分析力学总结报告#第一章#牛顿力学

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二. 矢量分析及其运动学应用

〈1〉矢量运算：$\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$.

- $A\cdot(B\times C) = C\cdot(A\times B) = B\cdot(C\times A) = (\quad V) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & \cdots & \vdots \\ C_x & \cdots & C_z \end{vmatrix}$.
- $(A\times B)\times C = (A\cdot C)B - (B\cdot C)A$ (projection).
- $(A\times B)\cdot(C\times D) = (A\cdot C)(B\cdot D) - (A\cdot D)(B\cdot C)$.
- $\vec{A} = (\vec{e}\cdot\vec{A})\vec{e} + (\vec{e}\times\vec{A})\times\vec{e}$
- 极×极＝轴　非极×轴＝极　(奇偶等积).

〈2〉矢性函数：$\vec{A}(t) = A_x(t)\vec{i} + A_y(t)\vec{j} + A_z(t)\vec{k}$.

- ✗导矢：$\frac{d}{dt}\vec{A} \overset{def}{=} \lim_{\Delta t\to 0}\frac{\vec{A}(t+\Delta t)-\vec{A}(t)}{\Delta t} \overset{cal}{=} A_x'(t)\vec{i} + A_y'(t)\vec{j} + A_z'(t)\vec{k}$.
  - 几何 → 切向矢量
- ✗ $|d\vec{r}| = |ds|$ → 切向 unit vector $\frac{d\vec{r}}{ds}$；$\left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}$.
- ✗ 导数公式：形式与证明完全同 calculus. 略.
- ✗ $|\vec{A}(t)| = const \iff \vec{A}\cdot\frac{d\vec{A}}{dt} = 0$
- 不定积分 $B'(t) = A(t) \Rightarrow B \overset{def}{=} \int A\,dt \overset{cal}{=} \int A_x\,dt\,\vec{i} + \cdots + \cdots$
- $\int \vec{A}\cdot\vec{B}'dt = \vec{A}\cdot\vec{B} - \int\vec{B}\cdot\vec{A}'dt$；$\int\vec{A}\times\vec{B}'dt = \vec{A}\times\vec{B} + \int\vec{B}\times\vec{A}'dt$.

〈3〉场论复习：

- 方向导数 $\frac{\partial u}{\partial l} \overset{d}{=} \lim_{M\to M_0}\frac{u(M)-u(M_0)}{\overline{MM_0}} \overset{c}{=} \frac{\partial u}{\partial x}\cos\alpha + \frac{\partial u}{\partial y}\cos\beta + \frac{\partial u}{\partial z}\cos\gamma$
  $\frac{\partial u}{\partial l} = \frac{du}{ds} = \frac{\partial u}{\partial s}$ (l 为 $\widehat{s}$ 切线).
- 梯度 $\text{grad}\,u \overset{d}{=} u_x\vec{i} + u_y\vec{j} + u_z\vec{k}$：$\frac{\partial u}{\partial l} = (\overrightarrow{\text{grad}}\,u)\cdot\vec{l} = |\vec{g}|\cos\langle\vec{g},\vec{l}\rangle$
  ⇒ grad 的方向：最大增长，大小：最大增长率
- 通量 $\Phi \overset{d}{=} \oint\vec{v}\cdot d\vec{S}$　散度 $\text{div}\,\vec{A} \overset{d}{=} \lim_{\Delta\Omega\to M}\frac{\Delta\Phi}{\Delta V} \overset{c}{=} \frac{\partial\vec{P}}{\partial x} + \frac{\partial\vec{Q}}{\partial y} + \frac{\partial\vec{R}}{\partial z}$.
- 环量 $\Gamma \overset{d}{=} \oint\vec{A}\cdot d\vec{l}$　旋度 $\text{rot}\,\vec{A} \overset{d}{=} \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$ → 在数值和方向上标出最大环量的向…

刚体

- 速度场 $\vec{v} = \vec{\omega}\times\vec{r} + \vec{v}_0$ 的旋度 $\mathrm{rot}\,\vec{v} = 2\vec{\omega}$.
- 调和场：$\mathrm{div}\vec{A} = \mathrm{rot}\vec{A} = 0$. 调和函数 $\vec{A} = \mathrm{grad}\,u$

  $\mathrm{div}(\mathrm{grad}\,u) = 0 \Rightarrow (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2})u = \Delta u = 0.$
- 平面调和场：$\vec{A} = P\vec{i} + Q\vec{j}$ (i) $\mathrm{rot}\vec{A} = 0 \Rightarrow A = -\mathrm{grad}\,v$

  $P = v_x,\ Q = -v_y$ (ii) $\mathrm{div}\vec{A} = 0 \Rightarrow \mathrm{rot}(-Q\vec{i} + P\vec{j}) = 0$

  $\Rightarrow -Q\vec{i} + P\vec{j} = \mathrm{grad}\,u \Rightarrow P = u_y,\ Q = -u_x \Rightarrow u_x = v_y\ \ u_y = -v_x$

  $\Rightarrow$ C-R equ, 调共轭调和.

(4) Hamilton 算子

$\boxed{\nabla \equiv \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}}$ $\Rightarrow$ 矢量和微分的双重性质.

- $\nabla(uv) = u\nabla v + v\nabla u.$

  $\vec{i}\frac{\partial uv}{\partial x} + \cdots + \cdots = \vec{i}(u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}) + \cdots = u(\vec{i}\frac{\partial v}{\partial x} + \cdots + \cdots) + v(\vec{i}\frac{\partial u}{\partial x} + \cdots + \cdots)$
- $\nabla\cdot(u\vec{A}) = \vec{i}\ \frac{\partial}{\partial x}(uA_x) + \cdots = \cdots = u(\nabla\cdot\vec{A}) + (\nabla u)\cdot\vec{A}$
- $\nabla\times(u\vec{A}) = \cdots = u(\nabla\times\vec{A}) + (\nabla u)\times\vec{A}.$
- $\nabla(\vec{A}\cdot\vec{B}) = \vec{i}A_x(\nabla\times B) + (A\cdot\nabla)B + B\times(\nabla\times A) + (B\cdot\nabla)A.$

  $= \vec{i}\frac{\partial A_xB_x}{\partial x} + \cdots = (A_x\frac{\partial B_x}{\partial x} + B_x\frac{\partial A_x}{\partial x})\vec{i} + \cdots$

  right $= (A_x\frac{\partial}{\partial x} + A_y\frac{\partial}{\partial y} + A_z\frac{\partial}{\partial z})(B_x\vec{i} + B_y\vec{j} + B_z\vec{k}) + A_x(\nabla\times B) + \cdots$

  $= (\vec{i}A_x\frac{\partial B_x}{\partial x} + \cdots + \cdots) + A_y\frac{\partial B_x}{\partial y}\vec{i} + \cdots =$ left.
- $\nabla\cdot(\vec{A}\times\vec{B}) = \nabla\cdot(\vec{A}_c\times\vec{B}) + \nabla\cdot(\vec{A}\times\vec{B}_c)$ c: 变常矢.

  $= \vec{B}_c\cdot(\nabla\times\vec{A}) - \vec{A}_c\cdot(\nabla\times\vec{B}) = B\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$
- $\nabla\times(\vec{A}\times\vec{B}) = \nabla\times(\vec{A}_c\times\vec{B}) + \nabla\times(\vec{A}\times\vec{B}_c)$ 技巧重要!

  $= \vec{A}_c(\nabla\cdot\vec{B}) - (\vec{A}_c\cdot\nabla)\vec{B} + (\vec{B}_c\cdot\nabla)\vec{A} - \vec{B}_c(\nabla\cdot\vec{A})$

  $= (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A).$
- $\nabla\times(\nabla u) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \end{vmatrix} = \vec{0}$ 梯无旋

  $\nabla\cdot(\nabla\times\vec{A}) = \sum\frac{\partial}{\partial x}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_y & A_z \end{vmatrix} + \cdots = 0.$ 旋无散
- $\nabla f(u,v) = f_u\nabla u + f_v\nabla v.$
- $\oint_S \vec{A}\cdot d\vec{S} = \iiint_\Omega (\nabla\cdot\vec{A})\,dV$; $\oint \vec{A}\cdot d\vec{l} = \iint_S (\nabla\times\vec{A})\,d\vec{S}$

<5> 正交曲线坐标系.

- $\vec{r}=\vec{r}(q_1,q_2,q_3)$ $\quad q_i=C_i \to$ 等值曲面 $\xrightarrow{交}$ 坐标曲线 (正交).
- $\vec{e}_i(q_1,q_2,q_3)$：q_i 坐标曲线上单位切向量.
- $d\vec{r}_i=\vec{i}dx+\vec{j}dy+\vec{k}dz=\vec{i}\frac{\partial x}{\partial q_i}dq_i+\vec{j}\frac{\partial y}{\partial q_i}dq_i+\vec{k}\frac{\partial z}{\partial q_i}dq_i$ (沿 q_i 线).

  $ds_i=\pm\sqrt{(\frac{\partial x}{\partial q_i})^2+(\frac{\partial y}{\partial q_i})^2+(\frac{\partial z}{\partial q_i})^2}\,dq_i \overset{d}{=} \pm H_i dq_i$ (Lamé 系数)
- $dV=H_1H_2H_3\,dq_1dq_2dq_3$ $\quad dS_{12}=dS_1dS_2=H_1H_2dq_1dq_2$

  $\frac{\partial \vec{r}}{\partial q_i}=H_i\vec{e}_i \Rightarrow \frac{\partial \vec{r}}{\partial q_i}\cdot\frac{\partial \vec{r}}{\partial q_j}=H_i^2\delta_{ij} \Rightarrow ds^2=\sum H_i^2dq_i^2=\sum ds_i^2$

  $\Rightarrow$ cal H_i : $dx^2+dy^2+dz^2=H_1^2dq_1^2+H_2^2dq_2^2+H_3^2dq_3^2$
- $$\begin{pmatrix}H_1\vec{e}_1\\H_2\vec{e}_2\\H_3\vec{e}_3\end{pmatrix}=\begin{bmatrix}\frac{\partial x}{\partial q_1}&\frac{\partial y}{\partial q_1}&\frac{\partial z}{\partial q_1}\\\frac{\partial x}{\partial q_2}&\frac{\partial y}{\partial q_2}&\frac{\partial z}{\partial q_2}\\\frac{\partial x}{\partial q_3}&\frac{\partial y}{\partial q_3}&\frac{\partial z}{\partial q_3}\end{bmatrix}\begin{pmatrix}\vec{i}\\\vec{j}\\\vec{k}\end{pmatrix}$$

  $d\vec{r}=H_1\vec{e}_1dq_1+H_2\vec{e}_2dq_2+H_3\vec{e}_3dq_3$

  $\frac{d\vec{r}}{dt}=H_1\vec{e}_1\dot{q}_1+H_2\vec{e}_2\dot{q}_2+H_3\vec{e}_3\dot{q}_3$

- $r=r_1(t)e_1+r_2(t)\vec{e}_2+r_3(t)\vec{e}_3 \Rightarrow \frac{d\vec{r}}{dt}=(\dot{r}_1\vec{e}_1+r_1\dot{\vec{e}}_1)+\cdots+\cdots$

  $\frac{d}{dt}\begin{pmatrix}\vec{e}_1\\\vec{e}_2\\\vec{e}_3\end{pmatrix}=\Omega\begin{pmatrix}\vec{e}_1\\\vec{e}_2\\\vec{e}_3\end{pmatrix}$ $\quad \hat{e}\hat{e}^T=E$

  $\Downarrow$ $\quad \Rightarrow \vec{0}=\frac{d}{dt}E=(\frac{d}{dt}\hat{e})\hat{e}^T+\hat{e}(\frac{d}{dt}\hat{e}^T)$

  Ω 反对称 for 正交单位基矢 $\quad =\Omega\hat{e}\hat{e}^T+\hat{e}(\hat{e}^T\Omega^T)$

  求 Ω : $\hat{e}=M\hat{e}_0$, $\dot{\hat{e}}=\frac{dM}{dt}M^{-1}\cdot\hat{e}$. $\quad =\Omega+\Omega^T$
- 示例 :

  Polar coord system.

  $\begin{cases}x=\rho\cos\theta\\y=\rho\sin\theta\end{cases}$

  $dx^2+dy^2=d\rho^2+\rho^2d\theta^2$.

  $\Rightarrow H_\rho=1,\ H_\theta=\rho.$

  $\begin{pmatrix}\vec{e}_\rho\\\rho\vec{e}_\theta\end{pmatrix}=\begin{bmatrix}\cos\theta&\sin\theta\\-\rho\sin\theta&\rho\cos\theta\end{bmatrix}\begin{pmatrix}\vec{i}\\\vec{j}\end{pmatrix}$.

  $\vec{r}=\rho\vec{e}_\rho$

  $d\vec{r}=\vec{e}_\rho d\rho+\rho\vec{e}_\theta d\theta$.

  $\frac{d\vec{r}}{dt}=\dot{\rho}\vec{e}_\rho+\rho\vec{e}_\theta\dot{\theta}$

  $\frac{d^2\vec{r}}{dt^2}=\ddot{\rho}\vec{e}_\rho+\ddot{\theta}\rho\vec{e}_\theta+\dot{\rho}\dot{\theta}\vec{e}_\theta+\cancel{\rho\dot{\theta}\dot{\vec{e}}_\theta}$
  $+\dot{\rho}\dot{\vec{e}}_\rho+\rho\dot{\theta}\dot{\vec{e}}_\theta$

  $M=\begin{bmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{bmatrix}$ $\quad M'=\begin{bmatrix}-s&c\\-c&s\end{bmatrix}\dot{\theta}$

  $\Omega=\begin{bmatrix}0&?\\-?&0\end{bmatrix}\dot{q},\ \Omega M=M^{-1}$.

  $\Rightarrow ?=1.\ \Rightarrow \dot{\vec{e}}_\rho=\dot{\theta}\vec{e}_\theta,\ \dot{\vec{e}}_\theta=-\dot{\theta}\vec{e}_\rho$

  $\Rightarrow \frac{d^2\vec{r}}{dt^2}=(\ddot{\rho}-\rho\dot{\theta}^2)\vec{e}_\rho+(\rho\ddot{\theta}+2\dot{\rho}\dot{\theta})\vec{e}_\theta$

Spherical Coord-System:

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta . \end{cases} \qquad \cancel{dx} = \frac{\partial(x,y,z)}{\partial(r,\theta,\varphi)} = \begin{bmatrix} \sin\theta\cos\varphi, & \sin\theta\sin\varphi, & \cos\theta \\ r\cos\varphi\cos\theta, & r\sin\varphi\cos\theta, & -r\sin\theta \\ -r\sin\theta\sin\varphi, & r\sin\theta\cos\varphi, & 0 \end{bmatrix}$$

$\Rightarrow H_r = 1,\ H_\theta = r,\ H_\varphi = r\sin\theta.$

$$\Rightarrow \begin{pmatrix} \vec{e}_r \\ \vec{e}_\theta \\ \vec{e}_\varphi \end{pmatrix} = \begin{bmatrix} \sin\theta\cos\varphi, & \sin\theta\sin\varphi, & \cos\theta \\ \cos\theta\cos\varphi, & \cos\theta\sin\varphi, & -\sin\theta. \\ \cancel{\sin\theta}-\sin\varphi, & \cancel{\sin\theta}\cos\varphi. & 0 \end{bmatrix} \begin{pmatrix} \vec{i} \\ \vec{j} \\ \vec{k} \end{pmatrix}$$

$\frac{d\vec{r}}{dt} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\dot{\varphi}\vec{e}_\varphi$. 又 $\frac{d\vec{r}}{dt} = (r\vec{e}_r)' = \dot{r}\vec{e}_r + r\dot{\vec{e}_r}$

$\Rightarrow \dot{\vec{e}_r} = \dot{\theta}\vec{e}_\theta + \dot{\varphi}\sin\theta\,\vec{e}_\varphi .$

$\Rightarrow \begin{cases} \dot{\vec{e}_\theta} = -\dot{\theta}\vec{e}_r + ⑦\vec{e}_\varphi \\ \dot{\vec{e}_\varphi} = -\dot{\varphi}\sin\theta\,\vec{e}_r - ⑦\vec{e}_\theta \end{cases}$ 又 $\dot{\vec{e}_\varphi} = \dot{\varphi}(-\cos\varphi\,\vec{i} - \sin\varphi\,\vec{j})$

(i方向) $-\dot{\varphi}\cos\varphi + \dot{\varphi}\sin\theta(\sin\theta\cos\varphi) = -⑦\cos\theta\cos\varphi$

$\Rightarrow ⑦ = \cos\theta\dot{\varphi}. \Rightarrow \Omega = \begin{bmatrix} 0 & \dot{\theta} & \dot{\varphi}\sin\theta \\ -\dot{\theta} & 0 & \dot{\varphi}\cos\theta \\ -\dot{\varphi}\sin\theta & -\dot{\varphi}\cos\theta & 0 \end{bmatrix}$. ※信息是冗余的，so不留优秀路.

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2\sin\theta\cos\theta)\vec{e}_\theta + (r\sin\theta\ddot{\varphi} + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\varphi}\dot{\theta}\cos\theta)\vec{e}_\varphi .$$

- $du = \frac{\partial u}{\partial q_1}dq_1$ (q_1坐标线上) $\Rightarrow \frac{du}{ds_1} = \frac{1}{H_1}\frac{\partial u}{\partial q_1}$ (方向导数).

推广得 $\boxed{\nabla = \vec{e}_1\frac{1}{H_1}\frac{\partial}{\partial q_1} + \vec{e}_2\frac{1}{H_2}\frac{\partial}{\partial q_2} + \vec{e}_3\frac{1}{H_3}\frac{\partial}{\partial q_3}}$

实际计算还需: $\frac{\partial(\vec{e}_1,\vec{e}_2,\vec{e}_3)}{\partial(q_1,q_2,q_3)}$

$\frac{\partial\vec{e}_i}{\partial q_i} = -\frac{\vec{e}_j}{H_j}\frac{\partial H_i}{\partial q_j} - \frac{\vec{e}_k}{H_k}\frac{\partial H_i}{\partial q_k}.$

$\frac{\partial\vec{e}_j}{\partial q_i} = \frac{\vec{e}_i}{H_j}\frac{\partial H_i}{\partial q_j}. \Rightarrow (\dot{q}_1,\dot{q}_2,\dot{q}_3)\frac{\partial(\vec{e}_1,\vec{e}_2,\vec{e}_3)}{\partial(q_1,q_2,q_3)} = \frac{d}{dt}(\vec{e}_1,\vec{e}_2,\vec{e}_3)$

散度: $\mathrm{div}\vec{A} = (\vec{e}_1\frac{1}{H_1}\frac{\partial}{\partial q_1} + \cdots + \cdots)\cdot(A_1\vec{e}_1 + A_2\vec{e}_2 + A_3\vec{e}_3)$

$$= \underbrace{\frac{1}{H_1H_2H_3}}_{\Delta V}\left[\frac{\partial}{\partial q_1}(\underbrace{H_2H_3}_{\Delta S}A_1) + \frac{\partial}{\partial q_2}(H_3H_1A_2) + \frac{\partial}{\partial q_3}(H_1H_2A_3)\right]$$

旋度: $\mathrm{rot}\vec{A} = \frac{1}{H_1H_2H_3}\begin{vmatrix} H_1\vec{e}_1 & H_2\vec{e}_2 & H_3\vec{e}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ H_1A_1 & H_2A_2 & H_3A_3 \end{vmatrix}$

$\Delta u = \nabla\cdot\nabla(u) = \frac{1}{H_1H_2H_3}\left[\frac{\partial}{\partial q_1}\left(\frac{H_2H_3}{H_1}\frac{\partial u}{\partial q_1}\right) + \cdots + \cdots\right].$

• 示例：

Polar CS：$H_r = 1,\ H_\theta = r,\ H_z = 1$

$\nabla = \vec{e_r}\frac{\partial}{\partial r} + \frac{1}{r}\vec{e_\theta}\frac{\partial}{\partial\theta} + e_z\frac{\partial}{\partial z}$

$\nabla\cdot A = \frac{1}{r}\cdot\left[\frac{\partial}{\partial r}(rA_r) + \frac{\partial}{\partial\theta}(A_\theta) + \frac{\partial}{\partial z}(rA_z)\right]$

$$\nabla\times A = \frac{1}{r}\begin{vmatrix} \vec{e_r}, & r\vec{e_\theta}, & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$

Spherical CS：$H_r = 1\quad H_\theta = r,\quad H_\varphi = r\sin\theta$. 后略.

• 引入 $F_i = H_i A_i$，$G_i = H_j H_k A_i$. $H = H_1H_2H_3$，$\vec{E_i} = H_i\vec{e_i}$

$\Rightarrow\ \nabla\cdot A = \frac{1}{H}\left(\sum\frac{\partial G_i}{\partial q_i}\right)\quad \nabla\times A = \frac{1}{H}\cdot\begin{vmatrix} E_1 & E_2 & E_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ F_1 & F_2 & F_3 \end{vmatrix}$

<6> 自然坐标.（甚少会用）

• $\vec{\tau} = \frac{d\vec{r}}{ds}$.　def $\vec{n} \parallel \left|\frac{d\vec{\tau}}{ds}\right|$，$\kappa = \frac{1}{\rho} = \left|\frac{d\vec{\tau}}{ds}\right|\ \Rightarrow\ \frac{d\vec{\tau}}{ds} = \frac{1}{\rho}\vec{n}$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = v\vec{\tau}\quad \vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{\tau} + v\frac{d\vec{\tau}}{dt} = \frac{dv}{dt}\vec{\tau} + \frac{v^2}{\rho}\vec{n}$.

• $\rho = 1\Big/\left|\frac{d^2\vec{r}}{ds^2}\right|\qquad y = f(x) \Rightarrow \frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\frac{1+f'}{\sqrt{1+f'^2}} = \frac{f''}{(1+f'^2)^{\frac{3}{2}}}$

Similarly for $r = r(\theta)$，$\begin{cases} x = x(t) \\ y = y(t) \end{cases}$，.etc.

• $\vec{b} = \vec{\tau}\times\vec{n}\qquad \frac{d}{ds}\begin{pmatrix}\vec{\tau}\\ \vec{n}\\ \vec{b}\end{pmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & ? \\ 0 & -? & 0 \end{bmatrix}\begin{pmatrix}\vec{\tau}\\ \vec{n}\\ \vec{b}\end{pmatrix} \Rightarrow$ def $\frac{d\vec{b}}{ds} = -k\vec{n}$

三. 质点(系)动力学

<1> 质点：有质量无大小的几何点 // 暗含了质量守恒的假设

参照系：一个刚体（有大小不变形）// 这里的定义是循环的，不如说"相信"其不变形"

坐标系：定量描述位置.

• $\vec{F} = m\vec{a}$　$\vec{F}$ 形式上没有意义！

经典力学中的"力"包含了以下假设：（都 to some extent 错误）

① F 存在于 2个物体之间（⇐线性）

② $\vec{F}_{12} = -\vec{F}_{21}$（⇐动量守恒 ⇐ 空间平移不变）

③ $\vec{F}_{12} \parallel \vec{r}_1 - \vec{r}_2$（⇐ 角动量守恒 ⇐ 空间转动不变）

④ $\vec{F} = \vec{F}(\vec{r})$.　⑤ $\vec{F}$ 独立于参照系（越来越奇怪了---）

<2> 质点系：相互耦合的若干质点的系统.

• $m_c = \sum m_i\quad r_c = \frac{\sum m_i r_i}{m_c}$　|　$\sum F_i^{(i)} = 0$.

$\sum r_i' = 0$.

- $\sum \vec{p}_i = \sum m_i \frac{d}{dt}\vec{r}_i = \frac{d}{dt}\sum m_i \vec{r}_i = \frac{d}{dt} m_c \vec{r}_c = \vec{p}_c$. $\sum \vec{p}_i' = 0$.

$$\sum E_{ki} = \sum \frac{1}{2} m_i \left(\frac{dr_i}{dt}\right)^2 = \sum \frac{1}{2} m_i (\vec{v}_c + \vec{v}_i')^2$$
$$= \frac{1}{2} m_{BC} v_c'^2 + \frac{1}{2} m_c v_c^2 + \sum m_i \vec{v}_c \cdot \vec{v}_i'$$
$$= E_{kc} + E_k' + \frac{d}{dt}(\sum m_i r_i') \cdot \vec{v}_c = E_{kc} + E_k'.$$

无关参考系, just 数学.

$$\sum \vec{L}_i = \sum m_i \vec{r}_i \times \vec{v}_i \quad (\text{with respect to } O).$$
$$= \sum m_i (\vec{r}_c + \vec{r}_i') \times (\vec{v}_c + \vec{v}_i').$$
$$= \sum m_i (\vec{r}_c \times \vec{v}_c + \vec{r}_i' \times \vec{v}_c + \vec{r}_c \times \vec{v}_i' + \vec{r}_i' \times \vec{v}_i')$$
$$= \sum m_i (\vec{r}_c \times \vec{v}_c) + \sum m_i (\vec{r}_i' \times \vec{v}_i') = \vec{L}_c + \vec{L}'.$$

- $\sum (\vec{F}_i^{(i)} + \vec{F}_i^{(e)}) = \sum \vec{F}_i^{(e)} = \vec{F}^{(e)}$. ($F = F_{外}$)

$$\sum \vec{F}_i \cdot d\vec{r}_i = \sum F_i^{(i)} d\vec{r}_i + \sum F_i^{(e)} d\vec{r}_i \quad (A = A_{内} + A_{外})$$
$$\sum \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times F_i^{(i)} + \sum (\vec{r}_c + r_i') \times F_i^{(e)}$$
$$= \sum_{i \neq j} (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij} \cancel{+ \sum \vec{r}_c \times \vec{F}^{(e)} + \sum \vec{r}_i' \times \vec{F}_i^{(e)}}$$
$$= \sum \vec{r}_i \times \vec{F}_i^{(e)}. \quad (M = M_{外}).$$

- 惯性系中:

$$\frac{d}{dt}\vec{p}_i = \vec{F}_i \Rightarrow \frac{d}{dt}\vec{p}_c = \vec{F}^{(e)} \Leftrightarrow \frac{d}{dt}\vec{p}' = 0.$$
$$\frac{d}{dt}\vec{L}_i = \vec{r}_i \times \vec{F}_i \Rightarrow \frac{d}{dt}(\vec{L}_c + \vec{L}') = \sum \vec{r}_i \times \vec{F}_i^{(e)}.$$
$$\Leftrightarrow \frac{d}{dt}\vec{L}' = \sum \vec{r}_i' \times \vec{F}_i^{(e)} = M', \quad \frac{dL_c}{dt} = \vec{r}_c \times \vec{F}^{(e)} = M_c.$$
$$\frac{d}{dt} T_i = \vec{F}_i \, d\vec{r}_i \Rightarrow d(T_c + T') = \sum \vec{F}_i \cdot d\vec{r}_c + \sum \vec{F}_i \, d\vec{r}_i'$$

$$dT_c = \vec{F}^{(e)} d\vec{r}_c \Rightarrow dT' = \sum F_i^{(e)} d\vec{r}_i' + \sum F_i^{(i)} d\vec{r}_i' = A_{内}' + A_{外}'$$

- 总结: 质心的特殊性质, 使得

$\frac{d}{dt}\vec{p}' = \sum F' (= 0)$; $\frac{d}{dt}\vec{L}' = M'$; $\frac{d}{dt} T' = A'$ 在非惯性(平动)系中成立.

(非惯性力的作用被抵消).

分析力学总结报告#二.三章# 拉格朗日力学. 李欣 520072910001

零.引言/章节大纲

第二章
- 1°分析力学概念
- 2° 静力学&虚功原理
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  - (i) 完整保守系
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- 2° 振动
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  - (iii) 受迫振动:DOF=1,2
  - (iv) 非线性振动

引用.课程PPT,《理论力学》-金尚年,Landau《力学》

一.分析力学概念.

1° 约束与约束反力

质点,受力(一般而言!) $\begin{cases}\text{主动力:已知,保守(一般!)}\\ \text{约束力:几何约束的影响,未知.被动} \to \text{Aim:从方程中消去}\end{cases}$

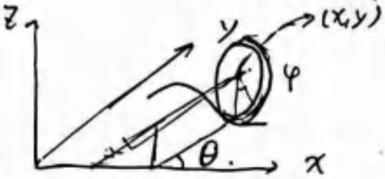
约束分类. $f_\alpha(r_1,r_2,\cdots,r_n,\dot r_1,\dot r_2,\cdots,\dot r_n,t)\le 0$ $\alpha=1,2,\cdots,k$.

- (i) 不含t:稳定/(不稳定) $\longrightarrow \frac{\partial f}{\partial t}=0$
- (ii) 不含 $\dot r$:几何约束/(运动约束) $\to \frac{\partial f}{\partial \dot r}=0$
- (iii) 取等:不可解约束(可解) $\longrightarrow$ 消去未知量,DOF=坐标数

完整约束(Hornonic constr-)
不可解几何约束

* 运动约束有时可化为几何约束:

轮, $\vec r=\vec r(x,y,\theta,\varphi)$.



- 纯滚动: $r\dot\varphi=\sqrt{\dot x^2+\dot y^2}$. $\begin{cases}\dot x=r\dot\varphi\sin\theta.\\ \dot y=r\dot\varphi\cos\theta.\end{cases}$
  (不可积微分约束)
- $\theta=\theta_0=constant$

$\Rightarrow\begin{cases}x-r\varphi\sin\theta_0=constant\\ y-r\varphi\cos\theta_0=constant\end{cases}\to$ 几何constraint.

分析力学的目标:分离几何约束与动力学约束

下无说明默认体系完整: $f_\alpha(r_1,r_2,\cdots,r_n,t)=0$.$(\alpha=1,2,\cdots,n)$ [1]

2° DOF与广义坐标.

DOF.S=3n−k. n:质点数,k:(不可解)约束条件数.

$\Rightarrow \vec r_i=\vec r_i(q_1,q_2,\cdots q_s,t)$. $\hat q$:广义坐标 $\longrightarrow$ 位形空间

要求: $\begin{cases}\text{(i) 唯一完全确定质点系位形}\\ \text{(ii) 独立 } \frac{\partial q_i}{\partial q_j}=0\,(i\neq j).\ \text{(自动满足约束方程)}.\end{cases}$

特点: $\begin{cases}\text{(i) 集体坐标}\\ \text{(ii) 不唯一-选取}\\ \text{(iii) 不必正交,不必也}\end{cases}$

二. 静力学 & 虚功原理.

1° 虚位移 $\delta \vec{r}$ ≡ 满足约束条件的假想无限小位移. ‖ 实位移 $d\vec{r}$: dt 时间内真实位移

性质 (i). 等时变分. $\boxed{\delta t = 0}$ [4] ＊ for 稳定约束: $d\vec{r} \in \{\delta \vec{r}\}$.

(ii). 不唯一. ‖ $d\vec{r}$ 唯一 ＊ 在可能位移中挑选真实的.

(iii). 无关受力. 无关运动方程

(iv). 满足约束条件: $\boxed{\sum_{i=1} \frac{\partial f_\alpha}{\partial r_i} \delta r_i = f_\alpha(\vec{r}+\delta\vec{r},t) - f_\alpha(\vec{r},t) = 0}$

2° 虚功原理 & 理想约束.

虚功 $\delta W = \vec{F} \cdot \delta \vec{r}$. 定义理想约束 (ideal constraint), 约束力总虚功 = 0.

$\boxed{\sum_i \vec{F}_{Ni} \cdot \delta \vec{r}_i = 0}$ [2] where $\vec{F}_N$: 表第 i 个质点所受约束力.

记主动力 $\vec{F}_i$, $\vec{F}_i + \vec{F}_{Ni} - m_i \ddot{\vec{r}}_i = 0$, 联立 [2],

$\Rightarrow \boxed{\sum_{i=1}^{n} (\vec{F}_i - m_i \ddot{\vec{r}}_i) \delta \vec{r}_i = 0 \quad [3]}$ 即 d'Alembert 方程 [条件 $\begin{cases} \text{ideal C.} \\ \text{惯性系} \end{cases}$]

3° 变换到广义坐标

$\vec{r}_i = \vec{r}_i(\vec{q}, t) \overset{[4]}{\Rightarrow} \boxed{\delta \vec{r}_i = \sum_{\alpha=1}^{s} \frac{\partial \vec{r}_i}{\partial q_\alpha} \cdot \delta q_\alpha}$ [5]

$\Rightarrow \delta W \overset{[2]}{=} \sum_{i=1}^{n} \vec{F}_i \delta \vec{r}_i \overset{[5]}{=} \sum_{i=1}^{n} \vec{F}_i \sum_{\alpha=1}^{s} \frac{\partial \vec{r}_i}{\partial q_\alpha} \delta q_\alpha = \sum_{\alpha=1}^{s} \left(\sum_{i=1}^{n} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \right) \delta q_\alpha$

定义广义力 $\boxed{Q_\alpha = \sum_{i=1}^{n} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_\alpha}}$ [6]. $\Rightarrow \boxed{\delta W = \sum_{\alpha=1}^{s} Q_\alpha \delta q_\alpha}$ [7].

$\Rightarrow Q_\alpha = \frac{\delta W}{\delta q_\alpha} \Rightarrow \boxed{Q = \nabla \cdot \delta W}$

静力学平衡条件: $\ddot{r}_i = 0 \Rightarrow \delta W = 0 \overset{[7]}{\Rightarrow} \sum_\alpha Q_\alpha \delta q_\alpha = 0$

q_α 的独立性 $\Rightarrow Q_\alpha = 0$ i.e. $\boxed{\sum_{i=1}^{n} \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} = 0, \alpha = 1, 2, \cdots s}$ [8] 平衡方程.

适用: 理想完整 C. 平衡. 惯.

若主动力保守, 即 $\boxed{\vec{F}_i = -\nabla_i V = -\frac{\partial V}{\partial r_i}}$ 代入 [6]

$\Rightarrow Q_\alpha = -\sum_{i=1}^{n} \nabla_i V \left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) = -\sum_{i=1}^{n} \frac{\partial V}{\partial r_i} \frac{\partial r_i}{\partial q_\alpha} = \cancel{\sum_i} -\frac{\partial V}{\partial q_\alpha}$ [17]

$\Rightarrow Q = -\nabla V$, 平衡条件: $\boxed{\frac{\partial V}{\partial q_\alpha} = 0, \alpha = 1, 2, \cdots, s}$. 条件: 理想完整保守系统. [9].

三. 理想完整系的 Lagrange 方程.

1° 广义速度与系统动能.

$\boxed{\frac{d}{dt}\vec{r}_i = \frac{\partial \vec{r}_i}{\partial t} + \sum_\alpha \frac{\partial \vec{r}_i}{\partial q_\alpha} \dot{q}_\alpha = \dot{\vec{r}}_i(q_\alpha, \dot{q}_\alpha, t)}$ 则两边求 $\frac{\partial}{\partial \dot{q}_\alpha}$ 得:

$\boxed{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} = \frac{\partial \vec{r}_i}{\partial q_\alpha} \quad [10]}$ (利用了: $\frac{\partial \dot{q}_\beta}{\partial q_\alpha} = 0$; $\frac{\partial \vec{r}_i}{\partial \dot{q}_\alpha} = 0 \Rightarrow \frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) = 0.$)

$\frac{\partial}{\partial \dot{q}_\alpha}\left(\frac{\partial \vec{r}_i}{\partial t}\right) = 0.$

对[11]两边同乘$\frac{\partial}{\partial q_\alpha}$得，

$$\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} = \frac{\partial^2 r_i}{\partial t \partial q_\alpha} + \sum_\beta \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_i}{\partial q_\beta}\dot{q}_\beta\right) = \boxed{\frac{\partial^2 r_i}{\partial q_\alpha \partial t}} + \sum_\beta \dot{q}_\beta \boxed{\frac{\partial^2 r}{\partial q_\alpha \partial q_\beta}}.$$

② 又有：$\frac{\partial r_i}{\partial q_\alpha} = \frac{\partial r_i}{\partial q_\alpha}(q,t) \Rightarrow$

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_\alpha}\right) = \boxed{\frac{\partial}{\partial t}\left(\frac{\partial r_i}{\partial q_\alpha}\right)} + \sum_\beta \boxed{\frac{\partial}{\partial q_\beta}\left(\frac{\partial r_i}{\partial q_\alpha}\right)}\cdot \dot{q}_\beta$$

$$\Rightarrow \boxed{\frac{\partial}{\partial q_\alpha}\left(\frac{d}{dt}\right) = \frac{d}{dt}\left(\frac{\partial}{\partial q_\alpha}\right)} \quad \text{for } f(q,t) \quad [12]$$

* 这是不平凡的，$\frac{\partial}{\partial \dot{q}_\alpha}$与$\frac{d}{dt}$对于$f(q,t)$没有这样的对易关系，

$T = \sum_i \frac{1}{2} m_i \dot{\vec{r}}_i^2 = T(q,\dot{q},t)$.

$$\begin{cases} \frac{\partial T}{\partial q_\alpha} = \sum_i m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} & [14] \\ \frac{\partial T}{\partial \dot{q}_\alpha} = \sum_i m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} = \sum_i m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \end{cases}$$

$$T = \sum_i \frac{m_i}{2}\left(\sum_\alpha \frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t}\right)\left(\sum_\beta \frac{\partial \vec{r}_i}{\partial q_\beta}\dot{q}_\beta + \frac{\partial \vec{r}_i}{\partial t}\right)$$

$$\begin{aligned} &= \sum_i \frac{1}{2} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2 \quad // T_0 \\ &+ \sum_i \sum_\alpha m_i \frac{\partial r_i}{\partial q_\alpha}\frac{\partial r_i}{\partial t}\dot{q}_\alpha \quad // T_1 \qquad [13] \\ &+ \sum_i \sum_{\alpha,\beta} \frac{1}{2} m_i \frac{\partial r_i}{\partial q_\alpha}\frac{\partial r_i}{\partial q_\beta}\dot{q}_\alpha \dot{q}_\beta \quad // T_2 \end{aligned}$$

2° 推导 ideal/Harmic 系统 Largrange Eq

将$\delta \vec{r}_i = \sum_\alpha \frac{\partial \vec{r}_i}{\partial q_\alpha}\delta q_\alpha$. 代入$\sum_i (\vec{F}_i - m_i \ddot{\vec{r}}_i)\delta \vec{r}_i = 0$.

$$\Rightarrow \sum_\alpha \left[\sum_i (\vec{F}_i - m_i \ddot{\vec{r}}_i)\frac{\partial \vec{r}_i}{\partial q_\alpha}\right]\delta q_\alpha = 0. \quad q_\alpha \text{独立} \Rightarrow \sum_i (\vec{F}_i - m_i \ddot{\vec{r}}_i)\frac{\partial \vec{r}_i}{\partial q_\alpha} = 0$$

$$\Rightarrow \boxed{\sum_i m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} - \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} = 0. \quad [15]}$$

$$\sum_i m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} = \sum_i \left[\frac{d}{dt} m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} - m_i \dot{\vec{r}}_i \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)\right]$$

$$\overset{[12]}{=} \frac{d}{dt}\sum_i m_i \dot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} - \sum_i m_i \dot{\vec{r}}_i \frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}.$$

$$\overset{[13]}{=} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} \overset{[15]}{=} \sum_i \vec{F}_i \frac{\partial \vec{r}_i}{\partial q_\alpha} = Q_\alpha.$$

即 $\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \quad [16]}$，IH系 Largrange Eq.

3° IHC系统 Largrange Eq.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} \overset{[17]}{=} -\frac{\partial V}{\partial q_\alpha}, \quad \frac{\partial V}{\partial \dot{q}_\alpha} = 0 \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0, \quad L = T - V} \quad [18]$$

应用：求球坐标系加速度表达：

$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (r^2\sin^2\theta)d\varphi^2$. $\quad \delta \vec{r} = \vec{e}_r \delta r + r\vec{e}_\theta \delta\theta + r\sin\theta\, \vec{e}_\varphi \delta\varphi$

$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2)$.

$$\begin{cases} \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = m\ddot{r} - m(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)r = \vec{F}\cdot\vec{e}_r \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = m(r^2\ddot{\theta} + 2r\dot{\theta}\dot{r}) - mr^2\sin\theta\cos\theta\,\dot{\varphi}^2 = r\vec{F}\cdot\vec{e}_\theta \\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\varphi}}\right) - \frac{\partial T}{\partial \varphi} = m(r^2\sin^2\theta\,\ddot{\varphi} + 2r\sin^2\theta\,\dot{\varphi}\dot{r} + 2r^2\sin\theta\cos\theta\,\dot{\varphi}\dot{\theta}) - 0 = r\sin\theta\,\vec{F}\cdot\vec{e}_\varphi. \end{cases}$$

$$\Rightarrow \ddot{\vec{r}} = \frac{\vec{F}}{m} = [\ddot{r} - r(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)]\vec{e}_r + [r\ddot{\theta} + 2\dot{\theta}\dot{r} - r\sin\theta\cos\theta\,\dot{\varphi}^2]\vec{e}_\theta + [r\sin\theta\,\ddot{\varphi} + 2\sin\theta\,\dot{\varphi}\dot{r} + 2r\cos\theta\,\dot{\varphi}\dot{\theta}]\vec{e}_\varphi$$

四、电磁场中 Lagrange 方程的广义势修正.

1° $\boxed{Q_\alpha = -\frac{\partial U}{\partial q_\alpha} + \frac{d}{dt}\frac{\partial U}{\partial \dot{q}_\alpha}}$ [19] $U = U(q,\dot{q}) \Rightarrow L = T - U$. 广义势.

2° Maxwell: $\begin{cases} \nabla\cdot B = 0 \\ \nabla\times E + \frac{\partial B}{\partial t} = 0. \end{cases}$ let $B = \nabla\times A \Rightarrow \frac{\partial B}{\partial t} = \nabla\times\frac{\partial A}{\partial t}$

$\Rightarrow \nabla\times\left(E + \frac{\partial A}{\partial t}\right) = 0$

$\Rightarrow E + \frac{\partial A}{\partial t} = -\nabla\varphi$. $\boxed{E = -\nabla\varphi - \frac{\partial A}{\partial t}}$ [20]

$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \overset{[20]}{=} q\left[-\nabla\varphi - \frac{\partial A}{\partial t} + \vec{v}\times(\nabla\times\vec{A})\right]$

$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + (\vec{v}\cdot\nabla)}$ [21]

$= q\left[-\nabla\varphi - \frac{\partial A}{\partial t} + \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A}\right]$

$\overset{[21]}{=} q\left[-\nabla\varphi - \frac{dA}{dt} + \nabla(\vec{v}\cdot\vec{A})\right]$ $\quad \nabla_v \equiv \frac{\partial}{\partial v_x}\vec{i} + \frac{\partial}{\partial v_y}\vec{j} + \frac{\partial}{\partial v_z}\vec{k}$.

$= -q\nabla(\varphi - \vec{v}\cdot\vec{A}) - q\frac{d\vec{A}}{dt}$. $\quad \nabla_v(\vec{v}\cdot\vec{A}) = \vec{A}$, $\nabla_v\varphi = 0$

$= -q\nabla(\varphi - \vec{v}\cdot\vec{A}) + q\frac{d}{dt}\left[\nabla_v(\varphi - \vec{v}\cdot\vec{A})\right] \Rightarrow \boxed{U = q(\varphi - \vec{v}\cdot\vec{A})}$ 满足 [19].

$\boxed{L = \frac{1}{2}mv^2 - q\varphi + q\vec{v}\cdot\vec{A}}$ [22] $\quad \boxed{\vec{p} = \nabla_v L = m\vec{v} + q\vec{A},\ H = \vec{p}\cdot\vec{v} - L = T + q\varphi}$

五、耗散函数 (dissipation func).

1° $f = -\lambda\dot{x}$. $\boxed{D = \frac{1}{2}\lambda\dot{x}^2} = \frac{1}{2}|f|\dot{x}$, $f = -\frac{\partial D}{\partial \dot{x}}$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = -\frac{\partial D}{\partial \dot{x}}$ $\quad dE = f\,dx = -2D\,dt \Rightarrow \boxed{-2D = \frac{dE}{dt}}$

2° 推广: $\boxed{D = \frac{1}{2}\sum \lambda_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta}$ $\quad f_\alpha = -\frac{\partial D}{\partial \dot{q}_\alpha} = -\sum_\beta \lambda_\beta\dot{q}_\beta$.

$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = -\frac{\partial D}{\partial \dot{q}_\alpha}}$ [23]

3° 电路对比: $L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0$

$T = \frac{1}{2}L\dot{q}^2$, $V = \frac{1}{2C}q^2$. $D = \frac{1}{2}R\dot{q}^2$, $\to$ [23]

耦合系统: $\boxed{T = \frac{1}{2}\sum L_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta.\ V = \frac{1}{2}\sum_{\alpha\beta}\frac{1}{C_{\alpha\beta}}q_\alpha q_\beta.\ D = \frac{1}{2}\sum_{\alpha\beta}R_{\alpha\beta}\dot{q}_\alpha\dot{q}_\beta}$

六、对称性与守恒律.

1° 力学守恒量. $f = f(q,\dot{q}_\alpha) \to$ Lagrange Eq 一次积分 $\to$ 运动积分.

DOF = S $\Rightarrow$ 2S 个方程 $\Rightarrow$ 消 t 得 2S-1. 至多有 2S-1 个独立的守恒量.

2° 广义动量. $p_\alpha \equiv \frac{\partial L}{\partial \dot{q}_\alpha} \Rightarrow \frac{\partial L}{\partial q_\alpha} = 0$ (对称) 时 $\frac{d}{dt}(p_\alpha) = 0$.

$\frac{\partial L}{\partial q_\alpha} = 0$: q_α 为循环坐标. $q = x$ $p_j = p_x$. $q = \theta$, $p_j = L$. $\Rightarrow$ 反映空间相应性.

3° 广义能量. 若 $\frac{\partial L}{\partial t} = 0$:

$\frac{dL}{dt} = \sum_\alpha \frac{\partial L}{\partial q_\alpha}\dot{q}_\alpha + \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}\ddot{q}_\alpha = \sum_\alpha \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right)\dot{q}_\alpha + \left(\frac{d}{dt}\dot{q}_\alpha\right)\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \sum_\alpha p_\alpha\dot{q}_\alpha$

$\Rightarrow \boxed{\sum_\alpha p_\alpha\dot{q}_\alpha - L = \text{const} \equiv H}$ [24].

4

$H = (\sum_\alpha p_\alpha \dot q_\alpha) - L \xrightarrow{V不含速度} \sum_\alpha \frac{\partial T}{\partial \dot q_\alpha}\dot q_\alpha - (T-V)$

$\overset{[13]}{=} \sum_\alpha \left(\frac{\partial T_2}{\partial \dot q_\alpha}\dot q_\alpha + \frac{\partial T_1}{\partial \dot q_\alpha}\dot q_\alpha + \frac{\partial T_0}{\partial \dot q_\alpha}\dot q_\alpha\right) - (T_2+T_1+T_0-V)$

$= 2T_2 + T_1 - T_2 - T_1 - T_0 + V = \boxed{T_2 - T_0 + V}$　$\frac{\partial L}{\partial t}$时, $T_0=T_1=0$, $H = T+V$.

4° 从对称性到守恒量.

刻画空间~~...~~不变性: 空间有一无限小平移 $\vec\varepsilon$ 或转动 $\delta\vec\varphi = \delta\varphi\,\vec e_n$ 时, $\delta L = 0$.

$\delta L = \sum_\alpha \left(\frac{\partial L}{\partial \dot q_\alpha}\delta\dot q_\alpha + \frac{\partial L}{\partial q_\alpha}\delta q_\alpha\right) \overset{类[2]}{=} \frac{d}{dt}\sum_\alpha\left(\frac{\partial L}{\partial \dot q_\alpha}\delta q_\alpha\right) \overset{[14]}{=} \frac{d}{dt}\left(\sum_\alpha\sum_i m_i\dot{\vec r}_i \frac{\partial \vec r_i}{\partial q_\alpha}\delta q_\alpha\right)$

$= \frac{d}{dt}\sum_i m_i \dot{\vec r}_i \left(\sum_\alpha \frac{\partial \vec r_i}{\partial q_\alpha}\delta q_\alpha\right) = \boxed{\frac{d}{dt}\left(\sum_i m_i\dot{\vec r}_i\cdot\delta\vec r_i\right) = \delta L}$

·空间平移: $\delta\vec r_i = \vec\varepsilon \Rightarrow \frac{d}{dt}\left(\sum_i m_i\dot{\vec r}_i\cdot\vec\varepsilon\right)=0 \Rightarrow \vec\varepsilon\cdot\frac{d\vec p}{dt}=0$.

·空间转动: $\delta\vec r_i = \delta\vec\varphi\times\vec r_i \Rightarrow \frac{d}{dt}\left[\sum_i m_i\dot{\vec r}_i\cdot(\delta\vec\varphi\times\vec r_i)\right]=0$

$\Rightarrow \frac{d}{dt}\left[\sum_{i=1}\vec r_i\times m_i\dot{\vec r}_i\right]\cdot\delta\vec\varphi = 0 \Rightarrow \delta\vec\varphi\cdot\frac{d\vec J}{dt}=0$.

·复合情况: 无限长均匀圆柱螺旋线. $\delta\vec r_i = \delta\vec\varphi\times\vec r_i + \frac{h}{2\pi}\delta\vec\varphi$.

$\Rightarrow \frac{d}{dt}\left[\sum m_i\dot{\vec r}_i\left(\delta\vec\varphi\times\vec r_i+\frac{h}{2\pi}\delta\vec\varphi\right)\right]=0 \Rightarrow \delta\vec\varphi\cdot\left[\frac{d\vec J}{dt}+\frac{h}{2\pi}\frac{d\vec p}{dt}\right]=0$

$\Rightarrow J_z + \frac{h}{2\pi}p_z = \text{const}$.

七. 二体问题.

1° 二体问题概述: $\vec r_{oc} = \frac{m_1\vec r_1+m_2\vec r_2}{m_1+m_2} \Rightarrow \begin{cases}\vec r_1 = \vec r_{oc}+\frac{m_2}{m_1+m_2}\vec r\\ \vec r_2 = \vec r_{oc} - \frac{m_1}{m_1+m_2}\vec r\end{cases}$ $(\vec r = \vec r_1-\vec r_2)$

$\Rightarrow T = \frac12 m_1\dot{\vec r}_1^2+\frac12 m_2\dot{\vec r}_2^2 = \frac12(m_1+m_2)\dot{\vec r}_{oc}^2+\frac12 m_r\dot{\vec r}^2$ $\left(\boxed{m_r = \frac{m_1m_2}{m_1+m_2}}\right)$.
↳ reduced mass.

$V = V^{(e)}(\vec r_{oc}) + V^{(i)}(\vec r)$ (assumption). $\Rightarrow L = L_o + L_i$,

$\boxed{L_o = \frac12(m_1+m_2)\dot{\vec r}_{oc}^2 - V^{(e)}(\vec r_{oc})}$　$\boxed{L_i = \frac12 m_r\dot{\vec r}^2 - V^{(i)}(\vec r)}$

2° 有心力场单粒子.

$\vec F = f(r)\vec e_r \Rightarrow$ 对力心角动量守恒 ⇒ 平面运动.
⇒ 面积速度 $\sigma = \frac12\vec r\times\vec v = \frac{J}{2m} = \text{Const}$.

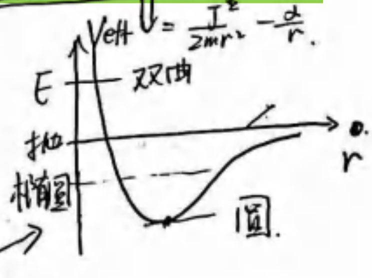
⇓ 保守力 ⇒ 机械能守恒

有效势: $E = \frac12 m(\dot r^2 + r^2\dot\theta^2) + V(r) = \frac12 m\dot r^2 + \boxed{\frac{J^2}{2mr^2}+V(r) \equiv V_{eff}}$

$V_{eff} = \frac{J^2}{2mr^2} - \frac{\alpha}{r}$.

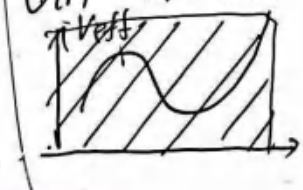
$\Rightarrow \dot r = \pm\sqrt{\frac{2}{m}(E-V_{eff}(r))}$　↗　$\theta = \int d\theta = \int\frac{J}{mr^2(t)}dt$. (通法, 一般算不动)

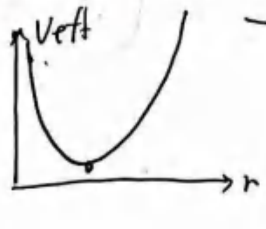
$\Rightarrow t = \pm\int\frac{\sqrt{\frac{m}{2}}\,dr}{\sqrt{E-V_{eff}(r)}}$



特别地, $V(r) = -\frac{\alpha}{r}$ 时. 解为圆锥曲线.

$V(r) \sim \alpha r^2$ 时.　两种稳定的情况!





Binet 公式：$\boxed{\dfrac{d^2u}{d\theta^2}+u+\dfrac{m}{J^2u^2}f\left(\dfrac{1}{u}\right)=0}$ [25]

derive：$\ddot r - r\dot\theta^2 = \frac{1}{m}f(r)$，$\begin{cases}\dfrac{dr}{dt} = \dot\theta\dfrac{dr}{d\theta} = \dfrac{J}{m}u^2\dfrac{d\frac{1}{u}}{d\theta} = -\dfrac{J}{m}\dfrac{du}{d\theta} \\ \dfrac{d^2r}{dt^2} = -\dfrac{J}{m}\dot\theta\dfrac{d^2u}{d\theta^2}\end{cases}$

$\Leftarrow$ $-\frac{J^2}{m^2}\frac{d^2u}{d\theta^2}u^2 - \frac{J^2}{m^2}u^3 - \frac{1}{m}f(r) = 0$

$f(r) = -\frac{\alpha}{r^2} \Rightarrow \frac{d^2u}{d\theta^2}+u-\frac{m\alpha}{J^2}=0 \Rightarrow \boxed{r(\theta)=\dfrac{p}{1+e\cos(\theta-\theta_0)}}$ $\boxed{p=\dfrac{J^2}{m\alpha},\quad e^2 = 1+\dfrac{2EJ^2}{m\alpha^2}}$

$a = \frac{1}{2}(r_{max}+r_{min}) = -\frac{2\alpha}{E}$，$b^2 = -\frac{\alpha}{E}p$，$T = \frac{S}{\sigma} = \frac{m}{J}2\pi ab$.

· 轨道稳定性分析：(i) 圆轨道：记 $F = -\alpha u^n$.

$\Rightarrow \frac{d^2u}{d\theta^2}+u = R^{n-3}u^{n-2}$. where R 为平衡位置，$R^{n-3} = \frac{m\alpha}{J^2}$.

取 $u = \frac{1}{R}+\varepsilon$. $\Rightarrow \frac{d^2\varepsilon}{d\theta^2}+(3-n)\varepsilon = 0$. Stable iff $3-n>0$，$n<3$.

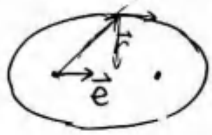
(ii) 一般轨道(*)：$(u_0^2+2u_0\varepsilon+\varepsilon^2)\left(\frac{d^2u_0}{d\theta^2}+\frac{d^2\varepsilon}{d\theta^2}+u_0+\varepsilon\right)+\frac{M}{J}F(u_0+\varepsilon) = 0$.

$\Rightarrow \frac{d^2\varepsilon}{d\theta^2}+A\varepsilon = 0$，$A = $ [很长一串]. $A>0$ 时 stable. $\Rightarrow -\frac{\alpha}{r}/r^2$：永远稳定.

· 轨道封闭性：封闭 $\Leftrightarrow$ $T(r)$: $T(\theta)$ 为有理数. $\longrightarrow$ 唯二的封闭条件.

· LRL vec：平方反比的特征.

$V = -\frac{\alpha}{r} \Rightarrow \boxed{\vec J\times\dot{\vec r}+\alpha\frac{\vec r}{r} = \text{Const vec}} = -\alpha\vec e$，$\boxed{\vec e^2 = \dfrac{2EJ^2}{m\alpha^2}+1}$



PF：$\left.\begin{array}{l} m\ddot{\vec r} = -\alpha\dfrac{\vec r}{r^3} \\ \dfrac{d}{dt}(\vec J\times\dot{\vec r}) = \vec J\times\ddot{\vec r}\end{array}\right] \Rightarrow \frac{d}{dt}(\vec J\times\dot{\vec r}) = -\frac{\alpha}{mr^3}\vec J\times\vec r$

$\vec J\times\vec r = (\sum m_i\vec r_i\times\dot{\vec r}_i)\times\vec r = m\dot{\vec r}(\vec r\cdot\vec r) - \vec r(m\dot{\vec r}\cdot\vec r) = mr^2\dot{\vec r} - mr\dot r\vec r$

$= mr(r\dot{\vec r} - \dot r\vec r) = \frac{m}{r^3}\left(\frac{r\dot{\vec r}-\dot r\vec r}{r^2}\right) = \frac{m}{r^3}\frac{d}{dt}\left(\frac{\vec r}{r}\right)$

$\Rightarrow \frac{d}{dt}(J\times\dot{\vec r}) = -\alpha\frac{d}{dt}\left(\frac{\vec r}{r}\right) \Rightarrow \vec e = -\frac{1}{\alpha}\vec J\times\dot{\vec r} - \frac{\vec r}{r} = \text{Const}$.

$\vec e^2 = \frac{1}{\alpha^2}(\vec J\times\dot{\vec r})^2 + \frac{2}{\alpha}(J\times\dot{\vec r})\cdot\frac{\vec r}{r}+1$ $\qquad (J\times\dot{\vec r})\cdot\vec r = -(\vec r\times\dot{\vec r})\cdot J = -\frac{J^2}{m}$

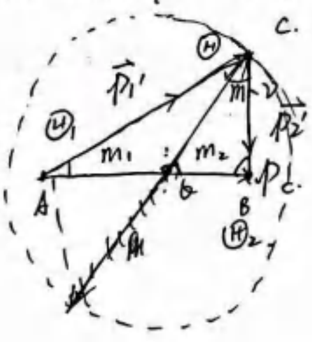
$= \frac{1}{\alpha^2}J^2\dot{\vec r}^2 + -\frac{2}{\alpha r}\cdot\frac{J^2}{m}+1 = \frac{2J^2}{m\alpha^2}\cdot\left(\frac{m}{2}\dot{\vec r}^2 - \frac{\alpha}{r}\right)+1 = \frac{2J^2E}{m\alpha^2}+1$. QED.

$\vec r\cdot\vec e = \frac{J^2}{m\alpha} - r = \text{Const} \Rightarrow r = \frac{J^2}{m\alpha}\frac{1}{1+e\cos\theta}$.

3° 弹性碰撞.

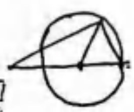
$\begin{cases} m_1\vec v_1+m_2\vec v_2 = m_1\vec v_1'+m_2\vec v_2' \\ \frac{1}{2}m_1\vec v_1^2+\frac{1}{2}m\vec v_2^2 = \frac{1}{2}m\vec v_1'^2+\frac{1}{2}m\vec v_2'^2\end{cases} \Rightarrow \boxed{\begin{cases}\vec v_1' = \dfrac{m_1\vec v_1+m_2\vec v_2}{m_1+m_2}+\dfrac{m_2}{m_1+m_2}|\vec v_1-\vec v_2|\vec e \\ \vec v_2' = \dfrac{m_1\vec v_1+m_2\vec v_2}{m_1+m_2}-\dfrac{m_1}{m_1+m_2}|\vec v_1-\vec v_2|\vec e\end{cases}}$ [26]

($\vec e$ 由弹性碰撞的细节决定).



B 核静止：B在圆上.

$m_1>m_2$：A在圆外.



Θ_1 有最大值. $\Theta>\frac{\pi}{2}$

$m_1<m_2$：A in

$\Theta_1\in[0,\pi]$，$\Theta<\frac{\pi}{2}$.

$m_1=m_2$：$\vec p_1\cdot\vec p_2 = 0$.

恢复系数 $e = \frac{I_2}{I_1}$.

$e = \frac{v_{2n}'-v_{1n}'}{v_{1n}-v_{2n}}$. $\in(0,1)$.

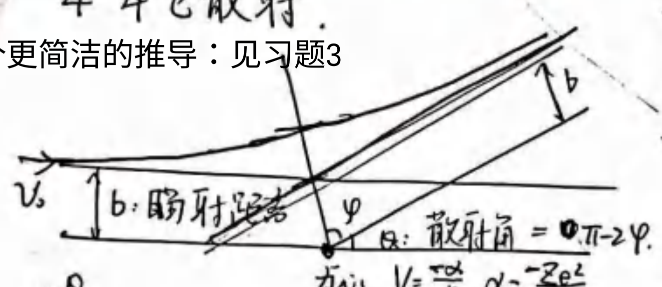
[26] 中 $\vec e$ 可修正为 $|\vec e| = e$.

$\tan\Theta_1 = \frac{m_2\sin\theta}{m_1+m_2\cos\theta}$，$\theta = \pi-2\Theta_2$

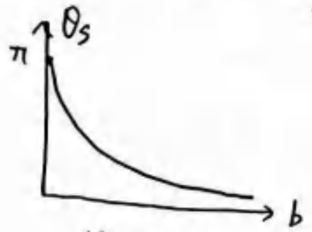
[28]

4° 库仑散射.

一个更简洁的推导：见习题3



方心: $V=\frac{-\alpha}{r}$. $\alpha=\frac{-Ze^2}{2\pi\varepsilon_0}$



$$\frac{d^2u}{d\theta^2}+u-\frac{m\alpha}{J^2}=0 \Rightarrow u=A\cos\theta+B\sin\theta+\frac{m\alpha}{J^2}.$$

$u(\pi)=0 \Rightarrow A=\frac{m\alpha}{J^2}.$

$\frac{1}{b}=u\frac{1}{\sin\theta}\Big|_{\theta=\pi}=B+A\left(\frac{1+\cos\theta}{\sin\theta}\right)=B+A\cot\frac{\theta}{2}$

$=B \Rightarrow \frac{1}{b}=B.$

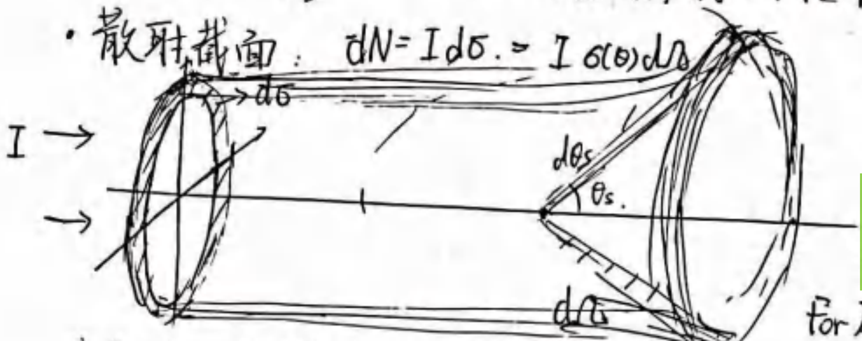
$u(\theta_s)=0 \Rightarrow \frac{m\alpha}{J^2}(1+\cos\theta_s)+\frac{1}{b}\sin\theta_s=0$

$\Rightarrow \boxed{\cot\frac{\theta_s}{2}=-\frac{J^2}{m\alpha b}=-\frac{mbv_0^2}{\alpha}}$ [27]

• 总散射截面σ_t: 能被散射的入射(x,y)坐标的集合.

对于截面为A的均匀全同粒子数，粒子被散射机率$\frac{\sigma_t}{A}$ (σ_t 100% A; 若 σ_t A>.)

• 散射截面: $dN=I\,d\sigma=I\,\sigma(\theta)\,d\Omega$



$$d\Omega=\frac{2\pi R\sin\theta R\,d\theta}{R^2}=2\pi\sin\theta\,d\theta.$$

$$d\sigma=2\pi b\,db$$

$$\boxed{\frac{d\sigma}{d\Omega}=\frac{b(\theta)}{\sin\theta}\left|\frac{db}{d\theta}\right|=\sigma(\theta)}\ (m^2/sr)$$

For 库仑: $\boxed{\sigma(\theta)=\left(\frac{\alpha}{4E}\right)^2\left(\sin\frac{\theta}{2}\right)^{-4}}$ [28]

Rutherford's Fomula.

应用: 测量 $dN, d\Omega \Rightarrow b(\theta) \Rightarrow V(r)$

[28] derive: $\cot\frac{\theta}{2}=-\frac{mv_0^2}{\alpha}b \Rightarrow -\frac{1}{2}\left(\sin\frac{\theta}{2}\right)^{-2}d\theta=-\frac{mv_0^2}{\alpha}db$

$\Rightarrow \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|=\frac{\frac{\alpha}{mv_0^2}\cot\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\cdot\left|\frac{\alpha}{2\sin^2\frac{\theta}{2}\,mv_0^2}\right|=\left[\frac{\alpha}{4(\frac{1}{2}mv_0^2)}\right]^2\csc^4\frac{\theta}{2}.$

修正到Lab系: $m_1\to m_r$, $v_0\to v'$, $\Theta_1=\arctan\frac{m_2\sin\theta}{m_1+m_2\cos\theta} \Rightarrow \sigma(\Theta_1)=f(E',\Theta_1)$

八. 振动问题

1° 单自由度微振动. 保守

$T\approx\frac{1}{2}m(q)\dot{q}^2$, $V=V(q_0)+\frac{\partial V}{\partial q}\Big|_{q=q_0}\times(q-q_0)+\frac{1}{2}\frac{\partial^2V}{\partial q^2}\Big|_{q=q_0}\times(q-q_0)^2$

($\frac{\partial V}{\partial q}\big|_{q=q_0}$ = 0 when 平衡; $\frac{\partial^2V}{\partial q^2}\big|_{q=q_0}$ >0 when 稳定)

$\tilde{m}=\frac{\partial^2T}{\partial\dot{q}^2}\Big|_{q=q_0}$, $\tilde{k}=\frac{\partial^2V}{\partial q^2}\Big|_{q=q_0} \Rightarrow m\ddot{q}+kq=0.$

$\boxed{\omega^2=\tilde{k}/\tilde{m}}$

2° 双自由度保守微振动.

记 $\hat{q}=\begin{pmatrix}q_1\\q_2\end{pmatrix} \Rightarrow T=\frac{1}{2}\dot{\hat{q}}^T A\dot{\hat{q}}$, $V=\frac{1}{2}\hat{q}'^T B\hat{q}'$. $(\hat{q}'=\hat{q}-\hat{q}_0)$

稳定: B正定 $A\ddot{\hat{q}}+B(\hat{q}-\hat{q}_0)=0.$

$\boxed{\det(B-\omega^2A)=0}$: 久期方程 ⇒ 2个正根 of ω^2 (一般无简并)

$\Rightarrow \hat{q}-\hat{q}_0=C_1\hat{q}_1\sin(\omega_1t+\alpha_1)+C_2\hat{q}_2\sin(\omega_2t+\alpha_2)$, $\hat{q}_1,\hat{q}_2$ 为 $(B-\omega^2A)\hat{q}_i$ 的解.

$$A_{ij}=\frac{\partial^2T}{\partial\dot{q}_i\partial\dot{q}_j},\quad B_{ij}=\frac{\partial^2V}{\partial q_i\partial q_j}$$

简正坐标: 对 A.B 同时对角化 ~~$U_1^TAU_1=\Lambda_A$ $U_2^TBU_2=\Lambda_B$~~.

$P_1^TAP_1=E$. $U_2^T(P_1^TBP_1)U_2=\Lambda_B$, $U_2^T(P_1^TAP_1)U_2=E$

$\Rightarrow P=P_1U_2$ 同时对角化 A,B $\Rightarrow \hat{q}'=P\hat{q}$: 简正模.

上述全部可推广至多自由度.

3° 受迫振动 ~~与~~ 阻尼振动.

·迫(单自由度): $\ddot{x}+\omega^2 x=\frac{f}{m}\cos(\gamma t+\beta)$.

$\Rightarrow x=C_1\cos(\omega t+\varphi)+\boxed{\frac{f}{m(\omega^2-\gamma^2)}\cos(\gamma t+\beta)}$ → 线性 → 不再微振动

$\begin{cases}\gamma=\omega \text{ 时}, & x=C'\cos(\omega t+\varphi)+\frac{f}{2m\omega}t\sin(\omega t+\beta)\\ \gamma=\omega+\varepsilon \text{ 时}, & \tilde{x}=(A+Be^{i\varepsilon t})e^{i\omega t}.\end{cases}$ $C=|\underset{ae^{i\alpha}}{A}+\underset{be^{i\beta}}{B}e^{i\omega t}|$

$C^2=a^2+b^2+2ab\cos(\varepsilon t+\beta-\alpha)$.

即 C 在 $|a-b|\sim|a+b|$ 间以 ε 周期变化, → Beat!

仅|B|的大小: $b=\left|\frac{f}{m}\frac{1}{(\omega^2-\gamma^2)}\right|=\frac{f}{m\cdot 2\varepsilon\omega}\propto\frac{1}{\varepsilon}$.

·阻(单): $\ddot{x}+\omega_0^2x+2\lambda\dot{x}=0$. ($f=-\alpha\dot{x}$, $\lambda=\frac{\alpha}{2m}$)

$x=\begin{cases}Ce^{-\lambda t}\cos(\omega t+\varphi), \ \omega=\sqrt{\omega_0^2-\lambda^2}, & \text{过阻尼}\\ C_1e^{-[\lambda-\sqrt{\lambda^2-\omega_0^2}]t}+C_2e^{-[\lambda+\sqrt{\lambda^2-\omega_0^2}]t} & \text{欠阻尼.}\\ (C_1+C_2t)e^{-\lambda t}. & \text{临界 damping}\end{cases}$

·综合性: $\ddot{x}+2\lambda\dot{x}+\omega_0^2x=\frac{f}{m}\cos(\gamma t)$. $\frac{f}{m}e^{i\gamma t}$

$x=Be^{i\gamma t}\Rightarrow B=\frac{f}{m}\frac{1}{\omega_0^2-\gamma^2+2i\lambda\gamma}$. $\Rightarrow\gamma=\sqrt{\omega_0^2-2\lambda^2}$ 振幅极值

$\omega_0>\lambda$时, $x=C_1e^{-\lambda t}\cdot\cos(\omega t+\varphi_1)+C_2\cos(\gamma t+\varphi_2)\xrightarrow{t\to\infty}C_2\cos(\gamma t+\varphi_2)$

·多自由度受迫振动: 简正坐标加单自由度即可.

4° 非线性振动(*)

普通微扰 → 久期项 → Lindstedt-Poicare方法. [示例]:

$\ddot{x}+x+\varepsilon x^3=0$ (Duffing 方程)

$x=x_0+\varepsilon x_1+\varepsilon^2x_2+\cdots$, $\tau=\omega t$, $\omega=\omega_0+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots$.

$\Rightarrow x_0=\cos\tau$; $x_1''+x_1=(2\omega_1-\frac{3}{4})\cos\tau-\frac{1}{4}\cos3\tau\Rightarrow\omega_1=\frac{3}{8}$.

$\Rightarrow\omega=1+\varepsilon\cdot\frac{3}{8}A+O(\varepsilon^2)$.

非线性振动与振幅有关, 可分解为基频与基频整数倍的谐频.