Discrete Math (Honor) 2021-Fall Homework-2: Solution

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Due: 2021.10.15 Friday in Class
(Please use A4 paper. Do not use exercise book!)

Problem 1. *(8 Points)*
Write down formulas for α and β, respectively, based on the following truth table.

P	Q	R	α	β
0	0	0	1	0
0	0	1	0	0
0	1	0	0	1
0	1	1	0	1
1	0	0	0	1
1	0	1	0	0
1	1	0	1	0
1	1	1	0	1

Answer:

1. $\alpha = (\neg P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge \neg R) = (P \leftrightarrow Q) \wedge \neg R$.
2. $$\begin{aligned}\beta =&(\neg P \wedge Q \wedge \neg R) \vee (\neg P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (P \wedge Q \wedge R) = (\neg P \wedge Q) \vee (P \wedge (Q \leftrightarrow R)) \\ =&(P \vee Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee R) = (P \vee Q) \wedge (\neg P \vee (Q \leftrightarrow R)) \\ =&(Q \wedge R) \vee (\neg R \wedge (P \oplus Q))\end{aligned}$$

Problem 2. *(16 Points)*
Write down both the Conjunctive Normal Form and the Disjunctive Normal Form for each of the following formulas.

1. $P \wedge (Q \vee (\neg P \vee R))$
2. $(P \wedge Q) \vee (\neg P \wedge Q \wedge R)$
3. $P \leftrightarrow (Q \rightarrow (Q \rightarrow P))$
4. $(P \rightarrow Q) \vee ((Q \wedge P) \leftrightarrow (Q \leftrightarrow \neg P))$

Answer:

1. CNF: $P \wedge (Q \vee \neg P \vee R)$ or $P \wedge (Q \vee R)$,
   DNF: $(P \wedge Q) \vee (P \wedge R)$ or $(P \wedge Q) \vee (\neg P \wedge Q \wedge R)$
2. CNF: $Q \wedge (P \vee R)$,
   DNF: $(Q \wedge P) \vee (Q \wedge R)$ or $(P \wedge Q) \vee (\neg P \wedge Q \wedge R)$
3. CNF: $Q \vee P$,
   DNF: $(P \wedge \neg Q) \vee (\neg P \wedge Q) \vee P$, $P \vee Q$ or $P \vee (\neg P \wedge Q)$
4. CNF: $(\neg P \vee Q)$,
   DNF: $(\neg P \wedge \neg Q) \vee (\neg P \wedge Q) \vee (P \wedge Q)$ or $\neg P \vee Q$

Problem 3. *(20 Points)*
Prove the following inferences.

1. $P \vee Q, P \to S, Q \to R \quad \vdash \quad S \vee R$
2. $P \to (Q \to R), \neg S \vee P, Q \quad \vdash \quad S \to R$
3. $P \vee Q \to R \wedge S, S \vee E \to U \quad \vdash \quad P \to U$
4. $\neg Q \vee S, (E \to \neg U) \to \neg S \quad \vdash \quad Q \to E$
5. $\neg R \vee S, S \to Q, \neg Q \quad \vdash \quad Q \leftrightarrow R$

Answer:
Note 1: only provide one possible answer; need to **prove any rules not mentioned in class before using it!**
Note 2: proof of the following rules (named casually) are omitted in this solution sheet:

- Transition rule (Trans):$P \to Q, Q \to R \vdash P \to R$
- And-intros rule (AI): $P \to R, Q \to R \vdash P \vee Q \to R$
- (Axiom) False-imply-all (Exfalso): $\vdash P \wedge \neg P \to Q, \vdash \mathbf{F} \to Q$

1. (a) $P \to S$
   (b) $S \to S \vee Q$
   (c) $Q \to S \vee Q$
   (d) $P \to S \vee Q$ (Trans a b)
   (e) $P \vee Q \to S \vee Q$ (AI c d)
   (f) $Q \to R$
   ... (similarly)
   (g) $S \vee Q \to S \vee R$
   (h) $P \vee Q$
   (i) $S \vee Q$ (MD b e)
   (j) $S \vee R$ (MD g i)
2. (a) S (conditional proof, into)
   (b) $\neg S \vee P$
   (c) $S \to P$ (replacement b)
   (d) P (MD c a)
   (e) $P \to (Q \to R)$
   (f) $Q \to R$ (MD e d)
   (g) Q
   (h) R (MD f g)
   (i) $S \to R$ (conditional proof, out)
3. (a) P (conditional proof, into)
   (b) $P \to P \vee Q$ (axiom)
   (c) $P \vee Q$ (MD b a)
   (d) $P \vee Q \to R \wedge S$
   (e) $R \wedge S$ (MD d c)
   (f) $R \wedge S \to S$ (axiom)

(g) S (MD f e)
(h) $S \to S \lor E$ (axiom)
(i) $S \lor E$ (MD h g)
(j) $S \lor E \to U$
(k) U (MD j i)
(l) $P \to U$ (conditional proof, out)

4. (a) Q (conditional proof, into)
   (b) $\neg Q \lor S$
   (c) $Q \to S$ (replacement b)
   (d) S (MD c a)
   (e) $\neg\neg S$ (replacement d, or MD axiom d, in classical logic)
   (f) $(E \to \neg U) \to \neg S$
   (g) $((E \to \neg U) \to \neg S) \to (\neg\neg S \to \neg(E \to \neg U))$
   (h) $\neg\neg S \to \neg(E \to \neg U))$ (MD, h e)
   (i) $\neg(E \to \neg U)$
   (j) $\neg(\neg E \lor \neg U)$ (replacement)
   (k) $E \land U$
   (l) $E \land U \to E$ (axiom)
   (m) E (MD l k)
   (n) $Q \to E$ (conditional proof, out)
   (o) i. $\neg R \lor S$
       ii. $R \to S$ (replacement)
       iii. $S \to Q$
       iv. $R \to Q$
       v. Q (conditional proof, into)
       vi. $\neg Q$
       vii. $Q \land \neg Q$ (e, f)
       viii. R (MD Exfalso g)
       ix. $Q \to R$ (conditional proof, out)
       x. $(Q \to R) \land (R \to Q)$
       xi. $Q \leftrightarrow R$ (replacement)

Problem 4. *(8 Points)*
Formalize the following inferences and then prove it.
If both Alice and Bob are from Shanghai, then Eve is from Beijing. If Eve is from Beijing, then she likes eating duck. But we know that Eve does not like eating duck and Alice is from Shanghai. Therefore, Bob is not from Shanghai.

Answer: Let: P: Alice is from Shanghai; Q: Bob is from Shanghai; R: Eve is from Beijing; S: Eve likes eating duck. We know: $P \land Q \to R$, $R \to S$, P, $\neg S$. We need to prove $\neg Q$.

1. $\neg S$

2. $R \to S$

3. $\neg R$

4. $(P \wedge Q) \to R$
5. $\neg(P \wedge Q)$
6. $\neg P \vee \neg Q$
7. P
8. $\neg Q$