# MA2104 Multivariable Calculus

**Angle:** Let $a$ and $b$ be two non-zero vectors. Let the angle $\theta$ between $a$ and $b$. Then,

- $|a||b|\cos\theta = a \cdot b$
- $|a \times b| = |a||b|\sin\theta$

Let $P_1$ and $P_2$ be two planes with normals $n_1$ and $n_2$ respectively, and $\theta$ the angle between them.

- $\theta$ = angle between $n_1$ and $n_2 = \cos^{-1}\dfrac{n_1 \cdot n_2}{|n_1||n_2|}$
- Angle between $P_1$ and line $= 90^o - \theta$ (angle between line and $n_1$)

This also implies that $|a \cdot b| \le |a||b|$

**Projection**: projection of $a$ onto $b$ is defined as $proj_b a = \dfrac{a \cdot b}{\|b\|^2} b$ and

$\|proj_b a\| = \dfrac{(a \cdot b)}{|a|} = |b|\cos\theta$

**Cross Product**: defined as $a \times b = \det([i, j, k],\ a,\ b)$

- $a \times a = 0$, and $a \times b = 0$ implies b is parallel to $a$
- $a \times b$ is a vector perpendicular to both $a$ and $b$. Meaning, $a \cdot (a \times b) = 0$ and $b \cdot (a \times b) = 0$
- $|a \times b|$ is the area of parallelogram formed by $a$ and $b$
- $|a \cdot (b \times c)|$ = volume of parallelopiped formed by vectors $a$, $b$ and $c$

- $a \times b = -b \times a$ by property of determinant
- $a \cdot (b \times c) = (a \times b) \cdot c$
- $a \times (b \times c) = (a \cdot c)\ b\ -\ (a \cdot b)\ c$ which is a vector in the plane generated by $b$ and $c$

**Lines:** A line in $R^3$ can be determined by 2 points, $p_0$ and $p_1$, or a point $p_0$ and direction $v = p_1 - p_0$

$$r = p_0 + t\vec{v} \text{ for some scalar parameter } t \in R$$

Letting $r = \langle x, y, z \rangle$, $p_0 = \langle x_0, y_0, z_0 \rangle$, $v = \langle a, b, c \rangle$ the parametric equations for $r$ is:

$x = x_0 + ta,\ y = y_0 + tb,\ z = z_0 + tc$

$\rightarrow \dfrac{x - x_0}{a} = \dfrac{y - y_0}{b} = z - z_0$ for $c = 0$

**Planes**: A plane in $R^3$ is defined by 3 collinear points on a plane, or a point $p_0 = \langle x_0, y_0, z_0 \rangle$ in the plane and a normal vector $\vec{n} = \langle a, b, c \rangle$

$$r \rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

**Curves:** Let the curve $r(t)$ be parameterised by

$r(t) = \langle x = f(t),\ y = g(t),\ z = h(t) \rangle$

- $\lim_{t\to a} r(t) = \langle \lim_{t\to a} f(t),\ \lim_{t\to a} g(t),\ \lim_{t\to a} h(t) \rangle$
- If the limits of $f(t)$, $g(t)$, $h(t)$ exist at $a$, then $r(t)$ is <u>continuous</u> at $a$
- A curve $r(t)$ is differentiable at $t$ if the limit $\dfrac{dr}{dt} = r'(t) = \lim_{h\to 0}\dfrac{r(t+h) - r(t)}{h}$ exists

<u>Unit Tangent:</u> $T(t) = \dfrac{r'(t)}{|r(t)|}$ is tangent to $r(t)$

<u>Unit Normal:</u> $N(t) = \dfrac{T'(t)}{|T(t)|}$ is orthogonal to $T(t)$

<u>Binormal Vector:</u> $B(t) = T(t) \times N(t)$ is orthogonal to both $T(t)$ and $N(t)$

For a differentiable vectors $u(t)$, $v(t)$ and real-valued function $f(t)$:

- $\dfrac{d}{dt}[f(t)u(t)] = f'(t)u(t) + f(t)u'(t)$
- $\dfrac{d}{dt}[u(t) \cdot v(t)] = u'(t) \cdot v(t) + u(t) \cdot v'(t)$
- $\dfrac{d}{dt}[u(t) \times v(t)] = u'(t) \times v(t) + u(t) \times v'(t)$
- $\dfrac{d}{dt}[u(f(t))] = f'(t)u'(f(t))$

**Arc Length:** The approximation of the arc length of a curve $r(t)$ is given as:

$$s(t) = \sum_{k=1}^{n} |r(t_{k+1}) - r(t_k)| \approx \sum_{k=1}^{n} |r'(t_{k+1})|(t_{k+1} - t_k) = \int_a^b |r'(t)|$$

Note that $s'(t) = |r'(t)| \rightarrow$ the greater $r(t)$ changes, the larger arc length

**Curvature:** $\kappa(t) = \left|\dfrac{dT}{ds}\right| = \left|\dfrac{dT/dt}{ds/dt}\right| = \dfrac{|T'(t)|}{|r'(t)|} = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$

shows the ratio of change in the gradient over change in the arc length

- For circles with radius $r$, curvature $\kappa = \dfrac{1}{r}$

**Osculating Circle:** circle that comes close to approximating curve at $t$. For a given point $r(a)$, osculating circle has radius $r = \dfrac{1}{\kappa}$ and centre $r(a) + \dfrac{N}{\kappa}$

<u>Parametric Form:</u> $(r(a) + \dfrac{N}{\kappa}) + r\cos(t)T + r\sin(t)N$

<u>**Continuity:**</u> a function $f$ is continuous if $\lim_{x\to a} f(x) = f(a)$, meaning that the value of the function $f$ at $a$ agrees with the limit as $f$ approaches $a$

- **Clairaut's Theorem:** if $f_{xy}$ and $f_{yx}$ are continuous on domain $D$, then $f_{xy} = f_{yx}$

<u>**Limits:**</u> Let the function $f$ be defined on domain $D$. Then,

$\lim_{x\to a} f(x) = L$ if for every number $\epsilon > 0$, there exists a $\delta > 0$ such that for $(x, y) \in D$ and $0 < \sqrt{(x-a)^2} < \delta$, then $|f(x, y) - L| < \epsilon$

- This means for every point $x$ that is within $\delta$ of $a$, the value of $f(x)$ should not be $\epsilon$ away from $L$. $\rightarrow$ Must choose arbitrary $\delta$ to fulfil this condition
- Limit exists <u>*only if $(x, y)$ approaches $(a, b)$ from every direction$\rightarrow$if limit $L$ is different when $(a, b)$ is approached in two different directions, then $f$ has no limit at $(a, b)$*</u>

<u>**Differentiability:**</u> partial derivatives at point does not determine continuity, but differentiability $\rightarrow$ continuity. Function $f$ is defined to be differentiable at $(a, b)$ if there exists a vector $\nabla f(a, b)$ such that

$$\lim_{a,b\to 0}\frac{f(x+a, y+b) - f(x, y) - \nabla f(x, y) \cdot \langle a, b \rangle}{|\langle a, b \rangle|} = 0.$$

$\nabla f(x, y)$ is the gradient vector of $f$ where $\nabla f(x, y) = \left\langle \dfrac{\delta f}{\delta x}, \dfrac{\delta f}{\delta y} \right\rangle$

- Alternatively, if $f_x$ and $f_y$ exist near $(a, b)$ and are continuous on $(a, b)$, then the function $f$ is differentiable at $(a, b)$

**Directional Directives:** Let $u = \langle a, b \rangle$ be a unit vector. The directional directive of $f$ at point $(x_0, y_0)$ in the direction of $u$ is defined as

$$D_u f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} = \nabla f(x_0, y_0) \cdot u$$

- Direction of steepest ascent at $(x_0, y_0)$ is $\dfrac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$

**Particle Motion:** if $r(t)$ is the position of a particle at time $t$, then:

- $r'(t) = v(t)$ represents the velocity of the particle
- $r''(t) = a(t) = a_T T + a_N N$ represents the acceleration
- Acceleration can be expressed over the tangential and normal components of $r(t)$ where:
  - $a_T = v'(t) = \dfrac{r'(t) \cdot r''(t)}{|r'(t)|}$, $a_N = \kappa v^2 = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|}$

**Tangent Plane:** Suppose a function $f$ is differentiable at $(a, b)$. Let $C_1$ be the curve of the intersection of surface $f$ with plane $y = b$. Then:

$r_1(t) = \langle t, b, f(t, b) \rangle$ represents value of $r(t)$ at $y = b$ as $x$ shifts and

$r_1'(t) = \langle 1,\ 0,\ f_x(t, b) \rangle$ is the direction vector in the $x$ direction of $f$ at $y = b$ since $z = f(x, y)$ only affected by change in $x$

Similarly, letting $C_2$ be the curve intersecting with $f$ with plane $x = a$

$r_2(t) = \langle a, t, f(a, t) \rangle$ represents value of $r(t)$ at $x = a$ as $y$ shifts and

$r_2'(t) = \langle 0,\ 1,\ f_y(a, t) \rangle$ is the direction vector in the $y$ direction of $f$ at $x = a$ since $z = f(x, y)$ only affected by change in $y$

Tangent plane normal $\vec{n} = r'(a) \times r'(b) = \langle -f_x(a, b),\ -f_y(a, b),\ 1 \rangle$

Tangent Plane Eqn: $-f_x(a, b)(x - a) - f_y(a, b)(y - b) + z - f(a, b) = 0$

$\rightarrow z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$

- For any curve $r(t)$ that passes through $(a, b)$ at $t = t_0$, then the tangent vector $r'(t_0)$ at $(a, b)$ is perpendicular to the normal above $\vec{n}$
- The gradient vector $\nabla f(a, b)$ is perpendicular to the tangent line $f(a, b) = k$ to the level curve $f(x, y) = k$ at the point $(a, b) \rightarrow$ value of $f$ does not change along the surface $f(a, b) = k$
- If a curve lies in the intersection of 2 level surfaces $f$ and $g$, it's tangent line lies in the tangent planes to both $f$ and $g \rightarrow$ tangent line is perpendicular to both $\nabla f$ and $\nabla g$

**Second Derivative Test:** $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}(a, b)^2$

- $D > 0,\ f_{xx} > 0 \rightarrow f(a, b)$ is local minima
- $D > 0,\ f_{xx} < 0 \rightarrow f(a, b)$ is local maxima
- $D < 0 \rightarrow f(a, b)$ is a saddle point

$\det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$

**Lagrange Multipliers:** Given objective function $f$, maximization/minimization of $f$ subject to constraints $g = k$ and $h = j$ occurs when $\nabla f = \lambda\nabla g + \mu\nabla h \rightarrow$ gradient of $f$ proportional to gradient of $g$ & $h$.

If the constraints are subject to $g \le k$ and $h \le j$, then

- Find critical points of $f$ that lie within the given inequality constraints
- Find boundary values: $\nabla f = \lambda\nabla g + \mu\nabla h$ for $g \le k$ and $h \le j$

**Multiple Integral:** A double integral can be formulated as an iterated integral by integrating with respect to either variable first.

$$\iint_R f(x, y)\,dA = \lim_{\triangle A_{ij}\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\triangle A_{ij} = \lim_{\triangle x_{ij}\to 0, \triangle y_{ij}\to 0}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\triangle x_{ij}\triangle y_{ij}$$

$$= \lim_{\triangle x_{ij}\to 0}\sum_{i=1}^{m}[\lim_{\triangle x_{ij}\to 0}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\triangle y_{ij}]\triangle x_{ij} = \int_a^b[\int_c^d f(x, y)\,dy]\,dx = \int_c^d[\int_a^b f(x, y)\,dx]\,dy$$

- If $f(x, y) > g(x, y)$ for all $(x, y) \in R$, $\iint_R f(x, y)\,dA > \iint_R g(x, y)\,dA$
- Some integrals are impossible to integrate as it is $\rightarrow$ Need to change the order of integration: $\int_a^b[\int_c^d f(x, y)\,dy]\,dx \rightarrow \int_c^d[\int_a^b f(x, y)\,dx]\,dy$

**Polar Coordinates:** Let $x = r\cos\theta$, $y = r\sin\theta$. Then we can express the polar coordinates of our function as such: $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$.

The area element $dA = (r\,d\theta)(dr)$ as change in the angle $\theta$ depends on $r$.

$$\iint_R f(x, y)\,dA = \int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)r\,d\theta\,dr; \{\alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

- Some integrals in $x$ and $y$ are better formulated as integral over polar coordinates for ease of integration

**Triple Integrals:** The bounded volume of $f(x, y, z)$ is

$$\iiint_E f(x,y,z)dV = \int_c^d \int_a^b [\int_{u1(x,y)}^{u2(x,y)} f(x,y,z)\ dz]\ dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} [\int_{u1(x,y)}^{u2(x,y)} f(r\cos\theta, r\sin\theta, z)\ dz]\ r dr d\theta$$

**Spherical Coordinates:** *Spherical coordinates* of a point $P$ is $(\rho, \theta, \phi)$ where $0 \le \phi \le \pi$ is the angle between $x$-axis and $y$-axis, $\phi \ge 0$ the angle between $xy$-plane and the $z$-axis. $\rho =$ the radial distance from origin.

- $x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi$
- $\rho = \sqrt{x^2+y^2+z^2},\ \theta = \tan^{-1}\frac{y}{x},\ \phi = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}}$

$$\iiint_E fx,y,z)dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

- Here, $\frac{\delta(x,y,z)}{\delta(\rho,\theta,\phi)} = \rho^2\sin\phi$ is the Jacobian matrix from rectangular to spherical coordinates

**Change of Variables:** Given a transformation $T(u,v) = x(u,v), y(u,v)$, the approximation to the area $R = \vec{a} \times \vec{b}$ bounded under the $xy$-plane by the area $S$ under the $uv$-plane can be modelled as follows using the curve $r(u,v) = <x(u,v), y(u,v)>$:

- $\vec{a} = r(u_0 + \triangle u, v_0) - r(u_0,v_0) = \frac{r(u_0 + \triangle u, v_0) - r(u_0,v_0)}{\triangle u}\triangle u = \frac{\delta r}{\delta u}(u_0,v_0)\triangle u = <\frac{\delta x}{\delta u}, \frac{\delta y}{\delta u}> \triangle u$
- Similarly, $\vec{b} = r(u_0, v_0 + \triangle v) - r(u_0,v_0) = \frac{\delta r}{\delta u}(u_0,v_0)\triangle v = <\frac{\delta x}{\delta v}, \frac{\delta y}{\delta v}> \triangle v$
- Then, the area of $R$ is the area of the parallelogram bounded by $\vec{a}$ and $\vec{b}$

  $R = |\vec{a} \times \vec{b}| = |<\frac{\delta x}{\delta u}, \frac{\delta y}{\delta u}> \triangle u \times <\frac{\delta x}{\delta v}, \frac{\delta y}{\delta v}> \triangle v| = |\frac{\delta(x,y)}{\delta(u,v)}|\triangle u \triangle v$

  where the Jacobian of the transformation $T$ is $\frac{\partial(x,y)}{\partial(u,v)} \stackrel{\text{def}}{=} \det\begin{pmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{pmatrix} \neq 0$

- Then, the double integral over the area $R$ can be derived as:

  $\iint_R f(x,y)dA = \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}, y_{ij})R_{ij} = \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}, y_{ij})|\frac{\delta(x,y)}{\delta(u,v)}|\triangle u \triangle v = \iint_S f(x(u,v), y(u,v))|\frac{\delta(x,y)}{\delta(u,v)}|du dv$

- The Jacobian matrix of the inverse transformation $T^{-1} = (\frac{\delta(x,y)}{\delta(u,v)})^{-1} = \frac{\delta(u,v)}{\delta(x,y)}$

  Then the change of variable from $f(u,v)$ to $f(x,y)$ is as such:

  $$\iint_S f(u,v)dA = \iint_R f(u(x,y), v(x,y))|\frac{\delta(u,v)}{\delta(x,y)}|dx dy$$

  And the Jacobian of $T = \frac{\delta(x,y)}{\delta(u,v)} = \frac{1}{\frac{\delta(u,v)}{\delta(x,y)}}$

**Integrals over Curves:**

Integral of scalar functions over a path: Given a curve $C = r(t)$ parametrised by $t$, and $s$ the arc length of the curve $C = r(t), a \le t \le b$:

$$\int_C f(x,y)ds = \int_a^b f(x(t), y(t))s'(t)dt = \int_a^b f(x(t), y(t))|r'(t)|dt$$

Integral of vector functions over a path: Given a curve $C = r(t)$ parametrised by $t$, for $a \le t \le b$, the line integral of $F$ along $C$ is:

$$\int_C F\ dr = \int_C F(r(t))\ dr = \int_a^b < P(x,y), Q(,xy) > \cdot r'(t)dt = \int_a^b F(r(t)) \cdot < x'(t), y'(t) > dt$$

$$= \int_a^b P(x(t), y(t)) \cdot x'(t)dt + \int_a^b Q(x(t), y(t)) \cdot y'(t)dt = \int_C Pdx + \int_C Qdy$$

Fundamental Theorem of Calculus for Line Integrals:

If $F$ is a *conservative* vector field, which implies that there exists a potential scalar function $f$ defined on the set $D$ such that $F = \nabla f$, then

$$\int_C F\ dr = \int_C \nabla f\ dr = \int_a^b \nabla f(r(t)) \cdot r'(t)\ dr = f(r(b)) - f(r(a))$$

Path Independence: Let $F$ is a conservative vector field. If 2 smooth curves $C_1$ and $C_2$ in $D$ with the same initial and end points, $t = a, t = b$, then $\int_{C_1} F\ dr = \int_{C_2} F\ dr \rightarrow$ Path integral does not depend on exact path

- If $\nabla \times F = 0$, then $F$ is a conservative vector field $\rightarrow F$ is path independent

Theorem 26: Let $F = < P(x,y), Q(x,y) >$ be a conservative vector field such that $F = \nabla f$ for some potential function $f$ which is continuous on $D$. Then, $f_{xy} = f_{yx} \rightarrow (f_x)_y = (f_y)_x \rightarrow P_y = Q_x$

- *Conversely, if both Theorem 26 and Path Independence does not hold, $F$ is not a conservative vector field*
- Sufficient condition for converse: If region $D$ is simply connected such that for any simple closed curve $C \in D$, $\int_C F\ dr = \iint_D (Q_x - P_y)dA = 0 \rightarrow$ inside of $C$ is also contained in $D$, then $F$ is conservative if $P_x = Q_y$

**Green's Theorem:** Let $C$ be a simple closed curve in the region $D$, and let $P$ and $Q$ have continuous first order partials. Then,

$$\int_C F\ dr = \int_C P\ dx + Q\ dy = (\int_{C_1} + \int_{C_2} + \int_{-C_3} + \int_{-C_4})P\ dx + Q\ dy$$

$$= \int_a^b P(t, g_1(t))\ dt + \int_a^b -P(t, g_2(t))\ dt + \int_{g_1(t}^{g_2(t)} Q(b,t)\ dt + \int_{g_1(t}^{g_2(t)} -Q(a,t)\ dt$$

$$= -\int_a^b \int_{g_1(t)}^{g_2(t)} P(t,y)\ dydt + \int_{g_1(t)}^{g_2(t)} \int_a^b Q(x,t)\ dxdt$$

$$= -\iint_D \frac{\delta P}{\delta y} + \iint_D \frac{\delta Q}{\delta x} = \iint_D (Q_x - P_y)dA$$

- First Vector Form(curl): $\int_C F \cdot dr = \iint_D (\nabla \times F) \cdot k\ dA$
- Second Vector Form(Divergence): $\int_C f \cdot ds = \int_C F \cdot n\ ds = \iint_D (\nabla \cdot F)\ dA$

  where $n$ is the normal to the parametrized curve $C = r(t) = \langle x(t), y(t), z(t)\rangle$
- The first and second vector forms of Green's Theorem allows us to perform integration in the original coordinates $(x, y, z)$ instead of the parameter $t$

**Curl:** The *curl* of $F$ is the vector field is given as $\nabla \times \mathbf{F} = \det\begin{pmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R\end{pmatrix}$

**Divergence:** The *divergence* of $F$ is the scalar function $\nabla \cdot F = (\frac{\delta}{\delta x}i + \frac{\delta}{\delta y}j + \frac{\delta}{\delta z}k) \cdot F = \frac{\delta P}{\delta x} + \frac{\delta Q}{\delta y} + \frac{\delta R}{\delta z}$

- Suppose that $f$ has continuous second order partials. Then $\nabla \times (\nabla \cdot f) = 0 = \nabla \times F$ if $F$ is conservative $\rightarrow$ can be used to prove conservative and path independence
- Suppose that $F = Pi + Qj + Rk$ where $P, Q, R$ are continuous with second order partials. Then, $\nabla \cdot (\nabla \times F) = 0$

**Integral over Surfaces:** Let $r(u,v) = \langle x, y, z\rangle$ be the parameterized curve mapping from $r: D \rightarrow R^3$. Then, the tangent plane to the surface at point $(u_0, v_0)$ is: $\vec{n} = \frac{\delta r}{\delta u}(u_0,v_0) \times \frac{\delta r}{\delta v}(u_0,v_0) = r_u(u_0,v_0) \times r_v(u_0,v_0)$

- Area of parametrised surface:

  $$\lim_{u\rightarrow 0, v \rightarrow 0} \sum |r_u \times r_v| \triangle u \triangle v = \iint_D |r_u \times r_v| dA$$

- Surface Integral of a scalar function: Given a scalar function $f$ and a surface $S$ parametrised by $r: D \rightarrow R^3$, then:

  $$\iint_S f(x,y,z)dS = \iint_D f(r)|r_u \times r_v| du dv$$

- Surface Integral of vector field: Given a continuous vector field $F$ on an orientable surface $S$ parametrised by $r: D \rightarrow R^3$, the *flux* of $F$ is:

  $$\iint_S F \cdot dS = \iint_S F \cdot n\ dS = \iint_S F \cdot \frac{r_u \times r_v}{|r_u \times r_v|}dS = \iint_D F \cdot (r_u \times r_v)\ dA$$

  where the outward pointing normal $\vec{n} = \frac{\langle x(t), -y(t)\rangle}{|r'(t)|}$

**Stokes Theorem:** Let $S$ be a surface bounded by a simple closed positively oriented curve $C$. Let $F$ be a vector field with continuous partial derivatives.

$$\int_C F \cdot dr = \iint_D (\nabla \times F) \cdot \vec{k}\ dA = \iint_S (\nabla \times F) \cdot \vec{n}\ dS = \iint_S (\nabla \times F) \cdot dS$$

**Divergence Theorem:** Let $E$ be a simple solid region, and $S$ be the bounding surface of $E$ with positive outward orientations. Let $F$ be a vector field with continuous partial derivatives.

$$\iint_S F \cdot dS = \iint_S F \cdot \vec{n}\ dS = \iiint_E (\nabla \cdot F)\ dV$$

- Given a closed region that comprises of 2 surfaces $S$ & $S'$, then by the Divergence Theorem $\iint_S F \cdot dS + \iint_{S'} F \cdot dS' = \iiint_E (\nabla \cdot F)\ dV$
- Given a vector field $F = \langle x, y\rangle$, we can split the vector field into 2 parts: $F = \langle x, y\rangle = \langle x_1 + x_2, y_1 + y_2\rangle = \langle x_1, y_1\rangle + \langle x_2, y_2\rangle = F_1 + F_2$
  - This can be useful for when we already have the result for one part(e.g $F_1$), or separation may produce conservative fields.