9.22

3. 用定义

$\forall \varepsilon>0$ $\exists N\in N$ 当 $n>N$

有 $\left|\frac{a_{n+1}}{a_n}-0\right|<\varepsilon$

$\because \frac{a_{n+1}}{a_n}>0 \Rightarrow \frac{a_{n+1}}{a_n}<\varepsilon$

取 $\varepsilon=1$ 则 $\exists n_0\in N$, $n>n_0$

$\frac{a_{n+1}}{a_n}<\varepsilon=1$

* 分正负讨论

4 $\because \lim\limits_{n\to\infty} x_n=a$

$\therefore \forall\varepsilon>0$, $\exists N\in N$, 当 $n>N$

有 $|x_n-a|<\varepsilon$

由绝对值不等式 $\big||x_n|-|a|\big|\le|x_n-a|<\varepsilon$

$\therefore \lim\limits_{n\to\infty}|x_n|=|a|$

反之不成立 eg. $x_n=(-1)^n$

6.(1) $\forall \varepsilon>0$ 取 $N=\max(20,[\frac{2}{\varepsilon}]+1)$

当 $n>N$，$\left|\frac{4n^2+n+9}{3n^2-8}-\frac{4}{3}\right|$

$$=\left|\frac{4n^2+n+9}{3n^2-8}-\frac{4(n^2-\frac{8}{3})}{3(n^2-\frac{8}{3})}\right|$$

$$=\left|\frac{n+9+\frac{32}{3}}{3n^2-8}\right|$$

$$<\left|\frac{n+20}{n^2}\right|<\left|\frac{2}{n}\right|<\varepsilon$$

□

(2) 首先 $\frac{a^{n+1}}{(n+1)!}-\frac{a^n}{n!}$

$$=\frac{a^{n+1}-a^n(n+1)}{(n+1)!}$$

$$=\frac{a^n(a-n-1)}{(n+1)!}$$

不妨 $a>0$，则当 $n>a-1$ $\frac{a^{n+1}}{(n+1)!}<\frac{a^n}{n!}$

取 $\forall \varepsilon>0$ 取 $N=\max([a]+1, \frac{a^{[a]+1}}{([a]+1)!}\cdot[\frac{a}{\varepsilon}]+1)$

则 $\left|\frac{a^n}{n!}-0\right|<\left|\frac{a^{[a]+1}}{([a]+1)!}\cdot\frac{a}{n}\right|<\varepsilon$

□

扫描全能王 创建

8. $\forall \varepsilon > 0$, 取 $N = \left[\dfrac{\log m}{\log(\frac{\varepsilon}{A}+1)}\right]+1$

则当 $n > N$ 有 $n > \dfrac{\log m}{\log(\frac{\varepsilon}{A}+1)}$

$\therefore (\frac{\varepsilon}{A}+1)^n > m$

则有 $\varepsilon > (\sqrt[n]{m}-1)A$

$\therefore \left|\sqrt[n]{a_1^n+a_2^n+\cdots+a_m^n}-A\right|$

$< \left|\sqrt[n]{m\cdot A^n}-A\right|$

$= \left|(\sqrt[n]{m}-1)A\right| < \varepsilon$ $\square$

9.2~~4~~

7.(2) $\frac{1}{5}$ 极限运算

(5) $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$

$=\frac{1}{2}\times\frac{3}{2}\times\frac{2}{3}\times\frac{4}{3}\times\cdots\times\frac{n-1}{n}\times\frac{n+1}{n}$

$=\frac{n+1}{2n}$

$\therefore$ ~~故~~ $\lim=\frac{1}{2}$

(9) 1 极限运算

(10)* 夹逼

$(n+1)^\alpha-n^\alpha=n^\alpha\left[\left(1+\frac{1}{n}\right)^\alpha-1\right]$

$\because \alpha\in(0,1)$ $\left(1+\frac{1}{n}\right)^\alpha<\left(1+\frac{1}{n}\right)$

$\therefore (n+1)^\alpha-n^\alpha<n^\alpha\cdot\frac{1}{n}=n^{\alpha-1}$

又 $(n+1)^\alpha-n^\alpha>0$

$\lim\limits_{n\to\infty} n^{\alpha-1}=0$

$\therefore \lim=0$

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(14) $\left(\frac{(2n-1)!!}{2n!!}\right)^2$

$= \frac{1\times3\times3\times5\times\cdots\times(2n-1)\times(2n-1)}{2\times2\times4\times4\times\cdots\times(2n)\times(2n)}$

而易知 $(2n-1)\times(2n+1)<(2n)^2$

$\therefore \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 < \frac{2n-1}{4n^2} < \frac{1}{2n}$

易知 $(2n-1)\times(2n-1) > (2n)\times(2n-2)$

$\therefore \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 > \cancel{\frac{2n-2}{2\times(2n)\times}} \ \frac{1}{4n}$

夹逼 $\lim = 1$

(16) 由 Stolz

$\lim\limits_{n\to\infty} \frac{a_1+2a_2+\cdots+na_n}{1+2+\cdots+n}$

$= \lim\limits_{n\to\infty} \frac{a_1+2a_2+\cdots+(n+1)a_{n+1}-a_1-2a_2-\cdots-na_n}{1+2+\cdots+n+1-1-2-\cdots-n}$

$= \lim\limits_{n\to\infty} \frac{(n+1)a_{n+1}}{n+1} = \lim\limits_{n\to\infty} a_{n+1} = a$

扫描全能王 创建

10. 例2 (~~2班~~) *证有极限!

$0 < x_{n+1} = 1 + \frac{x_n}{1+x_n} < 2$

下证 $x_{n+1} > x_n$

① $n=1$ $x_2 = 1 + \frac{x_1}{1+x_1} = \frac{3}{2} > 1$ 成立

② 假设当 $n=k$ $x_{k+1} > x_k$

③ 当 $n=k+1$

$$x_{k+2} - x_{k+1}$$

$$= 1 + \frac{x_{k+1}}{1+x_{k+1}} - \left(1 + \frac{x_k}{1+x_k}\right)$$

$$= \frac{x_{k+1}(1+x_k) - x_k(1+x_{k+1})}{(1+x_{k+1})\cdot(1+x_k)}$$

$$= \frac{x_{k+1} - x_k}{(1+x_{k+1})(1+x_k)} > 0$$

$$x_{k+2} > x_{k+1}$$ ✓

则 $\{x_n\}$ 单增有上界 $\Rightarrow \lim\limits_{n\to\infty} x_n = A$

$\therefore A = 1 + \frac{A}{1+A}$

$A = \frac{1+\sqrt{5}}{2}$

扫描全能王 创建

(3) ~~证明~~

$$x_{n+1}=\frac{1}{2}\left(x_n+\frac{1}{x_n}\right)\geq 1$$

$$x_{n+1}-x_n=\frac{1}{2x_n}-\frac{x_n}{2}$$

$$=\frac{1-x_n^2}{2x_n}\leq 0$$

单减有下界 $\Rightarrow \lim\limits_{n\to\infty}x_n=A$

$$A=\frac{1}{2}\left(A+\frac{1}{A}\right)$$

$$\Rightarrow A=1$$

9.27

9. 首先 $\lim\limits_{n\to\infty}\frac{a_1+a_2+\cdots+a_n}{n}=a$

~~证明~~ 由 Stolz定理易得

$$\lim_{n\to\infty}\sqrt[n]{a_1 a_2\cdots a_n}$$

$$=\lim_{n\to\infty}e^{\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}}$$

$$=e^{\lim\limits_{n\to\infty}\frac{\ln a_1+\ln a_2+\cdots+\ln a_n}{n}}$$

$$=e^{\ln a}=a$$

(1) Stolz：

$$\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}e^{\ln\sqrt[n]{a_n}}$$

$$=\lim_{n\to\infty}e^{\frac{\ln a_n}{n}}$$

由Stolz $\lim_{n\to\infty}\frac{\ln a_n}{n}=\lim_{n\to\infty}\frac{\ln a_{n+1}-\ln a_n}{n+1-n}$

$$=\lim_{n\to\infty}\ln\frac{a_{n+1}}{a_n}$$

$$=\ln a$$

$\therefore \lim_{n\to\infty}\sqrt[n]{a_n}=a$

(2) 令 $a_n=\frac{n!}{n^n}$

则 $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}$

$$=\lim_{n\to\infty}\frac{(n+1)\cdot n^n}{(n+1)^{n+1}}$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n=\frac{1}{e}$$

由(1) $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$.

10 (4) (2)在前面

下证 $x_n>1$

① $n=1$ $x_1>1$ 成立

② 设 $n=k$ $x_n>1$ 成立

则 $n=k+1$

$$x_{k+1}=\frac{3x_k+1}{x_k+3}$$

$$=3-\frac{8}{x_k+3}>1$$

$\therefore x_n>1$

下证 ~~x_n递减 即 $x_{n+1}<x_n$~~

$$x_{n+1}-x_n=\frac{3x_n+1}{x_n+3}-x_n$$

$$=\frac{1-x_n^2}{x_n+3}<0$$

$\therefore x_{n+1}<x_n$

$\therefore$ 单减有下界 $\Rightarrow \lim\limits_{n\to\infty}x_n=A$

$\therefore A=\frac{3A+1}{A+3}$ $\therefore A=1$

11. $b_{n+1}=\frac{a_n+b_n}{2}\geq\sqrt{a_nb_n}=a_{n+1}$

$\therefore$ 又 $b_1>a_1$ $\therefore$ $b_n>a_n$ 恒成立

$b_{n+1}-b_n=\frac{a_n-b_n}{2}<0$ $\therefore b_{n+1}<b_n$

$\therefore b_n$ 单减有下界0. $\therefore b_n$ 收敛

$\frac{a_{n+1}}{a_n}=\sqrt{\frac{b_n}{a_n}}>1$ $\therefore a_{n+1}>a_n$

$\because b_{n+1}<b_n$ $\therefore b_n\leq b_1$

又对 a_n 有 $a_n<b_n\leq b_1$

$\therefore a_n$ 单增有上界 b_1 $\therefore a_n$ 收敛

假设 $\lim\limits_{n\to\infty}a_n=a$ $\lim\limits_{n\to\infty}b_n=b$

由 $b_{n+1}=\frac{a_n+b_n}{2}$ 两边取极限有

$b=\frac{a+b}{2}$ $\therefore a=b$

□

16. 首先不难知 $\underbrace{x_n \downarrow \quad \lim_{n\to\infty} x_n = 0}_{*\text{proof}}$

令 $a_n = n$, $b_n = \frac{1}{x_n}$

知 $b_n \uparrow$ $\lim_{n\to\infty} b_n = +\infty$

~~由stolz~~ $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n}$

$= \lim_{n\to\infty} \frac{n+1-n}{\frac{1}{x_{n+1}} - \frac{1}{x_n}}$

$= \lim_{n\to\infty} \frac{x_{n+1}x_n}{x_n - x_{n+1}}$

$= \lim_{n\to\infty} \frac{x_n(1-x_n)\cdot x_n}{x_n - x_n(1-x_n)}$

$= \lim_{n\to\infty} (1-x_n)$

$= 1$

由stolz $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\frac{1}{x_n}} = \lim_{n\to\infty} n x_n = 1$

□

Ex. 闭区间套