CS2601 Linear and Convex Optimization
Homework 4

Due: 2022.10.31

1. Determine if the following functions are convex, concave, or neither. Show your arguments.

(a). $f(\boldsymbol{x}) = f(x_1, x_2, x_3) = x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + 2x_3^2$ on $\mathbb{R}^3$

(b). $f(\boldsymbol{x}) = f(x_1, x_2) = (x_1x_2)^{-2}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(c). $f(x_1, x_2) = x_1^2x_2^3$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(d). $f(x_1, x_2) = x_1^{1/2}x_2^{-1/2}$ on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(e). $f(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha}$, where $0 \le \alpha \le 1$, on $\mathbb{R}^2_{++} = \{(x_1, x_2) : x_1 > 0, x_2 > 0\}$

(f). $f(\boldsymbol{x}) = \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^5$ on $\mathbb{R}^n$

2. Determine if the following sets are convex. Show your arguments. You can use any results we have proved in class.

(a). $S = \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 \ge |x_1|\}$

(b). $S = \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 \ge x_1^3\}$

(c). $S = \{\boldsymbol{x} \in \mathbb{R}^2 : \boldsymbol{x} > \boldsymbol{0}, x_1 \log x_1 + x_2 \log x_2 \le 2\}$

(d). $S = \{\boldsymbol{x} \in \mathbb{R}^2 : \log(1 + \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\|^3) \le 3, x_1 \ge \log(1 + e^{x_1 + 5x_2})\}$

3. Given two probability distributions $\boldsymbol{x}, \boldsymbol{y} \in \Delta_{n-1}$, where Δ_{n-1} is the probability simplex, the **Kullback-Leibler (KL) divergence** between them is defined by

$$KL(\boldsymbol{x}\|\boldsymbol{y}) = \sum_{i=1}^{n} x_i \log \frac{x_i}{y_i}$$

Use the concavity of log to show that $KL(\boldsymbol{x}\|\boldsymbol{y}) \ge 0$. You can assume $\boldsymbol{x} > \boldsymbol{0}, \boldsymbol{y} > \boldsymbol{0}$.

4. Determine if the following optimization problems are convex optimization problems.

(a).

$$\begin{aligned}
\min_{x_1, x_2} \quad & x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 \\
\text{s.t.} \quad & x_1 e^{-(x_1+x_2)} \leq 0 \\
& x_1^2 - 3x_2 = 0
\end{aligned}$$

(b).

$$\begin{aligned}
\min_{x_1, x_2} \quad & x_1^2 + x_2^4 \\
\text{s.t.} \quad & (x_1 - x_2)^2 + 4x_1x_2 + e^{x_1+x_2} \leq 0 \\
& x_1^2 - 2x_1x_2 + x_2^2 + x_1 + x_2 \leq 0 \\
& 6x_1 - 7x_2 = 0
\end{aligned}$$