Algorithm Design and Analysis

Assignment 3

Deadline: Nov 20, 2022

1. (25 points) You are given n jobs, where each job j has its processing time p_j and weight w_j. We need to process all of them on one machine. We use C_j to denote j's completion time in a schedule. Our goal is to find the best schedule that minimizes the average weighted completion time (i.e., min $\frac{\sum_{j=1}^{n} w_j C_j}{n}$). Design a greedy algorithm for it.

2. (20 points) You are given n prices $p_1, p_2, \cdots, p_n$, where p_i represents the i-th day price of a stock. On the i-th day, you are allowed to do one of the following operations:

   - Buy one unit of stock and pay p_i. Your stock will increase by one.
   - Sell one unit of stock and get p_i if your stock is at least one. Your stock will decrease by one.
   - Do nothing.

   How to maximize the profit (the money left after n days)?

   Professor Tao thinks the question is easy and designs the following greedy algorithm: We enumerate from day 1 to n and maintain a min-heap. In each iteration, we do two operations:

   1. We insert p_i into the heap.
   2. Let q be the minimized price in the heap. If $q < p_i$, we increase $profit$ by $p_i - q$, pop q from the heap, and insert p_i into the heap.

   Finally, Professor Tao claims $profit$ is the maximized profit we can get. Do you think Professor Tao is correct? If yes, prove the correctness. Otherwise, give a counterexample.

3. (30 points) There are n players who need cakes, whose requests are $x_1, x_2, \cdots, x_n$. You are asked to buy one piece of cake and then cut it to satisfy everyone's request. However, The "cut" operation is lossy. Whenever you make a "cut" operation to size W cake, you will lose the p fraction of the cake. (E.g., if you cut a cake at the median with size 2 and $p = 0.5$, you will get two pieces of cake with size 0.5, and the loss is 1. When you cut the 0.5 size cake again, you will lose 0.25 again.)

   (a) (5 points) There are four players where $x_1 = x_2 = x_3 = x_4 = 1$, and $p = 0.5$. what is the minimum size? Write down how you cut the cake. You don't need to prove it.

   (b) (5 points) There are four players where $x_1 = x_2 = 1$, $x_3 = x_4 = 3$, and $p = 0.5$. What is the minimum size? Write down how you cut the cake. You don't need to prove it.

   (c) (20 points) Given n players, their requests, and the loss factor p, design an efficient algorithm to find the minimum size of cake you need to buy.

4. (25 points) You are given a fractional number p/q, where p and q are two positive integers and $p < q$. Can you design a greedy algorithm to decompose it into the following form: $p/q = 1/a_1 + 1/a_2 + \cdots + 1/a_k$, where $a_1 < a_2 < a_3 < \cdots < a_k$ are all positive integers? (For example, $2/3 = 1/2 + 1/6$ and $2/3 = 1/2 + 1/7 + 1/42$ are both valid decomposition, while $2/3 = 1/3 + 1/3$ is not.) Remember to analyze the time complexity. Your algorithm should at least terminate for every input. Notice that if you succeed, you prove the decomposition always exists for all fractional numbers.

5. How long does it take you to finish the assignment (including thinking and discussing)? Give a score (1,2,3,4,5) to the difficulty. Do you have any collaborators? Write down their names here.