4. Extreme values via differential calculus

- Critical points. Local extreme values. Second partials test.
- Absolute extrema.
- Extreme values with constraints. Lagrange multipliers.

Local max and local min.

DEFINITION A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.

Suppose that $f(x, y)$ has a local max/min at (a, b) and f is differentiable at (a, b).
Consider the function of one variable $g(x) = f(x, b)$.
Then g is differentiable at a and g has a local max/min at a.
From single variable calculus, we know that this means that $g'(a) = 0$.
Hence $f_x(a, b) = g'(a) = 0$. Similarly, $f_y(a, b) = 0$ as well.
So we have

Theorem 17. *Suppose that $f(x, y)$ is differentiable at (a, b). If f has a local max/min at (a, b), then $\nabla f(a, b) = \mathbf{0}$.*

As in the single variable case, we say that a point (a, b) is a *critical point* of $f(x, y)$ if $\nabla f(a, b) = \mathbf{0}$.
Thus, if f has a local max/min at (a, b) and f is differentiable at (a, b), then (a, b) is a critical point.

Example. Find and examine the critical point(s) of the functions

$$f(x,y) = x^2 + y^2 - 2x - 6y + 14 \text{ and } g(x,y) = y^2 - x^2.$$

$$f_x = 2x - 2,\ f_y = 2y - 6,\ g_x = -2x,\ g_y = 2y.$$

Critical point for f is $(1,3)$. Critical point for g is $(0,0)$.

Observe that $f(x,y) = (x-1)^2 + (y-3)^2 + 4$.
So it is clear that $(1,3)$ is a local min.

For any $a > 0$, $g(a,0) = a^2 > 0 = g(0,0)$ and $g(0,a) = -a^2 < 0 = g(0,0)$.
In any disk with center $(0,0)$, there are points where $g > g(0,0)$ and where $g < g(0,0)$.
So $(0,0)$ is neither a local max nor a local min for g.

A critical point of f that is neither a local max nor a local min is called a *saddle point*.

In the case of a function of 2 variables, we have a counterpart of the "second derivative test" (for local max/min) in the single variable case.

SECOND DERIVATIVES TEST Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

Remarks.

(1) Case (c) means that (a, b) is a saddle point.
(2) If $D = 0$, the second derivative test gives no information. The critical point (a, b) can be a local max, local min or a saddle point.
(3) One can remember D as the determinant

$$\det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Examples.

(1) Find and classify all critical points of the function

$$f(x, y) = x^4 + y^4 - 4xy + 1.$$

$$f_x = 4x^3 - 4y,\ f_y = 4y^3 - 4x.$$

Set both of these to 0 to find critical points.

$$x^3 = y \ \&\ y^3 = x \implies x^9 = x \implies x = 0, \pm 1.$$

So the critiacl points are: $(0, 0), (1, 1), (-1, -1)$.

$$f_{xx} = 12x^2,\ f_{xy} = -4,\ f_{yy} = 12y^2.$$

$D = 144x^2y^2 - 16$.

(a) At $(0, 0)$, $D = -16 < 0 \implies$ saddle point.

(b) At $(1, 1)$, $D = 144 - 16 > 0$, $f_{xx} = 12 > 0 \implies$ local min.

(c) At $(-1, -1)$, $D = 144 - 16 > 0$, $f_{xx} = 12 > 0 \implies$ local min.

(2) Find the shortest distance from the point $(1, 0, -2)$ to the plane $x + 2y + z = 4$.

If (x, y, z) is a point on the plane, its distance to $(1, 0, -2)$ is

$$\sqrt{(x-1)^2 + y^2 + (z+2)^2} = \sqrt{(x-1)^2 + y^2 + (6 - x - 2y)^2}.$$

So we want to minimize the function

$$f(x, y) = (x-1)^2 + y^2 + (6 - x - 2y)^2.$$

We have

$$f_x = 2(x-1) - 2(6 - x - 2y) = 4x + 4y - 14,$$

$$f_y = 2y - 4(6 - x - 2y) = 4x + 10y - 24.$$

Critical point: $x = \frac{11}{6}, y = \frac{5}{3}, z = \frac{-7}{6}$.

$$f_{xx} = 4, \quad f_{yy} = 10, \quad f_{xy} = 4.$$

So $D = 40 - 16 > 0$ and $f_{xx} > 0$. The critical point is a local min.

(3) A rectangular box without a lid has volume $2m^2$. Find the minimum amount of material used.

Let the sides of the base be x and y and the height be z. Then $xyz = 2 \iff z = \frac{2}{xy}$.

The amount of material used is

$$f(x, y) = xy + 2xz + 2yz = xy + \frac{4}{y} + \frac{4}{x}.$$

Then

$$f_x = y - \frac{4}{x^2}, \quad f_y = x - \frac{4}{y^2}.$$

Critical points:

$$y = \frac{4}{x^2} \text{ and } x = \frac{4}{y^2}$$

$$\implies y = \frac{y^4}{4}$$

$$\implies y^3 = 4 \text{ since } y \neq 0$$

$$\implies y = 4^{1/3}, \quad x = \frac{4}{y^2} = 4^{1/3}.$$

At the critical point $(4^{1/3}, 4^{1/3})$,

$$f_{xx} = \frac{8}{x^3} = 2, \quad f_{yy} = \frac{8}{y^3} = 2, \quad f_{xy} = 1.$$

Then $D = 4 - 1 > 0$ and $f_{xx} > 0$.

So f has a local min at $(4^{1/3}, 4^{1/3})$.

The minimum amount of material used is

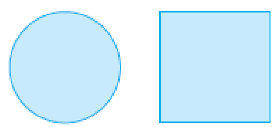
$$xy + \frac{4}{y} + \frac{4}{x} = 4^{1/3} \cdot 4^{1/3} + \frac{4}{4^{1/3}} + \frac{4}{4^{1/3}} = 3 \cdot 4^{2/3}.$$

Absolute max/min.

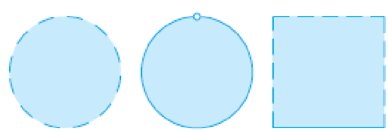
Suppose that $f(x,y) \leq f(a,b)$ for any (x,y) in the domain of f. Then f has an *absolute minimum* at (a,b). If $f(x,y) \geq f(a,b)$ for any (x,y) in the domain of f, then f has an *absolute maximum* at (a,b).

Example. The function $f(x,y) = x^2 + y^2$ has an absolute minimum (of value 0) at $(0,0)$. It has no absolute maximum.

A set D in $\mathbb{R}^2$ is *closed* if it includes all of its boundary points. D is *bounded* if it is contained in a disk of finite radius.



(a) Closed sets



(b) Sets that are not closed

Theorem 18. *Suppose that $f(x,y)$ is a continuous function whose domain D is a closed bounded set. Then f has both an absolute maximum and an absolute minimum on E.*

Note that f may have take its absolute max and/or absolute min at more than one point.

Example. Consider $f(x, y) = x^2 + y^2$ for $(x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1\}$.
f has an absolute minimum at $(0, 0)$.
f has an absolute maximum at any point (a, b) with $a^2 + b^2 = 1$.

To find the absolute maximum and minimum values of a continuous function f on a closed, bounded set D:

1. Find the values of f at the critical points of f in D.
2. Find the extreme values of f on the boundary of D.
3. The largest of the values from steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example. Find the absolute max/min and where they occur for the function

$$f(x,y) = x^2 - 2xy + 2y$$

on the rectangle $D = \{(x,y) : 0 \le x \le 3, 0 \le y \le 2\}$.

$f_x = 2x - 2y, \quad f_y = -2x + 2.$ Critical point $(1,1)$.

The 4 sides of the rectange D are

$$B : 0 \le x \le 3, y = 0, \quad T : 0 \le x \le 3, y = 2$$
$$L : x = 0, 0 \le y \le 2, \quad R : x = 3, 0 \le y \le 2.$$

So we need to find critical points for

$B : f(x) = x^2, 0 \le x \le 3, y = 0;$

critical point $(0,0)$; end point $(3,0)$,

$T : f(x) = x^2 - 4x + 4, 0 \le x \le 3, y = 2;$

critical point $x = 2$, $y = 2$; end points $(0,2)$, $(3,2)$,

$L : f(y) = 2y, 0 \le y \le 2, x = 0;$

no critical point; end points $(0,0)$, $(0,2)$,

$R : f(y) = 9 - 4y, 0 \le y \le 2, x = 3;$

no critical point; end points $(3,0)$, $(3,2)$.

Make a table

	(1,1)	(0,0)	(3,0)	(0,2)	(3,2)
f	1	0	9	4	1

Absolute max is 9, occuring at $(3, 0)$.
Absolute min is 0, occuring at $(0, 0)$

A proof of the second derivative for polynomials of degree 2.

Say

$$f(x,y) = ax^2 + bxy + cy^2 + dx + ey + g,$$

where a, b, c, d, e, g are constants. If (h,k) is a critical point, then

$$f_x(h,k) = 2ah + bk + d = 0, \quad f_y(h,k) = bh + 2ck + e = 0.$$

We can rewrite f as

$$f(x,y) = a[(x - h + \frac{b}{2a}(y-k)]^2 + \frac{4ac - b^2}{4a}(y-k)^2 + g' = 0,$$

where

$$g' = g - a(h + \frac{b}{2a}k)^2 - \frac{4ac - b^2}{4a}k^2.$$

Now

$$f_{xx} = 2a, \quad f_{yy} = 2c, \quad f_{xy} = b.$$

Hence $D = 4ac - b^2$.

If $D > 0$ and $f_{xx}(h,k) = 2a > 0$, then one can see from the equation above that

$$f(x,y) \geq g' = f(h,k) \text{ for any } (x,y).$$

Similarly, if $D > 0$ and $f_{xx}(h,k) = 2a < 0$, then one can see from the equation above that

$$f(x,y) \leq g' = f(h,k) \text{ for any } (x,y).$$

If $D < 0$,

$$f(h+\varepsilon, k) = a\varepsilon^2 + g'$$

$$f(h - \frac{b\varepsilon}{2a}, k + \varepsilon) = \frac{D}{4a}\varepsilon^2 + g'.$$

Since $a\varepsilon^2$ and $\frac{D}{4a}\varepsilon^2$ have different signs, one of these two values is $> g' = f(h,k)$ and the other is $< g' = f(h,k)$. This shows that (h,k) is a saddle point if $D < 0$.

Lagrange multipliers.
Consider this example which we have encountered before.

A rectangular box without a lid has volume $2m^2$. Find the minimum amount of material used.

It can be formulated as follows:

Minimize $A = xy + 2xz + 2yz$ subject to the condition $xyz = 2$.

The general form of such "constrained optimization" problem is

$$\text{Maximize/minimize } f(x, y, z) \text{ subject to } g(x, y, z) = k.$$

Recall that the equation $g(x, y, z) = k$ describes a level surface S of the function g.
Assume that the constrained optimization problem has a solution at $P(x_0, y_0, z_0)$.
Let $\mathbf{r}(t) = \langle u(t), v(t), w(t)\rangle$ be a curve on the surface S so that $\mathbf{r}(t_0)$ is the point P.
Then the function $h(t) = f(\mathbf{r}(t)) = f(u(t), v(t), w(t))$ has a max/min at $t = t_0$. Thus $h'(t_0) = 0$.
By the Chain Rule, this gives

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)u'(t_0) + f_y(x_0, y_0, z_0)v'(t_0) + f_z(x_0, y_0, z_0)w'(t_0) \\ &= \nabla f(P) \cdot \mathbf{r}'(t_0). \end{aligned}$$

This means that $\nabla f(P)$ is perpendicular to the tangent of any curve on S that passes through P.
So $\nabla f(P)$ must be perpendicular to the tangent plane to S at P.

Equivalently, $\nabla f(P)$ is a normal vector to the tangent plane to S at P.
Since S is a level surface of g, we know from before that $\nabla g(P)$ is a normal to the tangent plane to S at P.
Therefore, $\nabla f(P)$ and $\nabla g(P)$ are parallel, i.e., there is a constant λ so that

$$\nabla f(P) = \lambda \nabla g(P).$$

To summarize:

METHOD OF LAGRANGE MULTIPLIERS To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Example. Minimize $A = xy + 2xz + 2yz$ subject to the condition $xyz = 2$.

Let $g(x, y, z) = xyz$.

$\nabla A = \langle y+2z, x+2z, 2x+2z\rangle, \quad \nabla g(x, y, z) = \langle yz, xz, xy\rangle.$

We need to solve

$$\begin{cases} \langle y + 2z, x + 2z, 2x + 2y\rangle = \lambda\langle yz, xz, xy\rangle \\ xyz = 2 \end{cases}.$$

Or

$$\langle y + \frac{4}{xy}, x + \frac{4}{xy}, 2x + 2y\rangle = \lambda\langle \frac{2}{x}, \frac{2}{y}, xy\rangle.$$

Multiply the first component by x and the second component by y:

$$xy + \frac{4}{y} = 2\lambda = xy + \frac{4}{x}.$$

Thus $x = y$. Then

$$x + \frac{4}{x^2} = \frac{2\lambda}{x} \text{ and } 4x = \lambda x^2.$$

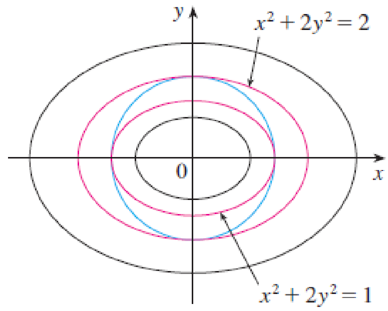
Hence $\lambda = \frac{4}{x}$ and it follows that

$$x + \frac{4}{x^2} = \frac{8}{x^2} \implies x = \frac{4}{x^2} \implies x = 4^{1/3}.$$

Then

$$y = x = 4^{1/3} \text{ and } z = \frac{2}{xy} = 4^{1/3}.$$

Example. Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.



First we find the critical point(s) in the inside of the circle.

$$f_x = 2x, \;\; f_y = 4y \implies \text{Critical point: } (0, 0).$$

On the boundary $x^2 + y^2 = 1$, we will solve the following problem by Lagrange multiplier:
Maximize/minize $f(x, y) = x^2 + 2y^2$ subject to $g(x, y) = x^2 + y^2 = 1$.

$$\nabla f = \lambda \nabla g \implies \langle 2x, 4y \rangle = \lambda \langle 2x, 2y \rangle.$$

From the first component, $2x = \lambda 2x \implies \lambda = 1$ or $x = 0$.
Case 1. $\lambda = 1$.
Then $4y = \lambda 2y = 2y \implies y = 0$.
Since $x^2 + y^2 = 1$, $x = \pm 1$. So we find two critical points $(\pm 1, 0)$ if $\lambda = 1$.
Case 2. $x = 0$.
Then $x^2 + y^2 = 1 \implies y = \pm 1$. So we find two critical points $(0, \pm 1)$ if $\lambda = 1$.

Now we check the critical points:

	(0,0)	(1,0)	(-1,0)	(0,1)	(0,-1)
f	0	1	1	2	2

So f has absolute max value of 2 at $(0, \pm 1)$ and absolute min value of 0 at $(0, 0)$.

Two constraints.

Suppose that the constrained optimization problem

$$\text{Max/min } f(x,y,z) \text{ subject to } \begin{cases} g(x,y,z) = k_1 \\ h(z,y,z) = k_2 \end{cases}$$

has a solution at $P(x_0, y_0, z_0)$.
Then there are constants λ and μ so that

$$\nabla f(P) = \lambda \nabla g(P) + \mu \nabla h(P).$$

Example. Maximize $f(x,y,z) = x + 2y + 3z$ on the intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Let $g(x,y,z) = x - y + z$ and $h(x,y,z) = x^2 + y^2$.

$$\nabla f = \langle 1,2,3\rangle, \quad \nabla g = \langle 1,-1,1\rangle, \quad \nabla h = \langle 2x, 2y, 0\rangle.$$

We need to solve the equation

$$\langle 1,2,3\rangle = \lambda\langle 1,-1,1\rangle + \mu\langle 2x,2y,0\rangle.$$

From the 3rd component, $\lambda = 3$. Then

$$\langle -2,5,0\rangle = \mu\langle 2x,2y,0\rangle \implies \mu x = -1 \text{ and } \mu y = \frac{5}{2}.$$

So

$$(-1)^2 + (\frac{5}{2})^2 = \mu^2(x^2+y^2) = \mu^2 \implies \mu = \pm\frac{\sqrt{29}}{2}.$$

Thus we find

$$(x,y) = \pm\frac{1}{\sqrt{29}}(-2,5).$$

The corresponding values of z can be obtained from $x - y + z = 1$.
To summarize, we find two critical points:
$(\pm\frac{-2}{\sqrt{29}}, \pm\frac{5}{\sqrt{29}}, 1 \pm \frac{7}{\sqrt{29}})$.
The values of f at these points are $3 \pm \sqrt{29}$.

Clearly, f has a max of $3+\sqrt{29}$ at $(\frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1+\frac{7}{\sqrt{29}})$ and a min of
$3-\sqrt{29}$ at $(\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1-\frac{7}{\sqrt{29}})$.

5. Multiple integrals

- Double integrals. Definition.
- Evaluation by repeated integration.
- Double integrals in polar coordinates.
- Triple integrals. Cylindrical and spherical coordinates.
- Change of variables in multiple integrals.

Double integrals.

Let

$$R = [a, b] \times [c, d] = \{(x, y) : a \leq x \leq b, c \leq y \leq d\}$$

be a rectangle and let $f : R \to \mathbb{R}$ be a function of 2 variables defined on R.
By a *partition* of R we mean a division of the rectangle into a grid

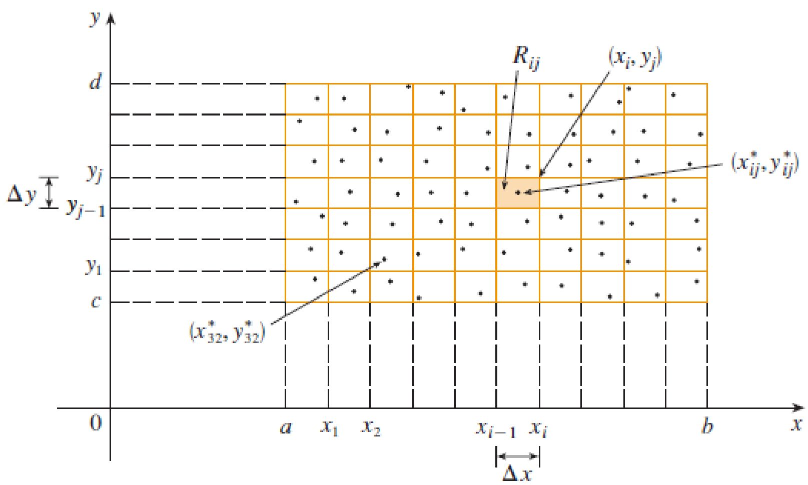
$$P = \{(x_i, y_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$$

where

$$a = x_0 < x_1 < \cdots < x_m = b, \quad c = y_0 < y_1 < \cdots < y_n = d.$$

The *mesh* of P is the number

$$\max\{x_i - x_{i-1}, y_j - y_{j-1} : 1 \leq i \leq m, 1 \leq j \leq n\}.$$



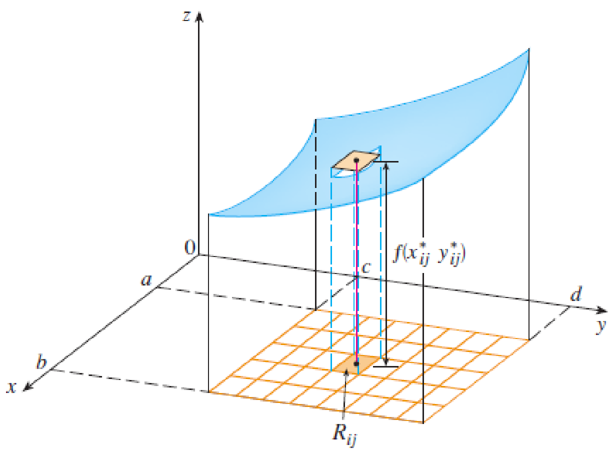
In each of the rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, choose a point (x_{ij}^*, y_{ij}^*) in R_{ij}.
Let $\Delta A_{ij} =$ area of $R_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$. The sum

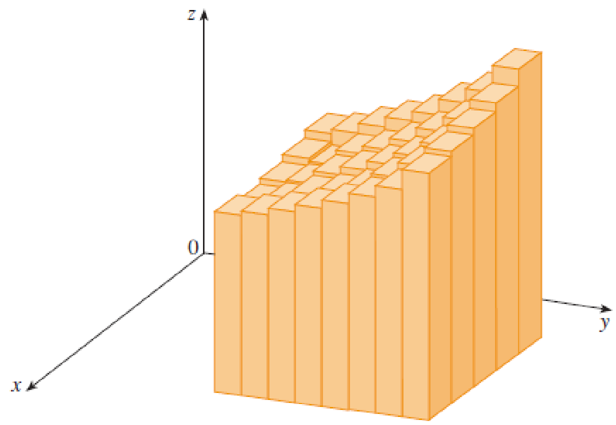
$$\sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A_{ij}$$

is called a *Riemann sum* of f on the rectangle R.

The *double integral of f over R*, if it exists, is the limit

$$\iint_R f(x,y)\,dA = \lim_{|P|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}.$$





Example. Consider the double integral $\iint_R xy\, dA$, where $R = [0,1] \times [0,2]$.
Divide R into a grid by subdividing $[0,1]$ into m equal pieces and $[0,2]$ into n equal pieces.
For each R_{ij}, choose (x_{ij}^*, y_{ij}^*) to be the center of R_{ij}.
Compute the Riemann sum with respect to this partition and choice of (x_{ij}^*, y_{ij}^*).
Find the limit of the Riemann sum as $m, n \to \infty$.

The (i,j) subrectangle is $[\frac{i-1}{m}, \frac{i}{m}] \times [\frac{2(j-1)}{n}, \frac{2j}{n}]$.
Then center of it is $(x_{ij}^*, y_{ij}^*) = (\frac{2i-1}{2m}, \frac{2j-1}{n})$.

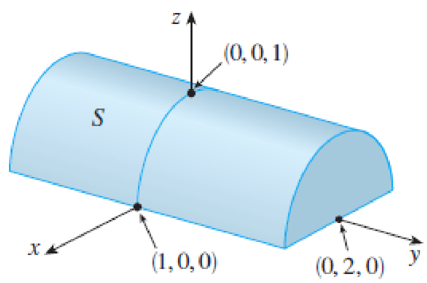
$$\begin{aligned}
&\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij} \\
&\quad = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{(2i-1)(2j-1)}{2mn} \cdot \frac{1}{m} \cdot \frac{2}{n} \\
&\quad = \frac{1}{m^2n^2}(2\sum_{i=1}^{m} i - \sum_{i=1}^{m} 1)(2\sum_{j=1}^{n} j - \sum_{j=1}^{n} 1) \\
&\quad = \frac{1}{m^2n^2}[(m(m-1) - m]\,[n(n-1) - n] \\
&\quad = (1 - \frac{2}{m})(1 - \frac{2}{n}).
\end{aligned}$$

Take limit as $m \to \infty$, $n \to \infty$, we get 1.

If $f(x, y) \geq 0$ for all (x, y) in R, then a Riemann sum represents an approximation to the volume of the solid lying between the xy-plane and the graph of f on the rectangle R.
As we take the limit $|P| \to 0$, the double integral $\iint_R f(x, y)\, dA$, if it exists, gives the volume of the solid.

If $f(x, y) = c$ is constant, then $\iint_R c\, dA =$ volume of solid of height c on rectangle $R = c$ Area of R.
In particular, $\iint_R 1\, dA =$ Area of R.

Example. Find the double integral $\iint_R \sqrt{1-x^2}\,dA$ by interpreting it as a volume.



The double integral has similar properties to one variable integrals.

Theorem 19. *Let R be a rectangle and suppose that $\iint_R f(x,y)\,dA$, $\iint_R g(x,y)\,dA$ both exist. Then*

(1) $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA.$

(2) *If c is a real number, then*
$$\iint_R cf(x,y)\,dA = c\iint_R g(x,y)\,dA.$$

(3) *If $f(x,y) \geq g(x,y)$ for all $(x,y) \in R$, then*
$$\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA.$$

Theorem 20. *Suppose that f is a continuous function on a rectangle R. Then $\iint_R f(x,y)\,dA$ exists.*

Iterated integrals.

Suppose that $f(x, y)$ is a function defined on a rectangle $R = [a, b] \times [c, d]$. Treating y (within $[c, d]$) as a constant, we may compute the 1 dimensional integral

$$\int_a^b f(x, y)\, dx.$$

The result will be a function $g(y)$ that depends on y, which we may integrate on the interval $[c, d]$:

$$\int_c^d g(y)\, dy = \int_c^d \Big[\int_a^b f(x, y)\, dx\Big]\, dy.$$

This is called an *iterated integral*.

We can also compute the iterated integral in the other order:

$$\int_a^b \Big[\int_d^c f(x, y)\, dy\Big]\, dx.$$

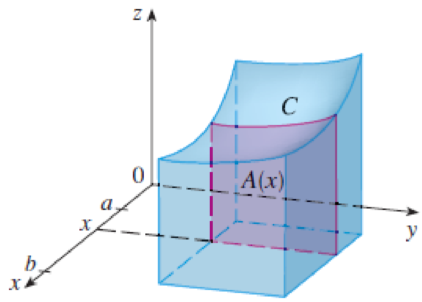
Example. Evaluate the iterated integrals

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx \quad \text{and} \quad \int_1^2 \int_0^3 x^2 y\, dx\, dy.$$

$$\int_0^3 \int_1^2 x^2 y\, dy\, dx = \int_0^3 x^2 \frac{y^2}{2}\Big|_{y=1}^{y=2} dx = \int_0^3 \frac{3x^2}{2}\, dy = \frac{27}{2}.$$

$$\int_1^2 \int_0^3 x^2 y\, dx\, dy = \int_1^2 \frac{x^3}{3}\Big|_{x=0}^{x=3} y\, dy = \int_1^2 9y\, dy = \frac{27}{2}.$$

Here is a visual interpretation of the iterated integral $\int_a^b \int_c^d f(x,y)\,dy\,dx$.



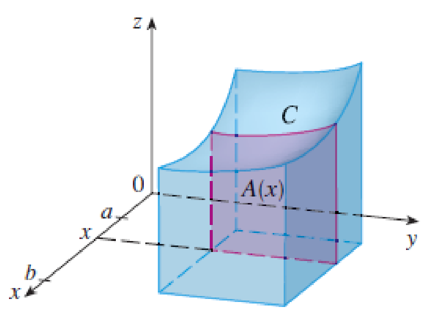
$$A(x) = \int_c^d f(x,y)\,dy.$$

The following important theorem tells us that double integrals can be computed as iterated integrals in either order.

Theorem 21. *Let $f(x,y)$ be a continuous function on the rectangle R. Then*

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dx\,dy.$$

Pictorially, this means that the volume of the solid shown below can be computed as $\int_a^b A(x)\,dx$, where $A(x)$ is the area of the cross section at x.
Similarly in the y direction.



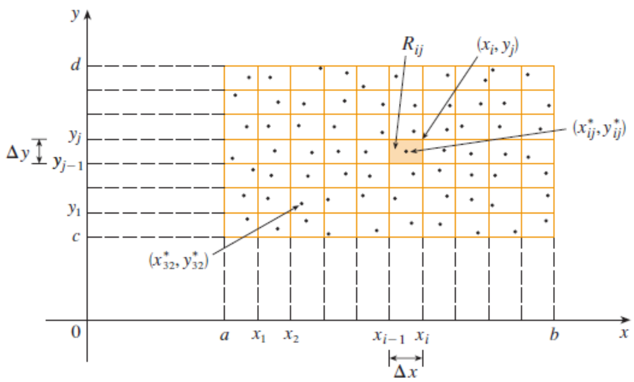
Example. Find the volume of the solid bounded on top by the elliptic paraboloid $x^2 + 2y^2 + z = 16$, on the bottom by the xy-plane and on the sides by the planes $x = 0$, $x = 2$ and $y = 0, y = 2$.

The picture is as follows.

Note that inside the square $0 \leq x \leq 2, 0 \leq y \leq 2$, z is always ≥ 0, which means above the xy-plane.
The volume is

$$\begin{aligned} V &= \int_0^2 \int_0^2 16 - x^2 - 2y^2 \, dx \, dy \\ &= \int_0^2 [16x - \frac{x^3}{3} - 2y^2 x]_{x=0}^{x=2} \, dy \\ &= \int_0^2 32 - \frac{8}{3} - 4y^2 \, dy = 48. \end{aligned}$$

Sketch of a proof of Theorem 21.



Suppose that in the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij},$$

we always choose (x_{ij}^*, y_{ij}^*) to be the center of the rectangle R_{ij}.

Then x_{ij}^* does not depend on j and y_{ij}^* does not depend on i.

Express ΔA_{ij} as $\Delta x_i \cdot \Delta y_j$. Then the Riemann sum is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i^*, y_j^*)\Delta x_i \cdot \Delta y_j = \sum_{j=1}^{n}[\sum_{i=1}^{m} f(x_i^*, y_j^*)\Delta x_i]\Delta y_j.$$

Now

$$\iint_R f(x,y)\,dA = \lim_{\Delta x_i \to 0, \Delta y_j \to 0} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A_{ij}$$

$$= \lim_{\Delta y_j \to 0} \sum_{j=1}^{n}\Big\{\lim_{\Delta x_i \to 0}[\sum_{i=1}^{m} f(x_i^*, y_j^*)\Delta x_i\Big\}\Delta y_j.$$

The inside sum is a Riemann sum for the integral $\int_a^b f(x, y_j^*)\,dx$. When we take limit $\Delta x_i \to 0$, the inside limit becomes $\int_a^b f(x, y_j^*)\,dx$.

Thus

$$\iint_R f(x,y)\,dA = \lim_{\Delta y_j \to 0} [\sum_{j=1}^{n} \int_a^b f(x, y_j^*)\,dx]\,\Delta y_j.$$

Now this sum is a Riemann sum for the integral

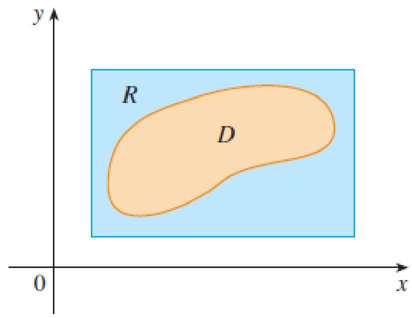
$$\int_c^d [\int_a^b f(x,y)\,dx]\,dy.$$

Taking the limit, we obtain

$$\iint_R f(x,y)\,dA = \int_c^d [\int_a^b f(x,y)\,dx]\,dy.$$

Double integrals over general regions.

Suppose that D is a set contained in a rectangle R and $f(x,y)$ is defined on D.



Define a new function F on R by

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is in } R \text{ but not } D. \end{cases}$$

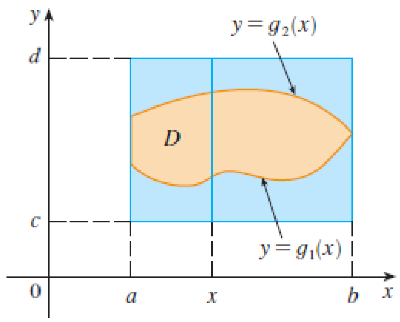
Then define

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA.$$

Type I region

A region D is a *Type I region* if it is the region between two continuous curves $y = g_1(x)$ and $y = g_2(x)$ for x between a and b.

$$D = \{(x, y) : g_1(x) \leq y \leq g_2(x), a \leq x \leq b\}.$$



For a Type I region D, we have

$$\begin{aligned}\iint_D f(x, y)\, dA &= \iint_R F(x, y)\, dA \\ &= \int_a^b \int_c^d F(x, y)\, dy, dx \\ &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.\end{aligned}$$

Example. Evaluate the double integral $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

First sketch the region D.

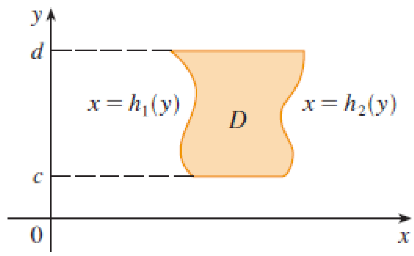
The curves intersect at $2x^2 = 1 + x^2 \implies x = \pm 1$.
Points of intersection: $(1, 2), (-1, 2)$.
The curve $y = 1 + x^2$ lies above $y = 2x^2$ when $-1 \le x \le 1$.

$$
\begin{aligned}
\iint_D (x+2y)\,dA &= \int_0^1 \int_{2x^2}^{1+x^2} x + 2y\,dy\,dx \\
&= \int_0^1 xy + y^2\Big|_{y=2x^2}^{y=1+x^2}\,dx \\
&= \frac{43}{60}.
\end{aligned}
$$

Type II region

A *Type II region* is a region of the form

$$D = \{(x, y) : h_1(y) \le x \le h_2(y), c \le y \le d\}.$$



For a Type II region,

$$\iint_D f(x, y)\, dA = \int_c^d \int_{g_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Example. Let D be the region bounded by the parabola $y^2 = 2x + 6$ and the straight line $x = y + 1$. Evaluate the integral $\iint_D xy\, dA$ both as a type I region and as a type II region.

Intersection: $2x + 6 = y^2 = (x - 1)^2$. Points of intersection: $(5, 4), (-1, -2)$.
The curve $x = 1 - y$ is to the right of $x = \frac{1}{2}(y^2 - 6)$ if $-2 \le y \le 4$. So

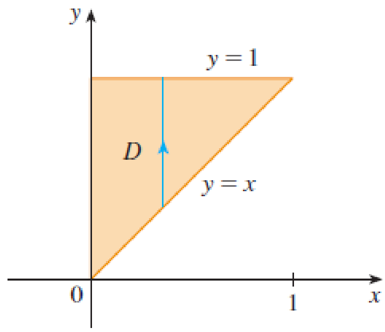
$$\iint_D xy\, dA = \int_{-4}^{2} \int_{\frac{1}{2}(y^2-6)}^{1-y} xy\, dx\, dy = -36.$$

To integrate in the other order, note that:
If $-3 \le x \le -1$, the curve on top is $y = \sqrt{2x + 6}$, the curve at the bottom is $y = -\sqrt{2x + 6}$.
If $-1 \le x \le 5$, the curve on top is $y = 1 - x$ and the one at the bottoom is $y = -\sqrt{2x + 6}$. So

$$\iint_D xy\, dA = \int_{-3}^{-1} \int_{-\sqrt{2x+6}}^{\sqrt{2x+6}} xy\, dy\, dx + \int_{-1}^{5} \int_{-\sqrt{2x+6}}^{1-x} xy\, dy\, dx = 0 + (-36) = -36.$$

Changing the order of integration

Example. Evaluate the iterated integral $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.



Use Theorem 21 to change the order of integration.

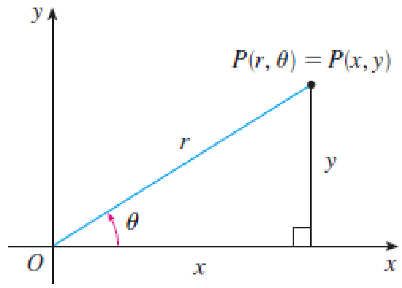
$$\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx = \iint_D \sin(y^2)\, dA = \int_0^1 \int_0^y \sin(y^2)\, dx\, dy.$$

The last integral is easy to compute.

$$\int_0^1 \int_0^y \sin(y^2)\, dx\, dy = \int_0^1 y \sin(y^2)\, dy = -\frac{1}{2} \cos(y^2)\Big|_0^1 = \frac{1 - \cos 1}{2}.$$

Polar coordinates.

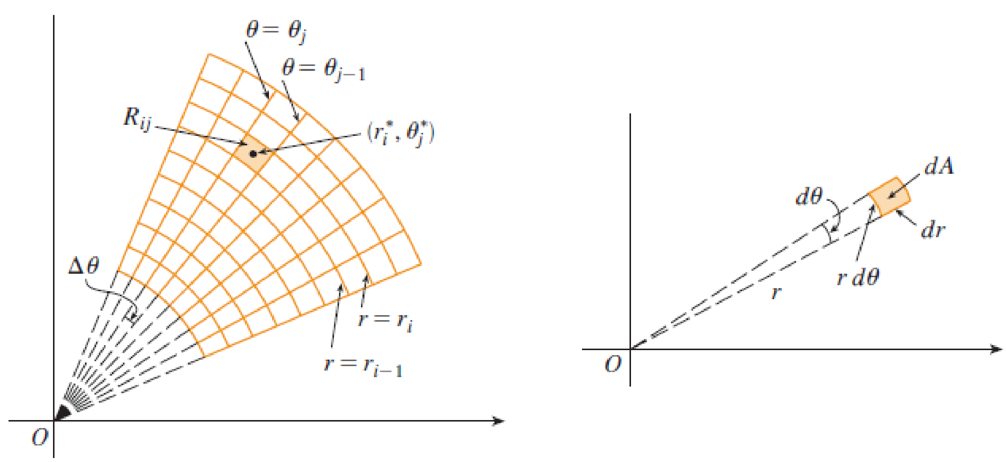
Recall that the *polar coordinates* (r, θ) of a point are given as in the following figure.



The polar coordinates and rectangular coordinates are related by

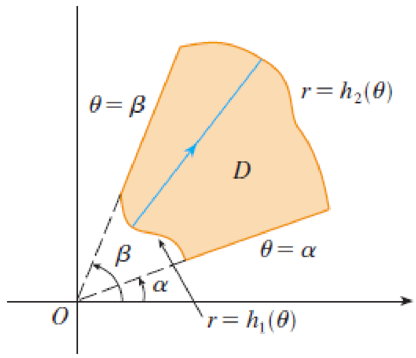
$$x = r\cos\theta, \quad y = r\sin\theta$$
$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}\frac{y}{x}.$$

To perform double integration in polar coordinates, we divided a region into a fan shaped grid instead of a rectangular grid.
The area element dA is a approximately $r\,d\theta\,dr = r\,dr\,d\theta$.



Suppose that a region D is bounded between curves $r = h_1(\theta)$ and $r = h_2(\theta)$ between the angles $\theta = \alpha$ and $\theta = \beta$, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$



Example. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$.

$$V = \iint_D z\,dA$$

where D is the region inside the circle

$$x^2 + y^2 = 2x \iff r^2 = 2r\cos\theta \iff r = 2\cos\theta.$$

$$V = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} (r^2) r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\,d\theta = \frac{3\pi}{2}.$$

Example. Use a double integral to find the area enclosed by one loop of the curve $r = \cos 2\theta$.

$\cos(2\theta) = 0 \implies 2\theta = \pm\frac{\pi}{2} + 2n\pi \implies \theta = \pm\frac{\pi}{4} + n\pi$, $n \in \mathbb{N}$.

$$A = \int_{-\pi/4}^{\pi/4} 1 \cdot r\, dr\, d\theta = \frac{\pi^2}{16}.$$

This is an important fact in probability theory:

Example. Show that the improper integral $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$.

Let $I = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx$. Then

$$
\begin{aligned}
I^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}}\,dy\right) \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2+y^2)}\,dx\,dy \\
&= \int_D e^{\frac{-1}{2}(x^2+y^2)}\,dA, \text{ where } D \text{ is the entire plane } \mathbb{R}^2.
\end{aligned}
$$

Evaluate the double integral in polar coordinates.

$$
\begin{aligned}
\int_D e^{\frac{-1}{2}(x^2+y^2)}\,dA &= \int_0^{2\pi}\int_0^{\infty} e^{\frac{-r^2}{2}} r\,dr\,d\theta \\
&= \int_0^{2\pi} \lim_{b\to\infty} \int_0^{b} e^{\frac{-r^2}{2}} r\,dr\,d\theta = \int_0^{2\pi} \lim_{b\to\infty} -e^{\frac{-r^2}{2}}\Big|_0^b\,d\theta \\
&= \int_0^{2\pi} 1\,d\theta = 2\pi.
\end{aligned}
$$

Hence $I = \sqrt{2\pi}$.

Remark. Consider the function $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.
Clearly $f(x) \geq 0$ for any x.
The fact above shows that $\int_{-\infty}^{\infty} f(x)\,dx$.
Hence f is a *probability density function*.
By definition, f is the probability density function of the *normal distribution* (of mean 0 and variance 1). [The bell curve.]
If a random event X is normally distributed with mean 0 and variance 1, then the probability that $(a \leq X \leq b)$ for some numbers a and b is given by $\int_a^b f(x)\,dx$.

Triple integrals.

Triple integrals can be defined and computed in much the same way as double integrals.
For example, if $B = [a, b] \times [c, d] \times [r, s]$ is a 3-dimensional box and f is a function on B, we can subdivide B into small rectangular boxes and form Riemann sums corresponding to each partition of B. The limit of the Riemann sums as the mesh of the partitions $\to 0$ is the triple integral

$$\iiint_B f(x, y, z)\, dV.$$

If E is a 3-dimensional region contained within a box B, and f is a function defined on B, then extend f to B by defining

$$F(x, y, z) = \begin{cases} f(x, y, z) & \text{if } (x, y, z) \text{ is in } E \\ 0 & \text{if } (x, y, z) \text{ is in } B \text{ but not } E. \end{cases}$$

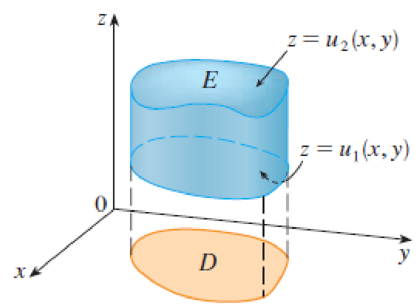
Then set

$$\iiint_E f(x, y, z)\, dV = \iiint_B F(x, y, z)\, dV.$$

Note that the integral $\iiint_E 1\, dV$ gives the volume of the region E.

Triple integrals via iterated integration.

As in the case for double integrals, triple integrals can be computed via iterated intergration.
Specifically, suppose that E is bounded by the surfaces $z = u_1(x,y)$ below and $z = u_2(x,y)$ above, and the projection (shadow) of E onto the xy-plane is the planar region D. We call this a *Type I region.* Then



$$\iiint_E f(x,y,z)\,dV = \iint_D \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\Big]\,dA.$$

The outer double integral $\iint_D \cdots\, dA$ is a double integral and so can be further reduced to an iterated integral using one of the methods in the last section.

Example. Evaluate the integral $\iiint_E xyz\,dV$, where E is the solid bounded above by the surface $z = \sqrt{y}$ and above the triangle in the xy-plane with vertices $(1,0), (2,1)$ and $(4,0)$.

The three boundary lines of E have equations $y = 0$, $y = x - 1$ and $y = 2 - \frac{x}{2}$.

$$\iiint_E xyz\,dV = \iint_D \Big[\int_0^{\sqrt{y}} xyz\,dz\Big]\,dA,$$

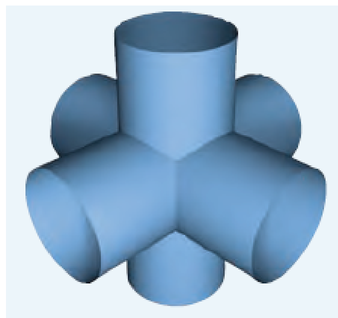
where D is the triangle with vertices $(1,0), (2,1)$ and $(4,0)$. Then

$$\iiint_E xyz\,dV = \int_0^1 \int_{1+y}^{4-2y} \Big[\int_0^{\sqrt{y}} xyz\,dz\Big]\,dx\,dy = \frac{11}{40}.$$

Example. Find the volume of the intersection of the cylinders

$$x^2 + y^2 \le 1, \quad x^2 + z^2 \le 1, \quad y^2 + z^2 \le 1$$

using triple integration.



Note that the restrictions on z are $z^2 \le 1 - x^2$ and $z^2 \le 1 - y^2$.

$$1 - x^2 = 1 - y^2 \iff y = \pm x$$

are a pair of lines crossing at the origin, which divides the circle $x + 2 + y^2 \le 1$ into 4 parts.
On the top and bottom parts, $1 - x^2 \ge 1 - y^2$. On the left and right parts, $1 - y^2 \ge 1 - x^2$.
So the volume is

$$\iiint_E 1\, dV = \iint_T \Big[\int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1\, dz\Big]\, dA$$
$$+ \iint_B \Big[\int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1\, dz\Big]\, dA + \iint_L \Big[\int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\Big]\, dA$$
$$+ \iint_R \Big[\int_{\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\Big]\, dA$$

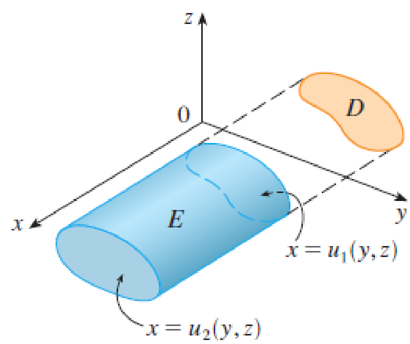
The double integrals are best performed in polar coordinates. For example,

$$\begin{aligned}\iint_R \Big[\int_{\sqrt{1-y^2}}^{\sqrt{1-y^2}} 1\,dz\Big]\,dA &= 2\iint_T \sqrt{1-y^2}\,dA\\ &= 2\int_{-\pi/4}^{\pi/4}\int_0^1 \sqrt{1-r^2\sin^2\theta}\,r\,dr\,d\theta\\ &= 2\int_{-\pi/4}^{\pi/4} \frac{\cos^3\theta-1}{3\sin^2\theta}\,d\theta\\ &= \frac{4-6\sqrt{2}}{3}.\end{aligned}$$

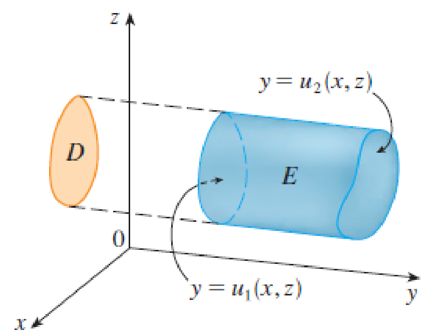
By symmetry, the other 3 integrals have the same value. So

$$V = \frac{8}{3}(2-3\sqrt{2}).$$

We can define Type II and Type III regions similarly, as suggested by the figure below



A type 2 region



A type 3 region

Similar reductions to iterated integrals apply.

Example. Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$, where E is the region bounded by the paraboloid $y = x^2+z^2$ and the plane $y = 4$.

Integrate on the variable y first.

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D \Big[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\Big]\,dA,$$

where D is the circle $x^2+z^2 \leq 4$. Thus

$$\iiint_E \sqrt{x^2+z^2}\,dV = \iint_D (4-(x^2+z^2))\sqrt{x^2+z^2}\,dy\big]\,dA.$$

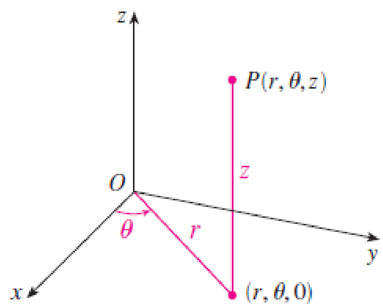
Using polar coordinates in the xz-plane, the integral becomes

$$\int_0^{2\pi}\int_0^2 (4-r^2)r\cdot r\,dr\,d\theta = \frac{384\pi}{5}.$$

As in the case for double integrals, judicious choice in changing the order of integration can help to evaluate a triple integral.

Cylindrical coordinates.

The *cylindrical coordinates* of a point P in $\mathbb{R}^3$ are the triple (r, θ, z) given in the following figure.

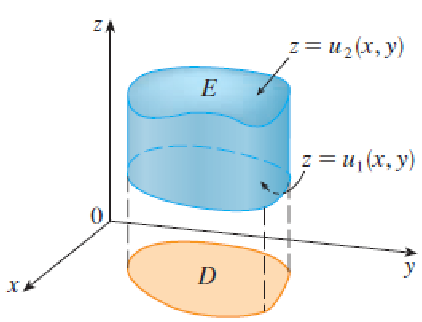


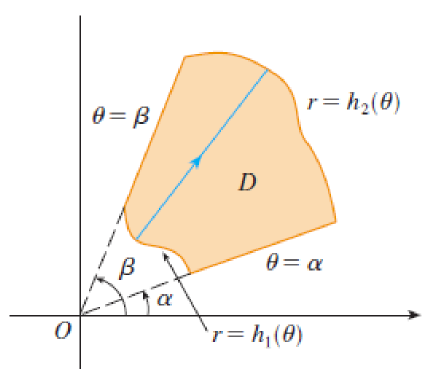
Clearly, this just amounts to the usual rectangular z coordinate in the vertical direction and the polar coordinates (r, θ) in the xy-plane.
Triple integral in cylindrical coordinates just means doing the z integral first followed by performing the remaining double integral in polar coordinates.

$$\iiint_E f(x, y, z)\, dV = \iint_D \Big[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\Big]\, dA$$

$$= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \Big[\int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\, dz\Big]\, r\, dr\, d\theta$$

provided E and its projection D onto the xy-plane are as shown

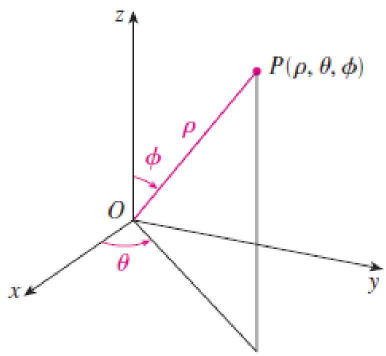




Spherical coordinates.

The *spherical coordinates* of a point P in $\mathbb{R}^3$ are the triple (ρ, θ, ϕ) given in the following figure.

$$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$$



We can see from the figure that if (x, y, z) are the rectangular coordinates of P, then

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi.$$

Exercise: Show that

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \theta = \tan^{-1}\frac{y}{x}, \quad \phi = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}}.$$

To gain familiarity with the spherical coordinate system, graph the "coordinate surfaces"

$$\rho = c, \quad \theta = c, \text{ and } \phi = c$$

for appropriate constants c.

Triple integrals in spherical coordinates.

The figure

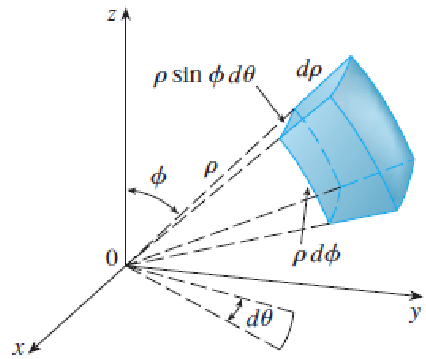


FIGURE 8
Volume element in spherical coordinates: $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

shows that the volume element dV is spherical coordinates is $\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.
So if E is a 3 dimensional region described in spherical coordinates by

$$E = \{(\rho, \theta, \phi) : \alpha \leq \theta \leq \beta, c \leq \phi \leq d, g_1(\theta, \phi) \leq \rho \leq g_2(\theta, \phi)\},$$

then

$$\iiint_E f(x, y, z) \, dV = \int_c^d \int_a^b \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi,$$

where f on the right has to be converted from rectangular to spherical coordinates.

Example. Find the volume of the "ice cream cone": the region above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2+y^2+z^2 = z$ by triple integration performed in spherical coordinates.

In spherical coordinates, the equations of the cone and the sphere are

$$\rho\cos\phi = \sqrt{\rho^2 \sin^2\phi} = \rho\sin\phi,\ \ \rho^2 = \rho\cos\phi.$$

When the two surfaces intersect,

$$\rho\sin\phi = \rho\cos\phi \implies \rho = 0 \text{ or } \phi = \pi/4.$$

For the triple integral $\iiint_E 1\, dV$,
ρ goes from 0 to the surface of the sphere: $\rho = \cos\phi$.
θ goes from 0 to 2π.
ϕ goes from 0 to $\pi/$.
Hence the volume is

$$\iiint_E 1\, dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{3\pi}{8}.$$

Example. Change the integral to spherical coordinates and evaluate it.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx.$$

In the integral, $\sqrt{x^2+y^2} \le z \le \sqrt{2-x^2-y^2}$.

$\sqrt{x^2+y^2} = z \implies \rho^2 \sin^2\phi = \rho^2\cos^2\phi \implies \rho = 0$ or $\phi = \dfrac{\pi}{4}$.

$z = \sqrt{2-x^2-y^2} \implies x^2+y^2+z^2 = 2 \implies \rho = \sqrt{2}$.

Similarly,

$$\begin{cases} 0 \le y \le \sqrt{1-x^2} \\ 0 \le x \le 1 \end{cases} \implies \begin{cases} 0 \le r \le 1 \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}.$$

Therefore,

$$\begin{aligned} &\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx \\ &\quad = \int_0^{\pi/4}\int_0^{\pi/2}\int_0^1 (\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi \\ &\quad = \frac{2}{3} - \frac{5}{6\sqrt{2}}. \end{aligned}$$

Change of variables in multiple integration.

Consider a double integral $\iint_D f(x,y)\,dA$. Let's say we make a "change of variables" or "substitution"

$$x = x(u,v), \quad y = y(u,v).$$

We want to find out how to compute the integral $\iint_D f(x,y)\,dA$ in terms of the variables (u,v).

Example. Let R be the square in the xy-plane with vertices $(1,0),(0,1),(-1,0)$ and $(0,-1)$. Consider the substitution

$$x = u+v, \quad y = u-v.$$

How to write

$$\iint_R f(x,y)\,dA = \iint_S \cdots\,du\,dv?$$

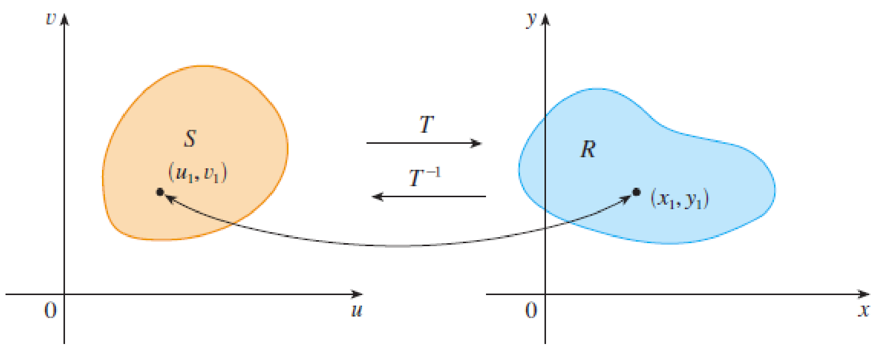
First, we need to find the region S in the (u,v)-plane corresponding to R in the (x,y)-plane. Then we need to determine the expression (in terms of u and v) in place of $\cdots$.

Change of variables as transformations.

In general, a pair of equations

$$x = x(u,v), \quad y = y(u,v)$$

can be viewed as a transformation T from the (u,v)-plane to the (x,y)-plane: $T(u,v) = (x(u,v), y(u,v))$.



We will require that T be a 1-to-1 transformation, so that there is an inverse transformation T^{-1} and S in the (u,v)-plane and R in the (x,y)-plane are related by

$$R = T(S) \iff T^{-1}(R) = S.$$

Note that the inverse T^{-1} can be expressed as a pair of functions

$$u = u(x,y), \quad v = v(x,y).$$

The second requirement is that the functions $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}, \frac{\partial y}{\partial u}, \frac{\partial y}{\partial v}$ exist and are continuous.
We say that T is a C^1 function.
Similarly, we also assume that T^{-1} is a C^1 function, i.e., $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous.

In this case, thinking of x as a function of (u, v) and each of u, v as a function of (x, y), we have by the Chain rule

$$\begin{aligned}
1 &= \frac{\partial x}{\partial x} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial x} \\
0 &= \frac{\partial x}{\partial y} = \frac{\partial x}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial x}{\partial v}\frac{\partial v}{\partial y} \\
0 &= \frac{\partial y}{\partial x} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial x} \\
1 &= \frac{\partial y}{\partial y} = \frac{\partial y}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial y}.
\end{aligned}$$

We can summarize the four equations as a matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}.$$

The matrix

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

is called the *Jacobian matrix* of the transformation T. From the above, we see that it is invertible and hence

$$\frac{\partial(x, y)}{\partial(u, v)} \stackrel{\text{def}}{=} \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \neq 0.$$

$\frac{\partial(x,y)}{\partial(u,v)}$ is called the *Jacobian* of the transformation T.
Note that from the above, the Jacobian matrix of the transformation T^{-1} is

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}^{-1}$$

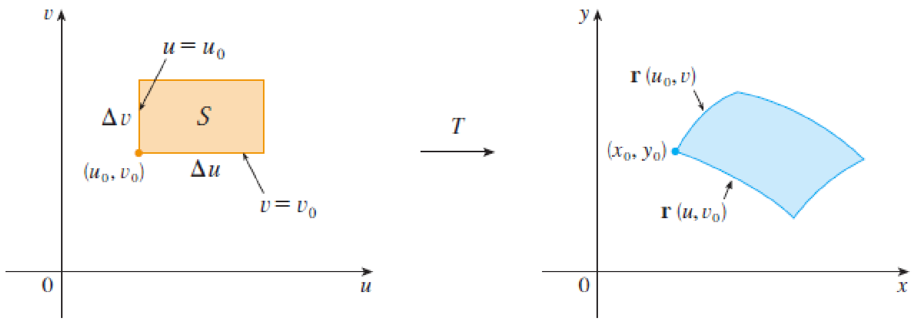
and the Jacobian of T^{-1} is

$$\frac{\partial(u, v)}{\partial(x, y)} = 1 \Bigg/ \frac{\partial(x, y)}{\partial(u, v)}.$$

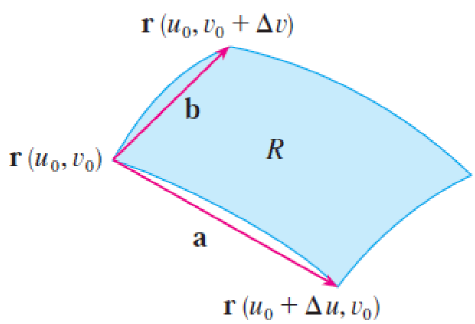
Back to integration.

Write the equations of the transformation T in vector form

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v)\rangle.$$



Given a small rectangle S in the uv-plane, let's try to estimate the area of its image (in blue) in the xy-plane.
The lower edge of the rectangle is transformed into a curve $\mathbf{r}(u, v_0)$ and the left edge into a curve $\mathbf{r}(u_0, v)$.
The blue figure can be approximated by a parallellogram with sides $\mathbf{a}$ and $\mathbf{b}$ as shown,



where

$$\begin{aligned}\mathbf{a} &= \mathbf{r}(u_0+\Delta u, v_0) - \mathbf{r}(u_0, v_0)\\ &= \frac{\mathbf{r}(u_0+\Delta u, v_0) - \mathbf{r}(u_0, v_0)}{\Delta u}\,\Delta u\\ &\approx \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0)\,\Delta u = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\rangle\,\Delta u.\end{aligned}$$

Similarly,

$$\mathbf{b} \approx \frac{\partial \mathbf{r}}{\partial v}(u_0, v_0)\,\Delta v = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v} \rangle\,\Delta v.$$

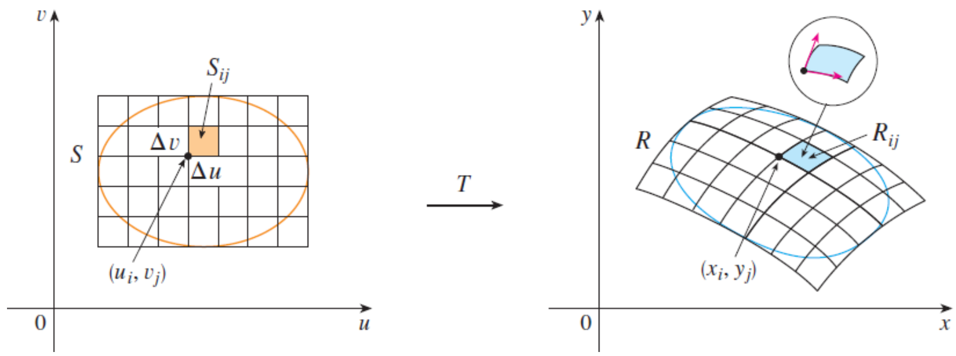
Treating $\mathbf{a}$ and $\mathbf{b}$ as vectors in $\mathbb{R}^3$:

$$\mathbf{a} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0 \rangle\,\Delta u,\ \ \mathbf{b} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0 \rangle\,\Delta v,$$

we find that the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$ is

$$\begin{aligned} |\mathbf{a} \times \mathbf{b}| &= \left| \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{pmatrix} \right| \Delta u\,\Delta v \\ &= \left| \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{pmatrix} \right| \Delta u\,\Delta v = \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \Delta u\,\Delta v. \end{aligned}$$

Now let's divide a region of integration S in the uv-plane into smalle rectangles S_{ij}. The image under the transformation $x = x(u, v)$, $y = y(u, v)$ is denoted by R_{ij}.



The double integral $\iint_R f(x, y)\, dA$ is approximately

$$\sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \text{ Area } (R_{ij})$$

$$\approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u_i\, \Delta v_j.$$

The second sum is a Riemann sum of the integral

$$\iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

Hence we obtain the change of variables formula for double integrals:

Theorem 22. *Suppose that T is a 1-to-1 C^1 transformation from a region S in the uv-plane to the region R in the xy-plane given by*

$$x = x(u, v), \quad y = y(u, v)$$

so that T^{-1} is also C^1. If f is a continuous function on R and R and S are type I or type II planar regions, then

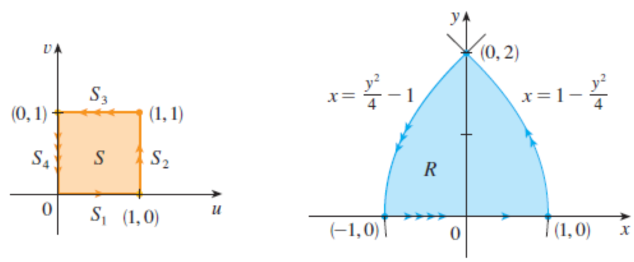
$$\iint_R f(x, y)\, dA = \iint_S f(x(u, v), y(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv.$$

Example. Use the change of variables $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral $\iint_R y\, dA$, where R is bounded by the x-axis and the parabolas $y^2 = 4-4x$ and $y^2 = 4+4x$, $y \geq 0$.

Note that

$$\begin{aligned} y^2 = 4 - 4x &\implies 4u^2v^2 = 4 - 4u^2 + 4v^2 \\ &\implies u^2(4+4v^2) = 4 + 4v^2 \implies u^2 = 1. \end{aligned}$$

Similarly, $y^2 = 4 + 4x \implies v^2 = 1$.



Consider the square S as shown in the (u, v)-plane.
On the side S_1, $0 \leq u \leq 1$ and $v = 0$.
Hence $0 \leq x = u^2 \leq 1$ and $y = 0$.
So the image of S_1 on the xy-plane is the line segment from $(0, 0)$ to $(1, 0)$.
On the side S_2, $u = 1$ and $0 \leq v \leq 1$.
Hence $y = 2v$ and $x = 1 - v^2 = 1 - (\frac{y}{2})^2$.
Thus $y^2 = 4 - 4x$ and $0 \leq y = 2v \leq 2$.
So the image of S_2 on the xy-plane is the part of the parabola $y^2 = 4 - 4x$ from $(1, 0)$ to $(0, 2)$.
Similarly we can consider the sides S_3 and S_4 and conclude that S in the uv-plane is mapped onto the region R.

$$\frac{\partial(x, y)}{\partial(u, v)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \det \begin{pmatrix} 2u & 2v \\ 2v & 2u \end{pmatrix} = 4(u^2 + v^2).$$

Therefore,

$$\begin{aligned}\iint_R y\,dA &= \iint_S 2uv|4(u^2+v^2)|\,du\,dv\\ &= \int_0^1\int_0^1 8uv(u^2+v^2)\,du\,dv = 2.\end{aligned}$$

Example. Evaluate the integral $\iint_R e^{\frac{x+y}{x-y}}\, dA$ where R is the trapezoidal region with vertices $(1,0), (2,0), (0-2), (0,-1)$.

Use the change of variables $u = x + y$, $v = x - y$.
It is easy to see that with this transformation, straight lines in the xy-plane are transformed into straight lines in the uv-plane.
We have the correspondence

(x,y)	(u,v)
$(1,0)$	$(1,1)$
$(2,0)$	$(2,2)$
$(0,-2)$	$(-2,2)$
$(0,-1)$	$(-1,1)$

Hence R is transformed into a trapezoid S in the uv-plane with the 4 vertices on the right.

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{pmatrix} = \frac{-1}{2}.$$

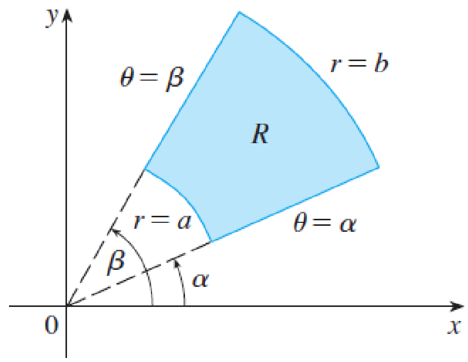
Therefore,

$$\begin{aligned} \iint_R e^{\frac{x+y}{x-y}}\, dA &= \iint_S \left|\frac{-1}{2}\right| e^{\frac{u}{v}}\, dA \\ &= \int_1^2 \int_{-v}^{v} \frac{1}{2} e^{\frac{u}{v}}\, du\, dv = \frac{3}{2}(e - e^{-1}). \end{aligned}$$

Polar coordinates is in fact a change of variables

$$x = r\cos\theta, \quad y = r\sin\theta.$$

If, e.g., S is a rectangle $\{(r,\theta) : a \le r \le b, \alpha \le \theta \le \beta\}$ in the (r,θ)-plane. Then the corresponding region R in the (x,y) plane is the fan shaped region:



Since

$$\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta}\end{pmatrix}\right| = \left|\det\begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta\end{pmatrix}\right| = r,$$

the change of variable formula gives

$$\iint_R f(x,y)\,dA = \iint_S f(r\cos\theta, r\sin\theta) r\,dr\,d\theta.$$

Three variables.

In three variables, a transformation or change of variables takes the form

$$x = x(u, v, w), \quad y = y(u, v, w), \quad z = z(u, v, w).$$

The Jacobian of the transformation is the 3×3 determinant

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} \stackrel{\text{def}}{=} \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}.$$

The change of variables formula takes the form

$$\iiint_R f(x, y, z)\, dV$$
$$= \iiint_S f(x(u, v, w), y(u, v, w), z(u, v, w)) \left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| du\, dv\, dw.$$

For example, for transformation from rectangular to spherical coordinates (ρ, θ, ϕ),

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

The Jacobian is

$$\det \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} = -\rho^2 \sin\phi.$$

Since $0 \leq \phi \leq \frac{\pi}{2}$, $\sin\phi \geq 0$. Thus

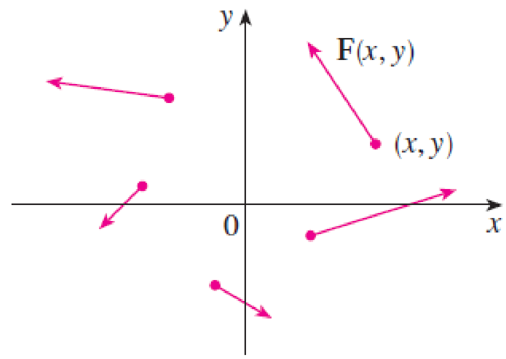
$$\left| \frac{\partial(x, y, z)}{\partial(u, v, w)} \right| = \rho^2 \sin\phi.$$

Therefore, the change of variables formula agrees with the previous formula obtained for integration in speherical coordinates.

6. Line and surface integrals

- Vector fields. Line integrals. Work done.
- Fundamental Theorem. Conservative vector fields.
- Green's Theorem.
- Divergence and curl.
- Parametrized surface.
- Surface integrals. The flux of a vector field.
- Stokes' Theorem.
- Divergence Theorem.

Let D be a set in $\mathbb{R}^2$. A *vector field* on D is a function $\mathbf{F}$ that assigns a two dimensional vector $\mathbf{F}(x, y)$ to each point (x, y) in D.



A vector field on a set E in $\mathbb{R}^3$ assigns a 3-dimensional vector $\mathbf{F}(x, y, z)$ to each point (x, y, z) in E.

Vector fields arise often in physics as force fields.

Example. Suppose that an object of mass M is placed at the origin in $\mathbb{R}^3$. The force exerted by M on an object of mass m placed at a position $\mathbf{x} = \langle x, y, z \rangle$ is

$$\mathbf{F}(\mathbf{x}) = -\frac{mMG}{|\mathbf{x}|^3}\mathbf{x},$$

where G is the universal constant of gravitation. $\mathbf{F}$ is the gravitational field due to M.

Example. Take any differentiable function $f(x, y)$ or $f(x, y, z)$. Then the *gradient vector field* of f is

$$\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$$

(add z coordinate if third dimension is present).
For example, consider the function

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \ (x, y, z) \neq 0.$$

Its gradient vector field is

$$\nabla f(x, y, z) = \frac{-1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}.$$

In particular, the gravitational field in a mass (at the origin) is a gradient vector field, namely ∇g, where $g(x, y, z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$.

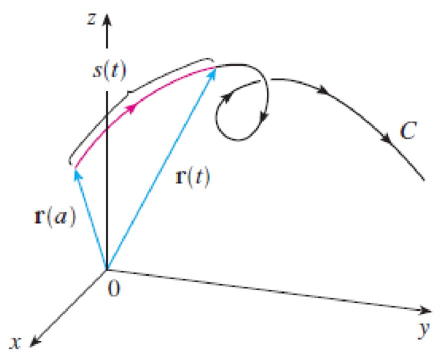
If a vector field $\mathbf{F}$ is a gradient vector field, say $\mathbf{F} = \nabla f$, then f is a *potential function* of $\mathbf{F}$. A vector field that is a gradient vector field is a *conservative vector field.*

Line integral or path integral of a scalar function.

Suppose that C is a smooth curve with parametrization

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\ \ a \le t \le b.$$

The discussion is the same if the third component is present. Recall that "smooth" means that $\mathbf{r}'(t)$ exists and is nonzero. Let $s(t)$ be the arc length measured along the curve C from the point $P = \mathbf{r}(a)$ on C to $\mathbf{r}(t)$.



If $f(x, y)$ is a function defined on C, let

$$\int_C f(x, y)\, ds \stackrel{\text{def}}{=} \int_a^b f(x(t), y(t))s'(t)\, dt.$$

From Chapter 2, we know that $s'(t) = |\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$. Hence

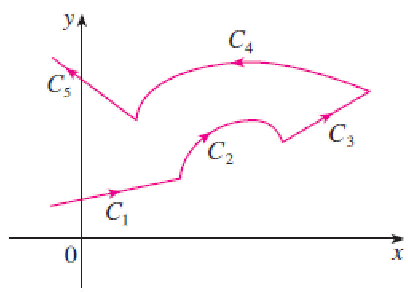
$$\begin{aligned}\int_C f(x, y)\, ds &= \int_a^b f(x(t), y(t))\, |\mathbf{r}'(t)|\, dt \\ &= \int_a^b f(x(t), y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\, dt.\end{aligned}$$

Example. Evaluate $\int_C (2 + x^2 y)\, ds$, where C is the upper half of the unit circle, traversed in the counterclockwise direction.

Parametrize C: $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$.

$$\begin{aligned}\int_C (2 + x^2 y)\, ds &= \int_0^\pi (2 + \cos^2 t \sin t) |\langle -\sin t, \cos t \rangle|\, dt \\ &= 2\pi + \frac{2}{3}.\end{aligned}$$

A *piecewise smooth curve* C is one that can be divided into a finite number of smooth curves $C_1, \ldots, C_n$, where the terminal point of C_i is the initial point of C_{i+1}.



For a piecewise smooth curve as above, we define

$$\int_C f(x, y)\, ds = \int_{C+1} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds.$$

Line integrals with respect to x and y along C.

We also define the line integrals with respect to x and y along a smooth curve C as

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))x'(t)\,dt,$$
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))y'(t)\,dt.$$

A sum of the form

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy$$

is usually abbreviated as $\int_C P(x,y)\,dx + Q(x,y)\,dy$.
Extension to piecewise smooth curves is straightforward.

Example. Evaluate the line integral $\int_C y^2\,dx + x\,dy$ if
(i) C is the straight line segment from $(-5,-3)$ to $(0,2)$;
(ii) C is the arc of the parabola $x = 4 - y^2$ from $(-5,-3)$ to $(0,2)$.

(i) Parametrize C: $\mathbf{r}(t) = \langle -5,-3\rangle + t\langle 5,5\rangle$, $0 \le t \le 1$.

$$\int_C y^2\,dx + x\,dy = \int_0^1 (-3+5t)^2 5 + (-5+5t)5\,dt = \frac{-5}{6}.$$

(ii) Parametrize C: $\mathbf{r}(t) = \langle 4-t^2, t\rangle$, $-5 \le t \le 2$.

$$\int_C y^2\,dx + x\,dy = \int_{-5}^{2} t^2(-2t) + (4-t^2)\cdot 1\,dt = \frac{1729}{6}.$$

Suppose that $\mathbf{F}$ is a continuous vector field defined on a smooth curve C. Parametrize C by $\mathbf{r}(t), a \leq t \leq b$. Then the *line integral of F along C* is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

If $\mathbf{F}$ is a force field and $\mathbf{r}$ is the position vector of a particle moving in the force field, then $\int_C \mathbf{F} \cdot d\mathbf{r}$ is defined to be the *work* done by $\mathbf{F}$ in moving the particle along the path C.

Recall that $\mathbf{T} = \frac{\mathbf{r}'}{|\mathbf{r}|}$ is the unit tangent of the curve C. Suppose we consider the scalar function $\mathbf{F} \cdot \mathbf{T}$ and its line integral along C,

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\, ds &= \int_a^b \big(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\big)|\mathbf{r}'(t)|\, dt \\ &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_C \mathbf{F} \cdot d\mathbf{r}.\end{aligned}$$

Also, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ and $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \leq t \leq b$, then we find that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b P(x(t), y(t), z(t))x'(t)\, dt + \int_a^b Q(x(t), y(t), z(t))y'(t)\, dt \\ &\quad + \int_a^b R(x(t), y(t), z(t))z'(t)\, dt \\ &= \int_C P\, dx + Q\, dy + R\, dz.\end{aligned}$$

Example. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y,z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$$

and let C be the parametrized path

$$x = t, \quad , y = t^2, \quad z = t^3, \quad 0 \le t \le 1.$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C xy\,dx + yz\,dy + zx\,dz$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle t^3, t^5, t^4 \rangle \cdot \langle 1, 2t, 3t^2 \rangle\,dt$$

$$= \int_0^1 t^3 + 5t^6\,dt = \frac{27}{28}.$$

Fundamental Theorem of Calculus for line integrals.

Recall that a vector field $\mathbf{F}$ is *conservative* if there is a scalar function f, a potential function of $\mathbf{F}$, so that $\mathbf{F} = \nabla f$. The importance of conservative vector fields comes from the following.

Theorem 23. *Let $\mathbf{F}$ be a conservative vector field on a region D, so that there is a potential function f defined on D such that $\mathbf{F} = \nabla f$ on the set D. If C is a smooth curve in D, parametrized by $\mathbf{r}(t)$, $a \le t \le b$, then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Proof. Consider the function $g(t) = f(\mathbf{r}(t)) = f(x(t), y(t), g(t))$. By the Chain Rule

$$g'(t) = f_x(\mathbf{r}(t))\, x'(t) + f_y(\mathbf{r}(t))\, y'(t) + f_z(\mathbf{r}(t))\, z'(t) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Therefore,

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_a^b g'(t)\, dt = g(b) - g(a) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \end{aligned}$$

$\square$

Example. The vector field $\mathbf{F}(\mathbf{x}) = -\frac{\mathbf{x}}{|\mathbf{x}|^3}$ is conservative and has potential function $f(\mathbf{x}) = \frac{1}{|\mathbf{x}|}$, $\mathbf{x} \neq \mathbf{0}$. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is *any* smooth path that does not pass through $\mathbf{0}$, begins at $(1, 2, 3)$ and ends at $(2, -1, 4)$.

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2, -1, 4) - f(1, 2, 3) = \frac{1}{\sqrt{21}} - \frac{1}{\sqrt{14}}.$$

Corollary 24. *(Path independence) Let* $\mathbf{F}$ *be a conservative vector field on a region* D. *If* C_1 *and* C_2 *are two smooth curves in* D *with the same initial points and same end points, then*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

The corollary can be used to show that some vector fields are not conservative.

Example. Show that the vector field

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

is not conservative on the set D consisting of $\mathbb{R}^2$ with the origin removed.

Let C_1 be the upper half of the unit circle from $(1, 0)$ to $(-1, 0)$ and let C_2 be the lower half of the unit circle from $(1, 0)$ to $(-1, 0)$.
Both are paths in D.
Parametrize C_1: $\mathbf{r}_1(t) = \langle \cos t, \sin t \rangle$, $0 \leq t \leq \pi$.
Parametrize C_2: $\mathbf{r}_2(t) = \langle \cos(-t), \sin(-t) \rangle$, $0 \leq t \leq \pi$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle \, dt = 1.$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \langle -\sin(-t), \cos(-t) \rangle \cdot \langle -\sin t, -\cos t \rangle \, dt = -1.$$

By Corollary 24, $\mathbf{F}$ is not conservative on D.

A *closed curve* is a curve whose initial and end points coincide.
A region D in the plane is *open* if it does not contain any of its boundary points.
A region D in the plane is *connected* if given any two points in D, there is a smooth curve lying in D that begins at one and ends at the other.
The next theorem says that path independence characterizes conservative vector fields.

Theorem 25. *Let* $\mathbf{F}$ *be a continuous vector field on an open connected region* D*. Then* $\mathbf{F}$ *is conservative if and only if for any smooth closed curve* C *in* D*, we have*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

The criterion for a conservative vector field given above is not convenient to check because it requires checking <u>all</u> smooth closed curves.

Suppose that $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a conservative vector field on a 2 dimensional region D.
Assume that the first order partial derivatives of P and Q exist and are continuous on D.
Since $\mathbf{F}$ is conservative, it has a potential function f: $\mathbf{F} = \nabla f$. So

$$P\mathbf{i} + Q\mathbf{j} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

Then

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 26. *Suppose that* $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ *is a conservative vector field on a 2 dimensional region* D. *Assume that the first order partial derivatives of* P *and* Q *exist and are continuous on* D. *Then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \text{ on the set } D.$$

Example. Show that the vector field

$$\mathbf{F}(x,y) = x^2y\mathbf{i} + xy^2\mathbf{j}$$

is not conservative on $\mathbb{R}^2$.

We have $P = x^2y$ and $Q = xy^2$.
Then $\frac{\partial P}{\partial y} = x^2 \neq y^2 = \frac{\partial Q}{\partial x}$.
Hence $\mathbf{F}$ is not conservative by Theorem 26.

However, the condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ is not enough to guarantee that $\mathbf{F}$ is conservative.

Example. Let

$$P(x,y) = -\frac{y}{x^2+y^2}, \quad Q(x,y) = \frac{x}{\sqrt{x^2+y^2}}.$$

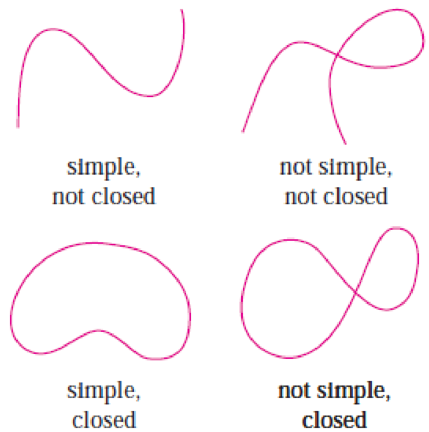
Show that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ on the set D consisting of $\mathbb{R}^2$ with the origin removed.
However, $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is <u>not</u> conservative on D.

$$\frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial Q}{\partial x}.$$

We will give a sufficient condition on the region D so that the converse of Theorem 26 holds, after a discussion of Green's Theorem.

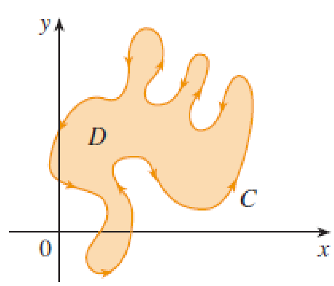
Simple closed curves.

A curve C in $\mathbb{R}^2$ is *closed* if it begins and ends at the same point. C is *simple* if it doesn't intersect itself between the initial and the end points. (No self-crossing.)



A simple closed curve divides the plane into three parts: the "inside", the "outside" and the curve itself. The *Jordan Curve Theorem* makes this statement precise; but we will just work with the intuitive notions.

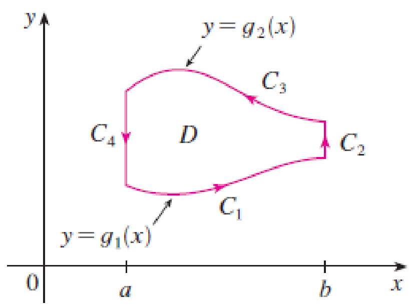
A simple closed curve C with parametrization $\mathbf{r}(t)$ is *positively oriented* if the direction of travel (for increasing t) is counterclockwise; equivalently, if the "inside" lies on the left while traveling along the curve in the direction of increasing t.



Positive orientation

Green's Theorem

As a special case of Green's Theorem, consider a region D that is both a Type I region and and Type II region. Let C be its positively oriented boundary.
Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ and that P and Q have continuous first order partials on an open set U containing D.
Note that $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy$.
Considering D as a Type I region, let us compute $\int_C P\,dx$.



C is the union of the curves C_1, C_2, C_3, C_4. Parametrize them by

$$\begin{aligned} C_1 &: \mathbf{r}_1(t) = t\mathbf{i} + g_1(t)\mathbf{j}, & a \le t \le b, \\ C_2 &: \mathbf{r}_2(t) = b\mathbf{i} + t\mathbf{j}, & g_1(b) \le t \le g_2(b), \\ -C_3 &: \mathbf{r}_3(t) = t\mathbf{i} + g_2(t)\mathbf{j}, & a \le t \le b, \\ -C_4 &: \mathbf{r}_4(t) = a\mathbf{i} + t\mathbf{j}, & g_1(a) \le t \le g_2(a). \end{aligned}$$

Here $-C_3$ means the curve C_3 traveled in the opposite direction. Same for $-C_4$.
Therefore,

$$\int_{C_1} P\,dx = \int_a^b P(t, g_1(t))\,dt, \quad \int_{C_2} P\,dx = 0$$

$$\int_{C_3} P\,dx = \int_a^b P(t, g_2(t))\,dt, \quad \int_{C_4} P\,dx = 0.$$

Hence

$$\int_C P\,dx = \int_a^b P(t, g_1(t)) - P(t, g_2(t))\,dt$$

$$= -\int_a^b \int_{g_1(t)}^{g_2(t)} \frac{\partial P}{\partial y}(t,y)\,dy\,dt = -\iint_D \frac{\partial P}{\partial y}\,dA.$$

Similarly, treating D as a Type II region gives

$$\int_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA.$$

Thus

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

The result holds for more general regions.

Theorem 27. *(Green's Theorem) Let C be a positively oriented smooth simple closed curve in $\mathbb{R}^2$. Suppose that P and Q have continuous first order partials on an open set containing D, which is the region consisting of C and its inside. Then*

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

Example. Evaluate the integral

$$\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy,$$

where C is the circle $x^2 + y^2 = 9$ with positive orientation.

The region D enclosed by C is the disk $\{(x,y) : x^2+y^2 \le 9\}$. By Green's Theorem,

$$\int_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$$

$$= \int_D (7-3)\,dA = 4 \cdot \text{ Area of } D = 36\pi^2.$$

If we choose P and Q so that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ and Green's Theorem applies, then we get

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 1\,dA = \text{ Area of } D.$$

For example, we can choose $P = -\frac{1}{2}y$ and $Q = \frac{1}{2}x$.

Corollary 28. *If C is a positively oriented simple closed curve in $\mathbb{R}^2$ and C encloses a region D, then*

$$\textit{Area of } D \; = \int_C \frac{-y}{2}\,dx + \frac{x}{2}\,dy.$$

Example. Use Green's Theorem to find the area inside the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad \text{where } a, b > 0.$$

We can parametrize the ellipse with positive orientation by

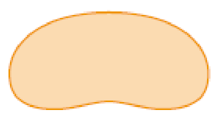
$$\mathbf{r}(t) = \langle a\cos t, b\sin t\rangle, 0 \le t \le 2\pi.$$

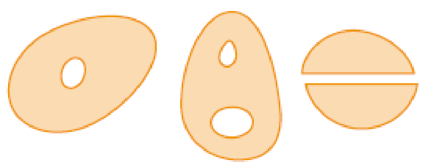
Then

$$\begin{aligned}
\text{Area of } D \; &= \int_C \frac{-y}{2}\,dx + \frac{x}{2}\,dy \\
&= \int_0^{2\pi} \frac{-b\sin t}{2}(-a\sin t) + \frac{a\cos t}{2}(b\cos t)\,dt \\
&= \int_0^{2\pi} ab\,dt = \pi ab.
\end{aligned}$$

Conservative vector fields again.

Recall that a vector field $\mathbf{F}$ is conservative on a region D if it has a potential function f on D, i.e., $\mathbf{F} = \nabla f$.
It was shown before that a vector field is conservative on D if and only if for any smooth closed curve C in D, we have $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$.
With the help of Green's Theorem, we now have a sufficient condition for a vector field to be conservative.
A region D in $\mathbb{R}^2$ is *simply connected* if for any simple closed curve C in D, the inside of C is also contained in D. A heuristic idea of simple connectedness is that the region D "has no holes".





Theorem 29. *Let D be a simply connected region in $\mathbb{R}^2$. Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field where P and Q have continuous first order partials. Then $\mathbf{F}$ is conservative on D if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.*

Proof. It is known from before that $\mathbf{F}$ conservative $\implies \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
Now suppose that $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. We show that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed curve C in D. Then we can conclude that $\mathbf{F}$ is conservative.

If C is a *simple* closed curve in D, then the region containing C and its inside are all in D. By Green's Theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dA = 0.$$

If C is a closed curve which may not be simple, we can decompose it into a union of simple closed curves $C_1, C_2, \ldots$. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \cdots = 0.$$

$\square$

Curl and divergence.

The curl and divergence are two operations of vector calculus that are of great importance in physical applications.

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field.
The *curl* of $\mathbf{F}$ is the vector field

$$\begin{aligned}\nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{pmatrix} \\ &= (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k}.\end{aligned}$$

The *divergence* of $\mathbf{F}$ is the scalar function

$$\nabla \cdot \mathbf{F} = (\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}) \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

Example. Let $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$. Compute $\nabla \times \mathbf{F}$ and $\nabla \cdot \mathbf{F}$.

$$\nabla \cdot \mathbf{F} = z + xz.$$
$$\nabla \times \mathbf{F} = \langle -x, 1, z \rangle.$$

There are some special combinations of gradient, curl and divergence that are of special interest.

Theorem 30. (1) *Suppose that f has continuous second order partials. Then $\nabla \times (\nabla f) = \mathbf{0}$.*
(2) *Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where P, Q, R has continuous second order partials. Then $\nabla \cdot (\nabla \times F) = 0$.*

Corollary 31. *If $\mathbf{F}$ is a conservative vector field that has continuous first order partials, then $\nabla \times \mathbf{F} = 0$.*

Reformulations of Green's Theorem

Suppose that $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field. We can think of it as a 3 dimensional vector field by putting the $\mathbf{k}$ component to be 0.
Let D be a region in the xy-plane and let C be its positively oriented boundary. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy.$$

Also

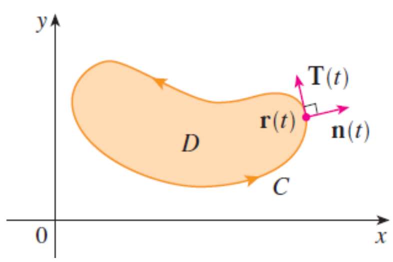
$$\nabla \times \mathbf{F} = \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{pmatrix} = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \cdot \mathbf{k}.$$

So Green's Theorem becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\,dA.$$

Recall that if $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ is a parametrization of C, then the unit tangent is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{x'(t)}{|\mathbf{r}'(t)|}\mathbf{i} + \frac{y'(t)}{|\mathbf{r}'(t)|}\mathbf{j}$$



and the outward pointing unit normal is

$$\mathbf{n}(t) = \frac{y'(t)}{|\mathbf{r}'(t)|} - \frac{x'(t)}{|\mathbf{r}'(t)|}.$$

Therefore, since $ds = |\mathbf{r}'(t)|\,dt$,

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int \langle P, Q\rangle \cdot \langle y'(t), -x'(t)\rangle\,dt$$
$$= \int Py'(t) - Qx'(t)\,dt = \int_C -Q\,dx + P\,dy.$$

By Green's Theorem, the last integral is equal to

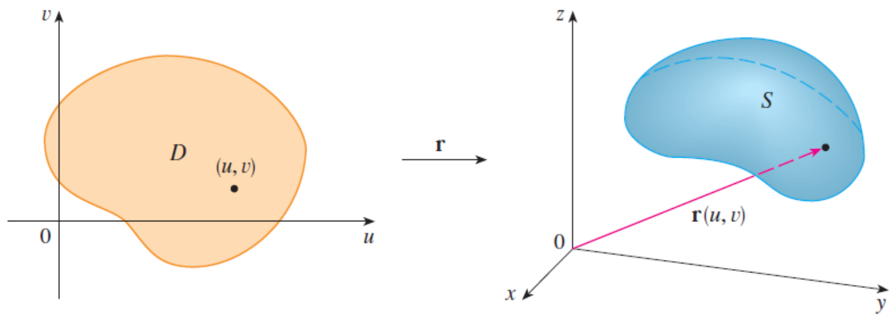
$$\iint_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA = \iint_D (\nabla\cdot\mathbf{F})\,dA.$$

So we arrive at another formulation of Green's Theorem

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D (\nabla\cdot\mathbf{F})\,dA,$$

where $\mathbf{n}$ is the outward pointing normal to the positively oriented boundary C of D.

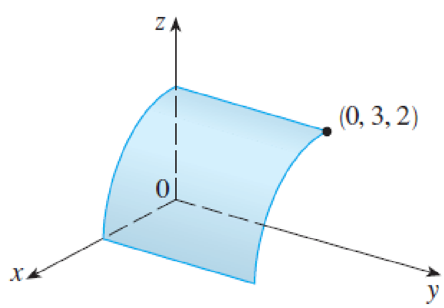
Parametric surfaces



In Chapter 2, we saw that curves can be parametrized as a vector function of 1 variable. Similarly, a vector function of two variables $\mathbf{r}(u, v)$ defined on a set D in $\mathbb{R}^2$ with values in $\mathbb{R}^3$ can be viewed as a parametrization of a surface in $\mathbb{R}^3$.

Example. Identify and sketch the parametric surface with equation

$$\mathbf{r}(u, v) = 2\cos u\mathbf{i} + v\mathbf{j} + 2\sin u\mathbf{k}, \;\; 0 \le u \le \frac{\pi}{2}, \;\; 0 \le v \le 3.$$



Example. Find two different parametrizations of the elliptic paraboloid $z = x^2 + 2y^2$.

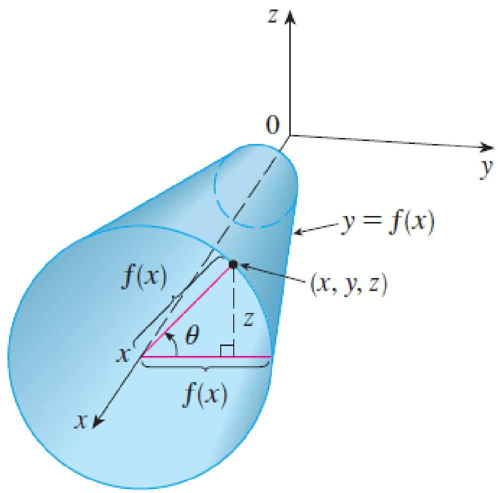
Using x, y as parameters, we have

$$\mathbf{r}_1(x, y) = \langle x, y, x^2 + 2y^2 \rangle.$$

Alternatively, parametrize the (x, y) coordinates "elliptically": $x = r\cos\theta$, $y = \frac{r}{\sqrt{2}}\sin\theta$.
Then $z = x^2 + 2y^2 = r^2$. So we have a parametrization

$$\mathbf{r}_2(r, \theta) = \langle r\cos\theta, \frac{r}{\sqrt{2}}\sin\theta, r^2 \rangle.$$

Example. Suppose that a curve $y = f(x)$, $a \leq x \leq b$, is revolved about the x-axis in 3 dimensional space. Parametrize the resulting surface.

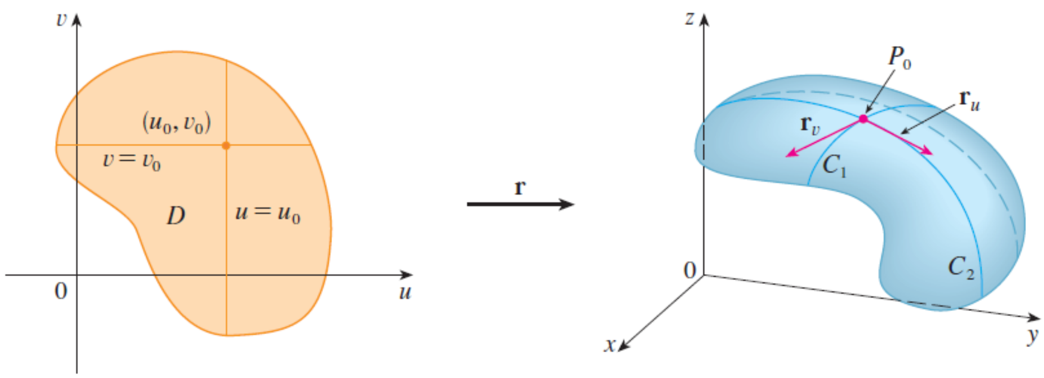


From the diagram, we use parameters x and θ.
Then $y = f(x)\cos\theta$, $z = f(x)\sin\theta$. So a parametrization is

$$\mathbf{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle, 0 \leq \theta \leq 2\pi, a \leq x \leq b.$$

Tangent plane to a parametrized surface

Suppose that a surface S is parametrized by a vector function $\mathbf{r} : D \to \mathbb{R}^3$.
Let $(u_0, v_0) \in D$ and the corresponding point on S be $P_0 = \mathbf{r}(u_0, v_0)$.



The vector functions $\mathbf{r}(u, v_0)$ and $\mathbf{r}(u_0, v)$ are two parametrized curves (C_1 and C_2 respectively) on S, both passing through P_0.
The tangent to C_1 at P_0 is

$$\frac{d}{du}\mathbf{r}(u, v_0)|_{u=u_0} = \frac{\partial \mathbf{r}}{\partial u}(u_0, v_0) = \mathbf{r}_u(u_0, v_0).$$

Similarly, the tangent to C_2 at P_0 is $\frac{\partial \mathbf{r}}{\partial v}(u_0, v_0) = \mathbf{r}_v(u_0, v_0)$.
If $\mathbf{r}_u(u_0, v_0)$ and $\mathbf{r}_v(u_0, v_0)$ are not parallel, then a normal to the tangent plane to S at P_0 is $\mathbf{r}_u(u_0, v_0) \times \mathbf{r}_v(u_0, v_0)$.
If this is the case at any (u_0, v_0), the parametrization is said to be *smooth*.

Example. Find the tangent plane to the parametric surface $\mathbf{r} = \langle u^2, v^2, u + 2v\rangle$ at the point $(1, 1, 3)$.

$$\mathbf{r}_u = \langle 2u, 0, 1\rangle|_{(1,1,3)} = \langle 2, 0, 1\rangle,$$
$$\mathbf{r}_v = \langle 0, 2v, 2\rangle|_{(1,1,3)} = \langle 0, 2, 2\rangle.$$

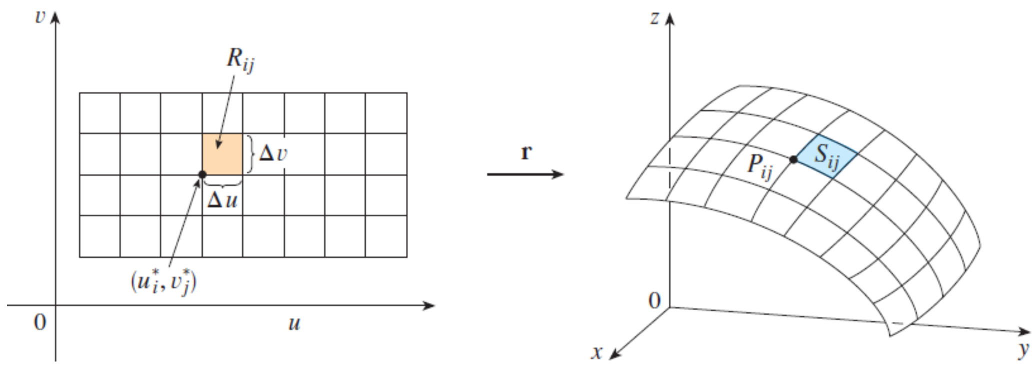
Normal to tangent plane

$$\langle 2, 0, 1\rangle \times \langle 0, 2, 2\rangle = \langle -2, -4, 4\rangle.$$

Equation of tangent plane: $-2(x-1)-4(y-1)+4(z-3) = 0$.

Surface area

Suppose that a surface S is parametrized by $\mathbf{r}(u, v)$ (defined on D) and we wish to find the area of the surface S. Subdivide D into a rectangular grid and consider the corresponding grid on S.



For each small rectangle R_{ij} as shown, we estimate the corresponding area S_{ij} by approximating it by a parallelogram. The edges of the parallelogram are

$$\mathbf{r}(u_i^* + \Delta u, v_j^*) - \mathbf{r}(u_i^*, v_j^*) \approx \mathbf{r}_u(u_i^*, v_j^*)\Delta u$$

and

$$\mathbf{r}(u_i^*, v_i^* + \Delta v) - \mathbf{r}(u_i^*, v_j^*) \approx \mathbf{r}_v(u_i^*, v_j^*)\Delta v$$

respectively.
Hence

$$\text{Area of } S_{ij} \approx |\mathbf{r}_u(u_i^*, v_j^*) \times \mathbf{r}_v(u_i^*, v_j^*)|\Delta u \Delta v.$$

Therefore, Area of S is approximately $\sum |\mathbf{r}_u \times \mathbf{r}_v|\Delta u \Delta v$. Taking limit, this Riemann sum becomes

$$\text{Area of } S = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA.$$

Example. Find the surface area of the sphere of radius a.

Parametrize the sphere: $\mathbf{r}(\theta, \phi) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$,
$0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$.

$|\mathbf{r}_\theta \times \mathbf{r}_\phi| = |\langle -a^2 \sin^2\phi \cos\theta, -a^2 \sin^2\phi \sin\theta, -a^2 \sin\phi \cos\phi \rangle| = a^2 \sin\phi$.
Area of sphere =

$$\iint_D |\mathbf{r}_\theta \times \mathbf{r}_\phi| \, dA = \int_0^\pi \int_0^{2\pi} a^2 \sin\phi \, d\theta \, d\phi = 4\pi a^2.$$

Example. A curve $y = f(x)$, $a \le x \le b$ is revolved about the x-axis in 3-space. Find the area of the resulting surface.

Parametrization:

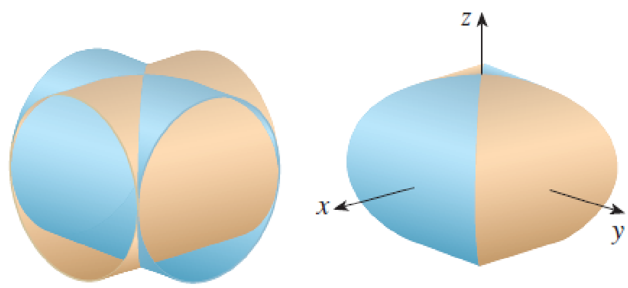
$$\mathbf{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta\rangle, 0 \le \theta \le 2\pi, a \le x \le b.$$

$$\begin{aligned}\mathbf{r}_x &= \langle 1, f'(x)\cos\theta, f'(x)\sin\theta\rangle\\ \mathbf{r}_\theta &= \langle 0, -f(x)\sin\theta, f(x)\cos\theta\rangle.\end{aligned}$$

$$\begin{aligned}\text{Area } &= \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\,dA\\ &= \int_a^b \int_0^{2\pi} f(x)\sqrt{1+(f'(x))^2}\,dx\,d\theta\\ &= 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx.\end{aligned}$$

Example.

The figure shows the surface created when the cylinder $y^2 + z^2 = 1$ intersects the cylinder $x^2 + z^2 = 1$. Find the area of this surface.



Where the blue and yellow cylinders intersect,

$$y^2 + z^2 = 1 = x^2 + z^2 \implies x^2 = y^2 \implies y = \pm x.$$

Consider the upper half (above the xy-plane) of the blue surface in the positive x-direction.
Parametrize it by $\mathbf{r}(x, y) = \langle x, y, \sqrt{1 - x^2}\rangle$, $(x, y) \in D$, where D is the triangle in the xy-plane with vertices $(0, 0)$, $(-1, -1)$ and $(1, 1)$.

$$|\mathbf{r}_x \times \mathbf{r}_y| = |\langle \frac{x}{\sqrt{1 - x^2}}, 0, 1\rangle| = \frac{1}{\sqrt{1 - x^2}}.$$

So the area of upper half of the blue surface in the positive x-direction is

$$\iint_D |\mathbf{r}_x \times \mathbf{r}_y|\, dA = \int_0^1 \int_{-x}^{x} \frac{1}{\sqrt{1 - x^2}}\, dy\, dx = 2.$$

The total surface area is 8 times as big so is equal to 16.

Surface integral of a scalar function

If $\mathbf{r} : D \to \mathbb{R}^3$ parametrizes a smooth surface S, then as we saw above, the area element dS on the surface S is given by $|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv$.
Hence, if f is a continuous scalar valued function on S, we define

$$\begin{aligned}\iint_S f(x,y,z)\, dS &= \iint_D f(\mathbf{r})|\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv \\ &= \iint_D f(\mathbf{r})|\mathbf{r}_u \times \mathbf{r}_v|\, dA.\end{aligned}$$

Example. Parametrize the unit sphere S: $x^2+y^2+z^2=1$ and evaluate $\iint_S x^2\, dS$.

Parametrize S:

$\mathbf{r}(\theta,\phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi.$

$$\begin{aligned}\iint_S x^2\, dS &= \int_0^{\pi}\int_0^{2\pi} (\sin\phi\cos\theta)^2 \sin\phi\, d\theta\, d\phi \\ &= \frac{4\pi}{3}.\end{aligned}$$

Example. Suppose that S is the graph of the function $z = g(x,y)$, $(x,y) \in D$, for some region D in the plane. Obtain a formula for the surface integral $\iint_S f(x,y,z)\,dS$ in terms of f and g.

$$\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle, (x,y) \in D.$$

Then

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{1 + g_x^2 + g_y^2}.$$

Hence

$$\iint_S f(x,y,z)\,dS = \iint_D f(x, yg(x,y))\,\sqrt{1 + g_x^2 + g_y^2}\,dA.$$

Orientable surfaces. Surface integral of a vector function.

Example. The *Möbius strip* is surface S parametrized by the function

$$\mathbf{r}(u,v) = (\cos v + u\cos\frac{v}{2}\cos v)\mathbf{i} + (\sin v + u\cos\frac{v}{2}\sin v)\mathbf{j} + u\sin\frac{v}{2}\mathbf{k},$$

for $-\frac{1}{2} \le u \le \frac{1}{2}$, $0 \le v \le 2\pi$.

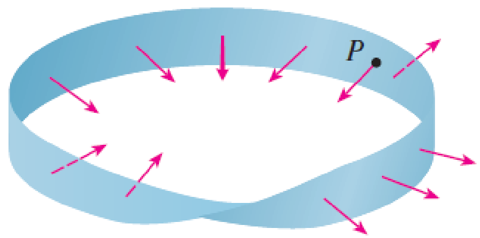
Notice that $\mathbf{r}(0,0) = \mathbf{r}(0,2\pi) = (0,1,0)$.
This means that the single point $P(2,2,0)$ on the Möbius strip has two representations in the parametrization.
Suppose we want to find the unit normal vector to the surface at point P.
Depending on whether we think of P as $\mathbf{r}(0,0)$ or as $\mathbf{r}(0,2\pi)$, we get

$$\frac{\mathbf{r}_u(0,0) \times \mathbf{r}_v(0,0)}{|\mathbf{r}_u(0,0) \times \mathbf{r}_v(0,0)|} = \mathbf{k}$$

or

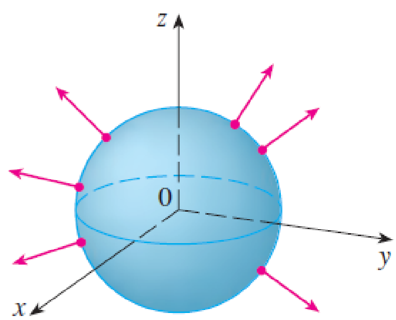
$$\frac{\mathbf{r}_u(0,2\pi) \times \mathbf{r}_v(0,2\pi)}{|\mathbf{r}_u(0,\pi) \times \mathbf{r}_v(0,2\pi)|} = -\mathbf{k}.$$

The fact that these disagree means that there is no consistent choice of a continuous normal vector field on the surface.
It can be demonstrated physically on a physical model of a Möbius strip.

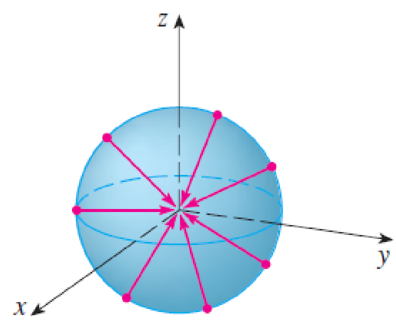


A surface, like the Möbius strip, on which there is no continuous choice of a normal vector field to the surface, is said to be *non-orientable*. Otherwise, it is *orientable*.

If we take a set E in 3 space and let S be its boundary, then S is an orientable surface. In this case, there are two possible choices of normal vectors (called *orientations*), one outward pointing and one inward pointing.
By convention, the outward pointing orientation is called the *positive orientation* and the inward pointing one the *negative orientation*.



Positive orientation



Negative orientation

Surface integral of a vector field

If $\mathbf{F}$ is a continuous vector field defined on an orientable surface S with parametrization $\mathbf{r}$, define the *surface integral*

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \mathbf{F} \cdot \Big(\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\Big)\, dS.$$

If S is the surface $\mathbf{r}(u, v)$ where (u, v) lies in the planar region D, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.$$

The surface integral is called the *flux* of the vector field $\mathbf{F}$ across the surface S.

Example. Find the flux of the vector field $z\mathbf{i}+y\mathbf{j}+x\mathbf{k}$ across the unit sphere $x^2+y^2+z^2=1$ with positive (outward pointing normal) orientation.

As before, we can parametrize the sphere by

$\mathbf{r}(\theta,\phi)=\langle\sin\phi\cos\theta,\sin\phi\sin\theta,\cos\phi\rangle,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi.$

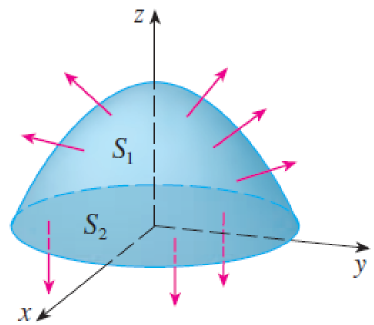
Then the normal

$$\mathbf{r}_\theta\times\mathbf{r}_\phi=\langle-\sin^2\phi\cos\theta,-\sin^2\phi\sin\theta,-\sin\phi\cos\theta\rangle$$

is <u>inward</u> pointing.
So we should use its negative instead. Then

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= -\iint_D \mathbf{F}\cdot(\mathbf{r}_\theta\times\mathbf{r}_\phi)\,dA\\ &= \int_0^\pi\int_0^{2\pi}\sin^3\phi\sin^2\theta\,d\theta\,d\phi=\frac{4\pi}{3}.\end{aligned}$$

Example. Find the flux of the vector field $y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ across the surface S which is the positively oriented boundary of the solid region enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.



Parametrization of S_1: $\mathbf{r}(x, y) = \langle x, y, 1 - x^2 - y^2 \rangle$, $(x, y) \in D = \{(x, y) : x^2 + y^2 \leq 1\}$.

$$\mathbf{r}_x \times \mathbf{r}_y = \langle 2x, 2y, 1 \rangle$$

is outward pointing.
Flux across S_1 is

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA \\ &= \iint_D 4xy + 1 - x^2 - y^2\, dA.\end{aligned}$$

Compute this integral using polar cooridinates.

$$\iint_D 4xy+1-x^2-y^2\, dA = \int_0^{2\pi} \int_0^1 [4r^2 \cos\theta \sin\theta + 1 - r^2] r\, dr\, d\theta = \frac{\pi}{4}.$$

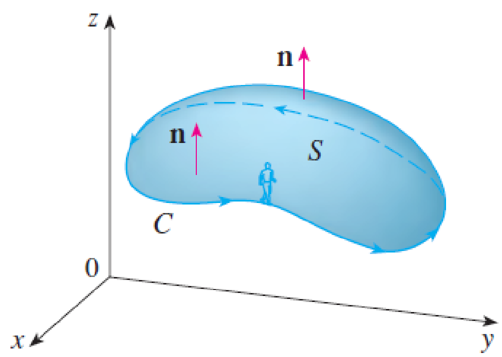
For S_2, use the parametrization $\mathbf{r}(x, y) = \langle x, y, 0 \rangle$, $(x, y) \in D$.
Since $\mathbf{r}_x \times \mathbf{r}_y = \langle 0, 0, 1 \rangle$ is inward pointing, we use its negative.
Then

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = -\iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA = 0.$$

Therefore, the total flux is $\frac{\pi}{4} + 0 = \frac{\pi}{4}$.

Stokes' Theorem

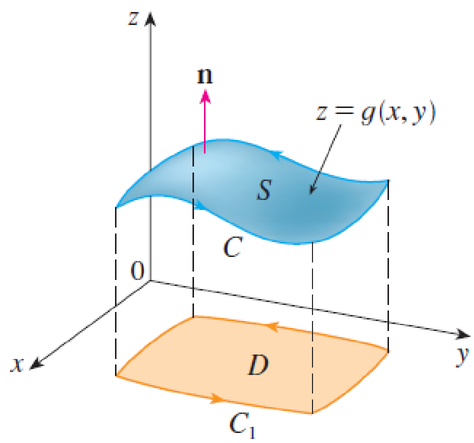
In this and the next section, we study two of the fundamental theorems of vector integration.
The first theorem we will look at is Stokes' Theorem, which relates the surface integral of the curl of a vector field on a surface S, i.e., $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ to the line integral of $\mathbf{F}$ on the boundary of S.
The situation is described in the following figure. Suppose that a parametrized surface S has orientation with unit normal vectors $\mathbf{n}$ as shown.
We give the boundary C of S the *positive orientation* as shown, so that traveling in the given direction of C (with "up" being the direction of $\mathbf{n}$) puts the surface S on the left hand side.



As a special case, suppose that S is the graph of a function $z = g(x, y)$, $(x, y) \in D$.
One can check that the vector function

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + g(u, v)\mathbf{k}$$

parametrizes S with upward orientation.



When the boundary C of S is given the positive orientation, then the projection of C onto the xy-plane is a simple closed curve C_1 traversed in the counterclockwise direction, i.e, positive direction in the xy-pplane.
Let $\langle x(t), y(t)\rangle$, $a \le t \le b$, be a parametrization of C_1. Then C can be parametrized by

$$C : \langle x(t), y(t), g(x(t), y(t))\rangle, a \le t \le b.$$

Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a vector field.
Then, after some calculations,

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \iint_D (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA \\ &= \iint_D [(Q_z - R_y)g_x - (P_z - R_x)g_y + (Q_x - P_y)]\, dA.\end{aligned}$$

And, using the Chain Rule,

$$
\begin{aligned}
\int_C \mathbf{F}\cdot\, d\mathbf{r} &= \int_a^b \mathbf{F}\cdot\langle x', y', g_x x' + g_y y'\rangle\, dt \\
&= \int_a^b (P + Rg_x)x' + (Q + Rg_y)y'\, dt \\
&= \int_{C_1} (P + Rg_x)\, dx + (Q + Rg_y)\, dy.
\end{aligned}
$$

The last term can be computed by Green's Theorem to give

$$\int_C \mathbf{F}\cdot\, d\mathbf{r} = \iint_D \big[\frac{\partial}{\partial x}(Q + Rg_y) - \frac{\partial}{\partial y}(P + Rg_x)\big]\, dA.$$

Now, it is important to remember that P, Q, R are functions of x, y, z and $z = g(x, y)$ is a function of x, y.
Hence to compute, for example, $\frac{\partial Q}{\partial x}$, we need to use the Chain Rule

$$\frac{\partial Q}{\partial x} = Q_x\frac{\partial x}{\partial x} + Q_y\frac{\partial y}{\partial x} + Q_z\frac{\partial z}{\partial x} = Q_x + Q_z g_x.$$

(Warning: The notation is a bit confusing, the Q on the left is the function $Q(x, y, g(x, y))$, while the Q on the right is the function $Q(x, y, z)$.)
Performing all calculations gives

$$
\begin{aligned}
&\int_C \mathbf{F}\cdot\, d\mathbf{r} \\
&\qquad = \iint_D [Q_x + Q_z g_x + (R_x + R_z g_x)g_y + Rg_{yx}] \\
&\qquad\quad - [P_y + P_z g_y + (R_y + R_z g_y)g_x + Rg_{xy}]\, dA \\
&\qquad = \iint_D [(Q_z - R_y)g_x - (P_z - R_x)g_y + (Q_x - P_y)\, dA.
\end{aligned}
$$

So we conclude that

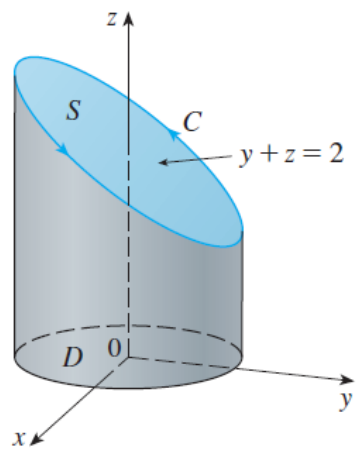
$$\iint_S (\nabla\times\mathbf{F})\cdot\, d\mathbf{S} = \int_C \mathbf{F}\cdot\, d\mathbf{r}.$$

This is a special case of Stokes' Theorem.

STOKES' THEOREM Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S}$$

Example. Evaluate $\int_C (-y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}) \cdot d\mathbf{r}$, where C is the intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$, oriented counterclockwise when viewed from above.



$$\nabla \times \mathbf{F} = \langle 0, 0, 1 + 2y \rangle.$$

Parametrize S by

$$\mathbf{r}(x, y) = \langle x, y, 2 - y \rangle, x^2 + y^2 \leq 1.$$

The normal is

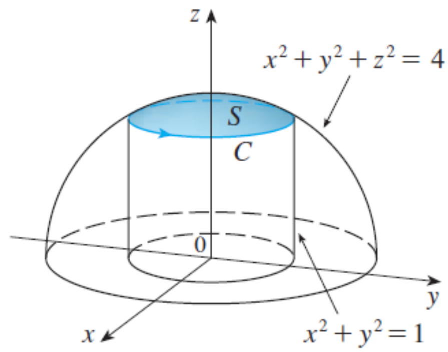
$$\mathbf{r}_x \times \mathbf{r}_y = \langle 0, 1, 1 \rangle$$

which points up; thus agrees with the orientation of C.

By Stokes' Theorem

$$
\begin{aligned}
\int_C (-y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}) \cdot \, d\mathbf{r} &= \iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S} \\
&= \iint_D \langle 0, 0, 1 + 2y \rangle \cdot \langle 0, 1, 1 \rangle \, dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 2r\sin\theta) r \, dr \, d\theta \\
&= \pi.
\end{aligned}
$$

Example. Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$, where $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + xy\mathbf{k}$, S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies within the cylinder $x^2 + y^2 = 1$, with upward orientation.



The curve C is $x^2 + y^2 = 1$, $z = 3$, oriented as shown. It can be parametrized by

$$\mathbf{r}(t) = \langle \cos t, \sin t, 3 \rangle, 0 \le t \le 2\pi.$$

Key observation:
By Stokes' Theorem

$$\begin{aligned}
\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \int_C \langle xz, yz, xy \rangle \cdot d\mathbf{r} \\
&= \int_C \langle 3\cos t, 3\sin t, \cos t \sin t \rangle \cdot \mathbf{r}'(t)\, dt \\
&= 0.
\end{aligned}$$

Divergence Theorem of Gauss' Theorem

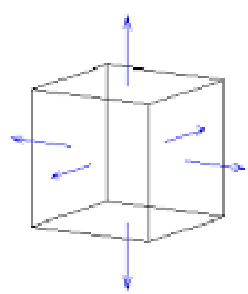
The final result we will look at is the Divergence Theorem which related a surface integral of a vector field over the boundary surface of a solid to the triple integral of the divergence over the solid.

THE DIVERGENCE THEOREM Let E be a simple solid region and let S be the boundary surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose component functions have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E (\nabla \cdot \mathbf{F})\, dV.$$

All solids regions encountered in practical applications, in particular, all solids that arise in this module, are "simple" so that the Divergence Theorem applies.
I will verify the Divergence Theorem for the case where E is a rectangular solid

$$E = \{(x, y, z) : a_1 \leq x \leq b_1, a_2 \leq y \leq b_2, a_3 \leq z \leq b_3\}.$$



Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$.
For example, the upper face S_{up} is parametrized by

$$\mathbf{r}(u, v) = (u, v, b_3),$$

where (u, v) lies in the rectangle $D = \{(u, v) : (a_1 \leq u \leq b_1, a_2 \leq v \leq b_2\}$. One can easily verify that with this parametrization,

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} = \mathbf{k},$$

which is the correct orientation. Hence

$$\iint_{S_{\text{up}}} \mathbf{F} \cdot \, d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \iint_D R(u, v, b_3) \, dA.$$

However, if we parametrize the lower face S_{low} by

$$\mathbf{r}(u, v) = (u, v, a_3), \text{ where } (u, v) \in D,$$

then the unit normal is $\mathbf{k}$, with is opposite to the orientation we want.

So with this parametrization,

$$\iint_{S_{\text{low}}} \mathbf{F} \cdot d\mathbf{S} = -\iint_D \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA = -\iint_D R(u, v, a_3) \, dA.$$

Similarly, computing the surface integral on the other 4 faces and summing, we find that with $S =$ boundary of E with outward orientation,

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \, d\mathbf{S} = \iint_D &(P(b_1, u, v) - P(a_1, u, v)) + \\
&(Q(u, b_2, v) - Q(u, b_1, v)) + (R(u, v, b_3) - R(u, v, a_3)) \, dA \\
= &\iint_D \int_{a_1}^{b_1} P_x(x, u, v) \, dx \, dA + \iint_D \int_{a_2}^{b_2} Q_y(u, y, v) \, dy \, dA + \\
&\iint_D \int_{a_3}^{b_3} R_x(u, v, z) \, dz \, dA \\
= &\iiint_E P_x \, dV + \iiint_E Q_y \, dV + \iiint_E R_z \, dV \\
= &\iiint_E (\nabla \cdot \mathbf{F}) \, dV.
\end{aligned}$$

Usually, the Divergence Theorem is applied to turn a surface integral into a volume (triple) integral.

Example. Find the flux of the vector field $\mathbf{F} = z\mathbf{i}+y\mathbf{j}+x\mathbf{k}$ across the unit sphere $x^2+y^2+z^2 = 1$ with outward oriented normal.

$$\nabla \cdot \mathbf{F} = 1.$$

By the Divergence Theorem

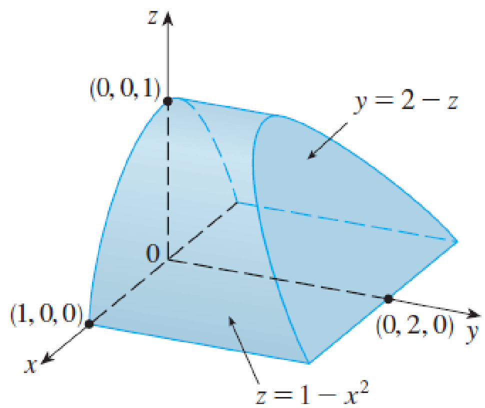
$$\text{Flux} = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 1\, dV = \text{ Volume of } E,$$

where E is the ball $x^2 + y^2 + z^2 \leq 1$.
So the flux is $\frac{4}{3}\pi$.

Example. Evaluate $\iint_S \mathbf{F} \cdot d\mathbf{S}$, where

$$\mathbf{F} = xy\mathbf{i} + (y^2 + e^{xz^2})\mathbf{j} + \sin(xy)\mathbf{k}$$

and S the boundary of the solid E bounded by the parabolic cylinder $z = 1 - x^2$ and the planes $y = 0$, $z = 0$ and $y + z = 2$.



$$\nabla \mathbf{F} = y + 2y + 0 = 3y.$$

So

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E 3y \, dV.$$

For the triple integral, integrate first in the y-direction,

$$\begin{aligned} \iiint_E 3y \, dV &= \int_0^1 \int_{-1}^1 \int_0^{2-z} 3y \, dy \, dx \, dz \\ &= 7. \end{aligned}$$

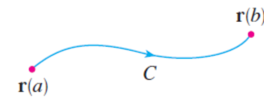
The table on the next page summarizes the main results of vector integration and shows that they are analogous to the Fundamental Theorem of Calculus in one variable integration.

Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

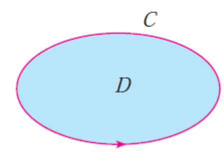
Fundamental Theorem for Line Integrals

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



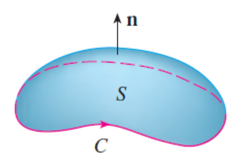
Green's Theorem

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_C P\,dx + Q\,dy$$



Stokes' Theorem

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$



Divergence Theorem

$$\iiint_E \operatorname{div} \mathbf{F}\,dV = \iint_S \mathbf{F} \cdot d\mathbf{S}$$