高等代数（荣誉）II

期末考试

1. 复数域上 A 相似于对角矩阵 $\Leftrightarrow$ 对 A 的任意特征值 λ_0，

$(\lambda_0 I - A)^2$ 和 $\lambda_0 I - A$ 的秩相等.

2. $\mathcal{A} \in \mathrm{End}\,V$ $\mathcal{A}^*$ 为 $\mathcal{A}$ 的对偶变换 $V/\mathbb{F}$ 为有限维向量空间，

则 $\mathcal{A} = 0$ 当且仅当 $\mathcal{A}^* = 0$

$$\mathcal{A} : U \to V \qquad A$$

$$\mathcal{A}^* : V^* \to U^* \qquad A^T$$

3. $\begin{bmatrix} A & B-A \\ 0 & B \end{bmatrix}$ 与 $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ 相似

$$\begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix}$$

4. $m_{\mathcal{A}}(\lambda) = f_{\mathcal{A}}(\lambda)$ if and only if for every $\mathcal{B}$ s.t. $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$, $\mathcal{B}$ is a poly of $\mathcal{A}$.

"$\Rightarrow$" If $f_{\mathcal{A}}(\lambda) = m_{\mathcal{A}}(\lambda)$ and $\mathcal{A}\mathcal{B} = \mathcal{B}\mathcal{A}$

Then $\exists \alpha \in V$ s.t. $V = \mathbb{F}[\mathcal{A}]\alpha$

so $\alpha, \mathcal{A}\alpha, \mathcal{A}^2\alpha, \cdots, \mathcal{A}^{n-1}\alpha$ is a basis of V

and $\exists\, g(\lambda) \in \mathbb{F}(\lambda)$ s.t. $\mathcal{B}\alpha = g(\mathcal{A})\alpha$

This implies that for $1 \le k \le n-1$

$$\mathcal{B}\mathcal{A}^k\alpha = \mathcal{A}^k\mathcal{B}\alpha = \mathcal{A}^k g(\mathcal{A})\alpha = g(\mathcal{A})\mathcal{A}^k\alpha$$

$$\Rightarrow \mathcal{B}\left(\sum_{i=1}^{n-1} a_i \mathcal{A}^i\alpha\right) = g(\mathcal{A})\left(\sum_{i=1}^{n-1} a_i \mathcal{A}^i\alpha\right)$$

$$\Rightarrow \mathcal{B} = g(\mathcal{A})$$

"$\Leftarrow$" Suppose that $m_{\mathcal{A}}(\lambda) \neq f_{\mathcal{A}}(\lambda)$

Then V has a basis $\{\alpha_1, \cdots, \alpha_n\}$ s.t.

$$\mathcal{A}(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{bmatrix} F_1 & & & \\ & F_2 & & \\ & & \ddots & \\ & & & F_s \end{bmatrix}$$

F_i - Frobenius block $1 \le i \le s$ $s \ge 2$ s.t. $m_{F_i}(\lambda) \mid m_{F_{i-1}}(\lambda)$

Let $B \in \mathrm{End}\, V$ s.t.

$$B(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) \underbrace{\begin{bmatrix} 0 & & & \\ & I & & \\ & & 0 & \\ & & & \ddots \\ & & & & 0 \end{bmatrix}}_{B}$$

then $AB = BA$ If $B = g(A)$ for some $g(\lambda) \in \mathbb{F}(\lambda)$

then $B = \begin{bmatrix} g(F_1) & & \\ & \ddots & \\ & & g(F_s) \end{bmatrix}$

$g(F_1) = 0$ $g(F_2) = I$

$\Rightarrow m_{F_1}(\lambda) \mid g(\lambda) \Rightarrow m_{F_2}(\lambda) \mid g(\lambda) \Rightarrow g(F_2) = 0$

contradiction!

5. $f(x_1,\cdots,x_n) = (x^T\ 1)\underbrace{\begin{bmatrix} A & b \\ b^T & a \end{bmatrix}}_{D}\begin{pmatrix} x \\ 1 \end{pmatrix}$

$$\begin{bmatrix} I & 0 \\ -b^TA^{-1} & 1 \end{bmatrix} D \begin{bmatrix} I & -A^{-1}b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & a-b^Ta^{-1}b \end{bmatrix}$$

Let $\begin{bmatrix} x \\ 1 \end{bmatrix} = \begin{bmatrix} I & -A^{-1}b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y \\ 1 \end{bmatrix}$

$$\Rightarrow f(x_1,\cdots,x_n) = y^{-1}Ay + a - b^TA^{-1}b$$

$$\geqslant a - b^TA^{-1}b$$

$$= a - b^TA^{-1}b \Leftrightarrow y=0$$

$$\Leftrightarrow x = -A^{-1}b$$

$A>0,\ B>0 \Rightarrow A = P^TP$

$AB = P^TPB = P^T PBP^T(P^T)^{-1}$??

设n阶实对称矩阵A是正定的.

(1) 证明: A的特征值均大于0.

(2) 设$f(x_1,\cdots,x_n)=x^TAx+2b^Tx+a$, $b\in\mathbb{R}^n$. 求$x=(x_1,\cdots,x_n)$的值使得$f(x_1,\cdots,x_n)$取最小值并求这个最小值.

(3) 已知A, B都是正定矩阵, 则AB的特征值都大于0.

λ - Matrices

$\mathbb{F}$ - field (ch$\mathbb{F}=0$, number field for example)

$\mathbb{F}[\lambda]$ - poly ring

$A(\lambda)_{m\times n} = (a_{ij}(\lambda))_{m\times n}$ — a λ-matrix, where $a_{ij}(\lambda)\in\mathbb{F}[\lambda]$

Ex $A(\lambda) = \begin{bmatrix} \lambda^2-1 & \lambda+1 \\ 1 & 2 \end{bmatrix}$

$\mathbb{F}^{m\times n} \subseteq \mathbb{F}[\lambda]^{m\times n}$

we can similarly define $r(A(\lambda))$, minors, ...

$A(\lambda)_{n\times n}$ is invertible if $\exists\ B(\lambda)\in\mathbb{F}[\lambda]$ s.t.

$A(\lambda)B(\lambda) = B(\lambda)A(\lambda) = I$

$A(\lambda)$ is invertible $\Leftrightarrow$ $r(A(\lambda))=n$ and $\det A(\lambda)\in\mathbb{F}\setminus\{0\}$

(I) $A(\lambda) \underset{col}{\overset{row}{\rightsquigarrow}} B(\lambda)$

(II) $A(\lambda) \rightsquigarrow \begin{pmatrix} cx & \overset{*}{\underset{*}{\cdots}} & cx \end{pmatrix}$ or $\begin{pmatrix} & cx & \\ * & \vdots & * \\ & cx & \end{pmatrix}$

(III) $A(\lambda) \rightsquigarrow \begin{pmatrix} a_{i1}(\lambda) & \cdots & \cdots & a_{in}(\lambda) \\ - & - & - & - \\ a_{j1}(\lambda)+k(\lambda)a_{i1}(\lambda) & \cdots & & a_{jn}(\lambda)+k(\lambda)a_{in}(\lambda) \end{pmatrix}$

$$A(\lambda) \rightsquigarrow \begin{bmatrix} * \\ 0 \end{bmatrix} \xrightarrow[\text{将次数最低的移到首位}]{\text{类似辗转相除}} \text{s.t. } a_{11}(\lambda) \mid a_{ij}(\lambda)$$

Notation For $1 \le k \le r = r(A(\lambda))$

k-determinant divisor $D_k(A(\lambda))$
行列式因子

$D_k(A(\lambda))$ = the monic maximal divisor of all the k-minors of $A(\lambda)$

Lemma (I) ~ (III) elementary transformations do not change D_k $1 \le k \le r = r(A(\lambda))$

(For $r < k \le \min\{m,n\}$, $D_k = 0$)

Thm $A(\lambda) \in \mathbb{F}[\lambda]^{m\times n}$

$$\rightsquigarrow \begin{bmatrix} \alpha_1(\lambda) & & & & & \\ & \alpha_2(\lambda) & & & & \\ & & \ddots & & & \\ & & & \alpha_r(\lambda) & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix} \qquad (1)$$

s.t. $\alpha_1(\lambda) \mid \alpha_i(\lambda)$ $i = 1, \cdots, r$ $\alpha_i(\lambda)$ - monic $1 \le i \le r$

$D_1 = d_1(\lambda) \quad D_2 = d_1(\lambda)\, d_2(\lambda)$

$D_i(\lambda) = d_1(\lambda) \cdots d_i(\lambda) \quad 1 \leq i \leq r$

$\Rightarrow$ (1) is uniquely determined by $A(\lambda)$

(1) is called the Smith (standard) form of $A(\lambda)$

$$d_i(\lambda) = \frac{D_i}{D_{i+1}}, \quad i = 1, \cdots, r$$

define $D_0 = 1$

$d_i(\lambda) \overset{1 \leq i \leq r}{—}$ invariant divisors of $A(\lambda)$

coro $\quad A(\lambda) \approx B(\lambda)$

$\Leftrightarrow$ They have the same determinant divisors

$\Longleftrightarrow$ They have the same invariant divisors.

Ex $\quad \lambda I - A \quad A \in \mathbb{F}^{n \times n} \quad \Rightarrow r(\lambda I - A) = n.$

$$\lambda I - A \rightsquigarrow \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & d_1(\lambda) & & \\ & & & & \ddots & \\ & & & & & d_m(\lambda) \end{bmatrix}$$

$$\Rightarrow \prod_{i=1}^{m} d_i(\lambda) = f_A(\lambda)$$

$$A = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \qquad \lambda I - A = \begin{bmatrix} \lambda & -1 & & \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ & & & \lambda \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & f_A(\lambda) \end{bmatrix}$$

We assume $\mathbb{F}$ is algebraic closed

Then $d_i(\lambda) = (\lambda - \lambda_1)^{k_{i1}} \cdots (\lambda - \lambda_s)^{k_{is}}$ $\quad i = 1, \cdots, r$

s.t. $0 \le k_{1j} \le k_{2j} \le \cdots \le k_{rj}$, $j = 1, 2, \cdots, s$

$$\left\{ (\lambda - \lambda_j)^{k_{ij}} \;\middle|\; k_{ij} \ge 1 \quad i = 1, 2, \cdots, r, \; j = 1, 2, \cdots, s \right\}$$

Ex $\quad A(\lambda) \rightsquigarrow \begin{bmatrix} \lambda & & & \\ & \lambda(\lambda-1)^2 & & \\ & & \lambda^2(\lambda-1)^2(\lambda+2) & \\ & & & 0 \end{bmatrix}$

$A(\lambda) \approx B(\lambda) \Leftrightarrow$ They have the same rank and the same elementary divisors

Ex $A(\lambda) \rightsquigarrow \begin{bmatrix} \lambda^2(\lambda+1) & 0 \\ 0 & \lambda(\lambda+1)^2 \end{bmatrix}$

Lemma If $(f_i, g_j) = 1$ $i, j = 1, 2$ Then

$$\begin{pmatrix} f_1 g_1 & 0 \\ 0 & f_2 g_2 \end{pmatrix} \approx \begin{pmatrix} f_1 g_2 & 0 \\ 0 & f_2 g_1 \end{pmatrix}$$

We need to prove $(f_1 g_1, f_2 g_2) = (f_1 g_2, f_2 g_1)$

Claim $(f_1 g_1, f_2 g_2) = (f_1, f_2)(g_1, g_2)$

Proof of the claim

Assume $(f_1 g_1, f_2 g_2) = d$ $(f_1, f_2) = d_1$ $(g_1, g_2) = d_2$

$\Rightarrow d_1 | f_1, d_1 | f_2, d_2 | g_1, d_2 | g_2$

$\Rightarrow d_1 d_2 | f_1 g_1, d_1 d_2 | f_2 g_2$

$\Rightarrow d_1 d_2 | d$

Since $(f_i, g_j) = 1$ $i, j = 1, 2$ we have $(d_1, d_2) = 1$

Notice $d | f_1 g_1 \Rightarrow d = d_1' d_2'$, $d_1' | f_1$, $d_2' | g_1$

$(f_1, g_2) = 1 \Rightarrow (d_1', g_2) = 1 \Rightarrow d_1' | f_2$

$\Rightarrow d_1' | d_1$ similar $d_2' | d_2 \Rightarrow d = d_1' d_2' | d_1 d_2$

Thus $d = d_1 d_2$

$$\lambda I - A \simeq \lambda I - B \iff A \sim B.$$

λ- matix

$$A(\lambda) \rightsquigarrow \begin{bmatrix} d_1(\lambda) & & & & \\ & \ddots & & & \\ & & d_r(\lambda) & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}$$

monic $\quad d_i(\lambda) \mid d_{i+1}(\lambda)$, $d_i(\lambda) \neq 0$, $r(A(\lambda)) = r$

$D_i(\lambda) = d_1(\lambda) \cdots d_i(\lambda)$

$\lambda I - A$, $A \in \mathbb{F}^{n \times n}$

$\lambda I - B$, $B \in \mathbb{F}^{n \times n}$

$\lambda I - A \sim \lambda I - B$ (称为A的不变因子……)

$\iff$ They have the same $\left\{ \begin{array}{l} \text{determinant} \\ \text{invariant} \\ \text{elementory} \end{array} \right.$ divisors

Theorem For $A, B \in \mathbb{F}^{n\times n}$, $\lambda I - A \approx \lambda I - B \iff A \sim B$

Lemma1 Let $A(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, and $A \in \mathbb{F}^{n\times n}$, then

最高次数对应需可逆，且计算方便.

$$A(\lambda) = (\lambda I - A)P(\lambda) + P \qquad (1)$$

$$A(\lambda) = Q(\lambda)(\lambda I - \overset{?}{B}) + Q \qquad (2)$$

for some $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$, $P, Q \in \mathbb{F}^{n\times n}$

Proof It is enough to prove (1) 待定系数法

We may assume

$$A(\lambda) = A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m \text{ for some } m \geq 0.$$

Suppose

$$A_0\lambda^m + A_1\lambda^{m-1} + \cdots + A_{m-1}\lambda + A_m = (\lambda I - A)(P_0\lambda^{m-1} + P_1\lambda^{m-2} + \cdots + P_{m-2}\lambda + P_{m-1}) + P$$

Then $P_0 = A_0$ $A_1 = -AP_0 + P_1$ $\cdots$ #

Proof of the Theorem

"$\Leftarrow$" Obvious

"$\Rightarrow$" Suppose $\lambda I - A \approx \lambda I - B$

Then $\exists\, U(\lambda), V(\lambda) \in GL_n(\mathbb{F}[\lambda])$

s.t. $\lambda I - A = U(\lambda)(\lambda I - B)V(\lambda)$ (3)

By Lemma1

$U(\lambda) = (\lambda I - A)P(\lambda) + P$ (4)

$V(\lambda) = Q(\lambda)(\lambda I - A) + Q$ (5)

for some $P(\lambda), Q(\lambda) \in \mathbb{F}[\lambda]^{n\times n}$ $P, Q \in \mathbb{F}^{n\times n}$

Applying (5) to (3) then

$U(\lambda)^{-1}(\lambda I - A) = (\lambda I - B)[Q(\lambda)(\lambda I - A) + Q]$

$\Rightarrow [U(\lambda)^{-1} - (\lambda I - B)Q(\lambda)](\lambda I - A) = (\lambda I - B)Q$

Let $T = U(\lambda)^{-1} - (\lambda I - B)Q(\lambda)$, $T \in \mathbb{F}^{n\times n}$ (6)

Then $T(\lambda I - A) = (\lambda I - B)Q$ (7)

By (7) $T = Q$, $TA = BQ = BT$

By (6) $I - U(\lambda)(\lambda I - B)Q(\lambda) = U(\lambda)T$

$\Rightarrow I = (\lambda I - A)V^{-1}(\lambda)Q(\lambda) + [(\lambda I - A)P(\lambda) + P]T$

$\Rightarrow I = (\lambda I - A)[V(\lambda)^{-1}U(\lambda) + P(\lambda)T) + PT$

$\Rightarrow V(\lambda)^{-1}U(\lambda) + P(\lambda)T = 0 \Rightarrow T \in Gl_n(\mathbb{F})$

$\Rightarrow A \sim B$.

Application

Ex $J = \begin{bmatrix} \lambda_0 & & & \\ 1 & \lambda_0 & & \\ & 1 & \ddots & \\ & & \ddots & 1 \ \lambda_0 \end{bmatrix}$ $\lambda I - J \rightsquigarrow \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 & \\ & & & & (\lambda-\lambda_0)^n \end{bmatrix}$

Ex $J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_s \end{bmatrix}$ $J_i = \begin{bmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & 1 & \ddots & \\ & & \ddots & 1 \ \lambda_i \end{bmatrix}_{m_i \times m_i}$

The elementary divisors of J : $(\lambda-\lambda_1)^{m_1}, (\lambda-\lambda_2)^{m_2}, \cdots (\lambda-\lambda_s)^{m_s}$

Let $A \in \mathbb{C}^{n\times n}$

Assume the elementary divisors of A are

$(\lambda-\lambda_1)^{m_1}, (\lambda-\lambda_2)^{m_2}, \cdots (\lambda-\lambda_s)^{m_s}$

Then $\lambda I - A \approx \lambda I - J \Rightarrow A \sim J$

Recall $A \in \mathbb{F}^{n\times n} \Rightarrow A \sim \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_s \end{bmatrix}$ $m_{F_i}(\lambda) | m_{F_{i+1}}(\lambda)$

Let $d_i(\lambda) = m_{F_i}(\lambda)$

Then $d_i(\lambda) | d_{i+1}(\lambda)$ $\prod_{i=1}^{s} d_i(\lambda) = f_A(\lambda)$

Ex Let $F = \begin{bmatrix} 0 & & & b_{n-1} \\ 1 & \ddots & & \vdots \\ & \ddots & & b_1 \\ & & 1 & b_0 \end{bmatrix}$ $\lambda I - F \rightsquigarrow \begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & |\lambda I - F| \end{bmatrix}$

$|\lambda I - F| = f_F(\lambda) = m_F(\lambda)$

Coro 2 For any $A \in \mathbb{F}^{n\times n}$ Assume that $d_1(\lambda), \cdots, d_s(\lambda)$ are the invariant divisors of A

Then $A \sim F = \begin{bmatrix} F_1 & & \\ & \ddots & \\ & & F_s \end{bmatrix}$?

where F_i $1 \le i \le s$ are Forbenious blocks s.t.

$f_{F_i}(\lambda) = m_{F_i}(\lambda) = d_i(\lambda)$

Coro 3 $A \in \mathbb{F}^{n\times n}$ $A \sim A^T$

Coro 4 $A, B \in \mathbb{F}^{n\times n}$ $\mathbb{F} \subseteq \overline{\mathbb{F}}$ an enterdid field of $\mathbb{F}$

Then $A \sim B$ in $\overline{\mathbb{F}} \Leftrightarrow A \sim B$ in $\mathbb{F}$

Let $V/\mathbb{F}$ be a v.s.

A bilinear form $(\cdot,\cdot)$ on V is called an altermate bilinear form (or altermating form).

if $(u,u)=0 \quad \forall u \in V$

$\Longrightarrow (u,v) = -(u,v)$

$\Longleftarrow \mathrm{ch}\mathbb{F} \neq 2$

Let V be f.d. $\quad \mathrm{Ch}\mathbb{F} \neq 2$

Give a basis $\{\alpha_1, \cdots, \alpha_n\}$ of V, we have

$G = ((\alpha_i, \alpha_j))_{i,j=1}^{n}$

$G^T = -G$

$|G^T| = |G| = (-1)^n |G|$

If n is odd, then $|G|=0$

We may assume $(\cdot,\cdot) \neq 0$

Then $\exists\, v_1, v_2$ s.t. $(v_1, v_2) = -(v_2, v_1) = 1$

It is obvious that $\{v_1, v_2\}$ are linearly independent

Let $L(v_1, v_2)^\perp = \{\alpha \in V \mid (\alpha, v_1) = (\alpha, v_2) = 0\}$

Then $\dim L(v_1, v_2)^\perp \geqslant n-2$

Furthermore, $L(v_1, v_2) \cap L(v_1, v_2)^\perp = \{0\}$

$\Rightarrow V = L(v_1, v_2) \oplus L(v_1, v_2)^\perp$ and $\dim L(v_1, v_2)^\perp = n-2$.

If $(\cdot,\cdot)\Big|_{L(v_1, v_2)^\perp} = 0$

Then $\exists$ a basis $v_1, v_2, v_3, \dots v_n$ s.t.

$$((v_i, v_j)) = \left[\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline & 0 & 0 \end{array}\right]$$

If $(\cdot,\cdot)\Big|_{L(\alpha_1, \alpha_2)^\perp} \neq 0$

Then $\exists \alpha_3, \alpha_4$ s.t. $(v_3, v_4) = -(v_4, v_3) = 1$

$\Rightarrow$ Theorem $\exists$ a basis $\{v_1, \dots, v_n\}$ of V s.t.

$$((v_i, v_j)) = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix}$$

In particular, if $(\cdot,\cdot)$ degenerate then $\dim V = 2m$

and $H = ((v_i, v_j)) = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$

$J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix} = ((u_i, u_j))$

$v_1, \underline{v_2}, v_3, \underline{v_4}, \cdots, v_{2m-1}, \underline{v_{2m}}$

辛空间 simplectice space

For any $A \in \mathbb{F}^{n \times n}$, A alternating

$\exists P \in GL_n(\mathbb{F})$ s.t.

$P^T A P = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$

Recall Given $A \in \mathbb{F}^{n\times n}$

$A^T = -A$ $\quad ch\mathbb{F} \neq 2$

$\exists P \in GL_n(\mathbb{F})$ s.t.

$$P^T A P = \begin{bmatrix} 0 & -I & \\ I & 0 & \\ & & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & -1 & & & \\ 1 & 0 & & & \\ & & 0 & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \end{bmatrix}$$

Let $T = (t_{ij})$ s.t. $t_{ji} = -t_{ij}$ and t_{ij} $i<j$ are interminates

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Then $\det T \in \mathbb{Z}[t] = \mathbb{Z}[t_{ij}, i<j] \subseteq \mathbb{Q}(t) = \left\{ \frac{f(t)}{g(t)} \,\middle|\, g(t) \neq 0 \right\}$

$t = (t_{12}, t_{13}, \cdots, t_{n-1\,n})$

Let $\mathbb{F} = \mathbb{Q}(t)$ $\quad T \in \mathbb{F}^{n\times n}$ (先打域，再回来)

$\exists P(t) \in Gl_n(\mathbb{F})$ s.t. $P(t)^T \, T P(t) = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} = J$

$\Rightarrow \det T \cdot \det P(t)^2 = 1 \Rightarrow \det T = \frac{1}{|P(t)|^2} = |P(t)|^{-2}$

又 $\det T \in \mathbb{Z}[t_{12}, t_{13}, \cdots, t_{n-1\,n}] = \mathbb{Z}[t]$

Define Pfaffian form of T denoted by $Pf(T)$ as follows:

$$Pf(T) = f(t) \text{ or } -f(t) \quad (唯一分解定理)$$

such that $Pf(T)\big|_{T=J} = 1$. (确定)

For $A \in \mathbb{R}^{n\times n}$ $A^T = -A$, (先考虑一般情况)

define $Pf(A) = Pf(T)\big|_{T=A}$ $(t_{ij} = a_{ij})$

Theorem For any $A \in \mathbb{R}^{n\times n}$, $A^T = -A$ and $U \in Gl_n(\mathbb{R})$

Application

We may assume $\mathcal{A}(u_1, \cdots, u_m) = (u_1, \cdots, u_m)A$

$$((\mathcal{A}u_i, \mathcal{A}u_j)) = A^T \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} A$$

$$(\mathcal{A}u_i, \mathcal{A}u_j) = (u_i, u_j) \Rightarrow A^T \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} A = J$$

Denote $Sp(V) = \{A \in Gl_n(V) \mid (Au, Av) = (u, v)\}$

辛群 The simplectic group of V

$Sp_{2n}(\mathbb{R}) = \{A \in Gl_n(\mathbb{R}) \mid A^T J A = J\} \subseteq Sl_n(\mathbb{R})$

$\mathbb{C}$ $\mathbb{C}$ $\mathbb{C}$

Fact $A \in Sp_{2n}(\mathbb{R}) \Rightarrow |A| = 1$

Let $U = A \in Sp_{2n}(\mathbb{R})$, then $A^T J A = J$

$Pf(A^T J A) = \det A \cdot Pf(J) \Rightarrow |A| = 1.$

The Proof of the Theorem

Let $\tilde{U} = (u_{ij})$ u_{ij} 未定元

Consider $Pf(\tilde{U}^T T \tilde{U})$

$Pf(\tilde{U}^T T \tilde{U}) = \det(U)\, Pf(T)$ or $-\det(\tilde{U}) Pf(T)$

It is known that

$Pf(\tilde{U}^T T \tilde{U})\big|_{\tilde{U}=I} = Pf(T)$

Then $Pf(\hat{U}^T T \hat{U}) = \det(\hat{U})\, Pf(T)$

$\Rightarrow$ For any $U \in Gl_n(\mathbb{R})$, $A \in \mathbb{R}^{n\times n}$, $A^T = -A$

$$Pf(U^T A U) = Pf(\tilde{U}^T T \tilde{U})\Big|_{\tilde{U}=U,\, T=A} = (\det U)\, Pf(A)$$

Another way to prove

$\det A = 1$ if $A \in Sp_{2n}(\mathbb{R})$

Proof We consider

$$\det(AJ + JA)$$

Notice that

$$A^T(AJ+JA) = (A^TA+1)J$$

A^TA+1 is positive-definite

$\Rightarrow |A^TA+1| > 0$

$\Rightarrow |A^T(AJ+JA)| = |A^TA+1| > 0$

It is enough to prove that $|AJ+JA| > 0$

We may assume that $A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$

Then $AJ+JA=\begin{bmatrix} A_2 & -A_1 \\ A_4 & -A_3 \end{bmatrix}+\begin{bmatrix} -A_3 & -A_4 \\ A_1 & A_2 \end{bmatrix}=\begin{bmatrix} A_2-A_3 & -A_1-A_4 \\ A_1+A_4 & A_2-A_3 \end{bmatrix}$

$$=\begin{bmatrix} P & -Q \\ Q & P \end{bmatrix}$$

$$\begin{vmatrix} P & -Q \\ Q & P \end{vmatrix}=\begin{vmatrix} P+\sqrt{-1}\,Q & \sqrt{-1}P-Q \\ Q & P \end{vmatrix}=\begin{vmatrix} P+\sqrt{-1}\,Q & 0 \\ 0 & P-\sqrt{-1}\,Q \end{vmatrix}$$

$$=|P+\sqrt{-1}\,Q|\cdot\overline{|P+\sqrt{-1}\,Q|}>0.$$

$\Rightarrow |A^T|=|A|>0.$

$\Rightarrow |A|=1.$

$M=\begin{bmatrix} X & Y \\ -\bar{Y} & \bar{X} \end{bmatrix}$ $X,Y\in M_n(\mathbb{C})$

四元数矩阵

Fact: $\det M\geq 0.$

① $\bar{M}J=JM$

② $\alpha \in \mathbb{C}^{2n}$ is

a generalized eigenvector of M

$\Rightarrow J\bar{\alpha}$ is a generalized eigenvector of M with $\bar{\lambda}_i$

$\alpha_1, J\bar{\alpha}_1, \alpha_2, J\bar{\alpha}_2, \cdots \alpha_s, J\bar{\alpha}_s$ linearly indep

1. Euclidean Space 欧氏空间 (over $\mathbb{R}$)

(Real inner space) 实内积空间

2. Unitary Space 酉空间 (over $\mathbb{C}$)

(Complex inner space) 复内积空间

Let $V/\mathbb{R}$ be a v.s. and $(\cdot|\cdot)$ a symmetric bilinear form on V. If $(\cdot,\cdot)$ is positive-definite, i.e. for any $0 \neq \alpha \in V$, $(\alpha|\alpha) > 0$, then we say $(\cdot|\cdot)$ gives an inner product over V, V is called Euclidean Space.

Ex1 $\mathbb{R}^n = \{(a_1, \cdots, a_n)^T \mid a_i \in \mathbb{R}\}$

Define for $\alpha = (a_1, \cdots, a_n)^T$, $\beta = (b_1, \cdots, b_n)^T$

$$(\alpha|\beta) = \sum_{i=1}^{n} a_i b_i$$

$$(\alpha|\alpha) = \sum_{i=1}^{n} a_i^2 \geq 0 \text{ and}$$

$$= 0 \Leftrightarrow \alpha = 0$$

$\Rightarrow \mathbb{R}^n$ is an Euclidean space

Let $A \in \mathbb{R}^{n\times n}$ be positive

For $\alpha, \beta \in \mathbb{R}^n$ as above, define

$$(\alpha|\beta) = \beta^T A \alpha = \alpha^T A \beta$$

$$\Rightarrow (\alpha|\alpha) = \alpha^T A \alpha \geq 0 \text{ and}$$

$$= 0 \Leftrightarrow \alpha = 0$$

Ex 2 $V = M_{m\times n}(\mathbb{R})$

For $A, B \in M_{m\times n}(\mathbb{R})$

define $(A|B) = tr(B^T A)$

$$(A|A) = tr(A^T A) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2 \geq 0.$$

and $=0 \Rightarrow a_{ij}=0 \quad i=1,\cdots,m \quad j=1,\cdots,n$

$\Rightarrow V$ is an inner space

Ex3 $\quad V=C[a,b]$ for $f(x),g(x)\in V$

define $(f(x)|g(x))=\int_a^b f(x)g(x)dx$

$\Rightarrow V$ is an inner space

Ex 4 $\quad V=\mathbb{R}[x]$ for $f(x),g(x)\in V$

define $(f(x)|g(x))=\int_0^1 f(x)g(x)dx$

Ex 5 $\quad l^2=\{(a_1,a_2,\cdots)\mid a_i\in\mathbb{R} \ \sum_{i=1}^{\infty}a_i^2<\infty\}$

For $\alpha=(a_1,a_2,\cdots) \quad \beta=(b_1,b_2,\cdots)$

define $(\alpha|\beta)=\sum_{i=1}^{\infty}a_ib_i$

Let $V/\mathbb{R}$ be an inner space for $\alpha \in V$, define
$\|\alpha\| = \sqrt{(\alpha|\alpha)}$, called the length of α.

Prop For $\alpha, \beta \in V$, we have $(\alpha|\beta)^2 \leq (\alpha|\alpha)(\beta|\beta)$
(Cauchy-Schwarz inquality) $\Leftrightarrow |(\alpha|\beta)| \leq \|\alpha\|\|\beta\|$
and $(\alpha|\beta)^2 = (\alpha|\alpha)(\beta|\beta)$ iff α, β Linearly dependent.

Proof It is obvious that for $\alpha=0$ or $\beta=0$, the Prop is true.
We may assume $\alpha \neq 0, \beta \neq 0$ We have

$$\left(\beta - \frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha \,\middle|\, \beta - \frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha\right) \geq 0$$

$$\Rightarrow (\beta|\beta) - \frac{(\alpha|\beta)^2}{(\alpha|\alpha)} - \frac{(\alpha|\beta)^2}{(\alpha|\alpha)} + \frac{(\alpha|\beta)^2}{(\alpha|\alpha)} \geq 0$$

$$\Rightarrow (\beta|\beta)(\alpha|\alpha) \geq (\alpha|\beta)^2$$

For non-zero $\alpha, \beta \in V$, define
$0 \leq \angle(\alpha, \beta) \leq \pi$ by $\cos\angle(\alpha, \beta) = \dfrac{(\alpha|\beta)}{\|\alpha\|\|\beta\|}$

For $\alpha, \beta \in V$, if $(\alpha|\beta)=0$, we say α and β are ?ortheginal to each other denoted by $\alpha \perp \beta$

For a subspace U of V, denote

$U^{\perp} = \{\alpha \in V / \alpha \perp \beta, \forall \beta \in U\}$

正交补

We now assume $\dim_{\mathbb{R}} V = n < \infty$

Let $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of V

Then $G((\alpha_i|\alpha_j)) > 0 \quad G^T = G$

$\forall \alpha, \beta \in V \quad \alpha = (\alpha_1, \cdots, \alpha_n) x \quad \beta = (\beta_1, \cdots, \beta_n) y$

Then $(\alpha|\beta) = x^T G y = y^T G x$

$G = I$ possible?

$G = I \Leftrightarrow (\alpha_i / \alpha_j) = \delta_{ij}$

Let $\{x_1, \cdots, x_n\} \subset V, x_i \neq 0$, be ortheginal system

such that $(x_i | x_j) = 0, i \neq j$

Then $\{x_1, \cdots, x_n\}$ are linearly independent. (反证法)

Is there a basis $\{x_1, \cdots, x_n\}$ of V such that

$$(x_i | x_j) = \delta_{ij} \text{?} \qquad \text{Yes}$$

Proof Let $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of V

Schmidt orthognalization

施密特正交化过程

Let $\eta_1 = \alpha_1$

$$\eta_2 = \alpha_2 - \frac{\alpha_2 | \eta_1}{(\eta_1 | \eta_1)} \eta_1$$

$$\eta_3 = \alpha_3 - \frac{\alpha_3 | \eta_1}{(\eta_1 | \eta_1)} \eta_1 - \frac{\alpha_3 | \eta_2}{(\eta_2 | \eta_2)} \eta_2$$

......

$$\eta_k = \alpha_k - \sum_{i=1}^{k-1} \frac{\alpha_k | \eta_i}{\eta_i | \eta_i} \qquad k = 4, \cdots n$$

$$L(\alpha_1, \cdots, \alpha_k) = L(\eta_1, \cdots, \eta_k)$$

$$\Rightarrow (\eta_1, \cdots, \eta_n) \text{ s.t. } (\eta_i | \eta_j) = \delta_{ij}$$

Let $\xi_i = \dfrac{\eta_i}{\|\eta_i\|}$, $i = 1, 2, \cdots, n$ Then $\{\xi_1, \cdots, \xi_n\}$ s.t.

$(\xi_i | \xi_j) = \delta_{ij}$

Recall Schmidt Orthogonalization

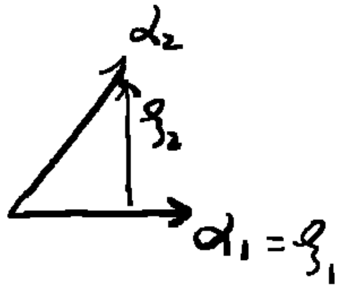
$V/\mathbb{R}$ $\quad \dim_{\mathbb{R}} V = n < \infty$

A basis $\{\alpha_1, \cdots, \alpha_n\}$ given

(A - positive definite $\Rightarrow A = P^T I P$)

Let

$$\begin{cases} \xi_1 = \alpha_1 \\ \xi_2 = \alpha_2 - \dfrac{(\alpha_2 | \xi_1)}{(\xi_1 | \xi_1)} \xi_1 \\ \vdots \\ \xi_i = \alpha_i - \displaystyle\sum_{k=1}^{i-1} \dfrac{(\alpha_i | \xi_k)}{(\xi_k | \xi_k)} \xi_k \qquad i = 3, \cdots, n \end{cases} \tag{1}$$



Let $\eta_i = \dfrac{\beta_i}{\|\beta_i\|} \Rightarrow (\eta_i | \eta_j) = \delta_{ij}$ orthonormal

$$L(\alpha_1, \cdots, \alpha_i) = L(\beta_1, \cdots, \beta_i)$$

$$\begin{cases} \alpha_1 = \|\beta_1\| \eta_1 \\ \alpha_2 = \|\beta_2\| \eta_2 + (\alpha_2 | \eta_1) \eta_1 \\ \vdots \\ \alpha_n = \|\beta_n\| \eta_n + \sum_{i=1}^{n-1} (\alpha_n | \eta_i) \eta_i \end{cases} \quad (2)$$

(2) $\Leftrightarrow$

$$(\alpha_1, \cdots, \alpha_n) = (\eta_1, \cdots, \eta_n) \underbrace{\begin{bmatrix} \|\beta_1\| & (\alpha_2, \eta_1) & \cdots & (\alpha_n, \eta_1) \\ & \|\beta_2\| & \cdots & (\alpha_n, \eta_2) \\ & & \ddots & \vdots \\ & & & \|\beta_n\| \end{bmatrix}}_{R} \quad (3)$$

$$((\alpha_i, \alpha_j)) = R^T \underbrace{((\eta_i, \eta_j))}_{I} R = R^T R$$

$G > 0$

$\forall \alpha, \beta \in V \quad \alpha = (\eta_1, \cdots \eta_n) X$

$\beta = (\eta_1, \cdots \eta_n) Y \Rightarrow (\alpha | \beta) = X^T Y = Y^T X$

Ex $V = \mathbb{R}^n$

$\Rightarrow (\alpha_1, \alpha_2, \cdots, \alpha_n) \in GL_n(\mathbb{R})$

Let $(\eta_1, \cdots, \eta_n) = Q \qquad (\eta_i | \eta_j) = \delta_{ij}$

(3) $\Leftrightarrow \quad A = QR$

$$Q = \begin{bmatrix} \eta_{11} & \eta_{12} & \cdots & \eta_{1n} \\ \eta_{21} & \eta_{22} & \cdots & \eta_{2n} \\ \vdots & \vdots & & \vdots \\ \eta_{n1} & \eta_{n2} & \cdots & \eta_{nn} \end{bmatrix}$$

$$(\eta_i | \eta_j) = \sum_{k=1}^{n} \eta_{ki} \eta_{kj} = \eta_j^T \eta_i = \eta_i^T \eta_j$$

$$\Rightarrow Q^T Q = \begin{bmatrix} \eta_1^T \\ \vdots \\ \eta_n^T \end{bmatrix} [\eta_1 \cdots \eta_n] = ((\eta_i^T | \eta_j))_{n \times n} = I$$

$\Rightarrow QQ^T = I$

Def. For $P \in \mathbb{R}^{n\times n}$, P is called an orthogonal matrix if $P^TP = PP^T = I$

If $PP^T = I$ Let $P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{pmatrix}$

Then $P^T = (P_1^T \cdots P_n^T)$

$\Rightarrow PP^T = I \Leftrightarrow \begin{pmatrix} P_1 \\ \vdots \\ P_n \end{pmatrix} (P_1^T \cdots P_n^T) = I$

$\Leftrightarrow P_i P_j^T = \delta_{ij}$

Homework : If P satisfies $P_i P_j^T = 0$ if $i \neq j$, is it true that $P^TP = \begin{bmatrix} c_1 & & 0 \\ & c_2 & \\ 0 & & \ddots \\ & & & c_n \end{bmatrix}$?

Let $\{\alpha_1, \cdots, \alpha_n\} \subset U$ linearly indep

$\Rightarrow \exists\ \eta_1, \cdots, \eta_r$ s.t. $(\eta_i | \eta_j) = \delta_{ij}$

s.t. $(\alpha_1, \cdots, \alpha_r) = (\eta_1, \cdots, \eta_r) R$.

Let $V=\mathbb{R}^n$, then $\underset{\overset{\shortparallel}{A}}{(\alpha_1,\cdots,\alpha_r)} \in \mathbb{R}^{n\times r}$

and $A=QR$, $Q\in\mathbb{R}^{n\times r}$

where $Q^TQ=I_{r\times r} \not\Rightarrow QQ^T=I$ if $r<n$.

— QR factorization (分解)

An application to the determinant of a matrix in $\mathbb{R}$.

Let $\{\alpha_1,\cdots,\alpha_n\}\subset\mathbb{R}$

Denote $A=(\alpha_1\ \alpha_2\ \cdots\ \alpha_n)\in\mathbb{R}^{n\times n}$

Then we have $|A|$

$$A=\underset{\overset{\shortparallel}{Q}}{(\eta_1\ \cdots\ \eta_n)}R \qquad R=\begin{bmatrix} \|\beta_1\| & & * \\ & \ddots & \\ 0 & & \|\beta_n\| \end{bmatrix}$$

$$|A|=|Q|\,|R| = \pm\underbrace{\|\beta_1\|\cdots\|\beta_n\|}_{\text{Volume}}$$

有向体积.

Let $U \subset V$ $\quad \dim_{\mathbb{R}} V = n < \infty$

Then $\exists$ a basis $(\alpha_1, \cdots, \alpha_r)$ of U s.t. $(\alpha_i | \alpha_j) = \delta_{ij}$

By Schmidt orthogonalization,

$\exists \alpha_{r+1}, \cdots, \alpha_n \in V$ s.t. $(\alpha_i | \alpha_j) = \delta_{ij}$ $\quad i, j = 1, 2, \cdots, n$

$\Rightarrow \alpha_j \in U^{\perp} = \{\alpha \in V \mid \alpha \perp U\}$

$j = r+1, \cdots, n$

$\Leftrightarrow L(\alpha_{r+1}, \cdots, \alpha_n) \subset U^{\perp}$

Assume $\alpha \in V$ s.t. $\alpha \perp U$

Then $\alpha = \sum_{i=1}^{n} x_i \alpha_i$ $\quad x_i \in \mathbb{R}$

and $(\alpha | \alpha_j) = 0$, $1 \le j \le r$

$\Updownarrow$

$\left(\sum_{i=1}^{n} x_i \alpha_i \middle| \alpha_j\right) = x_j = 0$, $1 \le j \le r$

$\Rightarrow \alpha = \sum_{i=r+1}^{n} x_i \alpha_i$

无限维

$(U^{\perp})^{\perp} \to U$ 的闭包

$\Rightarrow U^{\perp} = L(\alpha_{r+1}, \cdots, \alpha_n)$

$V = U \oplus \underset{\text{unique}}{U^{\perp}}$ $\quad$ LS, $\quad$ $V = U \oplus \underset{\text{not unique}}{U'}$

$U \oplus U^{\perp}$ $(U^{\perp})^{\perp} = U$.

(5) $\Leftrightarrow$ $\forall \alpha \in V$, $\exists \alpha_1 \in U$ $\alpha_2 \in U^{\perp}$

s.t. $\alpha = \alpha_1 + \alpha_2$, $\alpha_1 \perp \alpha_2$

Remark: If $\dim_{\mathbb{R}} V = \infty$, (5) does not always hold

Ex $V = C[-1,1]$, for $f(x), g(x) \in V$

Define

$$(f(x) | g(x)) = \int_{-1}^{1} f(x) g(x) dx$$

$\Rightarrow$ V is an Euclidean Space

Let $U = \{f(x) \in V \mid f(0) = 0\}$

$\Rightarrow U^{\perp} = \{0\}$, $U \subsetneq V$

Indeed, assume $f(x) \in U^{\perp}$, then

$x^2 f(x) \in U$, and $(x^2 f(x) | f(x)) = 0$

$\Rightarrow x f(x) = 0 \Rightarrow f(x) = 0$

$\not\Rightarrow$ $V = U \oplus U^{\perp}$

Ex $l^2 = \{ (a_1, a_2, \cdots) \mid a_i \in \mathbb{R}, \sum_{i=1}^{\infty} a_i^2 < \infty \}$

For $a = (a_1, \cdots)$, $b = (b_1, \cdots)$

$$(a \mid b) = \sum_{i=1}^{\infty} a_i b_i$$

Let $e_i = (\underbrace{0, \cdots, 0}_{i}, 1, 0, \cdots,)$, $i = 0, 1, \cdots$

$$e = (1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots) \in l^2$$

Let $V = L(e, e_1, e_2, \cdots)$

and $U = L(e_1, e_2, \cdots)$

It is easy to see $U^{\perp} = \{0\}$ $V \neq U$.

$A = (a_{ij})_{m \times n} \in \mathbb{R}^{m \times n}$

Then we have four vector spaces w.r.t A as follows

$N(A) = \{ x \in \mathbb{R}^n \mid Ax = 0 \} \subseteq \mathbb{R}^n$

$R(A) = \{ Ax \mid x \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

$N(A^T) = \{ y \in \mathbb{R}^m \mid A^T y = y^T A = 0 \} \subseteq \mathbb{R}^m$

$R(A^T) = \{ A^T y \mid y \in \mathbb{R}^m \} \subseteq \mathbb{R}^n$

$N(A), R(A^T) \subseteq \mathbb{R}^n$ - Euclidean space

$N(A^T), R(A) \subseteq \mathbb{R}^m$

Assume $r(A) = r$, then

$\dim N(A) = n-r$, $\dim R(A^T) = r$

$\mathbb{R}^n = N(A) \oplus R(A^T)$

Let $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$, $\alpha_i \in \mathbb{R}^{1\times n}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, then $Ax = 0 \Leftrightarrow \alpha_i x = 0$

$\Leftrightarrow (\alpha_i \mid x^T) = 0$, $1 \le i \le m$

$\Leftrightarrow \underset{\underset{L(\alpha_1, \alpha_2, \cdots, \alpha_m)}{\parallel}}{R(A^T)} \perp \alpha$

$\Rightarrow R(A^T) = N(A)^\perp$

$N(A) = R(A^T)^\perp$.

$\Leftrightarrow \mathbb{R}^n = N(A) \oplus R(A^T)$

Similarly, we have $\mathbb{R}^m = N(A^T) \oplus R(A)$

$\Leftrightarrow N(A^T)^\perp = R(A)$, $R(A)^\perp = N(A^T)$

Let U and V be Euclidean spaces, and

$\mathcal{A} \in \mathrm{Hom}_{\mathbb{R}}(U, V)$

Def $\mathcal{A}$ is called an isomorphism from U to V

if $\mathcal{A}$ is a bijection and for $\alpha, \beta \in U$,

$(\alpha | \beta) = (\mathcal{A}\alpha | \mathcal{A}\beta)$

In this case, we denote $U \cong V$

In particular, if $U = V$,

such $\mathcal{A}$ is called an automorphism of the

Euclidean space of U

Recall

Let $U_{/\mathbb{R}}$ and $V_{/\mathbb{R}}$ be Euclidean spaces, and

$\sigma \in \mathrm{Hom}_{\mathbb{R}}(U, V)$

If $\forall \alpha, \beta \in U$, we have $(\alpha | \beta) = (\sigma\alpha | \sigma\beta)$

and σ is a bijection

then we call σ an isomorphism from U to V

Denote $U \cong V$

In particular, if $U = V$, σ is called an isometry 等距变换 正交变换

of U. Denote

$$O(V) = \{\sigma \in \mathrm{End}\, V \mid \sigma \text{ is an isometry}\}$$

$O(V)$ is called the orthogonal group of U.

We now assume that V is f.d.

Then there exists an orthonormal basis

$\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ of V i.e. $(\alpha_i | \alpha_j) = \delta_{ij}$

Let $\mathcal{A} \in \operatorname{End} V$. If $\mathcal{A} \in O(V)$, then for any $\alpha, \beta \in V$, $(\alpha|\beta) = (\mathcal{A}\alpha|\mathcal{A}\beta)$

In particular,

$(\mathcal{A}\alpha_i|\mathcal{A}\alpha_j) = \delta_{ij}$

i.e. $\{\mathcal{A}\alpha_1, \cdots, \mathcal{A}\alpha_n\}$ is still an orthonormal basis of V

Conversely, if $(\mathcal{A}\alpha_i|\mathcal{A}\alpha_j) = \delta_{ij}$ $i,j = 1,2,\cdots,n$

Then $\mathcal{A}\alpha_i \neq 0$ $((\mathcal{A}\alpha_i|\mathcal{A}\alpha_i)=1)$

$\Rightarrow \{\mathcal{A}\alpha_1, \cdots, \mathcal{A}\alpha_n\}$ are linearly indep

$\Rightarrow \mathcal{A}$ is a bijection

$\Rightarrow \mathcal{A} \in O(V)$

We deduce that

Theorem Let V be f.d. Euclidean space, and $\mathcal{A} \in \operatorname{End} V$. Then $\mathcal{A} \in O(V)$ iff $\mathcal{A}$ changes orthonormal basis to an orthonormal basis

$\Leftrightarrow \forall \alpha, \beta \in V \quad (\alpha|\beta) = (\mathcal{A}\alpha|\mathcal{A}\beta)$

$\Leftrightarrow \forall \alpha \in V \quad (\mathcal{A}\alpha|\mathcal{A}\alpha) = (\alpha|\alpha)$

Remark : If V is infinite dimensional, then the fact that $(\mathcal{A}\alpha|\mathcal{A}\alpha) = (\alpha|\alpha)$ does not deduce that $\mathcal{A}$ is a bijection in general.

Ex $\quad V = l^2 = \{(a_1, a_2 \cdots) \mid a_i \in \mathbb{R}, \sum_{i=1}^{\infty} a_i^2 < \infty\}$

Let $\mathcal{A} \in \operatorname{End} V$ be defined as follows

$$\mathcal{A}(a_1, a_2 \cdots) = (0, a_1, a_2, \cdots)$$

Then $\forall \alpha, \beta \in V$

$$(\mathcal{A}\alpha|\mathcal{A}\beta) = (\alpha|\beta)$$

But $\mathcal{A}$ is not a bijection

Suppose $\dim V = n$

$\{\alpha_1, \cdots, \alpha_n\}$ a basis of V

s.t. $(\alpha_i|\alpha_j) = \delta_{ij}$

Let $\mathcal{A} \in \operatorname{End} V$ and $\mathcal{A}(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) A$

Then $((\mathcal{A}\alpha_i|\mathcal{A}\alpha_j)) = A^T \underbrace{((\alpha_i|\alpha_j))}_{= I} A = A^T A$

$\Rightarrow \mathcal{A} \in O(V) \Leftrightarrow A^T A = I \Leftrightarrow AA^T = I$

Denote $\underset{\text{实正交群}}{O(n)} = \{A \in \mathbb{R}^{n\times n} \mid A^T A = AA^T = I\}$

$O(V)$ - 样

Such A is called an orthogonal matrix (正交矩阵)

Ex $V = \mathbb{R}^2$

$$\mathcal{A}(e_1, e_2) = (e_1, e_2)\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Assume that $\mathcal{A} \in O(V) \subseteq \mathbb{R}^{n\times n} \subseteq \mathbb{C}^{n\times n}$

Let $\lambda \in \mathbb{C}$ and $\underset{\neq 0}{\alpha} \in \mathbb{C}^n$ be

such that

$$A\alpha = \lambda\alpha$$

Then $\bar{\alpha}^T A^T = \bar{\lambda}\bar{\alpha}^T$

$\Rightarrow \bar{\alpha}^T A^T A \alpha = \bar{\lambda}\lambda\bar{\alpha}^T\alpha$

$\Rightarrow \bar{\alpha}^T \alpha = \bar{\lambda}\lambda\bar{\alpha}^T\alpha$

We know that $\alpha^T\alpha > 0$

So $(1-\bar{\lambda}\lambda)\bar{\alpha}^T\alpha = 0$

$\Rightarrow \bar{\lambda}\lambda = 1 \Rightarrow |\lambda| = 1$

In particular, if $\lambda \in \mathbb{R}$, then $\lambda = 1$ or -1

Denote

特殊正交群

$$SO(n) = \{A \in O(n) \mid \det A = 1\} \subset O(n)$$

a Lie subgroup of $O(n)$

Next we introduce unitary space

$\mathbb{R}^n \rightsquigarrow \mathbb{C}^n$ – define inner produce?

Ex $\mathbb{C}^2$ $\alpha = \begin{pmatrix} 1 \\ \sqrt{-1} \end{pmatrix} \Rightarrow \alpha^T\alpha = 0$

$\mathbb{C}^n$ $\alpha \in \mathbb{C}^n \Rightarrow \bar{\alpha}^T\alpha \geq 0$

and $\bar{\alpha}^T\alpha = 0 \Leftrightarrow \alpha = 0$

Let $V/\mathbb{C}$ be a v.s and

$$(\cdot|\cdot): V \times V \to \mathbb{C}$$
$$(\alpha, \beta) \longmapsto (\alpha|\beta) \in \mathbb{C}.$$

If $(\cdot|\cdot)$ satisfies $\forall \alpha, \beta \in V$, $k_1, k_2 \in \mathbb{C}$

(1) $(k_1\alpha_1 + k_2\alpha_2 | \beta) = k_1(\alpha_1|\beta) + k_2(\alpha_2|\beta)$

(2) $(\beta|\alpha) = \overline{(\alpha|\beta)}$

$$\Downarrow$$

$$(\alpha | k_1\beta_1 + k_2\beta_2) = \overline{k_1}(\alpha|\beta_1) + \overline{k_2}(\alpha|\beta_2)$$

(3) $(\alpha|\alpha) \geq 0$, and $(\alpha|\alpha) = 0 \iff \alpha = 0$

Then we say $(\cdot|\cdot)$ is a complex inner product over V. V is called a complex inner space or unitary space (酉空间)

Ex $V = \mathbb{C}^n$ For $\alpha, \beta \in \mathbb{C}^n$

$(\alpha|\beta) := \overline{\beta}^T \alpha$ the standard inner product

$\Rightarrow \mathbb{C}^n$ is a unitary space

Ex $V = M_{m\times n}(\mathbb{C})$ For $A, B \in V$, define

$$(A|B) = tr(\bar{B}^T A) = tr(B^* A)$$

Notation: For $A \in \mathbb{C}^{m\times n}$, denote $A^* = \bar{A}^T$.

Then V is a unitary space

Ex V is the space of all continuous complex functions over $[a,b]$

Define

$$(f(x)|g(x)) = \int_a^b f(x)\overline{g(x)}\,dx \quad \text{for } f(x), g(x) \in V$$

Ex $V = l^2 = \{(a_1, a_2, \cdots) \mid a_i \in C, \sum_{i=1}^{\infty} a_i \bar{a}_i < \infty\}$

For $\alpha = (a_1, a_2, \cdots)$ $\beta = (b_1, b_2, \cdots)$

define $(\alpha|\beta) = \sum_{i=1}^{\infty} a_i \overline{b_i}$

We have the following Cauchy-Schwarz inequality.

$$|(\alpha|\beta)| \le \sqrt{(\alpha|\alpha)}\sqrt{(\beta|\beta)} = \|\alpha\|\,\|\beta\|$$

Proof We consider $\beta - \frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha$, and

$$\left(\beta - \frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha,\ \beta - \frac{(\beta|\alpha)}{(\alpha|\alpha)}\alpha\right)$$

$$= (\beta|\beta) - \frac{\beta|\alpha}{\alpha|\alpha}(\alpha|\beta) - \frac{(\alpha|\beta)}{(\alpha|\alpha)}(\beta|\alpha) + \frac{(\beta|\alpha)}{(\alpha|\alpha)}\frac{(\alpha|\beta)}{(\alpha|\alpha)}(\alpha|\alpha)$$

$$= (\beta|\beta) - \frac{(\beta|\alpha)(\alpha|\beta)}{(\alpha|\alpha)} \geq 0$$

$$\Rightarrow (\alpha|\beta)(\beta|\alpha) \leq (\alpha|\alpha)(\beta|\beta)$$

We may similarly define $\angle(\alpha, \beta)$ as follow: $(\alpha, \beta \neq 0)$

$$\cos \angle(\alpha, \beta) = \frac{|(\alpha|\beta)|}{\|\alpha\|\ \|\beta\|} \quad \left(\angle(\alpha, \beta) \in [0, \tfrac{\pi}{2}]\right)$$

In particular, if $(\alpha|\beta) = 0$, then we denote $\alpha \perp \beta$.

Recall for $A \in \mathbb{C}^{n \times n}$ s.t. $A^* = A$

$\exists\ P \in Gl_n(\mathbb{C})$ s.t. $P^*AP = \begin{bmatrix} I_p & & \\ & -I_q & \\ & & 0 \end{bmatrix}$

In particular, if A is positive-definite, then

$P^*AP = I$.

We also have Schmidt orthogonalization for a unitary space.

That is, if $\{\alpha_1, \cdots, \alpha_n\}$ is a basis of V (unitary)

Let $\xi_1 = \alpha_1$

$$\xi_2 = \alpha_2 - \frac{(\alpha_2|\xi_1)}{(\xi_1|\xi_1)}\xi_1$$

注意不能交换次序

$$\vdots$$

$$\xi_k = \alpha_k - \sum_{i=1}^{k-1} \frac{(\alpha_k|\xi_i)}{(\xi_i|\xi_i)}\xi_i \qquad 2 \le k \le n$$

$\Rightarrow (\xi_i|\xi_j) = 0, \ i \ne j.$

Let $\eta_i = \dfrac{\xi_i}{\|\xi_i\|}$, then $(\eta_i|\eta_j) = \delta_{ij}$, $i, j = 1, \cdots, n.$

and $L(\alpha_1, \cdots, \alpha_k) = L(\eta_1, \cdots, \eta_k)$

$$\Rightarrow (\alpha_1, \cdots, \alpha_n) = (\eta_1, \cdots, \eta_k)\begin{bmatrix} \|\xi_1\| & (\alpha_2|\eta_1) & \cdots & (\alpha_n|\eta_1) \\ & \|\xi_2\| & \cdots & \vdots \\ & & \ddots & \vdots \\ & & & \|\xi_n\| \end{bmatrix}$$

In particular, for linearly indep $\{\alpha_1, \cdots, \alpha_n\} \subset \mathbb{C}^n$,

let $A = (\alpha_1, \cdots, \alpha_r)$, then $\exists Q \in \mathbb{C}^{n\times r}$ s.t.

Q的列向量两两正交

$Q^*Q = I_r$ and $A = QR$ where $R = \begin{bmatrix} r_{11} & & \\ & \ddots & \\ 0 & & r_{rr} \end{bmatrix}$ $r_{ii} > 0$, $i = 1, \cdots, r$

Let $V/\mathbb{C}$ be a unitary space, and $\dim V = n$.

Assume $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of V.

Denote

$$G = (g_{ij})_{n\times n},$$

where $g_{ij} = (\alpha_j | \alpha_i)$ 为了表达方便

Then $G^* = G$ $\quad ((\alpha_i|\alpha_j) = \overline{(\alpha_j|\alpha_i)})$

and $G > 0$. (此处指正定)

$\forall\ \alpha = (\alpha_1, \cdots, \alpha_n)x \qquad \beta = (\alpha_1, \cdots, \alpha_n)y.$

Then
$$(\alpha|\beta) = \left(\sum_{i=1}^n x_i\alpha_i \Big| \sum_{j=1}^n y_j\alpha_j\right)$$
$$= \sum_{i=1}^n \sum_{j=1}^n x_i \overline{y_j}\, (\alpha_i|\alpha_j)$$
$$= \sum_{i=1}^n \sum_{j=1}^n \overline{y_j}\, (\alpha_j|\alpha_i) x_i$$
$$= y^* G x \qquad (\overline{y}^T G x)$$

suppose $\{\beta_1, \cdots, \beta_n\}$ is another basis of V.

Then we may assume

$$(\beta_1, \cdots, \beta_n) = (\alpha_1, \cdots, \alpha_n) P, \quad P \in Gl_n(\mathbb{C})$$

It is easy to deduce

$$((\beta_j|\beta_i))_{n\times n} = P^* G P$$

Let $\{\xi_1, \cdots, \xi_n\} \subset V$ s.t.

$$(\xi_i | \xi_j) = \delta_{ij}$$

Then $\forall \alpha \in V$ we have

1) $\alpha = \sum_{i=1}^{n} (\alpha|\xi_i)\xi_i$

2) $$(\alpha|\beta) = \left(\sum_{i=1}^{n} (\alpha|\xi_i)\xi_i, \sum_{j=1}^{n} (\beta|\xi_j)\xi_j\right)$$
$$= \sum_{i=1}^{n} (\alpha|\xi_i)\overline{(\beta|\xi_i)}$$

In particular,

$$(\alpha|\alpha) = \sum_{i=1}^{n} (\alpha|\xi_i)(\alpha|\xi_i) = \sum_{i=1}^{n} |(\alpha|\xi_i)|^2$$ 勾股定理

Definition

$\mathcal{A} \in \mathrm{End}\, V$ is called an isometry of V if $\mathcal{A} \in Gl(V)$

and $(\mathcal{A}\alpha|\mathcal{A}\beta) = (\alpha|\beta)$ (也称为酉变换)

Denote $U(V) = \{\mathcal{A} \in Gl(V) \mid \mathcal{A}$ is an isometry of $V\}$.

Assume $\dim V = n$ Then for $A \in \operatorname{End} V$, $A \in U(V) \Leftrightarrow$

A changes an orthonormal basis to an orthonormal basis.

$\Leftrightarrow$ The matrix of A under an orthonormal basis of V is a unitary matrix. (i.e. $A^*A = I = AA^*$

Denote

$$U(n) = \{ A \in \mathbb{C}^{n\times n} \mid A^*A = AA^* = I \}$$

A real Lie group, the unitary group of order n

Let $A \in U(n)$, and $\lambda \in \mathbb{C}$ $0 \neq \alpha \in \mathbb{C}^n$ s.t.

$$A\alpha = \lambda\alpha$$

Then $\alpha^* A^* = \bar{\lambda}\alpha^* \Rightarrow \alpha^* A^* A\alpha = \lambda\bar{\lambda}\alpha^*\alpha$

$\Rightarrow \lambda\bar{\lambda} = 1$

Denote

$$SU(n) = \{ A \in U(n) \mid \det A = 1 \}$$

— special unitary group

Ex $n=2$

$$SU(2) = \left\{ A \in \mathbb{C}^{2\times 2} \;\middle|\; A^* A = I \quad \det A = 1 \right\}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(2)$

$a, b, c, d \in \mathbb{C}$

$$A^* A = \begin{bmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} = \begin{bmatrix} x_1 + \sqrt{-1}\,x_2 & x_3 + \sqrt{-1}\,x_4 \\ -x_3 + \sqrt{-1}\,x_4 & x_1 - \sqrt{-1}\,x_2 \end{bmatrix}$$

$$= x_1 \underset{1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} + x_2 \underset{i}{\begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}} + x_3 \underset{j}{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}} + x_4 \underset{k}{\begin{bmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{bmatrix}}$$

$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j$

四元数除环 quatermim domain

and $\det A = a\bar{a} + b\bar{b} = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$

$$A \longleftrightarrow (x_1, x_2, x_3, x_4) \in S^3 = \left\{ (x_1, x_2, x_3, x_4) \;\middle|\; \sum_{i=1}^{4} x_i^2 = 1 \right\} \subset \mathbb{H}$$

$SU(2) \cong S^3$ as topological spaces

$SO(3)$?

Norm Spaces (赋范线性空间)

Def $V/\mathbb{R}(\mathbb{C})$

$\|\cdot\| : \quad V \to \mathbb{R}$

$\alpha \mapsto \|\alpha\|$

If $\|\cdot\|$ satisfies

1) $\|\alpha\| \geq 0$ and $\|\alpha\| = 0 \Rightarrow \alpha = 0$

2) $\|k\alpha\| = |k|\,\|\alpha\|$

3) $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$

Then $\|\cdot\|$ is called a norm over V, and V is called a norm space.

Ex If V is an inner space, then V is a norm space.

$$(\alpha+\beta|\alpha+\beta) = (\alpha|\alpha) + (\alpha|\beta) + (\beta|\alpha) + (\beta|\beta)$$
$$\leq \|\alpha\|^2 + 2\|\alpha\|\|\beta\| + \|\beta\|^2 = (\|\alpha\| + \|\beta\|)^2$$

Ex Let $V = C[a,b]$ Let $p > 1$ For $f(x) \in V$ define

$$\|f(x)\| = \left(\int_a^b |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

Lemma1 (Yong inequality)

For $a, b \in \mathbb{R}^+$, and $p, q > 1$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

Then $a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$.

Proof Let $\alpha = \frac{1}{p}$ $\beta = \frac{1}{q}$

consider $y = x^{\alpha}$ $x > 0$

$\Rightarrow y' = \alpha x^{\alpha - 1}$, $y'' = \alpha(\alpha - 1) x^{\alpha - 2} < 0$

consider $y = \alpha x + \beta$ $(1, 1)$

$\Rightarrow x^{\alpha} \leq \alpha x + \beta$ (凸函数)

Let $x = \frac{a}{b}$

$\Rightarrow \left(\frac{a}{b}\right)^{\frac{1}{p}} \leq \frac{1}{p} \frac{a}{b} + \frac{1}{q}$

$\Rightarrow a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$

Lemma 2 (Hölder inequality)

$$\int_a^b |f(x)g(x)|\,dx \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q dx\right)^{\frac{1}{q}} \quad \frac{1}{p}+\frac{1}{q}=1$$

Proof Let

$$a = \frac{|f(x)|^p}{\int_a^b |f(x)|^p dx} \qquad b = \frac{|g(x)|^q}{\int_a^b |g(x)|^q dx} \qquad \#$$

Lemma 3 (Minkowski inequality)

$$\left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{p}} \le \left(\int_a^b |f(x)|^p dx\right)^{\frac{1}{p}} + \left(\int_a^b |g(x)|^p dx\right)^{\frac{1}{p}}$$

Proof Let $q>1$ s.t. $\frac{1}{p}+\frac{1}{q}=1$

consider

$$\int_a^b |f(x)|\,|f(x)+g(x)|^{\frac{p}{q}} dx + \int_a^b |g(x)|\,|f(x)+g(x)|^{\frac{p}{q}} dx$$

$$\le \|f(x)\|_p \left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{q}} + \|g(x)\|_p \left(\int_a^b |f(x)+g(x)|^p dx\right)^{\frac{1}{q}}$$

离散型（作业）

1. Hölder 不等式

$$\sum_{j=1}^{\infty}|a_j b_j| \le \left(\sum_{i=1}^{\infty}|a_i|^p\right)^{\frac{1}{p}}\left(\sum_{i=1}^{\infty}|b_i|^q\right)^{\frac{1}{q}}$$

2. Minkowski 不等式

$$\left(\sum_{i=1}^{\infty}|a_i+b_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{\infty}|a_i|^p\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|b_i|^p\right)^{\frac{1}{p}}.$$

Ex $l^p=\{(a_1,a_2,\cdots) \mid \sum_{i=1}^{\infty}|a_i|^p<\infty\}$

Define $\|\cdot\|_p$ over l^p by $\|\alpha\|_p=\left(\sum_{i=1}^{\infty}|a_i|^p\right)^{\frac{1}{p}}$

$\Rightarrow$ l^p is a norm space

Remark If $p=2$ $\|\cdot\|$ is induced from the inner product given before

Q When is a norm $\|\cdot\|$ induced from an inner product?

Answer $\|\cdot\|$ satisfies $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$（充要条件）

?

inner product → norms → metrics (度量)

metric: for $u, v \in V$

1) $d(u,v) \geq 0$ $d(u,v) = 0 \Leftrightarrow u = v$

2) $d(u,v) = d(v,u)$

3) $\|u+v\| \leq \|u\| + \|v\|$

Ex $V = C[a,b]$

Define two metrics

完备 $d_1(f(x), g(x)) = \sup_{a \leq x \leq b} |f(x) - g(x)|$

不完备 $d_2(f(x), g(x)) = \int_a^b |f(x) - g(x)| dx$

Let $f_n(x) = \begin{cases} 1, & x \in (\frac{1}{2}+\frac{1}{n}, 1] \\ n(x-\frac{1}{2}), & x \in [\frac{1}{2}-\frac{1}{n}, \frac{1}{2}+\frac{1}{n}] \\ -1, & x \in [-1, \frac{1}{2}-\frac{1}{n}) \end{cases}$

Cauchy series

$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1, & x \in (\frac{1}{2}, 1] \\ 0, & x = \frac{1}{2} \\ -1, & x \in [-1, \frac{1}{2}) \end{cases}$

范数 → 度量 巴拿哈 内积 → 范数 → 度量 希尔伯特

有限维一定完备

Let V be f.d. i.e. $\dim_{\mathbb{F}} V = n < \infty$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms over V.

Then $\exists\ C_1, C_2 > 0$ s.t. for any $0 \neq \alpha \in V$

$$C_1 \leq \frac{\|\alpha\|_1}{\|\alpha\|_2} \leq C_2$$

(the equivalence of $\|\cdot\|_1$ and $\|\cdot\|_2$)

Proof Let $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of V

Then $\alpha = \sum_{i=1}^{n} x_i \alpha_i$

and $\|\cdot\|_1$ and $\|\cdot\|_2$ are functions of $x_1, \cdots, x_n$

Claim $\|\alpha\|_1 = f_1(x_1, \cdots, x_n)$

$\|\alpha\|_2 = f_2(x_1, \cdots, x_n)$ are continuous

For $\alpha = (\alpha_1, \cdots, \alpha_n) X$ $\beta = (\alpha_1, \cdots, \alpha_n) Y$

Then

$$\left| \|\alpha\|_1 - \|\beta\|_1 \right| \leq \|\alpha - \beta\|_1 = \left\| \sum_{i=1}^{n} (x_i - y_i) \alpha_i \right\|$$

$$\leq \sum_{i=1}^{n} |x_i - y_i| \, \|\alpha_i\|$$

$$\leq \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \sqrt{\sum_{i=1}^{n} \|\alpha_i\|^2}$$

$$\Rightarrow \quad \big| \|\alpha\|_1 - \|\beta\|_1 \big| \to 0 \quad \text{if} \quad \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \to 0$$

Consider

$$\left\{ \frac{f_1(x_1, \cdots, x_n)}{f_2(x_1, \cdots, x_n)} \;\middle|\; (x_1, \cdots, x_n) \in S^{n-1}, \; \sum_{i=1}^{n} x_i^2 = 1 \right\}$$ bounded closed

Then $\exists\, C_1 > 0,\; C_2 > 0$ s.t. $C_1 \le \dfrac{\|\alpha\|_1}{\|\alpha\|_2} \le C_2$

Ex $\quad V = M_n(\mathbb{F})$

A norm $\|\cdot\|$ over V is called matrix norm,

if $\|AB\| \le \|A\| \cdot \|B\|$

The norms we often use:

(1) Frobenius norm:

$$\|A\|_F = \sqrt{tr(A^* A)}$$

(2) Spectrum norm:

$$\|A\| = \sqrt{\rho(A^* A)}$$ (最大特征值)

(3) Operator norm:

$$\|A\| = \sup \left\{ \|AX\| \;\middle|\; \sum_{i=1}^{n} x_i^2 = 1 \right\} = \sup_{X \in \mathbb{C}^n} \frac{\|AX\|}{\|X\|}$$

Given $A \in \mathbb{F}^{n\times n}$, we have $A^0, A, A^2, \cdots$

Fact : $A^n \to 0 \iff \|A\| < 1$ for any norm $\|\cdot\|$ over $\mathbb{F}^{n\times n}$

Exercise : Let $A \in \mathbb{C}^{n\times n}$ and λ be an eigenvalues of A.

Then $\|\lambda\| \leq \|A\|$

Next we introduce normal operators on inner space.

Let V be an inner space, and $\mathcal{A} \in \operatorname{End} V$

Define $\mathcal{A}^* : V \to V$ by for any $\alpha, \beta \in V$,

$(\mathcal{A}\alpha | \beta) = (\alpha | \mathcal{A}^* \beta)$

First, for $\beta \in V$, $\mathcal{A}^*\beta$ is uniquely determined.

It follows from the fact that $(\cdot|\cdot)$ is non-degenerate

i.e. $(\alpha|\gamma) = 0$ for any $\alpha \in V \Rightarrow \gamma = 0$.

Secondly, $\mathcal{A}^* \in \operatorname{End} V$.

Indeed, for $k_1, k_2 \in \mathbb{F}$, $\beta_1, \beta_2 \in V$

we have $(\alpha | \mathcal{A}^*(k_1\beta_1 + k_2\beta_2)) = (\mathcal{A}\alpha | k_1\beta_1 + k_2\beta_2)$

$$= \bar{k}_1 (\mathcal{A}\alpha | \beta_1) + \bar{k}_2 (\mathcal{A}\alpha | \beta_2)$$

$$= \bar{k}_1 (\alpha | \mathcal{A}^*\beta_1) + \bar{k}_2 (\alpha | \mathcal{A}^*\beta_2)$$

$$= (\alpha | k_1 \mathcal{A}^*\beta_1) + k_2 (\alpha | \mathcal{A}^*\beta_2)$$

$$= (\alpha | k_1 \mathcal{A}^*\beta_1 + k_2 \mathcal{A}^*\beta_2), \quad \forall \alpha \in V$$

$$\Rightarrow \mathcal{A}^*(k_1\beta_1 + k_2\beta_2) = k_1\mathcal{A}^*\beta_1 + k_2\mathcal{A}^*\beta_2$$

$\mathcal{A}^*$ is called the adjoint (or conjugate) operator of $\mathcal{A}$.

We now assume that $\dim V = n < \infty$

Let $\{\alpha_1, \cdots, \alpha_n\}$ be a basis of V s.t. $(\alpha_i | \alpha_j) = \delta_{ij}$

$$\mathcal{A}(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) A$$

$$\mathcal{A}^*(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) B$$

Recall $(\mathcal{A}\alpha | \beta) = (\alpha | \mathcal{A}^*\beta)$

$$\Rightarrow (\mathcal{A}\alpha_i | \alpha_j) = (\alpha_i | \mathcal{A}^*\alpha_j)$$

$$\Rightarrow \left(\sum_{k=1}^{n} a_{ki}\alpha_k \Big| \alpha_j\right) = \left(\alpha_i \Big| \sum_{k=1}^{n} b_{kj}\alpha_k\right)$$

$$\Rightarrow a_{ji} = \overline{b_{ij}} \qquad \Rightarrow B = A^*.$$

Notations:

(1) If $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A}$ then we say $\mathcal{A}$ is normal (正规算子)

(2) If $\mathcal{A}^* = \mathcal{A}$ then we say $\mathcal{A}$ is self-adjoint (自伴随)

(3) If $\mathcal{A}^* = -\mathcal{A}$ then we say $\mathcal{A}$ is skey symmetric (V real inner space)

Skey-Hermite (V unitary space)

We now assume V is a f.d. unitary space.

Let $\mathcal{A} \in \operatorname{End} V$ and $U \subset V$ s.t. $\mathcal{A}U = U$

Let $\{\alpha_1, \cdots, \alpha_r\}$ be an orthonormal basis of U.

We extend it to an orthonormal basis of V

Then

$$\mathcal{A}(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n)\begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \qquad (2)$$

$$\Rightarrow \mathcal{A}^*(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n)\begin{bmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{bmatrix} \qquad (3)$$

Notice that $U^\perp = L(\alpha_{r+1}, \cdots, \alpha_n)$

(3) implies that $\mathcal{A}^* U \subset U^\perp$

We know that

$$AA^* = A^*A \iff AA^* = A^*A$$

Lemma (schur Lemma) Suppose $A \in \mathbb{C}^{n\times n}$

Then $\exists\ U \in U(n)$ s.t.

$$U^*AU = \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{bmatrix}$$

Proof If $n=1$, it is true.

We assume the statement is true for any $A \in \mathbb{C}^{m\times m}$, $m \le n-1$.

We now assume $A \in \mathbb{C}^{n\times n}$.

Then there exist $\alpha_1 \in \mathbb{C}^n$ and $\lambda_1 \in \mathbb{C}$ s.t.

$\|\alpha_1\| = 1$ and $A\alpha_1 = \lambda_1\alpha_1$

Extend α_1 to an orthonormal basis $\{\alpha_1, \cdots, \alpha_n\}$ of $\mathbb{C}^n$.

Then $U_1 = (\alpha_1, \cdots, \alpha_n) \in U(n)$

and $A(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) = \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & \\ \vdots & B \\ 0 & \end{array}\right]$

By inductive assumption,

$\exists\ U_2 \in \mathcal{U}(n-1)$ s.t.

$$U_2^* B_1 U_2 = \begin{bmatrix} \lambda_2 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Let $U = U_1 \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix}$

then $$U^* A U = \begin{bmatrix} 1 & \\ & U_2^* \end{bmatrix} \left[\begin{array}{c|c} \lambda_1 & * \\ \hline 0 & B_1 \end{array}\right] \begin{bmatrix} 1 & \\ & U_2 \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

#

For $A \in \mathbb{C}^{n\times n}$ A is called normal, if $AA^* = A^*A$

If $B = U^* A U$, $U \in \mathcal{U}(n)$

Then $B^* B = U^* A^* U \cdot U^* A U = U^* A^* A U$

$BB^* = U^* A A^* U$

$B^* B = BB^* \Longleftrightarrow AA^* = A^* A$

We now assume

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$$

Then $A^* A = \begin{bmatrix} \bar{a}_{11} & & 0 \\ \vdots & \ddots & \\ \bar{a}_{1n} & \cdots & \bar{a}_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ 0 & & a_{nn} \end{bmatrix}$

$$AA^* = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & \ddots & \vdots \\ & & a_{nn} \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & & \\ \vdots & \ddots & \\ \bar{a}_{1n} & \cdots & \bar{a}_{nn} \end{bmatrix}$$

$$\Rightarrow \quad A^*A = AA^* \quad \Leftrightarrow \quad A = \begin{bmatrix} a_{11} & & \\ & \ddots & \\ & & a_{nn} \end{bmatrix}$$

So we have

Theorem $A \in \mathbb{C}^{n\times n}$ is normal iff A is u-similar to a diagonal matrix, i.e.

$\exists\, U \in U(n)$ s.t.

$$U^* A U = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \qquad (6)$$

Let $U = (\alpha_1, \cdots, \alpha_n)$, then (4) is

$$A(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n)\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$\Leftrightarrow A\alpha_i = \lambda_i \alpha_i, \quad i = 1, \cdots, n$

$\Rightarrow A \in \mathbb{C}^{n\times n}$ is normal iff

A has n eigenvectors $\alpha_1, \cdots, \alpha_n$ such that $(\alpha_i | \alpha_j) = \delta_{ij}$

Theorem $\mathcal{A} \in \operatorname{End} V$ $(\dim V = n < \infty)$ is normal

iff $\mathcal{A}$ has n eigenvectors $\{\alpha_1, \cdots \alpha_n\}$

such that $(\alpha_i | \alpha_j) = \delta_{ij}$

Coro If $\mathcal{A}^* = \mathcal{A}$, $\mathcal{A}^* = -\mathcal{A}$, or $\mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = id$

then $\mathcal{A}$ has n eigenvectors $\alpha_1, \cdots, \alpha_n$ s.t. $(\alpha_i | \alpha_j) = \delta_{ij}$

Ex $A^* = A$ $\quad \exists\, U \in U(n)$ s.t. $A = U\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}U^*$

$\exists\, U \in U(n)$ s.t. $A = U\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}U^*$

$$\Rightarrow A^* = U\begin{bmatrix} \bar{\lambda}_1 & & \\ & \ddots & \\ & & \bar{\lambda}_n \end{bmatrix}U^* \Rightarrow \lambda_i = \bar{\lambda}_i \in \mathbb{R}$$

If $A^*A = AA^* = I$ then $\lambda_i \bar{\lambda}_i = 1$, $i=1,\cdots,n$

If $A^* = -A$ then $\lambda_i + \bar{\lambda}_i = 0 \Rightarrow \lambda_i \in \sqrt{-1}\,\mathbb{R}$, $i=1,\cdots,n$

Ex Let $A \in \mathrm{End}_{\mathbb{C}} V$ be normal

$U \subset V$ s.t. $AU \subset U$ $\exists \{\alpha_1, \cdots, \alpha_n\}$ of V s.t. $(\alpha_i|\alpha_j) = \delta_{ij}$

and $A(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}$

$$A^*(\alpha_1, \cdots, \alpha_n) = (\alpha_1, \cdots, \alpha_n) \begin{bmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{bmatrix}$$

$A^*A = AA^*$ implies that

$$\begin{bmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix} \begin{bmatrix} A_1^* & 0 \\ A_2^* & A_3^* \end{bmatrix}$$

$\Rightarrow A_1^* A_1 = A_1 A_1^* + A_2 A_2^*$

$\Rightarrow \mathrm{tr}(A_1^* A_1) = \mathrm{tr}(A_1 A_1^*) + \mathrm{tr}(A_2 A_2^*)$

$\Rightarrow \mathrm{tr}(A_2 A_2^*) = 0 \quad \Rightarrow A_2 = 0$.

$\Rightarrow A = \begin{bmatrix} A_1 & 0 \\ 0 & A_3 \end{bmatrix} \Rightarrow AU^\perp \subseteq U^\perp$

$V/\mathbb{C}$ or $\mathbb{R}$ — f.d. inner space

For $A \in \operatorname{End} V$, we have $A^* \in \operatorname{End} V$ s.t.

$(A\alpha|\beta) = (\alpha|A^*\beta)$, $\forall \alpha, \beta \in V$

Consider $(\cdot|\gamma)$, for fixed

$\gamma \in V$: $V \to \mathbb{F} = \mathbb{C}$ or $\mathbb{R}$

$\alpha \mapsto (\alpha|\gamma)$

$\Rightarrow (\cdot|\gamma) \in V^*$

donoted by f_γ

We have $\sigma: V \to V^*$

$\gamma \mapsto (\cdot|\gamma)$

Since $(\cdot|\cdot)$ is non-degenerate, it follows that σ is injective.

$\Rightarrow \sigma$ is isomorphism from V to V^*.

If $\mathbb{F} = \mathbb{R}$ A^* is the same

$(\mathcal{A}^*\alpha|\beta) = (\alpha|\mathcal{A}\beta)$

$(\mathcal{A}^*\alpha|\beta) = \overline{(\beta|\mathcal{A}^*\alpha)}$ $\qquad (\mathcal{A}^*)^* = \mathcal{A}$

$(\alpha|\mathcal{A}\beta) = \overline{(\mathcal{A}\beta|\alpha)}$

We know that

$$(\mathcal{A}\beta|\alpha) = (\beta|\mathcal{A}^*\alpha)$$

So $\quad (\mathcal{A}^*\alpha|\beta) = (\alpha|\mathcal{A}\beta)$

$\Rightarrow \quad \mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A}$ iff $(\mathcal{A}\alpha|\mathcal{A}\alpha) = (\mathcal{A}^*\alpha|\mathcal{A}^*\alpha)$

$\operatorname{Im}\mathcal{A}$, $\ker\mathcal{A}$, $(\operatorname{Im}\mathcal{A})^\perp$, $(\ker\mathcal{A})^\perp$ $\qquad$?

$\operatorname{Im}\mathcal{A}^*$, $\ker\mathcal{A}^*$, $(\operatorname{Im}\mathcal{A}^*)^\perp$, $(\ker\mathcal{A}^*)^\perp$

Recall

$$(\ker\mathcal{A})^\perp = \{f \in V^* \mid f(\alpha) = 0,\ \alpha \in \ker\mathcal{A}\}$$

By the fact that $(\mathcal{A}\alpha|\beta) = (\alpha|\mathcal{A}^*\beta)$ for any $\alpha, \beta \in V$

we have $\quad \operatorname{Im}\mathcal{A}^* \subset (\ker\mathcal{A})^\perp$

$\Rightarrow (\ker\mathcal{A})^\perp = \operatorname{Im}\mathcal{A}^*$

Recall $AA^* = A^*A \overset{\dim V = n}{\Longleftrightarrow}$

$\exists$ an orthonormal basis of V s.t.

under this basis the matrix of A is diagonal

$\Longleftrightarrow$ A is diagonalizable and there are n eigenvectors which are orthogonal to each other.

$\Rightarrow$ A is self-adjoint iff

A is normal and all eigenvalues are real.

A is isometry (等距的) iff

A is normal and $\lambda\bar{\lambda} = 1$

If $\mathbb{F} = \mathbb{R}$ $\quad A \in \mathbb{R}^{n\times n} \subseteq \mathbb{C}^{n\times n}$

A is normal $\Longleftrightarrow$ $A^*A = AA^*$

Assume $A \in \mathbb{R}^{n\times n}$ is normal, then

$\exists\ U = (\alpha_1, \cdots, \alpha_n) \in U(n)$ i.e. $\alpha_i^* \alpha_j = \delta_{ij}$

s.t. $U^*AU = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

Notice that

$$f_A(\lambda) = |\lambda I - A| \in \mathbb{R}[\lambda]$$

Then we have

$$U^* A U = \begin{bmatrix} \mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_r & & & & & \\ & & & \mu_{r+1} & & & & \\ & & & & \bar{\mu}_{r+1} & & & \\ & & & & & \ddots & & \\ & & & & & & \mu_{r+s} & \\ & & & & & & & \bar{\mu}_{r+s} \end{bmatrix} \quad (r+2s=n)$$

If $s=0$, then we may choose $U \in O(n)$,

$$U^T U = U U^T = I$$

If $s \neq 0$

Then we have

$$U = (\alpha_1 \cdots \alpha_2 \ \beta_1 \ \bar{\beta}_1 \cdots \beta_s \ \bar{\beta}_s)$$

$$(A\beta_i = \lambda_{r+i}\beta_i \Rightarrow A\bar{\beta}_i = \bar{\lambda}_{r+i}\bar{\beta}_i)$$

Notice that $\beta_i \perp \bar{\beta}_i$

We may assume that $\beta_i = \xi_i + \sqrt{-1}\,\eta_i$, $\xi_i, \eta_i \in \mathbb{R}^n$

$$\beta_i \perp \bar{\beta}_i \Leftrightarrow (\xi_i + \sqrt{-1}\,\eta_i \mid \xi_i - \sqrt{-1}\,\eta_i) = 0 \Leftrightarrow (\xi_i|\xi_i) = (\eta_i|\eta_i),\ (\xi_i|\eta_i) = 0.$$

$$\Rightarrow \quad (\xi_i|\xi_i) = (\eta_i|\eta_i) = \frac{1}{2} \qquad (\xi_i|\eta_i) = 0$$

since $(\beta_i|\beta_i) = (\xi_i|\xi_i) + (\eta_i|\eta_i)$

We consider $A\xi_i$, $A\eta_i$

Notice that

$$A(\xi_i + \sqrt{-1}\eta_i) = \mu_{r+i}(\xi_i + \sqrt{-1}\eta_i) \qquad (*)$$

Denote $\mu_{r+i} = a_i + \sqrt{-1}b_i$, $1\le i\le s$ $\quad a_i, b_i \in \mathbb{R}$

$$(*) \Leftrightarrow A(\xi_i + \sqrt{-1}\eta_i) = (a_i + \sqrt{-1}b_i)(\xi_i + \sqrt{-1}\eta_i)$$

$$\Leftrightarrow A\xi_i + \sqrt{-1}A\eta_i = a_i\xi_i - b_i\eta_i + \sqrt{-1}(b_i\xi_i + a_i\eta_i)$$

$$\Rightarrow A(\xi_i, \eta_i) = (\xi_i\ \eta_i)\begin{bmatrix} a_i & b_i \\ -b_i & a_i \end{bmatrix}$$

Let

$$P = (\alpha_1, \cdots, \alpha_r, \sqrt{2}\xi_1, \sqrt{2}\eta_1, \cdots, \sqrt{2}\xi_s, \sqrt{2}\eta_s)$$

Then $P \in O(n)$, i.e. $P^TP = I = PP^T$

and

$$AP = P\begin{bmatrix} \mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_r & & & & & \\ & & & a_1 & b_1 & & & \\ & & & -b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_s & b_s \\ & & & & & & -b_s & a_s \end{bmatrix} \text{ i.e. } P^TAP = \begin{bmatrix} \mu_1 & & & & & & & \\ & \ddots & & & & & & \\ & & \mu_r & & & & & \\ & & & a_1 & b_1 & & & \\ & & & -b_1 & a_1 & & & \\ & & & & & \ddots & & \\ & & & & & & a_s & b_s \\ & & & & & & -b_s & a_s \end{bmatrix}$$

In particular,

(1) If $A = A^* = A^T$ (real symmetric)

then $s = 0$, $U = P$

(2) If $A^T = -A$, then we have

$$P^TAP = \begin{bmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & b_1 & & & \\ & & -b_1 & 0 & & & \\ & & & & \ddots & & \\ & & & & & 0 & b_s \\ & & & & & -b_s & 0 \end{bmatrix} \qquad P \in O(n)$$

$\Rightarrow \exists\, C \in Gl_n(\mathbb{R})$ s.t.

$$C^TAC = \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & \ddots & & & & \\ & & & 0 & 1 & & \\ & & & -1 & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{bmatrix}$$

(If $r(A) = n$, then $n \in 2\mathbb{Z}_+$ and $C^TAC = \begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{bmatrix}$)

(3) If $A^TA = AA^T = I$, we have

$$P^TAP = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & & -1 & & & & & \\ & & & & \ddots & & & & \\ & & & & & -1 & & & \\ & & & & & & \begin{matrix} \cos\theta_1 & \sin\theta_1 \\ -\sin\theta_1 & \cos\theta_1 \end{matrix} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \begin{matrix} \cos\theta_s & \sin\theta_s \\ -\sin\theta_s & \cos\theta_s \end{matrix} \end{bmatrix}$$

In particular, if $n=3$, then $P^T A P$ is one of the following

$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ & \cos\theta_1 & \sin\theta_1 \\ & -\sin\theta_1 & \cos\theta_1 \end{bmatrix},$$

$$\begin{bmatrix} -1 & & \\ & \cos\theta_1 & \sin\theta_1 \\ & -\sin\theta_1 & \cos\theta_1 \end{bmatrix}$$

Ex $P \in \mathbb{C}^{n\times n}$, $P^2 = P$

Then P is diagonalizable.

And P is normal iff $P^* = P$

i.e. iff P is orthogonal projection (正交投影)

$P^2 = P$, $P^* = P \Leftrightarrow \exists\, U \in U(n)$ s.t. $U^* P U = \begin{bmatrix} I_r & \\ & 0 \end{bmatrix}$, $r = \operatorname{rank} P$

Next we introduce the spectral decomposition (谱分解) of a diagonalizable matrix.

Let $A \in \mathbb{C}^{n\times n}$, and A is diagonalizable

i.e. $\exists\, Q \in Gl_n(\mathbb{C})$ s.t.

$$Q^{-1} A Q = \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_s I_{n_s} \end{bmatrix}$$

Denote

$$Q = (Q_1 \ \ Q_2 \ \cdots \ \ Q_s)$$

$$Q^{-1} = \begin{pmatrix} \widetilde{Q}_1 \\ \vdots \\ \widetilde{Q}_s \end{pmatrix} \qquad Q_i \in \mathbb{C}^{n \times n_i}, \quad \widetilde{Q}_i \in \mathbb{C}^{n_i \times n}$$

Then $\quad Q_1\widetilde{Q}_1 + \cdots + Q_s\widetilde{Q}_s = I \qquad (1)$

$$Q^{-1}Q = \begin{pmatrix} \widetilde{Q}_1 \\ \vdots \\ \widetilde{Q}_s \end{pmatrix} (Q_1 \cdots , Q_s) = \begin{bmatrix} \widetilde{Q}_1 Q_1 & \cdots & \widetilde{Q}_1 Q_s \\ \vdots & & \vdots \\ Q_s Q_1 & \cdots & \widetilde{Q}_s Q_s \end{bmatrix} = I$$

$\Rightarrow \ \widetilde{Q}_i Q_i = I_{n_i} \quad , \quad \widetilde{Q}_i Q_j = 0 \quad , \ i \neq j \qquad (2)$

and

$$A = Q \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_s I_{n_s} \end{bmatrix} Q^{-1}$$

$$= (Q_1 \cdots Q_s) \begin{bmatrix} \lambda_1 I_{n_1} & & \\ & \ddots & \\ & & \lambda_s I_{n_s} \end{bmatrix} \begin{bmatrix} \widetilde{Q}_1 \\ \vdots \\ \widetilde{Q}_s \end{bmatrix}$$

$$= (\lambda_1 Q_1 \cdots \lambda_s Q_s) \begin{pmatrix} \widetilde{Q}_1 \\ \vdots \\ \widetilde{Q}_s \end{pmatrix}$$

$= \lambda_1 Q_1 \tilde{Q}_1 + \cdots \lambda_s Q_s \tilde{Q}_s$

Let $P_i = Q_i \tilde{Q}_i$, then by (1) and (2)

we have $\sum_{i=1}^{s} P_i = I$ (3)

$P_i^2 = P_i P_i = Q_i \tilde{Q}_i Q_i \tilde{Q}_i = P_i$

$P_i P_j = Q_i \tilde{Q}_i Q_j \tilde{Q}_j = 0, \ i \neq j$

$\Rightarrow \ P_i P_j = \delta_{ij} P_i$ (4)

$A = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_s P_s$ (5)

(3) ~ (5) is called the spectral composition of A.

Actually P_i is a polynomial of A

(3) ~ (5) $\Rightarrow A^2 = \lambda_1^2 P_1 + \cdots + \lambda_s^2 P_s$

$\Rightarrow \forall f(x) \in \mathbb{C}(\lambda)$, $f(A) = f(\lambda_1) P_1 + \cdots + f(\lambda_s) P_s$

$\exists f_i(\lambda) \in \mathbb{C}[\lambda]$ s.t. $f_i(\lambda_i) = 1$, $f_i(\lambda_j) = 0$

Then $f_i(A) = P_i$ $1 \le i \le s$

This decomposition is uniquely determined by A.

Recall

For $A \in \mathbb{C}^{n\times n}$ – diagonalizable

$$f_A(\lambda) = (\lambda-\lambda_1)^{n_1}\cdots(\lambda-\lambda_s)^{n_s}$$

Then $\quad A = \lambda_1 P_1 + \cdots + \lambda_s P_s \qquad (1)$

s.t. $\quad P_i P_j = \delta_{ij} P_i$

$\quad \sum_{i=1}^{n} P_i = I$

P_i - polynomials of A, $1 \le i \le s$

(1) – spectral decomposition of A

$\Rightarrow$ For any $g(x) \in \mathbb{C}[x]$, we have

$$g(A) = g(\lambda_1)P_1 + \cdots + g(\lambda_s)P_s$$

From the point of view of operators.

Let $\mathcal{A} \in \operatorname{End} V$ s.t. $\mathcal{A}$ is diagonalizable.

Then $\quad V = \bigoplus_{i=1}^{s} V_{\lambda_i}$, $\; V_{\lambda_i} = \{\alpha \in V \mid \mathcal{A}\alpha = \lambda_i \alpha\}$

For $\alpha \in V$, we have $\quad \alpha = \underset{\substack{| \\ \text{unique}}}{\sum_{i=1}^{s} \alpha_i}$, $\alpha_i \in V$?

Let $P_i \in V$ s.t. $P_i\alpha = \alpha_i$ $\quad 1 \le i \le s$

$\Rightarrow P_i \in \mathrm{End}\, V$ and $P_i^2 = P_i \quad P_iP_j = 0,\ i \ne j$

$\Rightarrow \mathcal{A} = \lambda_1 P_1 + \cdots + \lambda_s P_s \qquad (2)$

the spectral decomposition of $\mathcal{A}$.

We now assume $\mathcal{A}$ (A) is normal.

Then $P_i^2 = P_i$, $P_i^* = P_i$

Conversely, if $\mathcal{A}$ is diagonalizable, and in (2)

$P_i^* = P_i$ Then $\mathcal{A}\mathcal{A}^* = \mathcal{A}^*\mathcal{A}$

so we deduce that $\mathcal{A}$ is normal iff

$\mathcal{A}$ is diagonalizable and in (2) $P_i^* = P_i$ $\quad 1 \le i \le s$

For $A \in \mathbb{C}^{n\times n}$, A is normal

iff A is diagonalizable and in (1) $P_i^* = P_i$ $\quad 1 \le i \le s$

Singular Value Decomposition 奇异值分解

注：奇异值分解与满秩分解要求掌握

Recall (1) A^*A, AA^* are self-adjoint

(2) $r(A) = r(A^*) = r(A^*A) = r(AA^*)$?

(3) AA^* and A^*A have the same non-zero eigenvalues.

We denote $r(A) = r$.

Then A^*A and AA^* have ?

r non-zero eigenvalues $\lambda_1 > 0, \cdots, \lambda_r > 0$

and there exists an orthonormal basis of $\mathbb{C}^n$

$(\alpha_1, \cdots, \alpha_n)$ s.t.

$$A^*A(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n)\begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_r & \\ & & & 0 \end{bmatrix}$$

$\Rightarrow A^*A\alpha_i = \lambda_i\alpha_i$ (3)

$\Rightarrow AA^*A\alpha_i = \lambda_i A\alpha_i$ $1 \le i \le r$

By (3) $A\alpha_i \ne 0$, $1 \le i \le r$

Furthermore. $(A\alpha_i | A\alpha_j) = 0$, $i \ne j$ and $(A\alpha_i | A\alpha_i) = \lambda_i$.

Let $\beta_i = \frac{1}{\sqrt{\lambda_i}} \alpha_i \in \mathbb{C}^m \quad 1 \le i \le r$

Then $(\beta_i | \beta_j) = \delta_{ij}, \quad 1 \le i, j \le r$

Extend $\{\beta_1, \cdots, \beta_r\}$ to an orthonormal basis $\{\beta_1, \cdots, \beta_m\}$ of $\mathbb{C}^n$ s.t. $AA^* \beta_i = 0, \ r+1 \le i \le m$

$\Rightarrow A^* \beta_i = 0, \ r+1 \le i \le m$?

We deduce that

$$A(\alpha_1, \alpha_2 \cdots \alpha_n) = (A\alpha_1, A\alpha_2, \cdots, A\alpha_r, 0, \cdots, 0)$$

$$= (\sqrt{\lambda_1}\beta_1, \sqrt{\lambda_2}\beta_2, \cdots, \sqrt{\lambda_r}\beta_r, 0, \cdots, 0)$$

$$= (\beta_1, \cdots, \beta_m) \underbrace{\begin{bmatrix} \sqrt{\lambda_1} & & & & & \\ & \sqrt{\lambda_2} & & & & \\ & & \ddots & & & \\ & & & \sqrt{\lambda_r} & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0 \end{bmatrix}_{m \times n}}_{\Lambda}$$

Let $U = (\alpha_1, \cdots, \alpha_n)$

$V = (\beta_1, \cdots, \beta_n)$

Then $U^* U = I = UU^* \quad V^* V = I = VV^*$

and $AU = V\Lambda$ i.e. $A = V\Lambda U^*$ (S)

called the singular-value decomposition of A.

From (5) we immediately have the polar decomposition of $A \in \mathbb{C}^{n\times n}$

For $A \in \mathbb{C}^{n\times n}$, by (5),

$\exists\, U, V \in \mathcal{U}(n)$ s.t.

$$A = V \begin{bmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} U$$

$$= V \begin{bmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} V^* \underbrace{VU}_{\in \mathcal{U}(n)}$$

?

$$= VU\,U^* \begin{bmatrix} \sqrt{\lambda_1} & & & & & \\ & \ddots & & & & \\ & & \sqrt{\lambda_r} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} U$$

$\Rightarrow \exists\, Q_1, Q_2 \in \mathbb{C}^{n\times n}$ and $P \in \mathcal{U}(n)$

s.t. $A = Q_1 P = P Q_2$

唯一性 ?

and $Q_1^* = Q_1 \geq 0$, $Q_2^* = Q_2 \geq 0$

Recall for $A \in \mathbb{C}^{m\times n}$, we have $A = LR$

$L \in \mathbb{C}^{m\times r}$, $R \in \mathbb{C}^{r\times n}$, $r(A) = r$ 满秩分解

we have $A = UR$

s.t. $U^* U = I_r$ $R = \begin{bmatrix} r_{11} & & * \\ & \ddots & \\ 0 & & r_{nn} \end{bmatrix}$ $r_{ii} > 0$ 正交三角分解

For $A \in \mathbb{C}^{n\times n}$ $r(A) = n$ Then for some A,

$A = LU$ s.t. $L = \begin{bmatrix} l_{11} & & 0 \\ & \ddots & \\ * & & l_{nn} \end{bmatrix}$ $U = \begin{bmatrix} u_{11} & & * \\ & \ddots & \\ 0 & & u_{nn} \end{bmatrix}$

$l_{ii} \neq 0$, $u_{ii} \neq 0$, $i = 1, 2, \cdots, n$

$$L = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix}, \quad U = \begin{bmatrix} U_1 & U_2 \\ 0 & U_3 \end{bmatrix}$$

$L_1 \in \mathbb{C}^{k\times k}$ $|L_1| \neq 0$ $|U_1| \neq 0$

$$LU = \begin{bmatrix} L_1 & 0 \\ L_2 & L_3 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ 0 & U_3 \end{bmatrix} = \begin{bmatrix} L_1 U_1 & * \\ * & * \end{bmatrix} \quad |L_1 U_1| \neq 0$$

Generalized Inverse

Definition 1 For $A \in \mathbb{C}^{m \times n}$. X is called the generalized inverse of A, if $AX = P_{R(A)}$ — orthogonal projection

i.e. $(AX)^* = AX$ $XA = P_{R(X)}$ + $(XA)^* = XA$.

Definition 2 For $A \in \mathbb{C}^{m \times n}$, $X \in \mathbb{C}^{n \times m}$ is called a generalized inverse of A if X satisfies

(1) $AXA = A$

(2) $XAX = X$

(3) $(AX)^* = AX$

(4) $(XA)^* = XA$

The uniqueness of X

If $Y \in \mathbb{C}^{n\times m}$ s.t. $YAY = Y$, $AYA = A$, $(AY)^* = AY$, $(YA)^* = YA$.

Then $X = XAX = (XA)^* X = A^* X^* X$

$= A^* Y^* A^* X^* X = (YA)^* (XA)^* X = YAX$

$= YAYAX = Y (AY)^* (AX)^*$

$= Y (AXAY)^* = Y (AY)^* = YAY = Y.$

Denote such X by A^+

Existence of X

For $A \in \mathbb{C}^{m\times n}$, we have $A = V \Lambda U$

$$\Lambda = \begin{bmatrix} \sqrt{\lambda_1} & & & & \\ & \ddots & & & \\ & & \sqrt{\lambda_r} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}_{m\times n} \qquad V \in U(m),\ U \in U(n)$$

Then it is easy to check that

$$A^+ = U^* \begin{bmatrix} \sqrt{\lambda_1}^{-1} & & & & \\ & \ddots & & & \\ & & \sqrt{\lambda_r}^{-1} & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix}_{n\times m} V^*$$