MA2101 Linear Algebra 2 Cheat Sheet

Theorem 8.3.8: If W_1 and W_2 are 2 subspaces, then the intersection $W_1 \cap W_2$ is also a subspace

Theorem 8.3.12: If W_1 and W_2 are subspaces, then $W_1 + W_2 = \{w_1 + w_2 \,|\, w_1 \in W_1, w_2 \in W_2\}$ is also a subspace

Bases and Dimensions: A subset B of a vector space V is a basis if:

- B is a linearly independent set of vectors
- $V = \text{span}(B) \to V \subseteq \text{span}(B)$, $\text{span}(B) \subseteq V$
- A subset of less than $\dim(V)$ vectors cannot span V, and a subset of more than $\dim(V)$ vectors is linearly dependent $\to$ $|B| = \dim(V)$

Theorem 8.5.13: The following are equivalent:

1. B is a basis for V
2. B is linearly independent, and $|B| = \dim(V)$
3. B spans V and $|B| = \dim(V)$

Theorem 8.5.15: Let W be a subspace for V. Then:

- $\dim(W) \leq \dim(V)$
- $\dim(W) = \dim(V) \to W = V$ (Proof by contradiction)

Lemma 8.5.7: Let B be an ordered basis for V.

- For any $u, v \in V$, $[u]_B = [v]_B \to u = v$
- For any $u, v \in V$, $[u]_B$ and $[v]_B$ are linearly independent $\to u$ and v are linearly independent.
- $[c_1u_1 + \ldots + c_nu_n]_B = c_1[u_1]_B + \ldots + c_n[u_n]_B$
- $\text{span}(u_1, \ldots, u_n) \in V \to \text{span}([u_1]_B, \ldots [u_n]_B) \in V_B$

Direct Sums: $W_1 \oplus W_2 = \{w_1 + w_2 \,|\, w_1 \in W_1, w_2 \in W_2\}$ is a direct sum if every vector u can be uniquely expressed as $w_1 + w_2$

- $W_1 \cap W_2 = \{0\} \to W_1 \oplus W_2$
- If B_1 and B_2 are bases for W_1 and W_2, and $W_1 + W_2$ is a direct sum, then $B_1 \cup B_2$ is a basis for $W_1 \oplus W_2$
  - $\dim(W_1 \oplus W_2) = |B_1 \cup B_2| = |B_1| + |B_2| = \dim(W_1) + \dim(W_2)$

If $W_1 + W_2$ is not a direct sum, then:
$\dim(W_1 + W_2) = \dim(B_1 \cup B_2) = \dim(W_1) + dim(W_2) - \dim(W_1 \cap W_2)$

- $W_1 + W_2 + \ldots + W_n$ is a direct sum $\to (W_1 \cap W_2) = \{0\}$, $\ldots, (W_1 \cap W_2 \cap \ldots \cap W_{n-1}) \cap W_n = \{0\}$
- Let W be a subspace of V, and $W^\perp = \{u \,|\, u \cdot v = 0, \forall v \in W\}$. Then $W \oplus W^\perp$ is a direct sum, and $V = W \oplus W^\perp$

Linear Transformations: A mapping $T : V \to W$ is a linear transformation if and only if $T(au + bv) = aT(u) + bT(v)$

- $T(u) = Au = ([T(b_1)]\ [T(b_2)] \ldots [T(b_n)])[u]_B$ where B is a basis for V

Theorem 9.2.1: Let $T : V \to W$ be a transformation, and B, C be ordered bases for V and W respectively. Then,
$[T(u)]_C = [T]_{C,B}[u]_B = ([T(v_1)]_C\ [T(v_2)]_C, \ldots [T(v_n)]_C)[u]_B$ for $B = \{v_1, v_2, \ldots, v_n\}$

Theorem 9.2.6: Let B, C be two ordered bases for vector space V. For any vector $u \in V$, $[u]_C = [I_v]_{C,B}[u]_B$ where $[I_V]_{C,B} = ([I_v(v_1)]_C, [I_v(v_2)]_C, \ldots, [I_v(v_n)]_C) = ([v_1]_C, [v_2]_C, \ldots [v_n]_C)$ is the transition matrix from B to C.

- $[I_v]_{C,B} = ([I_v]_{B,C})^{-1}$

Composition of Linear Transformations: Let $T : V \to W$ and $S : U \to V$ be linear transformations $T \circ S(u) = T(S(u))$ is also a linear transformation.

- Let A, B, C be bases for U, V, W respectively.
  $[T \circ S]_{C,A} = [T(S(u))]_C = [T]_{C,B}[S(u)]_B = [T]_{C,B}[S]_{B,A}[u]_A$
  $\to [T \circ S]_{C,A} = [T]_{C,B}[S]_{B,A}$

Definition 9.3.9: For 2 matrices $A, B \in M_{n\times n}$, B is similar to A if there exists invertible matrix P such that $B = P^{-1}AP$.
Alternatively, matrices A and B are similar if:

1. $\det(A) = \det(B)$ & $tr(A) = tr(B)$
2. They share the same characteristic polynomial(same eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_k$)
3. They share the same dimension for each of the eigenspaces
4. The minimal polynomials of each Jordan Block in the JCF of both matrices have the same corresponding powers.
   $\to c_T(x) = \det(J_{\lambda_1} - \lambda_1 I)\det(J_{\lambda_1} - \lambda_2 I) \ldots \det(J_{\lambda_1} - \lambda_k I)$
   $= c_{E_{\lambda_1}}(x)c_{E_{\lambda_2}}(x) \ldots c_{E_{\lambda_k}}(x) = (x - \lambda_1)^{t_1}(x - \lambda_2)^{t_2} \ldots (x - \lambda_k)^{t_k} = 0$

Where $t_1, t_2 \ldots, t_k$ are the powers of the minimal polynomials of each Jordan Block.

Lemma 9.3.8: Let $T : V \to V$ be a linear operator, and B, C be 2 ordered bases for V. Then,
$[T]_B = [T]_{B,B} = [I_v \circ T \circ I_v]_{B,B} = [I_v]_{B,C}[T]_C[I_v]_{C,B} = P^{-1}[T]_C P$

Kernels & Ranges: Let $T : V \to W$ be a linear transformation

- $Ker(T) = \{u \,|\, u \in V, T(u) = 0\}$ is the kernel of T
  - $Ker(T)$ is also the null space of the standard matrix A
- $R(T) = \{T(u) \,|\, u \in V\}$ is the range of T
- $R(T)$ and $Ker(T)$ are subspaces of W, V respectively

Dimension Theorem for Linear Transformations:
$\text{rank}(T) + \text{nullity}(T) = \text{rank}([T]_{C,B}) + \text{nullity}([T]_{C,B}) = \dim(V)$

- If $T : V \to W$ is an isomorphism, then
  $nullity(T) + rank(T) = 0 + \dim(W) = \dim(V) \to \dim(V) = \dim(W)$

Isomorphisms: For a mapping $T : V \to W$, T is *injective* if every element $w \in R(T)$ there exist at most 1 element $v \in V$ such that $T(v) = w$. T is injective iff $Ker(T) = \{0\} \leftrightarrow nullity(T) = 0$

- T is <u>surjective</u> if every element $w \in W$ has at least 1 element $v \in V$ such that v is the pre-image of w by T. T is surjective iff $R(T) = W \leftrightarrow rank(T) = \dim(W)$
- T is bijective if T is both injective and surjective. If T is bijective, there exists an inverse map $T^{-1} : W \to V$ such that $T \circ T^{-1} = I$.
  If T is a linear transformation, T is an *isomorphism from V to W*.

Let B, C be ordered bases for V and W respectively. Then,

- T is an isomorphism if $[T]_{C,B}$ is invertible.
- if T is an isomorphism, then $[T^{-1}]_{C,B} = [T]_{B,C}$

Definition 9.6.10: If there exists an isomorphism T from V to W, then V is said to be *isomorphic* to W $\to V \cong W$

- Let $T : U \to V, S : V \to W$ be two isomorphisms on vector spaces. Then, $T \circ S$ is also an isomorphism from U to W, and that $U \cong V \cong W \to U \cong W$ by transitivity.

Theorem 9.6.13: V and W are isomorphic iff $\dim(V) = \dim(W)$

10.2 Multilinear Forms: Let $T : V^n \to F$ be a mapping such that $V^n = V \times V \times \ldots \times V = \{u_1, u_2, \ldots, u_n \,|\, u_i \in V\}$ be a product space of V. T is called a *multilinear form* on V if for each $1 \leq i \leq n$
$T(u_1, \ldots, u_{i-1}, av + bw, u_{i+1}, \ldots, u_n) =$
$aT(u_1, \ldots, u_{i-1}, v, u_{i+1}, \ldots, u_n) + bT(u_1, \ldots, u_{i-1}, w, u_{i+1}, \ldots, u_n)$

- For bilinear form $T(u, v) : F^2 \times F^2 \to F^2$, the transformation can be represented as $T(U, v) = u^T A_{2x2} v$ such that $A_{2\times 2}$ is a matrix where $a_{ij} = T(e_1, e_2)$ where e_i is the standard basis for F
- This can be extended to $[u]_B A[v]_b$ where $a_{ij} \in A = T(b_i, b_j)$ for $B = \{b_1, \ldots, b_n\}$ as a basis for V

For a multilinear form $T(u_1, u_2, \ldots, u_n)$ where $u_1, \ldots, u_n$ are linearly independent vectors, then $T(u_1, u_2, \ldots, u_n) = 0$

Diagonalization: Let $T : V \to V$ be a linear operator. Then, a non-zero vector u is said to be an *eigenvector* associated with the *eigenvalue* λ if $T(u) = \lambda u$.

- $T(u) = \lambda u \to (\lambda I_V - T)(u) = 0 \to u \in \ker(\lambda I_V - T) \leftrightarrow nullity(\lambda I_V - T) > 0$ which implies λ is an eigenvalue of T iff $\det(\lambda I_V - T) = 0$
- Then, $\ker(\lambda I_V - T) = E_\lambda(T) \to$ is the eigenspace associated with the eigenvalue λ

<u>T is diagonalizable if and only if:</u>

- $m_T(x) = (x - \lambda_1)(x - \lambda_2) \ldots (x - \lambda_k)$
- $\dim(E_{\lambda_i}(T)) = r_i$ for $i = 1,2,...,k$
- $V = E_{\lambda_1}(T) \oplus E_{\lambda_2}(T) \oplus \ldots \oplus E_{\lambda_k}(T)$

Definition 11.1.10: $T : V \to V$ is diagonalizable iff there exists an ordered basis B such that $[T]_B$ is a diagonal matrix, or if every vector v in B is an eigenvector of T

- If B, C are two ordered bases for T such that $[T]_B = P^{-1}[T]_C P$, $[T]_B u = P^{-1}[T]_C P u = \lambda u \to [T]_C(Pu) = \lambda(Pu), P = [I_V]_{C,B}$
- Similarly if the eigenbasis B for T is such that $|B| < \dim(V)$, T is not diagonalizable

Algorithm 11.1.12: To determine if a linear operator $T : V \to V$ is diagonalizable, and find it's eigenvectors and eigenvalues:

1. Find a basis C for T and let $A = [T]_C$ be the standard matrix
2. Find $c_T(x) = \det(\lambda I_V - T) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2} \ldots$
3. Find the eigenspace B_i, $(\lambda_i I_V - T)u = 0$ for each eigenvalue λ_i
4. Let the eigenbasis $B = B_1 \cup B_2 \cup \ldots \cup B_k$ be a basis for V

Invariant Subspace: A subspace W is called $T : V \to V$ *invariant* if $T(u)$ is contained in W $\forall u \in W$: $T[W] = \{T(u) \,|\, u \in W\} \subset W$
The *restriction* $T|_W : W \to W = T|_W(u) = T(u), \forall u \in W$

- $W = span\{u, T(u), T^2(u), \dots\}$ is a T-invariant subspace of V known as the T-cyclic subspace generated by u

Theorem 11.3.10: Let $T : V \to V$ be a linear operator, and $W = span\{u, T(u), T^2(u), \dots\}$ is a $T-$cyclic subspace.
- $\dim(W) = k$ where k is the smallest integer such that $T^k(u)$ is a linear combination of $u, T(u), \dots, T^{k-1}(u)$
- $\{u, T(u), T^2(u), \dots, T^{k-1}(u)\}$ is a basis for W
- $T^k(u) = a_0u \ + \ a_1T(u) \ + \dots + \ a_kT^{k-1}(u) \to$ $c_{T|_W}(x) = -a_0 - a_1x - \dots - a_{k-1}x^{k-1} - x^k$ is the characteristic polynomial for the $T|_W$
- The coordinates $[T^k(u)]_B$ can be used to recover the characteristic polynomial of $T|_W$

Jordan Canonical Forms: The Jordan Block of order t $J_t(\lambda)$ is an upper triangular matrix with the diagonal entries $= \lambda$, and the entry $a_{i+1,i} = 1$.

$$J_t(\lambda) = \begin{pmatrix} \lambda & 1 & & 0 \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

Theorem 11.6.4: For a linear operator $T : V \to V$, suppose the at the characteristic polynomial of T can be factored over the field F. Then, there exists a basis B such that $[T]_B = J$ where $\lambda_1, \lambda_2, ..$ are eigenvalues of T.

$$J = \begin{pmatrix} J_{t_1}(\lambda_1) & & & 0 \\ & J_{t_2}(\lambda_2) & & \\ & & \ddots & \\ 0 & & & J_{t_m}(\lambda_m) \end{pmatrix}$$

For each Jordan Block, the eigenspace associated with that eigenvalue has dimension 1
- Then, each Jordan Block within J is the standard matrix with respect to a basis C for $T-$cyclic subspace W: $[T|_W]_C$, since the image of the transformation lies within W.
- Let $W_1, W_2, \dots, W_k$ be $T-$invariant subspaces such that $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$, and let $C_t = \{v_1^{(t)}, v_2^{(t)}, \dots, v_m^{(t)}\}$ be a basis for W_t. Then, $B = C_1 \cup C_2 \cup \dots \cup C_k$ is a basis for V.

$$[T]_B = \left([T(v_1^{(1)})]_B \cdots [T(v_{n_1}^{(1)})]_B \ [T(v_1^{(2)})]_B \cdots [T(v_{n_2}^{(2)})]_B \cdots [T(v_1^{(k)})]_B \cdots [T(v_{n_k}^{(k)})]_B\right)$$
$$= \begin{pmatrix} A_1 & 0 & & 0 \\ 0 & A_2 & & 0 \\ & & \ddots & \\ 0 & 0 & & A_k \end{pmatrix}$$

$c_T(x) = \det(xI_n - [T]_B) = \det(xI_{m_1} - A_1)\det(xI_{m_2} - A_2)\dots\det(xI_{m_k} - A_k)$
$= c_{A_1}(x)c_{A_2}(x)\dots c_{A_k}(x) = c_{T|_{W_1}}(x)c_{T|_{W_2}}(x)\dots c_{T|_{W_2}}(x)$
- Supposed that $c_T(x)$ can be factorised over into linear factors. Then, there exists a basis B such that $[T]_B = J$

Theorem 11.4.4 (Cayley Hamilton Theorem): Let T be a linear operator and $[T]_B = L_A = A$. $c_T(T) = c_T([T]_B) = c_A(A) = 0_V$
- If a matrix A is invertible, we can derive from the characteristic polynomial $c_A(A) = 0$ that $A^{-1} = p(A)$

Minimal Polynomials: The minimal polynomial $m_T(x)$ of T is the polynomial such that:
1. $m_T(x)$ is monic(coefficient of the highest power is 1)
2. $m_T(x) = O_V$
3. If $p(x)$ is a non-zero polynomial such that $p(T) = 0_V$, then the degree of $p(x)$ must greater than or equal to that of $m_T(x)$

The minimal polynomial can be used to distinguish matrices with the same characters polynomial, but different Jordan Blocks.

Theorem 11.5.8: For $c_T(x) = (x - \lambda_1)^{r_1}(x - \lambda_2)^{r_2}\dots(x - \lambda_k)^{r_k}$ and $m_T(x) = (x - \lambda_1)^{s_1}(x - \lambda_2)^{s_2}\dots(x - \lambda_k)^{s_k}$, $1 \le s_i \le r_i$
This means $(x - \lambda_i)^{s_i}$ strictly divides $c_T(x)$: $c_T(x) = (x - \lambda_i)^{s_i}q(x)$
Define $K_{\lambda_i}(T) = \ker((T - \lambda_i I)^{s_i})$ as the generalised eigenspace.
- $V = K_{\lambda_1}(T) \oplus K_{\lambda_2}(T) \oplus \dots \oplus K_{\lambda_1 k}(T)$
- $E_{\lambda_i}(T) \subset K_{\lambda_i}(T)$
- $m_{T|_{K_{\lambda_i}(T)}}(x) = (x - \lambda_i)^{s_i}$ & $m_{T|_{K_{\lambda_i}(T)}}(x) = (x - \lambda_i)^{r_i}$
- $\dim(K_{\lambda_i}(T)) = r_i$

Corollary 11.5.11: Let W be a $T-$ invariant subspace. Then, if T is diagonalizable, then $T|_W$ is also diagonalizable
- If S and T are two diagonalizable linear operators, then $[S]_B$ and $[T]_B$ are diagonal matrices if and only if $S \circ T = T \circ S$

Bezout's Identity: If there exists polynomials $p(x), q(x)$, such that $\gcd(p(x), q(x)) = 1$, there exists polynomials $a(x), b(x)$ such that: $a(x)p(x) \ + \ b(x)q(x) = 1 \to a(T) \circ p(T) \ + \ b(T) \circ q(T) = I_V$
From this identity, we have that:
- $\ker(q(T)) \subset R(p(T))$ & $\ker(q(T)) \cap \ker(P(T)) = \{0\}$

If $p(x), q(x)$ are such that $p(x)q(x) = 0$ (E.g $m_T(x) = p(x)q(x) \vee c_T(x) = p(x)q(x)$), then:
- $\ker(q(T)) = R(p(T))$ & $V = \ker(q(T)) \oplus \ker(P(T))$

Inner Product Space: is a vector space equipped with an inner product that follows the following axioms:

IP1 For all $u, v \in V$, $\langle u, v\rangle = \overline{\langle v, u\rangle} \to \langle u, v\rangle = \langle v, u\rangle if u, v \in R$
IP2 For all $u, v, w \in V$, $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$
IP3 For all $c \in F$, $u, v \in V$, $\langle cu, v\rangle = c\langle u, v\rangle$ and $\langle u, cv\rangle = \bar{c}\langle u, v\rangle$
IP4 $\langle 0, 0\rangle = 0$, and for non-zero $u \in V$, $\langle u, u\rangle > 0$. Additionally, $\langle u, 0\rangle = \langle 0, u\rangle = 0$

Theorem 12.2.4:
1. $|0| = \langle 0, 0\rangle = 0$, and $|u| = \langle u, u\rangle > 0$, $u \ne 0$
2. For $c \in F$, $|cu| = \sqrt{\langle cu, cu\rangle} = \sqrt{c \cdot \bar{c}\langle u, u\rangle} = |c||u|$
3. **(Cauchy-Schwartz Inequality)** if u, v are linearly dependent vectors $\in V$, then $|\langle u, v\rangle| \le |u||v|$
4. **(Triangle Inequality)** For any $u, v \in V$, $|u + v| \le |u| + |v|$

Orthogonality: Vectors $u, v \in V$ are said to be orthogonal to each other if their inner product $\langle u, v\rangle = 0$. They are further said to be orthonormal if additionally their norm $|u| = 1$

Theorem 12.3.6: Let $B = \{v_1, v_2, \dots, v_n\}$ be an orthonormal basis for a vector space V. For any vector $u \in V$, $u = \langle u, v_1\rangle v_1 \ + \ \langle u, v_2\rangle v_2 \dots \ + \ \langle u, v_n\rangle v_n$

Orthogonal Complements: Let W be a subspace of the vector space V. The orthogonal complement of W is defined to be the set $W^\perp = \{v \in V \,|\, \langle u, v\rangle \ \forall u \in W\}$

Theorem 12.4.3:
1. $W^\perp$ is a subspace of V
2. $W \cap W^\perp = \{0\} \to W \oplus W^\perp$ is a direct sum
3. If W is finite dimensional, $V = W \oplus W^\perp$
4. If V is finite dimensional, $\dim(V) = \dim(W) \ + \ \dim(W^\perp)$

Gram-Schmidt Process: Suppose that $\{u_1, u_2, \dots, u_n\}$ is a basis for finite dimensions space V. Then,
$v_1 = u_1$, $v_2 = u_2 \ - \dfrac{\langle u_2, v_1\rangle}{\langle v_1, v_1\rangle}v_1$, … ,
$v_n = u_n - \dfrac{\langle u_n, v_1\rangle}{\langle v_1, v_1\rangle}v_1 - \dfrac{\langle u_n, v_2\rangle}{\langle v_2, v_2\rangle}v_2 - \dots - \dfrac{\langle u_n, v_{n-1}\rangle}{\langle v_{n-1}, v_{n-1}\rangle}v_{n-1}$ gives us an orthogonal basis $\{v_1, v_2, \dots, v_n\}$ for V. Additionally, $\{w_1, w_2, \dots, w_n\} = \{\dfrac{v_1}{|v_1|}, \dfrac{v_2}{|v_2|}, \dots, \dfrac{v_n}{|v_n|}\}$ is orthonormal basis

Best Approximations: Let V be an inner product space and W a subspace of V such that, $V = W \oplus W^\perp$. Then for any $u \in V$: $d(u, proj_W(u)) \le d(u, w)$ for $w \in W \to proj_W(u)$ is the best approximation of u onto W

Adjoint Operator: Let V be an inner product space and T a linear operator on V. The *adjoint* of T is the linear operator T^* such that $\langle T(u), v\rangle = \langle u, T^*(v)\rangle$, $\forall u, v \in V$
- The adjoint operator is unique if it exists
- If V is finite dimensional, then the adjoint always exists
- If B is an ordered orthonormal basis for V, $[T^*]_B = ([T]_B)^*$
- $\text{rank}(T) = \text{rank}(T^*)$ and $\text{nullity}(T) = \text{nullity}\ (T^*)$

Proposition 12.5.7: Let S, T be linear operators over V, S^*, T^* exists
- $(S + T)^*$ exists and $(S + T)^* = S^* + T^*$
- $(S \circ T)^*$ exists and $(S \circ T)^* = S^* \circ T^*$
- $(T^*)^*$ exists and $(T^*)^* = T$

Unitary and Orthogonal Diagonalization: For a linear operator T:
- T is a *self-adjoint* operator if $T = T^*$ and $[T]_B = [T]_B^* = [T^*]_B$
- T is *normal* operator if $T \circ T^* = T^* \circ T$, $[T]_B[T]_B^* = [T]_B^*[T]_B$
- T is a *unitary* operator if $T \circ T^* = T^* \circ T = I_V$, $\mathbb{F} = \mathbb{C}$ and *orthogonal* operator if $\mathbb{F} = \mathbb{R}$. Then, $[T]_B[T]_B^* = [T]_B^*[T]_B = I_V$

Theorem 12.5.11: The following are equivalent:
- T is unitary ($\mathbb{F} = \mathbb{C}$) or orthogonal ($\mathbb{F} = \mathbb{R}$)
- For all $u, v \in V$, $\langle T(u), T(v)\rangle = \langle u, v\rangle$
- For all $u \in V$, $|T(u)| = |u|$
- There exists an orthonormal basis $\{w_1, w_2, \dots, w_n\}$ such that $\{T(w_1), T(w_2), \dots, T(w_n)\}$ is also an orthonormal basis