' \text{ and } a_N = \kappa v^2.$$

a_T is the tangential component. a_N is the normal component.

Proof. Since $\mathbf{T} = \mathbf{r}'/|\mathbf{r}'|$,

$$\mathbf{r}' = |\mathbf{r}'|\mathbf{T} = (\mathbf{r}' \cdot \mathbf{r}')^{1/2}\mathbf{T}.$$

Therefore,

$$\begin{aligned}
\mathbf{r}'' &= (\mathbf{r}' \cdot \mathbf{r}')^{1/2}\mathbf{T}' + \frac{1}{2}(\mathbf{r}' \cdot \mathbf{r}')^{-1/2}(2\mathbf{r}' \cdot \mathbf{r}'')\mathbf{T} \\
&= |\mathbf{r}'|\mathbf{T}' + |\mathbf{r}'|^{-1}(\mathbf{r}' \cdot \mathbf{r}'')\mathbf{T} \\
&= |\mathbf{r}'|\,|\mathbf{T}'|\mathbf{N} + |\mathbf{r}'|^{-1}(\mathbf{r}' \cdot \mathbf{r}'')\mathbf{T}
\end{aligned}$$

Hence

$$a_T = |\mathbf{r}'|^{-1}(\mathbf{r}' \cdot \mathbf{r}'') \text{ and } a_N = |\mathbf{r}'|\,|\mathbf{T}'|.$$

Since $\kappa = |\mathbf{T}'|/|\mathbf{r}'|$,

$$a_N = \kappa|\mathbf{r}'|^2 = \kappa v^2.$$

Also $v = |\mathbf{v}| = (\mathbf{r}' \cdot \mathbf{r}')^{1/2}$. Thus

$$v' = \frac{1}{2}(\mathbf{r}' \cdot \mathbf{r}')^{-1/2}(2\mathbf{r}' \cdot \mathbf{r}'') = |\mathbf{r}'|^{-1}(\mathbf{r}' \cdot \mathbf{r}'') = a_T.$$

□

Theorem 9.

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$
$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

Proof. The first result is obtained above.
The second result follows from the expression for κ obtained in Theorem 7. □

Example. Determine the tangential and normal components of the acceleration at $t = 1$ for a particle whose position is given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$.

$$\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle, \quad \mathbf{r}''(t) = \langle 0, 2, 6t \rangle.$$
$$\mathbf{r}'(1) = \langle 1, 2, 3 \rangle, \quad \mathbf{r}''(1) = \langle 0, 2, 6 \rangle.$$
$$a_T = \frac{\mathbf{r}'(1) \cdot \mathbf{r}''(1)}{|\mathbf{r}'(1)|} = \frac{22}{\sqrt{14}}.$$
$$a_N = \frac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|} = \frac{|\langle 6, -3, 2 \rangle|}{\langle 1, 2, 3 \rangle|} = \sqrt{\frac{7}{2}}.$$

Planetary motion.

Consider the motion of a planet of mass m revolving around the sun (of mass M). Put the sun at the origin of our coordinate system, and denote the force of gravity acting on the planet by the sun by $\mathbf{F}$. Newton proposed the following laws: let $\mathbf{r}(t)$ be the position of the planet at time t, then

$$\begin{aligned} \text{Second law of motion:} \quad & \mathbf{F} = m\mathbf{a}, \\ \text{Law of universal gravitation:} \quad & \mathbf{F} = -\frac{GMm}{r^2}\,\mathbf{u}, \\ & \text{where } r = |\mathbf{r}| \text{ and } \mathbf{u} = \frac{\mathbf{r}}{r}. \end{aligned}$$

Combining the two laws, we see that the motion of the planet is described by the equation

$$\mathbf{r}'' = \mathbf{a} = -\frac{GM}{r^3}\,\mathbf{r}.$$

This is a differential equation for the vector function $\mathbf{r}$. We will solve it and prove Kepler's First Law of Planetary Motion: A planet revolves around the sun in an ellipse with the sun at a focus of the ellipse.

First

$$(\mathbf{r} \times \mathbf{r}')' = \mathbf{r} \times \mathbf{r}'' + \mathbf{r}' \times \mathbf{r}' = \mathbf{0}$$

since $\mathbf{r}$ is parallel to $\mathbf{r}''$.
Thus $\mathbf{r} \times \mathbf{r}'$ is a constant, denote it by $\mathbf{h}$.
In particular, $\mathbf{r}(t) \perp \mathbf{h}$ for any t. So that $\mathbf{r}(t)$ always lies on the plane perpendicular to $\mathbf{h}$.
We fix our coordinate system so that $\mathbf{h}$ is in the direction of $\mathbf{k}$. Then $\mathbf{r}$ and $\mathbf{r}'$ both lie in the (x, y)-plane.

Next

$$\begin{aligned}\mathbf{r}'' \times \mathbf{h} &= -\frac{GM}{r^3}\mathbf{r} \times (\mathbf{r} \times \mathbf{r}') \\ &= -\frac{GM}{r^3}[(\mathbf{r} \cdot \mathbf{r}')\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\mathbf{r}'].\end{aligned}$$

Differentiating $\mathbf{r} \cdot \mathbf{r} = r^2$ gives

$$2\mathbf{r} \cdot \mathbf{r}' = 2rr'.$$

Hence

$$\mathbf{r}'' \times \mathbf{h} = -\frac{GM}{r^3}[rr'\mathbf{r} - r^2\mathbf{r}'].$$

Now comes the crucial observation:

$$\Big(\frac{\mathbf{r}}{r}\Big)' = \frac{r\mathbf{r}' - r'\mathbf{r}}{r^2} = -\frac{1}{r^3}[rr'\mathbf{r} - r^2\mathbf{r}'].$$

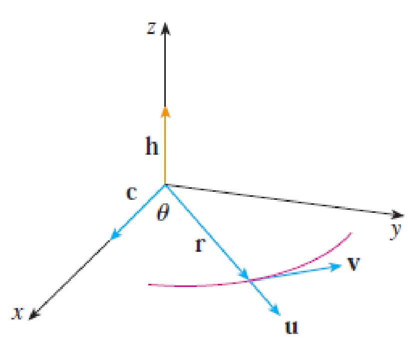
Therefore,

$$(\mathbf{r}' \times \mathbf{h})' = \mathbf{r}'' \times \mathbf{h} = GM\Big(\frac{\mathbf{r}}{r}\Big)'.$$

So there is a constant vector $\mathbf{c}$ so that

$$\mathbf{r}' \times \mathbf{h} = \Big(\frac{GM}{r}\Big)\mathbf{r} + \mathbf{c}.$$

Note that since $\mathbf{r}' \times \mathbf{h} \perp \mathbf{h}$, it is in the xy-plane, so is $\mathbf{r}$. Hence $\mathbf{c}$ is in the xy-plane. Choose our coordinate system so that $\mathbf{c}$ lies in the $\mathbf{i}$ direction. Denote by θ the angle made by $\mathbf{r}$ and $\mathbf{c}$.



Denote $|\mathbf{c}| = c$ and $|\mathbf{h}| = h$. Now

$$\begin{aligned}\mathbf{r}\cdot(\mathbf{r}'\times\mathbf{h}) &= (\frac{GM}{r})\mathbf{r}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{c}\\ &= GMr + rc\cos\theta\end{aligned}$$

and

$$\mathbf{r}\cdot(\mathbf{r}'\times\mathbf{h}) = (\mathbf{r}\times\mathbf{r}')\cdot\mathbf{h} = \mathbf{h}\cdot\mathbf{h} = h^2.$$

So

$$r(GM + c\cos\theta) = h^2.$$

Therefore, making the substitutions $e = \frac{c}{GM}$ and $d = \frac{h^2}{c}$,

$$r = \frac{h^2}{c\cos\theta + GM} = \frac{ed}{1 + e\cos\theta}.$$

Using the relations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we can express this equation in rectangular coordinates and see that it is the equation of a conic section (parabola, ellipse or hyperbola) with a focus at the origin. Since the orbit of a planet is a closed curve, it must be an ellipse.

$$\begin{aligned}& r = \frac{ed}{1 + e\cos\theta}\\ \Longleftrightarrow\ & r + er\cos\theta = ed\\ \Longleftrightarrow\ & \sqrt{x^2+y^2} + ex = ed\\ \Longleftrightarrow\ & \sqrt{x^2+y^2} = ed - ex\\ \Longleftrightarrow\ & (1-e^2)x^2 + 2de^2x + y^2 = e^2d^2\\ \Longleftrightarrow\ & \frac{(x-h)^2}{a^2} + \frac{y^2}{b^2} = 1,\end{aligned}$$

where

$$h = -\frac{e^2 d}{1-e^2}, \quad a^2 = \frac{e^2d^2}{(1-e^2)^2}, \quad b^2 = \frac{e^2d^2}{1-e^2}.$$

Center of ellipse is $(h, 0)$.
Foci of ellipse: $(h \pm c, 0)$, where

$$c^2 = a^2 - b^2 = \frac{e^4 d^2}{(1 - e^2)^2}.$$

Hence $c = -h$. Thus one of the foci is at $(0, 0)$.

3. Continuity and differentiability of functions of several variables.

- Equations and functions in several variables and their graphs.
- Limit and continuity of a function of several variables.
- Partial derivatives. Higher order partials. Equality of mixed partials.
- Differentiability. Gradient.
- Chain rule.
- Tangent plane. Tangents and normals to level curves and level surfaces.

In $\mathbb{R}^2$, an equation in two variables generally represent a curve.

Examples.

(1) $x + y = 1$ is a straight line in $\mathbb{R}^2$.
(2) $x^2 + y^2 = 1$ is the circled of radius 1 centered at the origin.

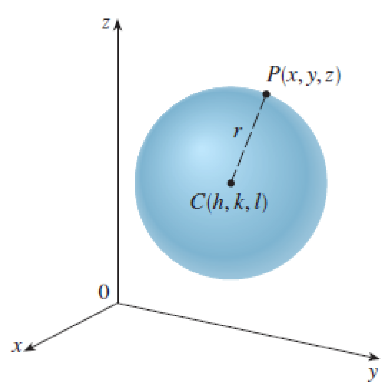
Simiarly, an equation in three variables in $\mathbb{R}^3$ generally represent a surface.

Examples.

(1) $x + y + z = 1$ is a plane in $\mathbb{R}^3$.
(2) If $C(h, k, l)$ is a given point and $r > 0$, then

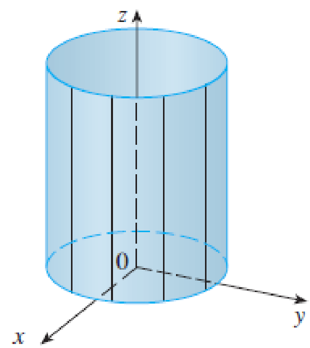
$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

is the sphere of radius r centered at C.



Warning: Sometimes one or more variables may be missing from an equation. In this case, one has to tell from the context whether the equation is supposed to describe something in $\mathbb{R}^2$ or $\mathbb{R}^3$. For example,

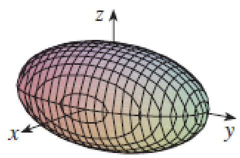
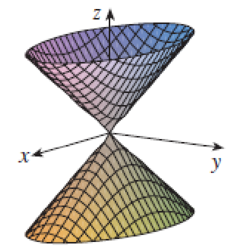
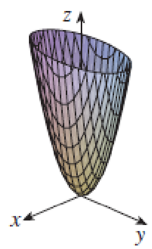
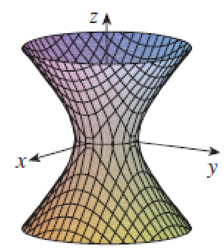
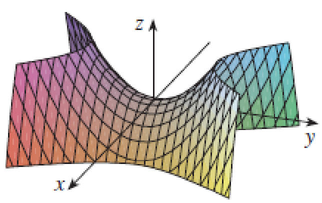
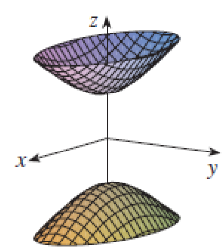
(1) $z = 3$ is the horizontal plane at height 3 in $\mathbb{R}^3$.
(2) $x^2 + y^2 = 1$ represents a circle if it is treated as an equation in $\mathbb{R}^2$; but it represents a cylinder if it is treated as an equation in $\mathbb{R}^3$.



$x^2 + y^2 = 1$

You are advised to familiarize yourself with the graphs of the *quardric surfaces* summarized in the table on the next page.

TABLE 1 Graphs of quadric surfaces

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ All traces are ellipses. If $a=b=c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x=k$ and $y=k$ are hyperbolas if $k\neq 0$ but are pairs of lines if $k=0$.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c<0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Horizontal traces in $z=k$ are ellipses if $k>c$ or $k<-c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Functions of two variables.

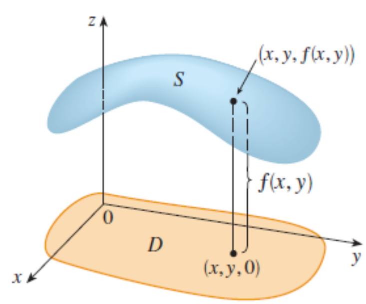
If D is a set in $\mathbb{R}^2$, a *function* $f : D \to \mathbb{R}$ is a rule that assigns a unique number $f(x, y)$ to every point $(x, y) \in D$. D is called the *domain* of f.
If the domain is not stated explicitly, we usually take it to be the largest set for which the formula $f(x, y)$ makes sense.

Example. The domain of the function given by the formula $f(x, y) = \sqrt{1 - x^2 - y^2}$ is the set

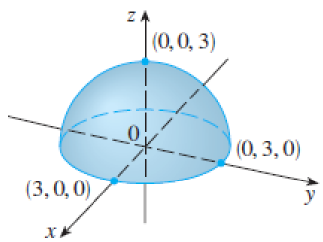
$$\{(x, y) : x^2 + y^2 \leq 1\}.$$

The *graph* of a function of two variables $f : D \to \mathbb{R}$ is the graph of the equation $z = f(x, y)$. It is a surface in $\mathbb{R}^3$. Precisely, the graph of f is the set

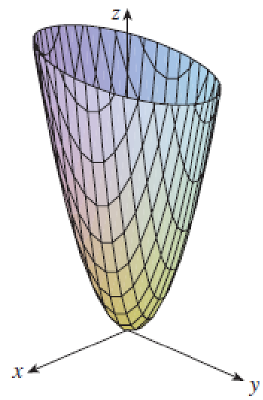
$$\{(x, y, z) : (x, y) \in D, z = f(x, y)\}.$$



Examples



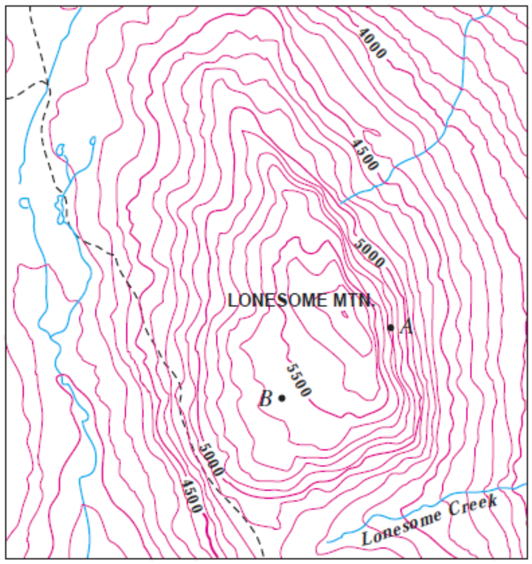
Graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$



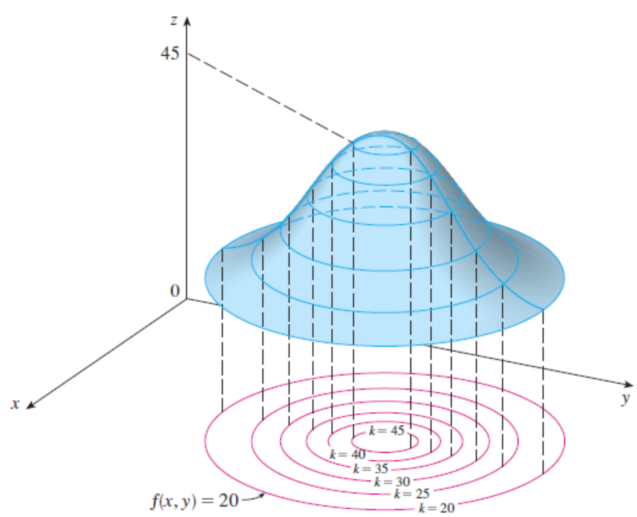
Graph of $h(x, y) = 4x^2 + y^2$

<u>Level curves</u>.

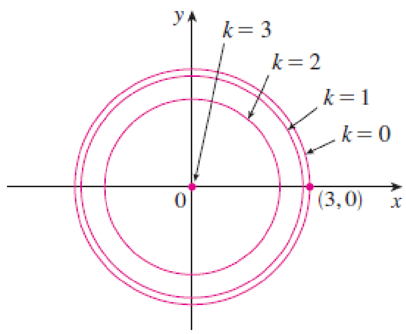
Another way to visualize a function $f(x, y)$ that requires only drawing graphs in two dimensions is to use *level curves*, as in topographical maps.



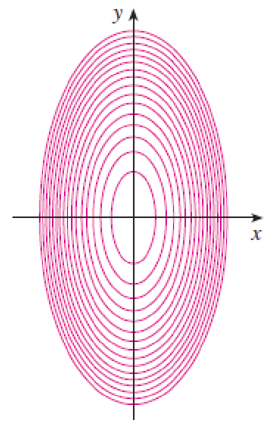
In this picture, the curves represent points at the same height. Similarly, for a function $f(x, y)$. The *level curve at value k* is the curve $f(x, y) = k$ (as a curve in $\mathbb{R}^2$).



Example. 2 sets of level curves.



$g(x, y) = \sqrt{9 - x^2 - y^2}$



$h(x, y) = 4x^2 + y^2$

<u>Functions of three variables</u>.

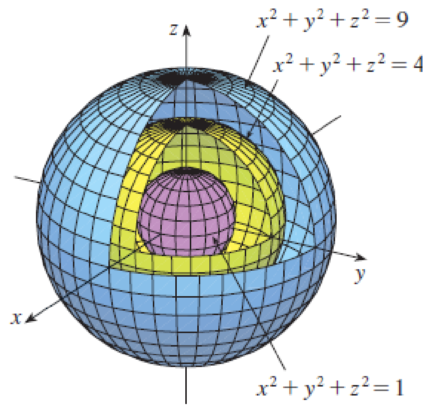
The above applies, with suitable changes, to functions of three variables $f(x, y, z)$. However, in this case, the graph of such a function is the graph of the equation in 4 variables:

$$w = f(x, y, z),$$

which cannot be drawn.
The *level surfaces* are of the form $f(x, y, z) = k$ for a constant k. They can help use visualize the behavior of the function f.

Example. Level surfaces of the function

$$f(x, y, z) = x^2 + y^2 + z^2.$$



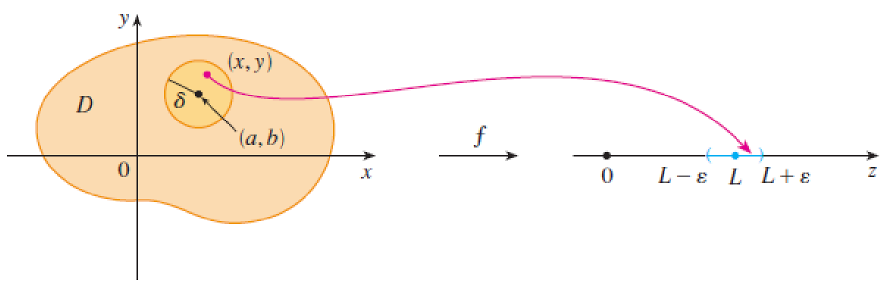
Limits and continuity.

DEFINITION Let f be a function of two variables whose domain D includes points arbitrarily close to (a, b). Then we say that the **limit of $f(x, y)$ as (x, y) approaches (a, b)** is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that

if $\quad (x, y) \in D \quad$ and $\quad 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \quad$ then $\quad |f(x, y) - L| < \varepsilon$



Example. Show that $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{\sqrt{x^2+y^2}} = 0$.

The key observation is that

$(|x|-|y|)^2 \geq 0 \implies x^2+y^2 \geq 2|x|\,|y| \implies \sqrt{x^2+y^2} \geq \sqrt{2|xy|}$.

Hence

$$\left|\frac{3x^2y}{\sqrt{x^2+y^2}}\right| \leq \frac{3|x^2y|}{\sqrt{2|xy|}} = \frac{3}{\sqrt{2}}|x|^{3/2}|y|^{1/2}.$$

Suppose that $\varepsilon > 0$ is given, choose $\delta = \frac{2^{1/4}}{3^{1/2}}\varepsilon^{1/2}$.

If $0 < \sqrt{x^2+y^2} < \delta$, then $|x|, |y| < \delta$. Hence

$$\left|\frac{3x^2y}{\sqrt{x^2+y^2}}\right| \leq \frac{3}{\sqrt{2}}|x|^{3/2}|y|^{1/2} < \frac{3}{\sqrt{2}}\delta^2 = \varepsilon.$$

It follows from definition of the limit that if $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then the limit of f along any path that approaches (a,b) must all be the same value L. Therefore,

Theorem 10. *Suppose that C_1 and C_2 are two paths that approach (a,b). If*

$$f(x,y) \to L_1 \text{ as } (x,y) \to (a,b) \text{ along } C_1,$$
$$f(x,y) \to L_2 \text{ as } (x,y) \to (a,b) \text{ along } C_2.$$

and $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Example. Determine if the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ exists.

If (x,y) approaches $(0,0)$ along the line $y=0$, then

$$\lim_{x\to 0} \frac{xy}{x^2+y^2}\Big|_{y=0} = \lim_{x\to 0} 0 = 0.$$

If (x,y) approaches $(0,0)$ along the line $y=x$, then

$$\lim_{x\to 0} \frac{xy}{x^2+y^2}\Big|_{y=x} = \lim_{x\to 0} \frac{x^2}{x^2+x^2} = \frac{1}{2}.$$

Since the two limits are different along different paths that approach $(0,0)$, the limit $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ does not exist.

Warning. Suppose that two approaches to (a,b) are tried and it is found that $f(x,y)$ has the same limit along these approaches. We cannot conclude that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists. This is because we have not tried all ways to reach the point (a,b). In fact, there are so many ways to reach (a,b), we cannot try them all.

Example. Determine if the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exists.

Suppose we let (x, y) approach $(0, 0)$ along any straight line $y = bx$, where b is a constant. Then

$$\lim_{x\to 0} \frac{xy^2}{x^2+y^4}\Big|_{y=bx} = \lim_{x\to 0} \frac{b^2x^3}{x^2+b^4x^4} = \lim_{x\to 0} \frac{b^2x}{1+b^4x^2} = 0.$$

However, this is <u>not</u> enough to show that the limit exists! In fact, if we (x, y) approach $(0, 0)$ along the parabola $x = y^2$, then

$$\lim_{y\to 0} \frac{xy^2}{x^2+y^4}\Big|_{x=y^2} = \lim_{y\to 0} \frac{y^4}{y^4+y^4} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}.$$

Since this is a different answer than the one above, we can conclude that the limit $\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ does not exist.

Remark. When choosing a path, make sure that it does approach the point where the limit it to be taken.

Continuity.

DEFINITION A function f of two variables is called **continuous at** (a, b) if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

We say f is **continuous on** D if f is continuous at every point (a, b) in D.

Examples.

(1) A polynomial function (in the variables x, y) is continuous on $\mathbb{R}^2$.
(2) A rational function (i.e., a quotient of polynomials) is continuous on its domain (that is, as long as it is defined).
(3) Sums, differences and products of continuous functions are continuous.
(4) A quotient of continuous functions is continuous at all points where the denominator is nonzero.

Examples

(1) $f(x, y) = 2xy + x^y - 3x^2y^5$ is continuous on $\mathbb{R}^2$.
(2) $g(x, y) = \frac{x^3y - 2xy^4}{x^2 + y^4}$ is continuous at all points (x, y) where $x^2 + y^4 \neq 0$, i.e., at all points except $(0, 0)$.

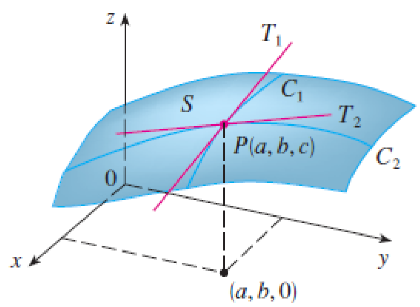
Partial derivatives.

Let $f(x,y)$ be a function of two variables. The *partial derivative of f with respect to x at a point* (a,b) is defined to be
$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0}\frac{f(a+h,b)-f(a,b)}{h}.$$
In practice, it means that we differentiate f treating x as the variable and y as a constant. Some other notation for the partial derivative are
$$\frac{\partial f}{\partial x}(a,b) = f_x(a,b) = D_x f(a,b).$$

Example. Let $f(x,y) = \sin\left(\frac{x}{1+y}\right)$. Find $\frac{\partial f}{\partial x}$ at $(x,y) = (1,2)$.
$$\frac{\partial f}{\partial x}(1,2) = \frac{1}{1+y}\cos\left(\frac{x}{1+y}\right)\Big|_{(1,2)} = \frac{1}{3}\cos\frac{1}{3}.$$

Example. Suppose that z is defined implicitly in terms of x, y by the equation
$$x^3 + y^3 + z^3 + 6xyz = 1.$$
Find $\frac{\partial z}{\partial x}$.
$$\begin{aligned} & 3x^2 + \frac{\partial z^3}{\partial x} + 6xy\frac{\partial z}{\partial x} + 6yz = 0 \\ \iff & 3x^2 + 3z^2\frac{\partial z}{\partial x} + 6xy\frac{\partial z}{\partial x} + 6yz = 0 \\ \iff & \frac{\partial z}{\partial x} = \frac{-x^2-2yz}{z^2+2xy}. \end{aligned}$$

Geometrically, let C_1 be the curve that is the intersection of the vertical plane $y = b$ with the graph of f. Then $\frac{\partial f}{\partial x}(a, b)$ is the slope of the curve C_1 at $(x, y) = (a, b)$.



Partial derivative with respect to y is defined similarly. If f is a function of 3 variables x, y, z, then we can also define $\frac{\partial f}{\partial z}$.

Higher order partials.

Let $f(x, y)$ be a function of 2 variables. (Similar discussion applies to functions of 3 variables.) The partial derivative f_x is also a function of 2 variables. So we can take its partial derivatives with respect to both x and y.

$$\frac{\partial f_x}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right).$$

This is usually abbreviated to

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}.$$

Similarly, we have 3 other possible *second order partials*

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy},$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}.$$

Example. Find all 4 second order partials of the function $f(x, y) = x^3 + x^2y^3 - 2y^2$.

$$f_x = 3x^2 + 2xy^3, \quad f_y = 3x^2y^2 - 4y$$

$$f_{xx} = 6x + 2y^3, \quad f_{yx} = 6xy^2$$

$$f_{xy} = 6xy^2, \quad f_{yy} = 6x^2y - 4.$$

In this case, we find that the two *mixed partials* f_{xy} and f_{yx} are equal. In fact, it happens generally under some mild hypothesis.

CLAIRAUT'S THEOREM Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

Differentiability.

Existence of partial derivatives at a point is in fact a rather weak condition.

Example. Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Then both $f_x(0,0)$ and $f_y(0,0)$ exist (in fact, both are 0). But f is not even continuous at $(0,0)$.

By analogy with the 1 variable situation, we expect that function that is differentiable at a point should at least be continuous there. So the function f in the example should not be differentiable at $(0,0)$ even though it has both partials there.

How to define differentiability for a function of 2 (or 3) variables?

Recall that in the 1 variable case, we have

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

which can be rewritten as

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0.$$

This formula can be suitably generalized to the higher variable case.

Let $f(x, y)$ be a function of 2 variables. Then f is *differentiable at* (a, b) if there is a vector, denoted $\nabla f(a, b)$, so that

$$\lim_{(h,k)\to(0,0)} \frac{f(a+h, b+k) - f(a, b) - \nabla f(a, b) \cdot \langle h, k\rangle}{|\langle h, k\rangle|} = 0.$$

Note that the vector $\nabla f(a, b)$ must have the same number of components as the number of variables of f. It is called the *gradient vector* of f at the point (a, b), also denoted by $\mathbf{grad}\, f$.

Theorem 11. *If $f(x, y)$ is differentiable at (a, b), then*

$$\nabla f(a, b) = \langle f_x(a, b), f_y(a, b)\rangle.$$

Proof. In the equation above, let $(h, k) \to (0, 0)$ along the positive x-axis; that is, fix $k = 0$ and let $h \to 0$ through positive values. Write $\nabla f(a, b) = \langle u, v\rangle$. Then we have

$$\lim_{h\to 0} \frac{f(a+h, b+0) - f(a, b) - \langle u, v\rangle \cdot \langle h, 0\rangle}{|\langle h, 0\rangle|} = 0.$$

So

$$\begin{aligned} &\lim_{h\to 0} \frac{f(a+h, b) - f(a, b) - uh}{h} = 0 \\ \Longrightarrow \quad &\lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} = u. \end{aligned}$$

But the limit on the left is $f_x(a, b)$. So $u = f_x(a, b)$. Similarly, $v = f_y(a, b)$. $\square$

Example. Let $f(x,y) = xy + \sin x$. Show that f is differentiable at any point $(a,b) \in \mathbb{R}^2$.

$$f_x(a,b) = b + \cos a, \quad f_y(a,b) = a.$$

So we need to show that

$$(*) \qquad \lim_{(h,k)\to(0,0)} \frac{f(a+h,b+k) - f(a,b) - \langle b+\cos a, a\rangle \cdot \langle h,k\rangle}{|\langle h,k\rangle|} = 0.$$

We have

$$\begin{aligned}
&\Big|\frac{f(a+h,b+k) - f(a,b) - \langle b+\cos a, a\rangle \cdot \langle h,k\rangle}{|\langle h,k\rangle|}\Big| \\
&= \Big|\frac{hk + \sin(a+h) - \sin a - h\cos a}{\sqrt{h^2+k^2}}\Big| \\
&\le \frac{|hk|}{|h|} + \frac{|\sin(a+h) - \sin a - h\cos a|}{|h|} \\
&\le |k| + \Big|\frac{\sin(a+h) - \sin a}{h} - \cos a\Big|
\end{aligned}$$

Now

$$\lim_{h\to 0} \frac{\sin(a+h) - \sin a}{h} = \frac{d}{dx}\sin x\Big|_{x=a} = \cos a.$$

Hence

$$\lim_{k\to 0} |k| = 0 \text{ and } \lim_{h\to 0}\Big|\frac{\sin(a+h) - \sin a}{h} - \cos a\Big| = 0.$$

So equation $(*)$ holds.

Directional derivatives

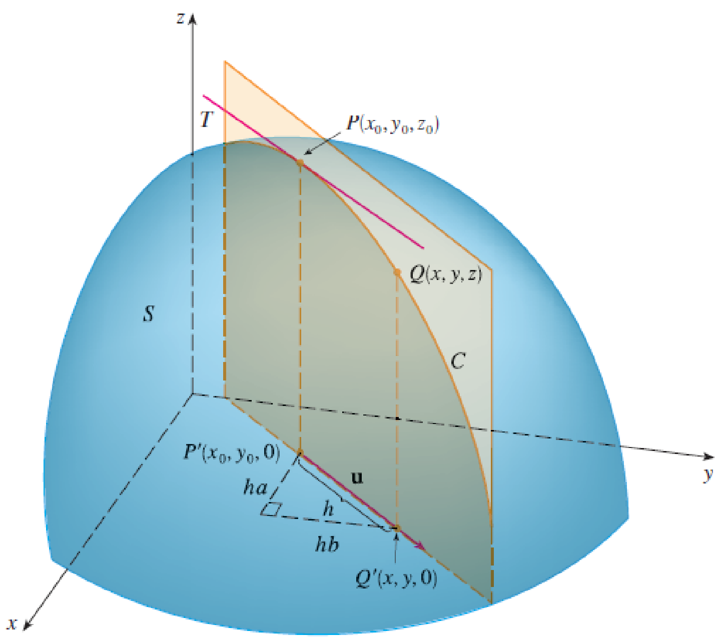
Let $\mathbf{u} = \langle a, b \rangle$ be a unit vector, i.e., $|\mathbf{u}| = 1$. Let $f(x, y)$ be a function of 2 variables. The *directional derivative of f at (x_0, y_0) in the direction* $\mathbf{u}$ is defined to be

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if the limit exists.
Observe that

$$D_{\mathbf{i}} f(x_0, y_0) = f_x(x_0, y_0) \quad D_{\mathbf{j}} f(x_0, y_0) = f_y(x_0, y_0).$$

$D_{\mathbf{u}} f(x_0, y_0)$ is the rate of change (or the slope of the tangent line) of the graph of f restricted to the direction $\mathbf{u}$ at the point (x_0, y_0).



Similar to the case for partial derivatives, we have

Theorem 12. *Let* $\mathbf{u}$ *be a unit vector and let* $f(x, y)$ *be differentiable at* (x_0, y_0). *Then* $D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0)\cdot\mathbf{u}$.

Corollary 13. *Suppose that* $f(x, y)$ *is differentiable at* (x_0, y_0) *and* $\nabla f(x_0, y_0) \neq \mathbf{0}$. *Then at* (x_0, y_0), f *increases fastest in the direction*

$$\nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|.$$

This means that $D_{\mathbf{u}}f(x_0, y_0)$ *is the largest (among all unit vectors* $\mathbf{u}$*) when* $\mathbf{u} = \nabla f(x_0, y_0)/|\nabla f(x_0, y_0)|$.

Proof.

$$D_{\mathbf{u}}f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u} = |\nabla f(x_0, y_0)|\,|\mathbf{u}|\cos\theta,$$

where θ is the angle between $\nabla f(x_0, y_0)$ and $\mathbf{u}$.
So $D_{\mathbf{u}}f(x_0, y_0)$ is maximized when $\theta = 1$, i.e, when $\mathbf{u}$ is in the same direction as $\nabla f(x_0, y_0)$. In this case, since $\mathbf{u}$ is a unit vector,

$$\mathbf{u} = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}.$$

$\square$

Example. Let $f(x, y) = x \sin y - \frac{1}{xy}$. Find the direction in which f increases fastest at the point $(1, \frac{\pi}{2})$. What is the maximum rate of increase?

$$\nabla f(1, \frac{\pi}{2}) = \langle \sin y + \frac{1}{x^2 y}, x\cos y + \frac{1}{xy^2}\rangle\big|_{(1,\frac{\pi}{2})} = \langle 1 + \frac{2}{\pi}, \frac{4}{\pi^2}\rangle.$$

The direction of maximum increase is

$$\mathbf{u} = \frac{\nabla f(1, \frac{\pi}{2})}{|\nabla f(1, \frac{\pi}{2})|} = \frac{\langle \pi^2 + 2\pi, 4\rangle}{\sqrt{(\pi^2 + 2\pi)^2 + 16}}.$$

The maximum rate of increase is

$$D_{\mathbf{u}}f(x_0, y_0) = |\nabla f(x_0, y_0)|\,|\mathbf{u}| = |\nabla f(x_0, y_0)| = \frac{\sqrt{(\pi^2 + 2\pi)^2 + 16}}{\pi^2}.$$

Proving the differentiability of a function using the definition can be challenging. The following result gives a handy sufficient criterion for f to be differentiable at some point.

THEOREM If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example. Consider $f(x, y) = xe^{xy}$. Then

$$f_x(x, y) = xye^{xy} + e^{xy} \quad f_y(x, y) = x^2e^{xy}.$$

Both of these partials exist and are continuous on $\mathbb{R}^2$. Hence f is differentiable on $\mathbb{R}^2$.

Chain rule

One of the most useful rules of differentiation is the Chain rule. Recall that in the 1 variable case, it goes as follows.

If $y = f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then

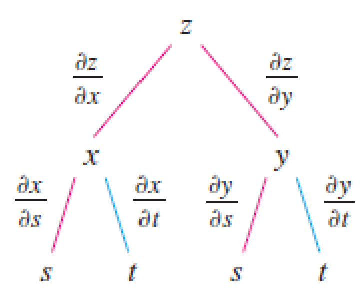
$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

For functions of more than one variable, there are different versions of the chain rule. Here is one example.

THE CHAIN RULE Suppose that $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are differentiable functions of s and t. Then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \qquad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

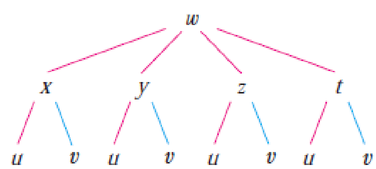
A good mnemonic device is the following tree diagram.



To obtain the correct chain rule for $\frac{\partial z}{\partial s}$, e.g., find all branches of the tree that begins at z and ends at a point s. Multiply together terms on the same branch, then sum the results for all branches linking z and s.

The tree mnemonic works for any number of variables (if it is set up right!).

Example. Suppose $w = f(x, y, z, t)$ is differentiable and $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$ and $t = t(u, v)$ are differentiable functions of (u, v). Write the chain rule for $\frac{\partial w}{\partial v}$.



From the tree diagram above, we see that

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial v}.$$

EXAMPLE If $u = x^4y + y^2z^3$, where $x = rse^t$, $y = rs^2e^{-t}$, and $z = r^2s\sin t$, find the value of $\partial u/\partial s$ when $r = 2$, $s = 1$, $t = 0$.

$$\begin{aligned}\frac{\partial u}{\partial s} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial s} \\ &= (3x^3y)(re^t) + (x^4 + 2yz^3)(2rse^{-t}) + (3y^2z^2)(r^2\sin t).\end{aligned}$$

When $r = 2, s = 1, t = 0,$

$$x = 2, \quad y = 2, \quad z = 0.$$

Substituting into the above expression gives

$$\frac{\partial u}{\partial s} = (48)(2) + (16 + 0)(4) + (0)(0) = 160.$$

Implicit differentiation.

An equation such as $x^3 + y^3 + z^3 + 6xyz = 1$ can be viewed as defining z implicitly as a function of x and y.
Generally, suppose we have the equation $F(x, y, z) = 0$, where z is viewed as a function of (x, y), and x, y are viewed as independent variables. Differentiating with respect to x, the chain rule gives

$$F_x + F_z \frac{\partial z}{\partial x} = 0.$$

So $F_x + F_z \frac{\partial z}{\partial x} = 0$, which gives

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}.$$

Example. Find $\frac{\partial z}{\partial x}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

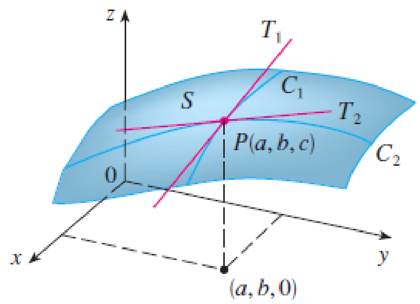
$$F_x = 3x^2 + 6yz \text{ and } F_z = 3z^2 + 6xy.$$

Hence

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy}.$$

Tangent plane

Suppose that $f(x, y)$ is differentiable at (a, b).



Parametrize the curve C_1, the intersection of the graph of f with the plane $y = b$ by $\mathbf{r_1}(t) = \langle t, b, f(t, b)\rangle$. The tangent to $\mathbf{r_1}$ at the point $P(a, b, f(a, b))$ is

$$\mathbf{r_1}'(a) = \langle 1, 0, f_x(a, b)\rangle.$$

Similarly, if C_2 is the intersection of the graph of f with the plane $x = a$, parametrized as $\mathbf{r_2}(t) = \langle a, t, f(a, t)\rangle$, then the tangent at $P(a, b, f(a, b))$ is $\mathbf{r_2}'(b) = \langle 0, 1, f_y(a, b)\rangle$. The plane passing through P determined by the vectors $\mathbf{r_1}'$ and $\mathbf{r_2}'$ is called the *tangent plane* to the graph of f at P. A normal to the tangent plane is

$$\mathbf{r_1}'(a) \times \mathbf{r_2}'(b) = \langle -f_x(a, b), -f_y(a, b), 1\rangle.$$

Hence an equation for the tangent plane is

$$-f_x(a, b)(x - a) - f_y(a, b)(y - b) + z - f(a, b) = 0$$

or

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example. Find an equation of the tangent plane to $f(x, y) = 4 - x^2 - 2y^2$ at $(1, 1, 1)$.

$$f_x(1, 1) = -2x|_{x=1} = -1, \quad f_y(1, 1) = -4y|_{y=1} = -4.$$

Also, $f(1, 1) = 1$. Hence equation of the tangent plane is

$$z = 1 - (x - 1) - 4(y - 1) \iff z = 6 - x - 4y.$$

The next result explains why the tangent plane is worthy of its name.

Theorem 14. *Let $f(x, y)$ be differentiable at (a, b) and let $\mathbf{n}$ be a normal to the tangent plane at $P(a, b, f(a, b))$. Suppose that $\mathbf{r}$ is a curve that lies on the graph of f, passes through the point P at $t = t_0$ and $\mathbf{r}'(t_0)$ exists. Then $\mathbf{r}'(t_0) \perp \mathbf{n}$.*

Proof. Express

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle.$$

Since $\mathbf{r}$ lies on the graph of f,

$$z(t) = f(x(t), y(t)) \text{ for all } t.$$

Differentiate with respect to t, using the Chain Rule on the fight hand side.

$$z'(t) = f_x((x(t), y(t))x'(t) + f_y((x(t), y(t))y'(t).$$

At point t_0, $x(t_0) = a$ and $y(t_0) = b$. Thus

$$\begin{aligned}
&z'(t_0) = f_x(a, b)x'(t_0) + f_y(a, b)y'(t_0) \\
\iff &\langle -f_x(a, b), -f_y(a, b), 1\rangle \cdot \langle x'(t_0), y'(t_0), z'(t_0)\rangle = 0 \\
\iff &\mathbf{n} \cdot \mathbf{r}'(t_0) = 0 \\
\iff &\mathbf{n} \perp \mathbf{r}'(t_0).
\end{aligned}$$

$\square$

The same idea proves the following theorems as well.

Theorem 15. *Let $f(x, y)$ be differentiable at (a, b) and let $k = f(a, b)$. Then the gradient $\nabla f(a, b)$ is perpendicular to the (tangent line to the) level curve $f(x, y) = k$ at the point (a, b).*

Proof. Parametrize the level curve as $\mathbf{r}(t) = \langle x(t), y(t)\rangle$ so that $\mathbf{r}(t_0) = \langle a, r\rangle$. Then

$$f(x(t), y(t)) = k.$$

Differentiate with respect to t using the Chain Rule.

$$f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t) = 0.$$

At $t = t_0$,

$$\begin{aligned} & f_x(a, b)x'(t_0) + f_y(a, b)y'(0) = 0 \\ \iff & \nabla f(a, b) \cdot \mathbf{r}'(t_0) = 0 \\ \iff & \nabla f(a, b) \cdot \perp \text{ direction vector of level curve.} \end{aligned}$$

□

Example. Take $f(x, y) = x^2 + y^2$. Verify that $\nabla f(a, b)$ is perpendicular to the level curve of f that passes through (a, b).

The level curve passing through (a, b) is the circle $x^2 + y^2 = a^2 + b^2$ centered at the origin.

$$\nabla f(a, b) = \langle 2x, 2y\rangle|_{(a,b)} = \langle 2a, 2b\rangle.$$

$\nabla f(a, b)$ is parallel to the radius vector from the origin to the point (a, b). So it is perpendicular to the circle at (a, b).

Theorem 16. *Let $f(x, y, z)$ be differentiable at (a, b, c) and let $k = f(a, b, c)$. Then the gradient $\nabla f(a, b)$ is perpendicular to the (tangent plane to the) level surface $f(x, y, z) = k$ at the point (a, b).*

Example. Take $f(x, y, z) = x^2 + y^2 - 2z$. Verify that $\nabla f(a, b, c)$ is perpendicular to the level surface of f that passes through (a, b, c).

Let $k = a^2 + b^2 - 2c$. The level surface passing through (a, b, c) is

$$x^2 + y^2 - 2z = k \iff z = \frac{1}{2}(x^2 + y^2 - k) = g(x, y).$$

We have

$$g_x(a, b) = a, \quad g_y(a, b) = b.$$

By the above, a normal to the tangent plane is

$$\mathbf{n} = \langle -g_x(a, b), -g_y(a, b), 1\rangle = \langle -a, -b, 1\rangle.$$

Now

$$\nabla f(a, b) = \langle 2a, 2b, -2\rangle = -2\mathbf{n}.$$

Hence $\nabla f(a, b) \parallel \mathbf{n}$ and so $\nabla f(a, b)$ is perpendicular to the tangent plane.

Remark. The line passing through a point P and perpendicular to the tangent plane at P is called the *normal line*. I leave the derivation of the equations of the normal line to you.