Chap 14

多元函数微分学

Chap 14 — 1

偏导数与全微分

14.1.1 偏导数

一. 定义

设 $f(x,y)$ 在 $U(P_0(x_0,y_0))$ 有定义. 仅给 x 以增量 Δx 相应有函数的增量(对 x 偏增量)

$$\Delta_x z = f(x_0+\Delta x, y_0) - f(x_0, y_0)$$

函数 f 在点 (x_0,y_0) 处对 x 的偏导数

$$f_x(x_0,y_0) \overset{\text{def}}{=} \lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0)-f(x_0,y_0)}{\Delta x}$$

◆ 偏导数也可记为 $\left.\dfrac{\partial f}{\partial x}\right|_{(x_0,y_0)}$

◆ 对变量y的偏导数类似;

◆ 可偏导: 两个偏导数都存在.

◆ 偏导(函)数: $f_x(x,y),\ f_y(x,y)$ or $\dfrac{\partial f(x,y)}{\partial x}, \dfrac{\partial f(x,y)}{\partial y}$

二. 偏导数的求法

$$\boxed{(1)\quad \left.\frac{\partial f}{\partial x}\right|_{(x_0,\ y_0)} = \left.\frac{\mathrm{d}f(x,y_0)}{\mathrm{d}x}\right|_{x=x_0},\quad \left.\frac{\partial f}{\partial y}\right|_{(x_0,\ y_0)} = \left.\frac{\mathrm{d}f(x_0,y)}{\mathrm{d}y}\right|_{y=y_0}}$$

(2) $f_x(x_0, y_0) = f_x(x, y)\Big|_{(x_0, y_0)}$, $f_y(x_0, y_0) = f_y(x, y)\Big|_{(x_0, y_0)}$

例 求函数 $z(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}$ 的偏导数 $z_x(0,1), z_y(0,1)$.

例 求函数$u = x^y$ $(x > 0)$的偏导数.

例 设 $f(x, y) = \mathrm{e}^{x+y}\left[x^{\frac{1}{3}}(y-1)^{\frac{1}{3}} + y^{\frac{1}{3}}(x-1)^{\frac{2}{3}}\right]$

求 $f_x(0,1)$, $f_y(0,1)$. (历年试题)

三. 连续与可偏导

- **可偏导未必连续**

例　考察　$f(x,y)=\begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

在(0,0)的情况.

- **连续未必可偏导**

例　考察 $f(x,y) = |x| + |y|$ 在(0,0)的情况.

四、偏导数的几何意义

曲面$z = f(x, y)$与平面$y = y_0$的交线

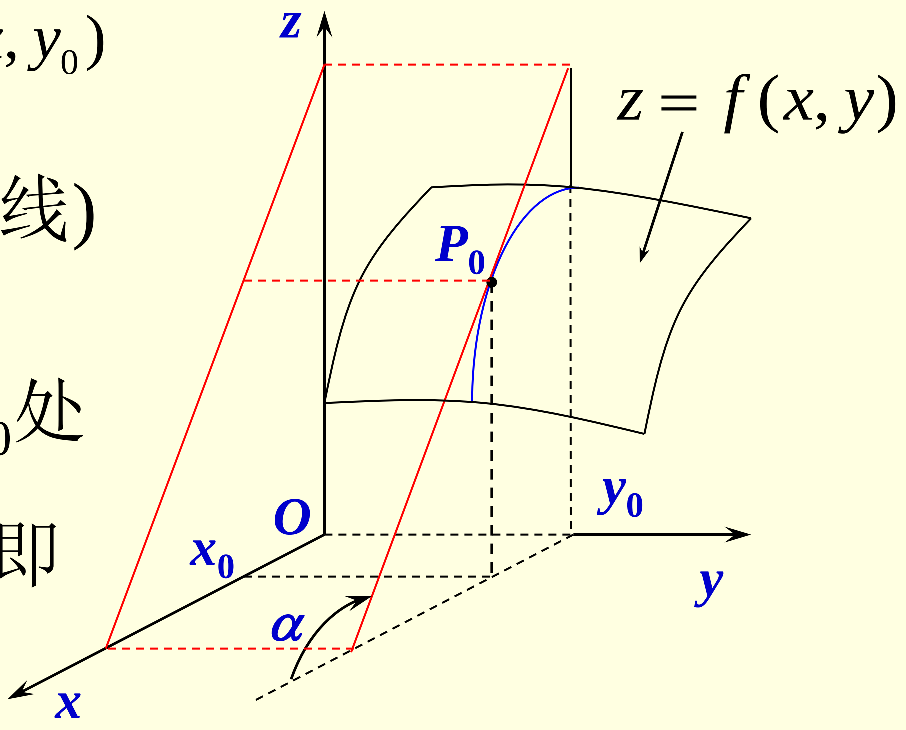
$$\begin{cases} z = f(x, y) \\ y = y_0 \end{cases} \Rightarrow \quad z = f(x, y_0)$$

(平面$y = y_0$上的曲线)

$f_x(x_0,y_0)$是该曲线在P_0处

的切线关于x轴的**斜率**. 即

$$\boxed{f_x(x_0, y_0) = \tan\alpha}$$



例 求曲线 $\begin{cases} z = \dfrac{x^2 + y^2}{4} \\ y = 4 \end{cases}$ 在点(2,4,5)处的切线及其

与x轴的夹角.

14.1.2 全微分

一、定义

一元情形: 若 $\Delta f = A\cdot\Delta x + o(\Delta x)$, 则称 f 在x_0可微,

并把线性主部$A\cdot\Delta x$称为 f 在x_0处的微分, 记为

$$\mathrm{d}f\big|_{x=x_0} = A\cdot\Delta x$$

二元情形：对函数$z=f(x,y)$, 若**全增量**

$$\Delta z = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$$

$$= A\cdot\Delta x + B\cdot\Delta y + o(\rho)$$

其中A, B是常数, $\rho=\sqrt{(\Delta x)^2+(\Delta y)^2}$, 则称 f 在(x_0,y_0)**可微**.

并把$A\cdot\Delta x + B\cdot\Delta y$称为 f 在(x_0,y_0)处的**全微分**. 记为

$$\boxed{\mathrm{d}z\Big|_{(x_0,y_0)} \equiv \mathrm{d}f\Big|_{(x_0,y_0)} = A\cdot\Delta x + B\cdot\Delta y}$$

若f 在区域D内处处可微, 则称f 是D内的**可微函数**.

二、可微、连续与可偏导

- **可微必连续**
- **可微必可偏导**, 且若

$$\boxed{\begin{aligned} &\mathrm{d}f\Big|_{(x_0,y_0)} = A\cdot\Delta x + B\cdot\Delta y \\ \Rightarrow\quad & f_x(x_0,y_0) = A,\ f_y(x_0,y_0) = B \end{aligned}}$$

全微分公式

$$\boxed{\mathrm{d}f(x,y)=f_x(x,y)\mathrm{d}x+f_y(x,y)\mathrm{d}y}$$

例 求函数 $z=x^y$ 在点(1,1)处的全微分.

例 求函数 $z=\arctan\frac{y}{x}$ 的全微分.

➢ 有连续偏导数必可微

结论

$$\text{偏导数连续} \Rightarrow \text{可微} \Rightarrow \begin{cases} \text{连续} \\ \text{可偏导} \end{cases}$$

例 考察函数

$$f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0), \\ 0, & (x,y)=(0,0)\end{cases}$$

在(0,0)处的可微性、其偏导数在(0,0)的连续性.

三、全微分的几何意义

因为 $\Delta z = z - z_0 = f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0)$

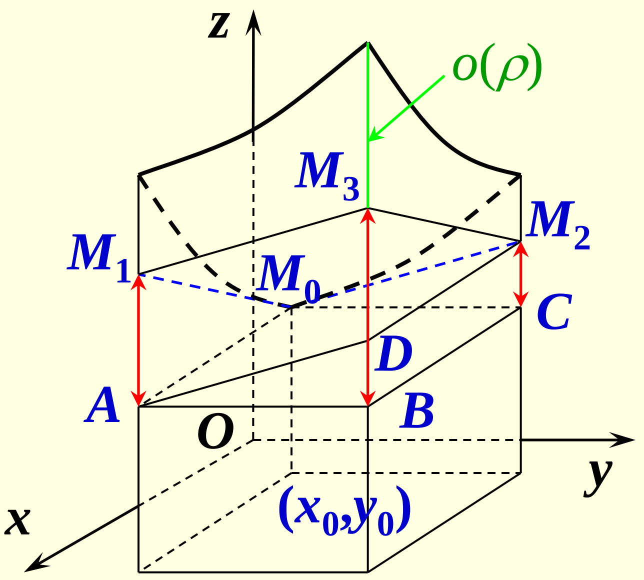
$$\mathrm{d}z\big|_{(x_0,y_0)} = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

切线 M_0M_1

$$\begin{cases} z - z_0 = f_x(x_0,y_0)(x-x_0) \\ y = y_0 \end{cases}$$

切线 M_0M_2

$$\begin{cases} z - z_0 = f_y(x_0,y_0)(y-y_0) \\ x = x_0 \end{cases}$$



故曲面 $z=f(x,y)$ 在 M_0 点有切平面

$$\boxed{z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)}$$

Chap14 — 2

复合函数微分法

14.2.1 复合函数的偏导数

定理 设 $u=u(x,y)$, $v=v(x,y)$在(x,y)可偏导, $z=f(u,v)$在相应的(u,v)处可微, 则复合函数$z=f(u(x,y),v(x,y))$在(x,y)处可偏导, 且

$$\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x},\quad \frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$$

- **链法则**

$$\begin{tikzcd} & u \arrow[r] \arrow[dr] & x \\ z: f \arrow[ur] \arrow[dr] & & \\ & v \arrow[r] \arrow[uur] & y \end{tikzcd}$$

$$z_x=f_u u_x+f_v v_x$$

$$z_y=f_u u_y+f_v v_y$$

- **矩阵形式**

$$(z_x \quad z_y)=(f_u \quad f_v)\begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}$$

➢ 推广 设向量值函数

$$\boldsymbol{f}(u,v)=\begin{pmatrix} f_1(u,v) \\ f_2(u,v) \end{pmatrix}, \qquad \boldsymbol{g}(x,y)=\begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix}$$

偏导数连续, 则 $\boldsymbol{f}\circ\boldsymbol{g}=\begin{pmatrix} z_1(x,y) \\ z_2(x,y) \end{pmatrix}$ 的**Jacobi矩阵**

$$\boldsymbol{D}_{\boldsymbol{f}\circ\boldsymbol{g}}(x,y)\overset{\text{def}}{=}\begin{pmatrix} \dfrac{\partial z_1}{\partial x} & \dfrac{\partial z_1}{\partial y} \\ \dfrac{\partial z_2}{\partial x} & \dfrac{\partial z_2}{\partial y} \end{pmatrix}=\begin{pmatrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_1}{\partial v} \\ \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_2}{\partial v} \end{pmatrix}\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}=\boldsymbol{D}_{\boldsymbol{f}}(u,v)\cdot\boldsymbol{D}_{\boldsymbol{g}}(x,y)$$

➢ **想一想** $\boldsymbol{f}$ 为m维k元, $\boldsymbol{g}$为k维n元向量值函数的情形

例 设函数$u = f(x, y, z)$可微，而$z = z(x, y)$可偏导.

求复合函数$u = f(x, y, z(x, y))$对x的偏导数.

例 设 $f(u,v) \in C^{(1)}$, $f(x,x^2) = x^3$, $f_u(x,x^2) = x^2 - 2x^4$,

求 $f_v(x,x^2)\ (x \neq 0)$. **(历年试题)**

例 设$f(x, y)$在$(0,0)$点可微，且$f(0,0) = 0$, $f_x(0,0) = 1$,

$f_y(0,0) = 2$. 求极限 $\lim\limits_{x \to 0}[1 + f(x,2x)]^{\frac{1}{x}}$

例 设 $z=f(x^2-y^2,\varphi(xy))$, f,φ 可微, 求 $x\dfrac{\partial z}{\partial x}-y\dfrac{\partial z}{\partial y}$.

(历年试题)

例 设函数$z=f(x,y)$可微, 作变换 $x=r\cos\theta, y=r\sin\theta$

试将 $\left(\dfrac{\partial z}{\partial x}\right)^2+\left(\dfrac{\partial z}{\partial y}\right)^2$

化为以r, θ为变量的形式.

14.2.2 一阶全微分形式的不变性

函数 $z = f(u, v)$的全微分

$$\mathrm{d}z = \frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v$$

若u, v又是x, y的可微函数$u = u(x, y), v = v(x, y)$, 则

复合函数 $z = f(u(x, y), v(x, y))$ 的全微分

$$\mathrm{d}z = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y$$

$$= \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}\right)\mathrm{d}x + \left(\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y}\right)\mathrm{d}y$$

$$=\frac{\partial f}{\partial u}(\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y)+\frac{\partial f}{\partial v}(\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y)$$

注意到 $\mathrm{d}u=\frac{\partial u}{\partial x}\mathrm{d}x+\frac{\partial u}{\partial y}\mathrm{d}y,\qquad \mathrm{d}v=\frac{\partial v}{\partial x}\mathrm{d}x+\frac{\partial v}{\partial y}\mathrm{d}y$

从而 $$\mathrm{d}z=\frac{\partial f}{\partial u}\mathrm{d}u+\frac{\partial f}{\partial v}\mathrm{d}v$$

因此, 对于函数$z=f(u,v)$, 无论u,v是自变量还是函数, 都有

$$\mathrm{d}z=\frac{\partial f}{\partial u}\mathrm{d}u+\frac{\partial f}{\partial v}\mathrm{d}v$$

全微分运算法则

(1) $\mathrm{d}(u \pm v) = \mathrm{d}u \pm \mathrm{d}v$

(2) $\mathrm{d}(uv) = v\mathrm{d}u + u\mathrm{d}v$

(3) $\mathrm{d}(\frac{u}{v}) = \frac{v\mathrm{d}u - u\mathrm{d}v}{v^2} \quad (v \neq 0)$

例 设 $z = \arctan\frac{y}{x}$，求z_x, z_y.

Chap14 — 3

高阶偏导数与全微分

14.3.1 高阶偏导数

$f(x,y)$在某邻域内的偏导数$f_x(x,y)$, $f_y(x,y)$的偏导数称为 f 的**二阶偏导数**. 记为

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right),\ f_{xy}=\frac{\partial^2 f}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{yx}=\frac{\partial^2 f}{\partial y\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right),\ f_{yy}=\frac{\partial^2 f}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

类似可定义三阶偏导数, 例如

$$f_{xxy}=\frac{\partial^3 f}{\partial x^2\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right)$$

例 求函数 $z=\ln x+\mathrm{e}^{y}\sin x$ 的所有二阶偏导数.

问题: 混合偏导数是否总与求导次序无关?

例 设 $f(x,y)=\begin{cases} xy, & |x|\geq|y| \\ -xy, & |x|<|y| \end{cases}$, 求 $f_{xy}(0,0)$, $f_{yx}(0,0)$.

分析 $$f_{xy}(0,0)=\lim_{y\to 0}\frac{f_x(0,y)-f_x(0,0)}{y}$$

$(y\neq 0)$ $$f_x(0,y)=\lim_{x\to 0}\frac{f(x,y)-f(0,y)}{x}$$

$$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}$$

定理 若$f(x,y)$的两个二阶混合偏导数在(x,y)连续, 则

$$f_{xy}(x,y)=f_{yx}(x,y)$$

例 函数 $z=f(xy,\dfrac{x}{y})$, f 有连续二阶偏导数, 求

$$\frac{\partial^2 z}{\partial x^2},\ \frac{\partial^2 z}{\partial x\partial y}.$$

例 设 $u=x-2y, v=x+3y$, 取u,v为新自变量, 变换方程

$$6\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\partial y}-\frac{\partial^2 z}{\partial y^2}=0.$$

14.3.2 高阶全微分

设 $f(x,y)$ 可微，则 $\mathrm{d}z=\dfrac{\partial z}{\partial x}\mathrm{d}x+\dfrac{\partial z}{\partial y}\mathrm{d}y\overset{\text{def}}{=}\left(\dfrac{\partial}{\partial x}\mathrm{d}x+\dfrac{\partial}{\partial y}\mathrm{d}y\right)z$

二阶全微分

$$\mathrm{d}^2z=\mathrm{d}(\mathrm{d}z)=\mathrm{d}\left(\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y\right)=\mathrm{d}\left(\frac{\partial z}{\partial x}\right)\cdot\mathrm{d}x+\mathrm{d}\left(\frac{\partial z}{\partial y}\right)\cdot\mathrm{d}y$$

$$=\left(\frac{\partial^2 z}{\partial x^2}\mathrm{d}x+\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}y\right)\mathrm{d}x+\left(\frac{\partial^2 z}{\partial y\partial x}\mathrm{d}x+\frac{\partial^2 z}{\partial y^2}\mathrm{d}y\right)\mathrm{d}y$$

$$=\frac{\partial^2 z}{\partial x^2}\mathrm{d}x^2+2\frac{\partial^2 z}{\partial x\partial y}\mathrm{d}x\mathrm{d}y+\frac{\partial^2 z}{\partial y^2}\mathrm{d}y^2\overset{\text{def}}{=}\left(\frac{\partial}{\partial x}\mathrm{d}x+\frac{\partial}{\partial y}\mathrm{d}y\right)^2 z$$

n阶全微分

$$\mathrm{d}^n z \overset{\text{def}}{=} \left(\frac{\partial}{\partial x} \mathrm{d}x + \frac{\partial}{\partial y} \mathrm{d}y \right)^n z$$

$$= \left(\sum_{k=0}^{n} C_n^k \frac{\partial^n}{\partial x^k \partial y^{n-k}} \mathrm{d}x^k \mathrm{d}y^{n-k} \right) z$$

注意 高阶全微分不再具有形式不变性！