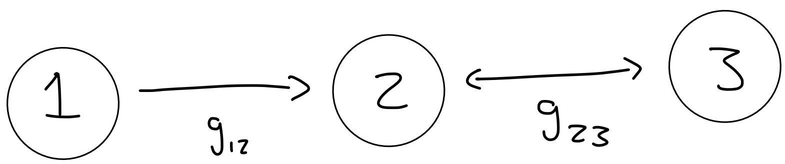
An Analytical approximate solution to coupled network of economies.

→ Here we will try to find the approximate relative coupling from a driving economy to those indirectly connected to it.

- In the computational model, the Kaldor Business Model was used, but here we will simplify this by just using van der pol oscillator since we are interested in the general behavior of the interactions.



- Three 2^{nd} order non-linear eqns. — (van der pol)

$$① \quad \ddot{x}_1 - \mu_1(1 - x_1^2)\,\dot{x}_1 + x_1 = 0$$

$$② \quad \ddot{x}_2 - \mu_2(1 - x_2^2)\,\dot{x}_2 + x_2 = g_{12}(x_1 - x_2) + g_{23}(x_3 - x_2)$$

$$③ \quad \ddot{x}_3 - \mu_3(1 - x_3^2)\,\dot{x}_3 + x_3 = g_{23}(x_2 - x_3)$$

This term will depend on the coupling between x_1 and x_2

- Let's assume small oscillations about the equilibrium $(x < 1)$ —

$$① \approx \ddot{x}_1 - \mu_1\dot{x}_1 + x_1 = 0 \quad \leftarrow 2^{nd} \text{ order ode (homogenous)}$$

$$② \approx \ddot{x}_2 - \mu_2\dot{x}_2 + x_2 = g_{12}(x_1 - x_2) + g_{23}(x_3 - x_2) \quad \leftarrow 2^{nd} \text{ order ode (nonhomogenous)}$$

$$③ \approx \ddot{x}_3 - \mu_3\dot{x}_3 + x_3 = g_{23}(x_2 - x_3) \quad \leftarrow 2^{nd} \text{ order ode (non homogenous)}$$

· Solving each of these —

(1)
$$\left.\begin{aligned} & r^2 - \mu_1 r + 1 = 0 \\ & r_\pm = \frac{\mu_1}{2} \pm \frac{1}{2}\sqrt{\mu_1^2 - 4} \end{aligned}\right] \longrightarrow x_1 = A_1 e^{r_1^+ t} + B_1 e^{r_1^- t}$$

(2)
$$\ddot{x}_2 - \mu_2 \dot{x}_2 + x_2 = g_{12}(x_1 - x_2) + g_{23}(x_3 - x_2)$$
$$\ddot{x}_2 - \mu_2 \dot{x}_2 + x_2(1 + g_{12} + g_{23}) = g_{12} x_1 + g_{23} x_3$$

General soln.
$$\left.\begin{aligned} & r^2 - \mu_2 r + (1 + g_{12} + g_{23}) = 0 \\ & r_\pm = \frac{\mu_2}{2} \pm \frac{1}{2}\sqrt{\mu_2^2 - 4(1 + g_{12} + g_{23})} \end{aligned}\right] \longrightarrow x_2 = A_2 e^{r_2^+ t} + B_2 e^{r_2^- t}$$

Ansatz to non-homogenous solns.
$$x_2 = \overbrace{A_2 e^{r_2^+} + B_2 e^{r_2^- t}}^{\text{homogenous soln to } x_2} + g_{12}\overbrace{\left[A_1 e^{r_1^+ t} + B_1 e^{r_1^- t}\right]}^{\text{soln to } x_1} + g_{23}\underbrace{\overbrace{\left[A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}\right]}^{\text{homogenous soln to } x_3}}_{\text{First order approximation}}$$

→ There will be no "back interaction"

(3)
$$\ddot{x}_3 - \mu \dot{x}_3 + x_3 = g_{23}(x_2 - x_3)$$
$$\ddot{x}_3 - \mu \dot{x}_3 + x_3(1 + g_{23}) = g_{23} x_2$$

General soln.
$$\left.\begin{aligned} & r^2 - \mu r + (1 + g_{23}) = 0 \\ & r_\pm = \frac{\mu}{2} \pm \frac{1}{2}\sqrt{\mu^2 - 4(1 + g_{23})} \end{aligned}\right] \longrightarrow x_3 = A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}$$

Ansatz to non-homogenous solns.
$$x_3 = \overbrace{A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}}^{\text{homogenous soln to } x_2}$$
$$+ g_{23}\left\{ \overbrace{A_2 e^{r_2^+} + B_2 e^{r_2^- t}}^{\text{homogenous soln to } x_2} + g_{12}\overbrace{\left[A_1 e^{r_1^+ t} + B_1 e^{r_1^- t}\right]}^{\text{soln to } x_1} + g_{23}\overbrace{\left[A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}\right]}^{\text{homogenous soln to } x_3} \right\}$$

• So the approximate solns are —

$$x_1 = A_1 e^{r_1^+ t} + B_1 e^{r_1^- t}$$

$$x_2 = \overbrace{A_2 e^{r_2^+} + B_2 e^{r_2^- t}}^{\text{homogenous soln to } x_2} + g_{12}\Big[\overbrace{A_1 e^{r_1^+ t} + B_1 e^{r_1^- t}}^{\text{soln to } x_1}\Big] + g_{23}\Big[\overbrace{A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}}^{\text{homogenous soln to } x_3}\Big]$$

$$x_3 = \overbrace{A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}}^{\text{homogenous soln to } x_2}$$

$$+ g_{23}\Big\{\overbrace{A_2 e^{r_2^+} + B_2 e^{r_2^- t}}^{\text{homogenous soln to } x_2} + g_{12}\Big[\overbrace{A_1 e^{r_1^+ t} + B_1 e^{r_1^- t}}^{\text{soln to } x_1}\Big] + g_{23}\Big[\overbrace{A_3 e^{r_3^+ t} + B_3 e^{r_3^- t}}^{\text{homogenous soln to } x_3}\Big]\Big\}$$

→ Since $g_{12} < 1$ and $g_{23} < 1$, we can see that the interaction between node 1 and 3 is lessened. The further the node is (in terms of network links) the less the master (drive) node will impact it.