Weak and strong synchronization of chaos

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(Received 23 August 1996)

It is shown that synchronization in unidirectionally coupled chaotic systems develops in two stages as the coupling strength is increased. The first stage is characterized by a weak synchronization, i.e., a response system subjected to a driving system undergoes a transition and exhibits a behavior completely insensitive to initial conditions. Further increase of the coupling strength causes the dimension decrease of the overall dynamics and leads finally to a strong synchronization. In this stage, the dimension of the strange attractor in the full phase space of the two systems saturates to the dimension of the driving attractor. [S1063-651X(96)51211-0]

PACS number(s): 05.45.+b

A generic feature of nonlinear systems exhibiting chaotic motions is the extreme sensitivity to initial conditions. This feature, known as the ''butterfly effect,'' would seem to defy synchronization among dynamical variables in coupled chaotic systems. Nonetheless, coupled systems with certain properties of symmetry may exhibit synchronized chaotic motions. Most frequently a situation is studied where the complete system consists of *coupled identical subsystems*. Many different examples of such a type have been introduced [1,2]. In these examples, the synchronization is easy to detect. It appears as an actual equality of the corresponding variables of the coupled systems as they evolve in time. Geometrically, this implies a collapse of the overall evolution onto the identity hyperplane in the full phase space. We refer to this type of synchronization as a *conventional synchronization* (CS).

A more complicated situation arises when *coupled nonidentical chaotic systems* are investigated. For essentially different chaotic systems, the phase space does not contain any trivial invariant objects to which one can expect a collapse of the overall evolution. The central questions in this case are how to generalize a mathematical definition of chaotic synchronization for such systems and how to detect it in a real experimental situation.

Recently Rulkov *et al.* [3] considered this problem for the case of forced synchronization. This implies that the full system consists of an *autonomous driving* subsystem unidirectionally linked to a *response* subsystem. *Generalized synchronization* (GS) was taken to occur if there is a map Φ from the trajectories $X(t)$ of the attractor in the driving space **D** to the trajectories $Y(t)$ in the response space **R**; $Y(t)=\Phi(X(t))$. For nonidentical driving and response systems, this map differs from identity and this complicates the detection of the GS. To recognize the GS at a real experimental situation, Rulkov *et al.* [3] suggested a practical algorithm based on the assumption that Φ is a *smooth* (differentiable) map. The algorithm was tested on artificially constructed examples with *a priori* known map Φ.

In this paper, we consider both identical and essentially different coupled chaotic systems and show that in both cases the onset of synchronization is characterized by an *unsmooth* map Φ that becomes smooth only at sufficiently large coupling strength. We call the synchronization characterized by a *smooth* and an *unsmooth map* Φ *a strong* (SS) and *weak* (WS) synchronization, respectively. The CS is a particular case of the SS.

Next, unidirectionally coupled chaotic systems are considered which are of the following principal form:

$$\dot{X}=F(X), \tag{1a}$$

$$\dot{Y}=G(Y)+kP(X,Y). \tag{1b}$$

Here $X\equiv\{x_1,x_2,\ldots,x_d\}$ and $Y\equiv\{y_1,y_2,\ldots,y_r\}$ denote state vectors in d-dimensional space **D** and r-dimensional space **R**, respectively. F and G define the vector fields of the driving and response systems. P denotes a coupling term and k is a scalar parameter defining the coupling strength.

One can show that there exists some map Φ (not necessarily smooth) between X and Y if under the action of driving perturbations the response system ''forgets'' its initial conditions, i.e., when the response system becomes a ''*stable*'' *system* [4]. This suggests the following physical criterion to detect the GS in an experiment. Suppose that we can construct an auxiliary response system $Y'\in\mathbf{R}'$ identical with Y and link it to the driving system X in the same way as Y is linked to X [5]:

$$\dot{Y}'=G(Y')+kP(X,Y'). \tag{2}$$

The GS between X and Y occurs if there is the CS between Y and Y'. To show that CS between Y and Y' results in relationship $Y=\Phi(X)$, let us denote the solution of Eqs. (1) by $X(t)=\Psi_x(X_0,t)$ and $Y(t)=\Psi_y(X(t),Y_0,t)$, where $X=X_0$ and $Y=Y_0$ are the initial conditions at $t=0$. The CS between Y and Y' implies $\lim_{t\to\infty}\|Y-Y'\| =\lim_{t\to\infty}\|\Psi_y(X(t),Y_0,t)-\Psi_y(X(t),Y_0',t)\|=0$ for arbitrary initial conditions Y_0 and Y_0'. From this follows that Ψ_y is asymptotically independent of Y_0. At $t\to\infty$, Ψ_y is also independent directly on time t. Indeed, let $\widetilde{Y}_0'=\Psi_y(X(\tilde{t}),Y_0',\tilde{t})$

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1063-651X/96/54(5)/4508(4)/$10.00

be the state of the system Y' at an intermediate time $\tilde{t}<t$. Then the state of the system Y' at time t can be expressed as $Y'(t)=\Psi_y(X(t),\tilde{Y}'_0,t-\tilde{t})$ and the synchronization condition becomes $\lim_{t\to\infty}\|\Psi_y(X(t),Y_0,t)-\Psi_y(X(t),\tilde{Y}'_0,t-\tilde{t})\|=0$ for any $\tilde{t}<t$. It follows that at $t\to\infty$, Ψ_y is independent of both Y_0 and explicit time t. Thus, in the limit $t\to\infty$, we obtain a relationship between X and Y in the following form: $Y=\lim_{t\to\infty}\Psi_y(X(t),Y_0,t)\equiv\Phi(X(t))$.

Note that CS between Y and Y' does not guarantee the smoothness of Φ. Ding *et al.* [6] have shown that unsmooth maps do not preserve the dimension of strange attractors. A simple example of this type is the Weierstrass function $u=F_w(v)\equiv\sum_{n=1}^{\infty}\cos(n^{\beta}v)/n^{\alpha}$, specifying a continuous but not differentiable map of points on the v axis (with the dimension equal to 1) to points on the Weierstrass curve $v\to[v,u=F_w(v)]$ with a fractal dimension between 1 and 2 for typical α and β satisfying $1<\alpha<\beta$. Thus, for unsmooth map Φ, we can expect that the dimension of a strange attractor in the whole phase space $\mathbf{D}\oplus\mathbf{R}$ is larger than the dimension of driving attractor in $\mathbf{D}$ space. For smooth Φ, we can expect that these two dimensions are equal in magnitude.

Due to the identity of the original [Eq. (1b)] and the auxiliary [Eq. (2)] response systems, an extended phase space $\mathbf{D}\oplus\mathbf{R}\oplus\mathbf{R}'$ contains an invariant manifold $Y'=Y$. The stability of this defines the condition of synchronization between Y and Y'. The limit $\delta Y=Y'-Y\to 0$ leads to the variational equation of the response system

$$\frac{d\,\delta Y}{dt}=\delta Y\frac{\partial}{\partial Y}\{G(Y)+kP(X,Y)\} \tag{3}$$

defining r conditional Lyapunov exponents λ_j^R $j=1,2,\ldots,r$ [1]. A necessary condition for synchronization is $\lambda_j^R<0$, $j=1,2,\ldots,r$ [7].

In the following, we illustrate some properties of GS with specific examples. As usual in such problems, we start with a discrete time system. As a first simple example, two coupled identical one-dimensional maps are considered,

$$x(i+1)=f(x(i)), \tag{4a}$$

$$y(i+1)=f(y(i))+k\{f(x(i))-f(y(i))\}, \tag{4b}$$

with the following auxiliary response map:

$$y'(i+1)=f(y'(i))+k\{f(x(i))-f(y'(i))\}. \tag{5}$$

At any coupling strength k, Eqs. (4) have an invariant manifold $y=x$ and hence admit the CS, which in this case is equivalent to the SS. Figure 1 shows the phase portraits of the system for the logistic map $f(x)=4ax(1-x)$ in x-y and y-y' coordinates at $a=1$ and various values of parameter k. With the increase of k, the synchronization occurs first between y and y' and later on between x and y. Thus, the WS is observed even for identical systems and it precedes the SS. The thresholds of WS and SS are determined by two different Lyapunov exponents, namely, the conditional Lyapunov exponent,

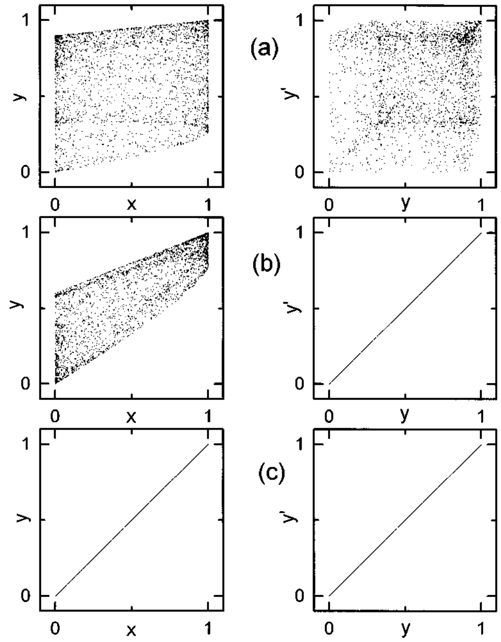

FIG. 1. x-y and y-y' phase portraits of coupled logistic maps described by Eqs. (4) and (5) with $f(x)=4ax(1-x)$ at $a=1$ and various values of the coupling strength k: (a) $k=0.1$, unsynchronized state; (b) $k=0.4$, WS; (c) $k=0.6$, SS.

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$$\lambda^R=\ln(1-k)+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln|f'(y(i))| \tag{6}$$

defining the stability of the invariant manifold $y'=y$, and the transverse Lyapunov exponent of the invariant manifold $y=x$,

$$\lambda_0=\ln(1-k)+\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln|f'(x(i))|. \tag{7}$$

The dependence of these exponents on k is shown in Fig. 2(a). $\lambda^R(k)$ becomes zero at two characteristic values of the coupling strength k_w and k_s, corresponding to the thresholds of WS and SS, respectively. Above the last threshold $k>k_s$, these two exponents coincide, $\lambda_0(k)=\lambda^R(k)$. For the logistic map, Eq. (7) transforms to $\lambda_0(k)=\ln(1-k)+\lambda^D$, where $\lambda^D=\ln 2$ is the Lyapunov exponent of the driving system and the threshold of SS is equal $k_s=1-\exp(-\lambda^D)=0.5$.

In a real experiment, the CS between systems Y and Y' will be partially disturbed by noise and a small mismatch between parameters of these systems. These factors will result in a finite amplitude of the deviation $Y'-Y$. The rms of this deviation $s=\sqrt{\langle(Y'-Y)^2\rangle}$ depends on the amplitude of noise α_n as $s\propto\alpha_n^{\gamma}$. In the case of Eqs. (4), and (5), $\gamma\approx 0.12$ for the WS and $\gamma=1$ for the SS [the inset in Fig. 2(a)]. The same scaling laws are observed for s vs Δa where Δa is the deviation between parameters of systems y and y' [$a=1$ for Eqs. (4) and $a=1-\Delta a$ for Eq. (5)]. Thus the WS is much more sensitive ($\gamma<1$) to noise and parameter deviation than the SS ($\gamma=1$).

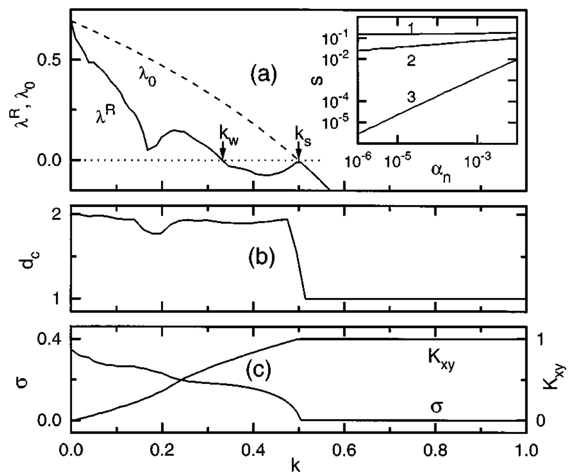

FIG. 2. (a) Lyapunov exponents λ^R and λ_0; (b) correlation dimension d_c of attractor in the x-y plane, and (c) thickness σ and cross correlator K_{xy} for coupled logistic maps as functions of coupling strength k. λ^R and λ_0 are calculated from Eqs. (6) and (7), respectively. d_c is determined from $N=50\,000$ data points $[x(i),y(i),i=1,\ldots,N]$. The thickness σ is calculated as follows. In an $\epsilon=0.001$ proximity of a given point $x=x(i)$, a local linear interpolation of the map $y=\Phi(x)$ is applied using a least squares fit. The local mean square deviations are averaged over $N_1=5000$ arbitrarily chosen reference points on the strange attractor and the root of this value is chosen as the thickness σ. The whole number of the data points is $N=50\,000$. The inset in (a) shows the deviation s vs amplitude of noise α_n: (1) unsynchronized state at $k=0.3$; (2) WS at $k=0.4$; (3) SS at $k=0.6$. At every iterate, random numbers uniformly distributed in the interval $[-\alpha_n/2,\alpha_n/2]$ have been added to the variables of Eqs. (4) and (5).

The WS observed with the help of an auxiliary response system y' may show no evidence in x-y coordinates. At $k_w<k<k_s$ there exists a relationship $y=\Phi(x)$; however, this map is unsmooth and has a fractal structure [Fig. 1(b), left]. The correlation dimension [8] d_c of an attractor lying in the x-y plane does not exhibit any characteristic changes at the threshold k_w [Fig. 2(b)]. An abrupt dimension decrease is observed only at the threshold k_s, where Φ is turned to identity. At the threshold of WS, there are no characteristic changes in cross correlator K_{xy} between x and y [Fig. 2(c)], although here this correlation is rather large, $K_{xy}(k_w)\approx 0.71$. To estimate the smoothness of the map Φ, we calculated its mean local "thickness" σ defining the deviation of points lying on the map from its local linear interpolation [9] [Fig 2(c)]. The WS also shows no evidence of this characteristic. The thickness σ decreases abruptly only at $k=k_s$ like the dimension d_c. At $k>k_s$, Φ becomes a smooth map; the thickness σ turns to zero and the global dimension d_c becomes equal to the dimension d_c^D of the strange attractor of the driving system, $d_c=d_c^D=1$.

Although this example is based on an uninvertable logistic map, the same effects are observed in coupled invertable Henon maps.

As a second example, we present the GS in essentially different time-continuous systems:

$$\dot{x}_1=-\alpha\{x_2+x_3\}, \tag{8a}$$

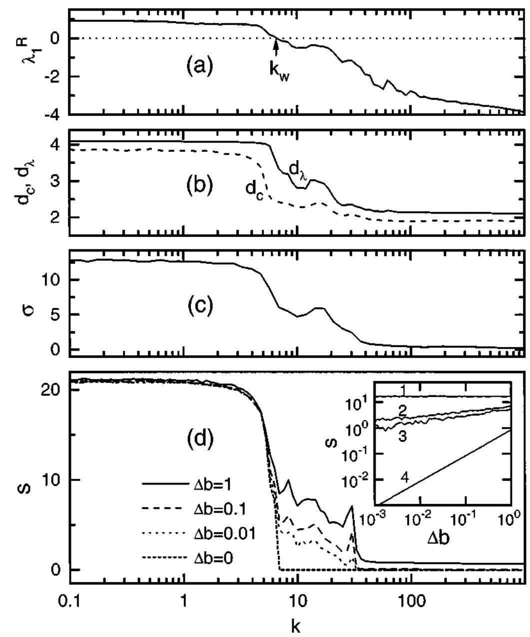

FIG. 3. (a) Maximal conditional Lyapunov exponent λ_1^R; (b) correlation d_c and Lyapunov d_λ dimension of attractor in $\mathbf{D}\oplus\mathbf{R}$ space; (c) thickness σ, and (d) deviation s for coupled Rössler and Lorenz systems as functions of coupling strength k. d_c and σ are calculated from $N=50\,000$ data points $[X(i\Delta t), Y(i\Delta t),i=1,\ldots,N]$ with $\Delta t=0.5$. σ is calculated similarly as in the first example, except that a local linear interpolation of Φ is performed in a high-dimensional space. Here $\epsilon=0.01$ defines a local parallelepiped $|x_j-x_j(i\Delta t)|/L_j<\epsilon$, $j=1,2,3$ around a reference point $X(i\Delta t)$, where L_j is the size of the attractor along the j coordinate. σ is averaged over $N_1=5000$ reference points, $b=28$, $\alpha=6$. The inset in (c) shows s vs deviation of the parameter Δb: (1) unsynchronized state at $k=5$; (2) WS at $k=10$; (3) WS at $k=20$; (4) SS at $k=50$.

$$\dot{x}_2=\alpha\{x_1+0.2x_2\}, \tag{8b}$$

$$\dot{x}_3=\alpha\{0.2+x_3(x_1-5.7)\}, \tag{8c}$$

$$\dot{y}_1=10(-y_1+y_2), \tag{9a}$$

$$\dot{y}_2=by_1-y_2-y_1y_3+kx_2, \tag{9b}$$

$$\dot{y}_3=y_1y_2-8/3y_3. \tag{9c}$$

These equations describe the coupling of the Rössler [10] [Eqs. (8), driving] and the Lorenz [11] [Eqs. (9), response] systems. The multiplier α is introduced to control the characteristic time scale of the driving system. The perturbation kx_2 is applied only to the second equation of the Lorenz system and does not contain any feedback term. In addition to Eqs. (8) and (9), we consider an auxiliary response system that is equivalent to the system of Eqs. (9) except that the variables y_i are replaced with y_i'

Despite the lack of any symmetry in Eqs. (8) and (9) admitting the CS, the GS in the form of WS and SS can be still observed in this system. As in the first example, the GS can be easily detected with the help of the auxiliary response system as the CS betwen Y and Y'. The threshold of WS is determined by $\lambda_1^R(k)=0$ and is equal to $k_w\approx 6.66$ [Fig. 3(a)].

In this model, the onset of WS is characterized by considerable decrease of both the dimension [Fig. 3(b)] and the thickness of the map [Fig. 3(c)]. At large $k \gtrsim 40$, the global dimension of the strange attractor saturates to the dimension of the driving attractor, and the thickness saturates to zero. These features indicate the smoothness of Φ and hence the onset of SS.

Generally the threshold of SS can be estimated from the Kaplan-Yorke conjecture [12] similarly as Badii *et al.* determined the condition at which a linear low-pass filter does not influence the dimension of filtered chaotic signals [13]. If we indicate with $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_{d+r}$ the whole spectrum of Lyapunov exponents of the original system Eqs. (1), the global Lyapunov dimension d_λ is given by $d_\lambda = l + \Sigma_{j=1}^{l} \lambda_j / |\lambda_{l+1}|$, where l is the largest integer for which the sum over j is non-negative. This spectrum consists of d Lyapunov exponents $\lambda_1^D \geqslant \lambda_2^D \geqslant \cdots \geqslant \lambda_d^D$ of the driving system that are independent of k and r conditional Lyapunov exponents $\lambda_1^R(k) \geqslant \lambda_2^R(k) \geqslant \cdots \geqslant \lambda_r^R(k)$ that depend on k. The SS occurs if the response system has no effect on the global Lyapunov dimension. This leads to the condition $\lambda_1^R(k) < \lambda_m^D$ where m is the minimal integer for which $\Sigma_{j=1}^{m} \lambda_j^D < 0$. This condition provides that the global Lyapunov dimension d_λ is equal to the Lyapunov dimension d_λ^D of the driving system, $d_\lambda = d_\lambda^D$

If the driving system is presented by a three-dimensional flow, this condition becomes $\lambda_1^R < \lambda_3^D$. For the system of Eqs. (8), we have $\lambda_1^D = 0.408$, $\lambda_2^D = 0$, and $\lambda_3^D = -37.656$. Because of the large negative value of λ_3^D the condition $\lambda_1^R(k) < \lambda_3^D$ is not achieved even for very large $k \simeq 1000$. However, the global dimension $d_\lambda(k)$ goes close to the dimension $d_\lambda^D = 2 + \lambda_1^D / |\lambda_3^D| \approx 2.01$ of the driving attractor at a smaller value of k when $\lambda_1^R(k) \lll -\lambda_1^D$, and the thickness $\sigma(k)$ determining the smoothness of the map Φ becomes small before the above threshold of SS is reached.

Figure 3(d) shows the influence of a small mismatch between parameters of systems Y and Y' in the case of Eqs. (8) and (9). The parameter b of the system Y' is replaced by $b + \Delta b$. For finite Δb, the two pronounced thresholds in the dependence $s = \sqrt{\langle (Y' - Y)^2 \rangle}$ vs k related to the onset of WS and SS are observed. The last threshold is conditioned by different sensitivities of WS and SS to the parameter deviation; $s \propto \Delta b^\gamma$ with $\gamma \approx 0.2$ for the WS and $\gamma = 1$ for the SS. These different scaling laws ($\gamma < 1$ for the WS and $\gamma = 1$ for the SS) can serve as practical critera to distinguish between the WS and SS in experiment.

In conclusion, the two stages of GS, namely, the WS and SS, can be distinguished in unidirectionally coupled chaotic systems. They are related to the existence of a smooth (SS) and an unsmooth (WS) map Φ between variables of the response X and driving Y systems $[Y = \Phi(X)]$ and can be detected with the help of an auxiliary response system.

The author thanks M. Ding and E. Kostelich for useful discussions. This research was partially supported by the EEC under Contract No. CIPDCT94-0011.

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