Discrete Math (Honor) 2021-Fall Homework-7

Instructor: Xiang YIN
Due: 2021.12.03 Friday in Class
(Please use A4 paper. Do not use exercise book!)

Problem 1. *(8 Points)*
For any relations R and S. Prove that $R \circ S \subseteq S$ if and only if $t(R) \circ S \subseteq S$.

Answer:

- $\Rightarrow$, if $R \circ S \subseteq S$:
  For any $< x, z > \in t(R) \circ S$, exists some y s.t. $< x, y > \in t(R) \wedge < y, z > \in S$
  From $< x, y > \in t(R)$, we have two cases:
  - $< x, y > \in R$
  - Exists finite $t_1, t_2, ..., t_k (k \geq 1)$ s.t. $< x, t_1 > \in R \wedge < t_1, t_2 > \in R \wedge ... \wedge ... < t_k, y > \in R$.
    Then we have $< t_k, z > \in R \circ S$, and from $R \circ S \subseteq S$ we immediately get $< t_k, z > \in S$
    Similarly, $< t_{k-1}, z > \in S, ... < t_1, z > \in S, < x, z > \in S$
- $\Leftarrow$, if $t(R) \circ S \subseteq S$:
  For any $< x, z > \in R \circ S$, exists some y s.t. $< x, y > \in R \wedge < y, z > \in S$
  Then $< x, y > \in t(R)$, then $< x, z > \in t(R) \circ S$. Therefore, $< x, z > \in S$

Problem 2. *(5 Points)*
Let $A = \{a, b, c, d\}$ and $R = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle c, d\rangle, \langle d, c\rangle, \langle d, d\rangle\} \subseteq A \times A$ be a relation on A. Determine whether or not R is an equivalence relation; if so, determine the quotient set of A.

Answer:
Yes.
The quotient set of A:
$\{\{a, b\}, \{c, d\}\}$

Problem 3. *(15 Points)*
Let $R_1 \subseteq A \times A$ and $R_2 \subseteq A \times A$ be two non-empty equivalence relations on A. Determine whether or not each of the following relations is an equivalence relation. If so, prove it; otherwise, provide an counter example.

1. $(A \times A) - R_1$
2. $(R_1)^2$
3. $R_1 - R_2$
4. $r(R_1 - R_2)$
5. $t(R_1 \cup R_2)$

Answer:

1. No. Suppose $A = \{a\}, R = \{< a, a >\}$
2. Yes. Prove R_1^2 is symmetric, transitive and reflexive respectively

3. No. Suppose any $R_1 = R_2$

4. No. Suppose $A = \{a, b, c\}, R_1 = \{< a, a >, < b, b >, < c, c >< a, b >, < b, a >, < a, c >, < c, a >, < b, c >, < c, b >\}$, $R_2 = \{< a, a >, < b, b >, < c, c >, < a, b >, < b, a >\}$ then $R_1 - R_2 = \{< a, c >, < c, a >, < b, c >, < c, b >\}$
$r(R_1 - R_2) = \{< a, a >, < b, b >, < c, c >< a, c >, < c, a >, < b, c >, < c, b >\}$ is not transitive $(< b, c >, < c, a >\in r(R_1 - R_2))$

5. Yes.

Problem 4. *(8 Points)*
Let $A = \{a, b, c, d, e, f\}$ and $R = \{\langle a, b\rangle, \langle a, c\rangle, \langle a, f\rangle, \langle b, c\rangle, \langle c, d\rangle, \langle c, e\rangle, \langle c, f\rangle, \langle d, e\rangle, \langle e, f\rangle\} \subseteq A \times A$.
1. Determine if $s(r(R))$ is a tolerance relation.
2. Find all maximal tolerance classes of $s(r(R))$.

Answer:

1. Yes.

2. $\{a, b, c\}, \{a, c, f\}, \{c, d, e\}, and\ \{c, f, e\}$

Problem 5. *(8 Points)*
Let $\langle S, \leq_1\rangle$ be $\langle T, \leq_2\rangle$ be two posets. Prove that $\langle S \times T, \leq\rangle$ is also a poset, where $\leq$ is defined by: $\langle s, t\rangle \leq \langle u, v\rangle$ iff $s \leq_1 u$ and $t \leq_2 v$.

Answer:

- Reflexive: ... (obvious)

- Transitive: ... (easy)

- Anti-symmetric: For any s, t, u, v s.t. $< s, t >\leq< u, v >$ and $< u, v >\leq< s, t >$. we have $s \leq_1 u \wedge u \leq_1 s$ and also $t \leq_2 v \wedge v \leq_2 t$, so $s = u$ and $t = v$, and $< u, v >=< s, t >$.

Problem 6. *(15 Points)*
Read the following definitions carefully and then answer the questions.

Let $\langle A, \leq\rangle$ be a poset and $B \subseteq A$ be a subset. We say

- B is a **Chain** if any two elements in B are comparable. The number of elements in B is called the *length* of the chain;

- B is a **Anti-Chain** if any two elements in B are incomparable. The number of elements in B is called the *length* of the chain.

where "x and y are incomparable" means that "neither $x \leq y$ nor $y \leq x$".

Now, let $\langle 2^{\{a,b,c\}}, \subseteq\rangle$ be a poset. Then
1. Write down two different chains with length 4.
2. Write down an anti-chains with length 3.
3. The *lower set* of an anti-chain B is defined as $L_B = \{x : x \in A \wedge (\exists y \in B)(x \leq y)\}$. What is the lower set of the anti-chain you provided in the above problem.

Answer:

1. $\{\} \subseteq \{a\} \subseteq \{a, b\} \subseteq \{a, b, c\}$

2. $\{a, b\}, \{b, c\}, \{a, c\}$

3. $\{\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}$