Now utilizing Limit cycles (Following Chapter 4.3)

$$\dot{x} = 2\mu\dot{x}(1-\beta x^2) - \omega_0^2 x$$

<u>Flow</u>:

$$(*)\quad \begin{aligned} \dot{x} &= \omega_0 v \\ \dot{v} &= 2\omega_0 v(1-\beta^2 x^2) - \omega_0 x \end{aligned}$$

Fixed Point @ $(x^*, v^*) = 0$

$$J = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 2\mu\omega_0 \end{pmatrix} \quad \left.\begin{aligned} \tau &= 2\mu\omega_0 \\ \Delta &= \omega_0^2 \end{aligned}\right] \rightarrow \lambda = \mu\omega_0 \pm i\omega_0\sqrt{1-\mu^2}$$

$\longrightarrow$ Approaches limit cycle.

- We can let $\mu \rightarrow 0$ and go into a rotating soln.

$$\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} \cos\omega_0 t & \sin\omega_0 t \\ -\sin\omega_0 t & \cos\omega_0 t \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}$$

- Substituting this into $(*)$

$$\dot{w} = 2\mu\omega_0(-ucs + ws^2)\left[1-\beta^2(u^2c^2 + 2uwcs + w^2s^2)\right]$$

$$\dot{u} = 2\mu\omega_0(-us^2 + wcs)\left[1-\beta^2(u^2c^2 + 2uwcs + w^2s^2)\right]$$

$$c = \cos\omega_0 t, \quad s = \sin\omega_0 t$$

$\longrightarrow$ We can average these equations over a single period –

$$\dot{u} = \mu\omega_0 u\left[1 - \frac{\beta^2}{4}(u^2 - w^2)\right]$$

$$\dot{w} = \mu\omega_0 w\left[1 - \frac{\beta^2}{4}(u^2 - w^2)\right]$$

$\longrightarrow$ Substituting to radial coordinates

$$\left.\begin{aligned} \dot{r} &= \mu\omega_0 r\left(1 - \frac{\beta^2}{4}r^2\right) \\ \dot{\theta} &= \omega_0 \end{aligned}\right] \rightarrow \text{First order approximation to the vdp.}$$

- Now we can consider a low order approximation.

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$

- Fixed Point. $r^* = 1$ — linearize about the fixed point.

$$\eta = r - 1 \rightarrow r = 1 + \eta$$

2nd order ~ 0

$$\dot{\eta} \cong (1+\eta)\left[1-(1+\eta)^2\right] = (1+\eta)\left[\cancel{1} - \cancel{1} - 2\eta - \eta^2\right]$$
$$\cong (1+\eta)(-2\eta) \cong -2\eta$$

$$\rightarrow \dot{\eta} = -2\eta \rightarrow \eta = \eta_0 e^{-2t}$$
$$\rightarrow r \cong 1 + \eta_0 e^{-2t}$$

- Now we can couple 3 of these low order vdp's —

$$r_1 = 1 + \eta_1 e^{-2t}$$
$$r_2 = 1 + \eta_2 e^{-2t} - g_{12}(r_1 - r_2) - g_{23}(r_3 - r_2)$$
$$r_3 = 1 + \eta_3 e^{-2t} - g_{23}(r_2 - r_3)$$

Adding coupling terms.

To first order $r_2(r_1)$

r_2 is a function of r_1

$$r_3 \sim 1 + \eta_3 e^{-2t} - g_{23}\left[1 + \eta_2 e^{-2t} - \underbrace{g_{12}(r_1 - r_2)}_{\text{depends on } r_1} - g_{23}(r_3 - r_2) - r_3\right]$$

depends on r_1 → but to a lower degree because g_{12} and $g_{23} < 1$

- We can draw a similar conclusion as we previously did about the coupling of the 1st and 3rd vdp's.