Day 10.08

P80 4: 用确界原理证明致密性定理

证明：

设数列$\{x_n\}$为有界数列. 定义数集$A=\{x|\{x_n\}$中大于x的点有无穷多个$\}$.
$\because \{x_n\}$有界,$\therefore A$有上界且非空. 由确界原理可得,$\exists a$, 使得$a=supA$.
故$\forall \epsilon>0, a-\epsilon$不是$A$的上界, 则$\{x_n\}$中大于$a-\epsilon$的项有无穷多个.
故$(a-\epsilon,a+\epsilon)$中有$\{x_n\}$中无穷多项, 即$\forall\epsilon>0,\exists n>N$, 使得$x_n\in(a-\epsilon,a+\epsilon)$.
取$\epsilon=1,\exists n_1$, 使得$x_{n_1}\in(a-1,a+1)$, 即$|x_{n_1}-a|<1$.
取$\epsilon=\frac{1}{2},\exists n_2>n_1$, 使得$|x_{n_2}-a|<\frac{1}{2}$.
$\cdots\cdots$
取$\epsilon=\frac{1}{k},\exists n_k>n_{k-1}$, 使得$|x_{n_k}-a|<\frac{1}{k}$.
由此得到x_n的子列x_{n_k}，当$k\to\infty, x_{n_k}\to a$.

证毕

P57 2. (2)

证明：

$$\left|\frac{x^2+1}{3x^2-x+1}-\frac{1}{3}\right|=\left|\frac{x-2}{3(3x^2-x+1)}\right|<\left|\frac{x-1}{2x^2-x-1}\right|=\left|\frac{1}{2x+1}\right|<\left|\frac{1}{x}\right|$$

(注意到上方第一个不等式在$x>2$时恒成立)

故$\forall\epsilon>0,\exists X=max(2,\frac{1}{\epsilon})>0,\forall x\in(X,+\infty):\left|\frac{x^2+1}{3x^2-x+1}-\frac{1}{3}\right|<\left|\frac{1}{x}\right|<\epsilon$

证毕

P57 2. (3)

证明：

$$\left|\frac{1}{x}-\frac{1}{a}\right|=\left|\frac{x-a}{xa}\right|$$

$\because x\to a,\therefore$ 不妨设 $|x|\in(\left|\frac{a}{2}\right|,|2a|)$

$$\therefore\left|\frac{1}{x}-\frac{1}{a}\right|<\left|\frac{2(x-a)}{a^2}\right|$$

$$\therefore\forall\epsilon>0,\exists\delta=\frac{a^2\epsilon}{2},\forall x\in D\cap\mathring{U}(a,\delta):\left|\frac{1}{x}-\frac{1}{a}\right|<\left|\frac{2(x-a)}{a^2}\right|<\epsilon$$

证毕

P57 2. (4)

证明：

$$\left|\sqrt{x}-\sqrt{a}\right|=\left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right|$$

当$a=0$, 上式 $=\left|\sqrt{x}\right|$

则$\forall\epsilon>0,\exists\delta=\epsilon^2,\forall x\in D\cap\mathring{U}(0,\delta),\left|\sqrt{x}-\sqrt{a}\right|=\left|\sqrt{x}\right|<\epsilon$

当$a>0,\because x\to a$, 不妨$x\in(\frac{a}{4},4a)$

$$\therefore\left|\sqrt{x}-\sqrt{a}\right|=\left|\frac{x-a}{\sqrt{x}+\sqrt{a}}\right|<\left|\frac{x-a}{\frac{2}{3}\sqrt{a}}\right|$$

$$\therefore\forall\epsilon>0,\exists\delta=\frac{2}{3}\epsilon\sqrt{a},\forall x\in D\cap\mathring{U}(a,\delta),\left|\sqrt{x}-\sqrt{a}\right|<\left|\frac{x-a}{\frac{2}{3}\sqrt{a}}\right|<\epsilon$$

证毕

P 57 4. (2)

$$\lim_{x\to x_0^+} f(x) = \frac{\pi}{2}$$
$$\lim_{x\to x_0^-} f(x) = -\frac{\pi}{2}$$

Day 10.11

Ex 1.

证明：

$\because \lim_{n\to\infty} x_n = x_0, \therefore \forall\epsilon > 0, \exists N, \forall n > N, |x_n - x_0| < \epsilon$

$\because x_n < x_0, \therefore$ 上式可简化为$x_0 - x_n < \epsilon$

取$\epsilon = 1, \exists N_1, \forall n > N_1, x_0 - x_n < \epsilon = 1$

令$n_1 = N_1 + 1,$ 则 $|x_{n_1} - x_0| < \epsilon = 1$

取$\epsilon = \min(\frac{1}{2}, x_0 - x_{n_1}),$ 则$\exists N_2, \forall n > N_2, x_0 - x_n < x_0 - x_{n_1},$ 即$x_{n_1} < x_n$

令$n_2 = \max(n_1 + 1, N_2 + 1),$ 有$x_{n_2} > x_{n_1}$且 $|x_{n_2} - x_0| < \epsilon \le \frac{1}{2}$

取$\epsilon = \min(\frac{1}{3}, x_0 - x_{n_2}), \ldots\ldots$

由此得到$\{x_n\}$子列$\{x_{n_k}\}$，且由构造方式易知$\{x_{n_k}\}$满足：

$(1)\forall n_k, x_{n_k} < x_{n_{k+1}}, (2)\forall n_k, |x_{n_k} - x_0| < \frac{1}{k}$

则该子列严格递增，下证该子列收敛到x_0

$\forall\epsilon > 0, \exists K = [\frac{1}{\epsilon}] + 1, \forall k > K, |x_{n_k} - x_0| < \frac{1}{k} < \epsilon$

证毕

Ex 2.

证明：

$\forall 0 < \epsilon < b - a, \because f(x)$在$[a, b]$严格递增， $\therefore f(b-\epsilon) < f(b), f(b) - f(b-\epsilon) > 0$

$\because \lim_{n\to\infty} f(x_n) = f(b), \therefore \forall\delta > 0, \exists N, \forall n > N, |f(x_n) - f(b)| < \delta$

对上述$\epsilon,$ 取$\delta = f(b) - f(b-\epsilon),$ 则$\exists N_1, \forall n > N_1, |f(x_n) - f(b)| = f(b) - f(x_n) < \delta = f(b) - f(b-\epsilon)$

则$f(b-\epsilon) < f(x_n), \because f(x)$在$[a, b]$严格递增$, \therefore b - \epsilon < x_n$即 $|x_n - b| < \epsilon$

综上$, \forall 0 < \epsilon < b - a, \exists N_1, \forall n > N_1, |x_n - b| < \epsilon,$ 则 $\lim_{n\to\infty} x_n = b$

证毕