# Discrete Math (Honor) 2021-Fall Homework-5

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**Due: 2021.11.12 Friday in Class**
(Please use A4 paper. Do not use exercise book!)

**Problem 1.** *(12 Points)*
Write the Skolem normal form for each of the following formulae.

1. $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
2. $(\forall x)(P(x) \to (\exists y)Q(x,y)) \vee (\forall z)R(z)$

**Answer**

1. Step 1:
$(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
$(\forall x)(\neg P(x) \vee Q(x)) \to (\neg(\exists x)P(x) \vee (\exists x)Q(x))$
$(\forall x)(\neg P(x) \vee Q(x)) \to ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
$\neg(\forall x)(\neg P(x) \vee Q(x)) \vee ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
$(\exists x)\neg(\neg P(x) \vee Q(x)) \vee ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
$(\exists x)(P(x) \wedge \neg Q(x)) \vee ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
$(\exists x)(\forall y)(P(x) \wedge \neg Q(x)) \vee (\neg P(y) \vee (\exists x)Q(x))$
$(\exists x)(\forall y)(\exists z)(P(x) \wedge \neg Q(x)) \vee (\neg P(y) \vee Q(z))$

   Step 2:
$(\forall y)(\exists z)(P(u) \wedge \neg Q(u)) \vee (\neg P(y) \vee Q(z))$
$(\forall y)(P(u) \wedge \neg Q(u)) \vee (\neg P(y) \vee Q(f(y)))$
or $(\forall y)(P(a) \to Q(a)) \to (P(y) \to Q(f(y)))$

2. Step 1:
$(\forall x)(P(x) \to (\exists y)Q(x,y)) \vee (\forall z)R(z)$
$(\forall x)(\neg P(x) \vee (\exists y)Q(x,y)) \vee (\forall z)R(z)$
$(\forall x)(\exists y)(\neg P(x) \vee Q(x,y)) \vee (\forall z)R(z)$
$(\forall x)(\exists y)(\neg P(x) \vee Q(x,y) \vee (\forall z)R(z))$
$(\forall x)(\exists y)(\forall z)(\neg P(x) \vee Q(x,y) \vee R(z))$

   Step 2:
$(\forall x)(\forall z)(\neg P(x) \vee Q(x, f(x)) \vee R(z))$

**Problem 2.** *(8 Points)*
Formalize the following inference and prove it **by resolution**.

*All SJTU students are smart. Bob is both an SJTU student and an NBA player. Therefore, some NBA player is smart.*

**Answer:** We define following predicates $a$ : "Bob" $P(x)$ :" $x$ is an SJTU student." $Q(x)$ : " $x$ is smart." $R(x)$ : "$x$ is an NBA player." We need to prove

$$(\forall x)(P(x) \to Q(x)) \wedge P(a) \wedge R(a) \Rightarrow (\exists y)(R(y) \wedge Q(y))$$

Write the Skolem normal form of $\alpha \wedge \neg\beta$

$$\begin{aligned}
&(\forall x)(P(x) \to Q(x)) \wedge P(a) \wedge R(a) \wedge \neg(\exists y)(R(y) \wedge Q(y)) \\
=&(\forall x)(\neg P(x) \vee Q(x)) \wedge P(a) \wedge R(a) \wedge (\forall y)(\neg R(y) \vee \neg Q(y)) \\
=&(\forall x)(\forall y)((\neg P(x) \vee Q(x)) \wedge (\neg R(y) \vee \neg Q(y)) \wedge P(a) \wedge R(a))
\end{aligned}$$

The clause set is $S = \{\neg P(x) \vee Q(x), \neg R(y) \vee \neg Q(y), P(a), R(a)\}$

$$\left.\begin{array}{r}\neg P(x) \vee Q(x) \\ P(a)\end{array}\right\} \xrightarrow{\sigma=\{x/a\}} \neg Q(a), \qquad \left.\begin{array}{r}Q(a) \\ \neg R(y) \vee \neg Q(y)\end{array}\right\} \xrightarrow{\sigma=\{y/a\}} \neg R(a)$$

This gives us a contradiction $R(a) \wedge \neg R(a)$

**Problem 3.** *(12 Points)*
Determine whether each of the following propositions is true or false.

1. $\emptyset \subseteq \emptyset$
2. $\emptyset \in \emptyset$
3. $\emptyset \subseteq \{\emptyset\}$
4. $\emptyset \in \{\emptyset\}$
5. $\{\emptyset\} \subseteq \{\emptyset\}$
6. $\{\emptyset\} \in \{\emptyset\}$
7. $\{\emptyset\} \subseteq \{\{\emptyset\}\}$
8. $\{\emptyset\} \in \{\{\emptyset\}\}$
9. $\{a, b\} \in \{a, b, \{a, b\}\}$
10. $\{a, b\} \subseteq \{a, b, \{a, b\}\}$
11. $\{a, b\} \in \{a, b, \{\{a, b\}\}\}$
12. $\{a, b\} \subseteq \{a, b, c, \{\{a, b\}\}\}$

**Answer:** 1. T 2. F 3. T 4. T 5. T 6. F 7. F 8. T 9. T 10. T 11. F 12.T

**Problem 4.** *(6 Points)*
Let $A = 2^{2^{2^{\emptyset}}}$. Determine whether each of the following propositions is true or false.

1. $\emptyset \in A$
2. $\emptyset \subseteq A$
3. $\{\emptyset\} \in A$
4. $\{\emptyset\} \subseteq A$
5. $\{\{\emptyset\}\} \in A$
6. $\{\{\emptyset\}\} \subseteq A$

**Answer:**
$2^{\emptyset} = \{\emptyset\}$
$2^{\{\emptyset\}} = \{\emptyset, \{\emptyset\}\}$
$2^{\{\emptyset,\{\emptyset\}\}} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

1. T 2. T 3. T 4. T 5. T 6. T

**Problem 5.** *(6 Points)*
Write down the following sets by listing their elements.

1. $2^{\{\emptyset,\{1,\{2\}\}\}}$
2. $\bigcup\{\{a, b\}, \{\{a\}, \{b\}\}, \{a, \{b\}\}, \{\{a\}, b\}\}$

3. $\bigcap\{2^{\emptyset}, 2^{2^{\emptyset}}, 2^{2^{2^{\emptyset}}}\}$

**Answer:**

1. $\{\emptyset, \{\emptyset\}, \{\{1, \{2\}\}\}, \{\emptyset, \{1, \{2\}\}\}\}$
2. $\{a, b, \{a\}, \{b\}\}$
3. $\{\emptyset\}$

**Problem 6.** *(8 Points)*
Find two sets $A$ and $B$ such that $(\bigcap A) \cap (\bigcap B) \neq \bigcap(A \cap B)$.
Find two sets $C$ and $D$ such that $(\bigcap C) \cap (\bigcap D) = \bigcap(C \cap D)$.

**Answer:**
Example:
$A = \{\emptyset, \{\emptyset\}\}, B = \{\{\emptyset\}\}$
any $C = D$ is an example

**Problem 7.** *(10 Points)*
Let $A$ be a family of sets. Prove that $A$ is a transitive set *if and only if* $\bigcup A \subseteq A$.

**Answer:**

1. $\Rightarrow$: suppose A is a transitive set.
For any $x \in \cup A$, exists some $y$ such that $x \in y$ and $y \in A$. Since A is transitive, we have $x \in A$, therefore, $\bigcup A \subseteq A$.
2. $\Leftarrow$: suppose $\bigcup A \subseteq A$.
For any x, y such that $x \in y$ and $y \in A$, then we have $x \in \cup A$. Therefore, A is transitive.