hw9ans.pdf

1. The Lagrangian is

$$\mathcal{L}(x,\mu) = x_1^2 + (x_2-1)^2 + \mu_1(x_1 - x_2 - 1) + \mu_2((x_1-1)^2 + x_2^2 - 1)$$

case I: $\mu_1 = \mu_2 = 0$

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2x_1 = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2-1) = 0 \end{cases}$$

$x_1 = 0,\ x_2 = 1$

which violates the second restriction.

case II: $\mu_1 > 0, \mu_2 = 0$

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2x_1 + \mu_1 = 0 \\ \partial_{x_2}\mathcal{L} = 2x_2 - 2 - \mu_1 = 0 \\ \partial_{\mu_1}\mathcal{L} = x_1 - x_2 - 1 = 0 \end{cases} \qquad \begin{cases} x_1 = 1 \\ x_2 = 0 \\ \mu_1 = -2 \end{cases}$$

which violates $\mu_1 \geq 0$

case III: $\mu_1 = 0, \mu_2 > 0$

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2x_1 + 2\mu_2(x_1-1) = 0 \\ \partial_{x_2}\mathcal{L} = 2(x_2-1) + 2\mu_2 x_2 = 0 \\ \partial_{\mu_2}\mathcal{L} = (x_1-1)^2 + x_2^2 - 1 = 0 \end{cases} \qquad \begin{cases} x_1 = 1 - \frac{\sqrt{2}}{2} \\ x_2 = \frac{\sqrt{2}}{2} \\ \mu_2 = \sqrt{2} - 1 \end{cases}$$

This result is valid

case IV: $\mu_1 > 0, \mu_2 > 0$

$$\begin{cases} \partial_{x_1}\mathcal{L} = 2x_1 + \mu_1 + 2\mu_2(x_1-1) = 0 \\ \partial_{x_2}\mathcal{L} = 2x_2 - 2 - \mu_1 + 2\mu_2 x_2 = 0 \\ x_1 - x_2 - 1 = 0 \\ (x_1-1)^2 + x_2^2 = 0 \end{cases} \qquad \begin{cases} x_1 = 1 - \frac{\sqrt{2}}{2} \\ x_2 = -\frac{\sqrt{2}}{2} \\ \mu_1 = 2 \\ \mu_2 = -1 \end{cases}$$

violating $\mu_2 \geq 0$,

In conclusion, the x^* is $\begin{bmatrix} 1-\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}$, μ^* is $\begin{bmatrix} 0 \\ \sqrt{2}-1 \end{bmatrix}$, $\min f(x) = 3 - 2\sqrt{2}$. However, chatgpt thinks the solution is $\begin{bmatrix} x \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, which is wrong.

2. $\min\ x_1^2 + x_2^2$

s.t. $(x_1-1)^2 + (x_2-1)^2 - 1 \leq 0$

$(x_1-1)^2 + x_2^2 - 1 \leq 0$

$$\mathcal{L}(x,\mu) = x_1^2 + x_2^2 + \mu_1((x_1-1)^2 + (x_2-1)^2 - 1) + \mu_2((x_1-1)^2 + x_2^2 - 1)$$

① $x^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $g_1(x^{(1)}) = -1$, $\mu_1 = 0$

$\partial_x \mathcal{L} = \begin{pmatrix} 2x_1 + 2\mu_2(x_1-1) \\ 2x_2 + 2\mu_2 x_2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any μ_2. $x^{(1)}$ is not an optimal solution.

② $x^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $g_2(x^{(2)}) = 1$, $\mu_2 = 0$

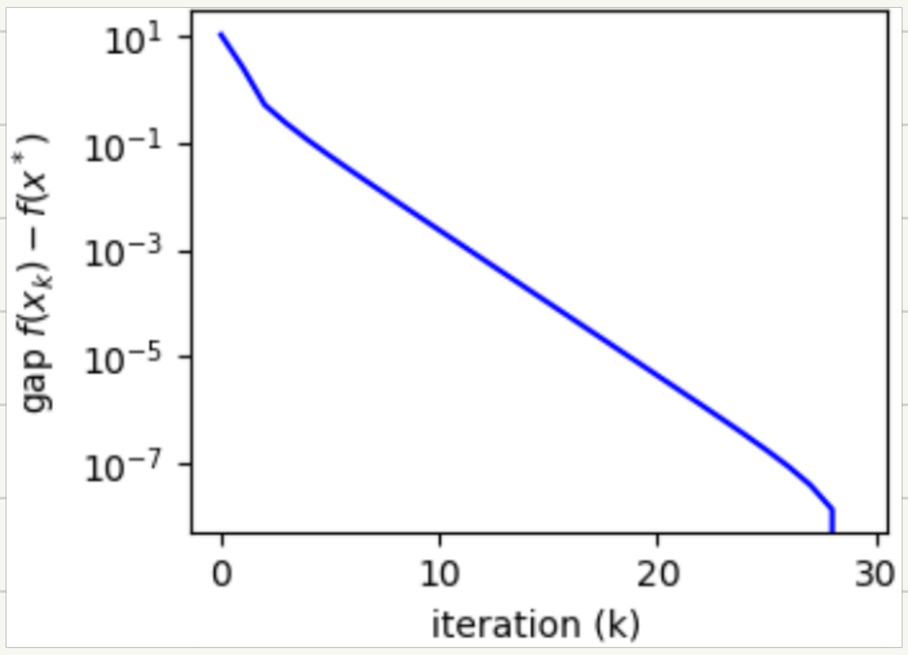
$\partial_x \mathcal{L} = \begin{pmatrix} 2x_1 + 2\mu_1(x_1-1) \\ 2x_2 + 2\mu_1(x_2-1) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for any μ_1, $x^{(2)}$ is not an optimal solution.

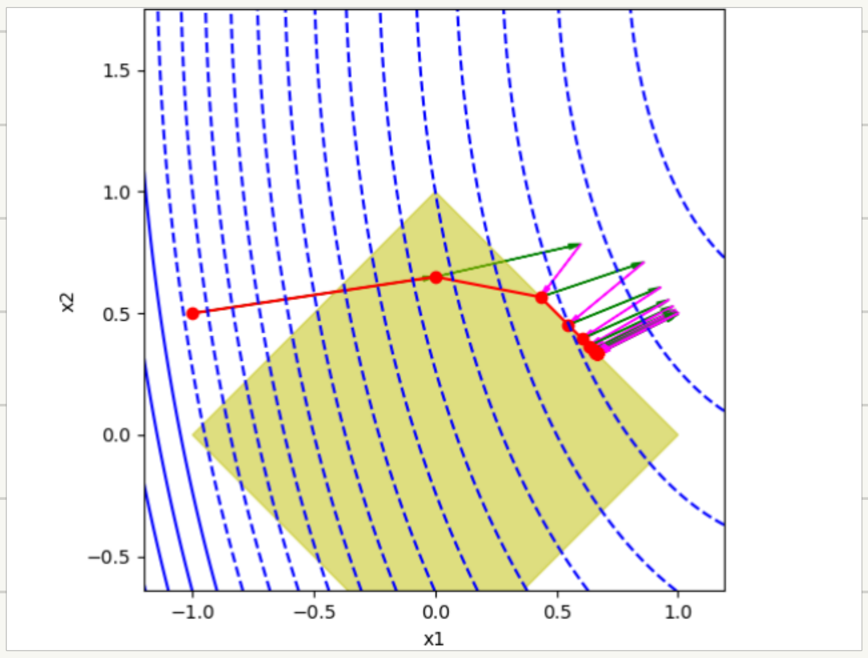
③ $x^{(3)} = \begin{pmatrix} 1-\frac{\sqrt{2}}{2} \\ 1-\frac{\sqrt{2}}{2} \end{pmatrix}$, $g_2(x^{(3)}) \neq 0$, $\mu_2 = 0$

$\partial_x \mathcal{L} = 0, \ \mu_1 = \sqrt{2}-1$

$x^{(3)}$ is the optimal solution.

3. when $t=1$, the number of interations is 29, the solution given is (0.66666666, 0.33333334) which is approximately $(\frac{2}{3}, \frac{1}{3})$, the optimal value is around 4.778





4. (1) $\mathcal{L}(x, \lambda) = e^{2x_1} + e^{x_2} + e^{x_3} + \lambda(x_1 + x_2 + x_3 - 1)$

$$\begin{cases} 2e^{2x_1} + \lambda = 0 \\ e^{x_2} + \lambda = 0 \\ e^{x_3} + \lambda = 0 \\ x_1 + x_2 + x_3 = 1 \end{cases} \qquad \begin{cases} x_1 = -0.07726 \\ x_2 = 0.5386 \\ x_3 = 0.5386 \\ \lambda = -1.7136 \end{cases}$$

$\min f(x) \simeq 4.284$

(2) $$\begin{bmatrix} \nabla^2 f(x_k) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} d \\ \lambda \end{bmatrix} = \begin{bmatrix} -\nabla f(x_k) \\ 0 \end{bmatrix}, \ A = (1, 1, 1),$$

$$\begin{pmatrix} 4e^{2x_1} & & & 1 \\ & e^{x_2} & & 1 \\ & & e^{x_3} & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \lambda \end{pmatrix} = \begin{pmatrix} -2e^{2x_1} \\ -e^{x_2} \\ -e^{x_3} \\ 0 \end{pmatrix}$$

$$\lambda = -\frac{5}{2}\left(\frac{1}{4e^{2x_1}} + \frac{1}{e^{x_2}} + \frac{1}{e^{x_3}}\right)^{-1}$$

$$d = \begin{pmatrix} -\frac{1}{4e^{2x_1}}(\lambda + 2e^{2x_1}) \\ -\frac{1}{e^{x_2}}(\lambda + e^{x_2}) \\ -\frac{1}{e^{x_3}}(\lambda + e^{x_3}) \end{pmatrix}$$

(3) result:

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iteration 0: [0. 1. 0.]
iteration 1: [-0.11369186  0.56845929  0.54523257]
iteration 2: [-0.09499115  0.5534111   0.54158005]
iteration 3: [-0.08601059  0.54598966  0.54002094]
iteration 4: [-0.0816069   0.5423022   0.53930469]
iteration 5: [-0.07942602  0.54046402  0.538962  ]
iteration 6: [-0.07834074  0.53954628  0.53879446]
iteration 7: [-0.07779938  0.53908775  0.53871163]
iteration 8: [-0.07752902  0.53885857  0.53867046]
iteration 9: [-0.07739392  0.53874399  0.53864993]
iteration 10: [-0.07732639  0.53868671  0.53863968]
iteration 11: [-0.07729263  0.53865807  0.53863455]
iteration 12: [-0.07727575  0.53864376  0.538632  ]
iteration 13: [-0.07726731  0.5386366   0.53863072]
iteration 14: [-0.07726309  0.53863302  0.53863008]
iteration 15: [-0.07725887  0.53862944  0.53862944]
optimal value: 4.28414144031119
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