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On the tidal force of the Moon on the Earth

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Abstract
Here, first, the static formulation of tidal forces with completely spherical symmetry is reviewed (a differential calculus-based method is additionally introduced for readers with more familiarity with/interest in mathematical tools). Then, the result is generalized to the tidal force of the Moon acting on the Earth by considering the rotational dynamics and the (real) oblate spheroidal shape of the Earth.

1. Introduction

Clearly, human beings have been familiar with tidal forces for thousands of years and scientists have known the analytical forms of these forces for hundreds of years (after the discovery of Newtonian analytical mechanics). One of the applications of Newtonian general theory of gravity, among many other important applications/consequences/predictions and conclusions in analytical mechanics, celestial mechanics and almost all different fields of physics, is the analytical formulation of tidal forces.

Tidal forces, in addition to being of theoretical significance, are of great importance and application in our life and in a wide variety of studies. Almost all the introductory textbooks on astronomy contain some parts related to tides. The readers who do not have any previous familiarity with tidal forces are recommended to read at least one of these books before reading this work (one may also visit the internet sites [1–3]). A detailed discussion by emphasizing the usual misconceptions on tidal forces can be found in [4]. The paper in *American Journal of Physics* entitled 'A dynamical picture of the oceanic tides' [5] is also recommended. In what follows, among several other applications, some effects of tidal forces are introduced.

(1) While we are having high tides on the side of the Earth closest to the Moon, we also have high tides on the side of the Earth farthest from the Moon. The difference in distance, and thus the difference in gravitational pull, causes the same exact movement of water. So, there are two high tides every day.

(2) On the Earth's surface (especially in the oceans), lunar tides are approximately twice the solar tides and during the lunar month the strength of the tides changes by a factor of approximately 3 to 1. An especially small ocean tide on the Earth which occurs at the first or third quarter of a lunar month when the moon is at right angles to the Sun–Earth line is called a neap tide. Especially large ocean tides on the earth which occur at full or new moon when the Moon and Sun are collinear with the Earth are called spring tides.

(3) *Tidal friction/tidal locking*. One of the most interesting/important consequences of tidal forces is so-called gravitational (tidal) locking, which is the reason for the synchronization of the rotational and orbital motions of the Moon in the Earth–Moon system [6]. The Earth is slowing down (length of day increases approximately 0.002 s per century); the Moon is slowly moving away from the Earth. Eventually, the Earth will become tidally locked to the Moon.

(4) *Roche limit*. If a satellite is too close to a planet, the tidal forces can exceed the self-gravitation of the satellite (possible dissociation to smaller parts). The minimum distance from an object where this occurs depends on the density of the satellite (Roche limit) [7]. Saturn's rings, which are at distances less than this limit, may be considered as an (some) example(s).

(5) *The effects of tidal forces on atmospheric processes* [8]. Just as our Moon and Sun cause ocean tides, they also cause tides in the solid earth and in the atmosphere. There is a semi-diurnal tide (twice per day) which results from a combination of lunar and solar gravity, and the curved path in space the Earth follows as a result. There is also a once a day (diurnal) component which results when the Moon and Sun are not directly over the equator. Because the Moon revolves around the Earth once per lunar month, there is also a semi-monthly and monthly component to the tides. These components are more or less visible over the entire oceans depending on local influences such as the shape of shoreline, depth of local ocean and so forth. We expect to observe all these same tidal components in the atmosphere as well, but there are some differences. The atmosphere, being a fluid, is affected by the Moon, resulting in an atmospheric tide, a wave that propagates through the atmosphere. However, the increase in atmospheric pressure that occurs at the front edge of the wave is so slight that it is difficult to detect from the myriad of other waves that are always present in the atmosphere. An interesting effect of atmospheric tides is in the period of rotation of the planet Venus that has a much denser atmosphere than the Earth [9].

(6) Land masses also have a very small but detectable tidal effect. High precision gravitometers and distance measuring instruments are able to detect these land tides. In particular, new satellites that can measure the topology of the Earth show unmistakable 'ups and downs' due to the gravitational pull of the Moon. It has been hypothesized that these small shifts might be correlated with earthquakes and/or volcanic activity. Meanwhile, the effect of tidal forces on land masses is very noticeable for some of the other bodies in our solar system. For example, the tidal forces exerted by Jupiter on its moon Io (and by the other Jovian moons on Io) cause considerable heating of Io's interior, leading to volcanism and causing the surface of that moon to now look like a large cheese pizza.

(7) The effects of tidal forces on an elastic satellite in a closed orbit [10], on sea levels [11], may be considered as some other examples.

(8) Tidal forces are not restricted only to the height of the sea/ocean. Underwater earthquakes, slumps and volcanic eruptions can produce tidal waves called *tsunami*. Although these waves have low amplitude in the open sea/ocean, when they approach shallow water they

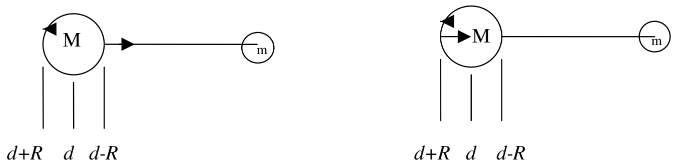

Figure 1. The Earth–Moon system: tidal forces at two equitorial points (not to correct relative scale).

suddenly break and change to high tides. The destructive *tsunami* of 26 December 2004 in the Indian Ocean may be considered as an example.

Gravitational forces are proportional to the inverse square of distances; tidal forces as differential gravitational forces should have a functional form of proportionality to the inverse cubic of distances. In the next section (the first subsection), a simple derivation of these forces is introduced. For readers more familiar with/interested in mathematical tools, a differential calculus-based derivation of tidal forces is introduced in subsection 2.2. In section 3, the tidal force corresponding to the Earth–Moon system by considering the rotational dynamics and the oblate spheroidal shape of the Earth is calculated.

2. Analytical formulation of tidal forces

2.1. Simple particular configurations

Tidal forces arise because gravitational force at surface points of a non-point mass (e.g. the Earth) differs from the force at its centre of mass. Before finding the general form of the tidal forces and as a simple particular example, consider the tidal forces at two equatorial points as in figure 1. The gravitational difference force (i.e. the tidal force) at any (both) of these points is a maximum. At both points it is directed away from the centre of M (the force is always parallel or anti-parallel to the direction to m).

The gravitational force of the mass m (e.g. the Moon) acting on elements of unit mass of M (e.g. the Earth) is $f(d) = -\frac{Gm}{d^2}$ (In fact, this is a force per unit mass of the Earth's elements). Thus, the gravitational difference (tidal) force T is

$$T = f(d \mp R) - f(d) = -\frac{Gm}{(d \mp R)^2} - \left(-\frac{Gm}{d^2}\right) \tag{1}$$

the binomial expansion of the term $\left(\frac{Gm}{(d\mp R)^2}\right)$ is

$$\left(\frac{Gm}{(d \mp R)^2}\right) = \left(\frac{Gm}{d^2}\right)\frac{1}{\left(1 \mp \left(\frac{R}{d}\right)\right)^2} = \left(\frac{Gm}{d^2}\right)\left(1 \mp 2\left(\frac{R}{d}\right) + O\left(\frac{R}{d}\right)^2\right) \tag{2}$$

where by $O\left(\frac{R}{d}\right)^2$ we mean the terms of second and higher order of $\left(\frac{R}{d}\right)$.

In the limit of $R \ll d$ (for the Earth–Moon system this is a very good approximation), the terms of second and greater order of $\left(\frac{R}{d}\right)$(i.e. $O\left(\frac{R}{d}\right)^2$ of equation (2)) are all negligible and thus it is enough to work with the tidal force T up to its first-order term ($T|1^{\mathrm{st}}$ order). Therefore, by means of (1) and (2), we find

$$T|1^{\mathrm{st}}\ \text{order} = \mp\frac{2GmR}{d^3}. \tag{3}$$

In the next subsection (for a more simple and geometrical approach, see [7]), the generalized form of the above relation corresponding to the configuration in figure 2 will be found.

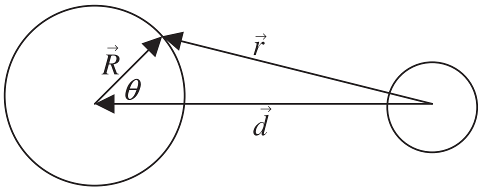

Figure 2. The Earth–Moon system (not to correct relative scale).

2.2. A differential calculus-based derivation of tidal forces for general configuration(s)

Tidal forces occur from differential gravity forces created on an object because of the difference in distance on either side of say, a comet from a planet or sun. The object under consideration experiences a ‘difference force’. Thus, the corresponding mathematical model of our problem is: what is the functional form of this ‘difference force’?

Because tidal forces do not uniquely correspond to gravitational forces and for simplicity, we work with forces of arbitrary functional forms having necessary mathematical conditions such as the differentiability of these functions with respect to their arguments (we know that the famous inverse square law of gravitational forces, except at the origin which is excluded from theory at first, has a differentiable functional form), we can simply consider gravitational forces (e.g. the Earth–Moon system).

In the language of differential calculus, the problem is translated as: if we know the function $f(\vec{x})$ at two known points $\vec{d}$ and $\vec{r}$, what will be the ‘difference function’ between these two points? (In the famous problem of the Earth–Moon system, $\vec{d}$ is a fixed vector between centre to centre of the Earth–Moon system (directed from the Moon to the Earth) and $\vec{r}$ is a variable vector from the centre of the Moon to any arbitrary point on the surface of the Earth. Clearly, $\vec{d} = \vec{r} - \vec{R}$ where $\vec{R}$ is the radial vector of the Earth (see figure 2)). Suppose the difference vector between the two points $\vec{d}$ and $\vec{r}$ be specified by vector $\vec{R}$; then

$$f(\vec{r}) - f(\vec{d}) = f(\vec{d} + \vec{R}) - f(\vec{d}). \tag{4}$$

By means of the differential calculus of three-dimensional functions (see the appendix), we have

$$f(\vec{d} + \vec{R}) = \mathrm{e}^{(\vec{R}\cdot\vec{\nabla}_d)} f(\vec{d}) \tag{5}$$

where the operator $\vec{\nabla}_d$ differentiates with respect to $\vec{d}$.

Insertion of (4) in (5) leads to

$$f(\vec{r}) - f(\vec{d}) = \mathrm{e}^{(\vec{R}\cdot\vec{\nabla}_d)} f(\vec{d}) - f(\vec{d}), \tag{6}$$

or

$$f(\vec{r}) - f(\vec{d}) = (\mathrm{e}^{(\vec{R}\cdot\vec{\nabla}_d)} - \hat{I}) f(\vec{d}). \tag{7}$$

Based on the above definition of the vectors $\vec{d}$ and $\vec{R}$, and according to figure 2, the angle between them is $\pi - \theta$. Thus

$$\vec{R}\cdot\vec{\nabla}_d = (\cos(\pi - \theta))R\frac{\partial}{\partial d} = -(\cos\theta)R\frac{\partial}{\partial d}. \tag{8}$$

The gravitational force of the moon acting on elements of unit mass of the Earth is $f(\vec{d}) = -\frac{GM_m}{d^2}$ (M_m is the mass of the Moon). Thus, by means of (7) and (8), we find the differential force T_θ (the tidal force of the Moon on an element of the Earth of unit mass at a latitude angle θ) as

$$T_\theta = \left\{\sum_{n=1}^{\infty} \frac{1}{n!}\left[-(\cos\theta)R\frac{\partial}{\partial d}\right]^n\right\}\left(-\frac{GM_m}{d^2}\right). \tag{9}$$

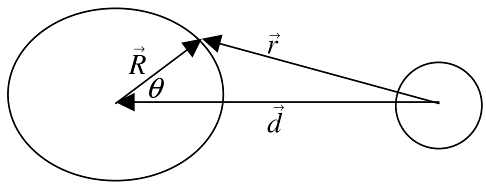

Figure 3. The Earth (oblate spheroidal)–Moon system (not to correct relative scale).

In the limit of $R \ll d$ (as explained in the previous subsection), it is enough to consider only the $n = 1$ term of the above sum and set aside the higher order negligible terms. So

$$T_\theta|1^{\text{st}} \text{ order} = -(\cos\theta)R\frac{\partial}{\partial d}\left(-\frac{GM_m}{d^2}\right) = \frac{-2GM_mR\cos\theta}{d^3}. \tag{10}$$

This is the generalized (static with complete spherical symmetry) form of (3) that is usually calculated via geometrical methods.

Clearly the above relation describes a static situation. To include the real three-dimensional configuration and rotational dynamics of the earth, we should enter the angle of longitude ϕ ($\phi = \omega t$, where ω is the angular velocity of the Earth's rotation). To do this, it is enough to write the dot product $(\vec{R} \cdot \vec{\nabla})$ in three dimensions as

$$\vec{R} \cdot \vec{\nabla}_d = (\cos\phi)(\cos(\pi - \theta))R\frac{\partial}{\partial d} = -(\cos\phi)(\cos\theta)R\frac{\partial}{\partial d} \tag{11}$$

where we have assumed that the angle of longitude of the Earth's surface point on the line joining centre to centre of the Earth–Moon is $\phi = 0$ at $t = 0$.

Therefore, the more complete form of (10) is

$$T_{(\theta,\phi)}|1^{\text{st}} \text{ order} = \frac{-2GM_mR(\cos\theta)(\cos\phi)}{d^3} = \frac{-2GM_mR(\cos\theta)(\cos\omega t)}{d^3}. \tag{12}$$

The term $(\cos\omega t)$ shows us that the tidal force (at any fixed point) becomes maximum/minimum twice a day.

3. The tidal force of the Moon on the rotating Earth with an oblate shape

We know that not only the planet Earth but also all other planets and their moons do not have complete spherical shapes for several reasons. The main reason for the Earth is that since it is not a perfect rigid body (its interior parts contain liquid metals of high densities), its rotation has made it elongated at its equatorial parts over hundreds of millions of years. Because of this fact, a coefficient of oblateness ε is defined as

$$\varepsilon = \frac{R_e - R_p}{R_e} \tag{13}$$

where R_e and R_p are the radii of the Earth at its equatorial and pole points, respectively. Full spherical symmetry occurs when $\varepsilon = 0$.

Now, we want to consider the tidal force of the Moon acting on the Earth with an oblate shape (see figure 3). Although one may study this problem using oblate spheroidal coordinates [12] and/or observe some more professionally (mathematically more advanced) related works (e.g. [13]), we can solve the problem by a simple modification of (12) by emphasizing the point that the radial vector of the earth $\vec{R}$ now changes not only in direction (as in the case of complete spherical symmetry) but also in its magnitude based on the relation

$$R = \sqrt{R_e^2 \cos^2\theta + R_p^2 \sin^2\theta}. \tag{14}$$

Therefore, we find the following expression for the first-order term of the tidal force of the Moon acting on unit mass element of the Earth with an oblate shape

$$T_{(\theta,\phi)}\Big|_{\text{oblate}}^{1^{\text{st}}\text{ order}} = \frac{-2GM_m\left(\sqrt{R_e^2\cos^2\theta + R_p^2\sin^2\theta}\right)(\cos\theta)(\cos\phi)}{d^3}, \tag{15}$$

or

$$T_{(\theta,\phi)}\Big|_{\text{oblate}}^{1^{\text{st}}\text{ order}} = \frac{-2GM_m\left(\sqrt{R_e^2\cos^2\theta + R_p^2\sin^2\theta}\right)(\cos\theta)(\cos\omega t)}{d^3}. \tag{16}$$

In terms of the coefficient of oblateness ε introduced above, we have

$$T_{(\theta,\phi)}\Big|_{\text{oblate}}^{1^{\text{st}}\text{ order}} = \frac{-2GM_m R_e(\sqrt{1-2\varepsilon\sin^2\theta + \varepsilon^2\sin^2\theta})(\cos\theta)(\cos\omega t)}{d^3}. \tag{17}$$

Clearly, we can recover (12) by letting $R_e \to R_p$ in (16) or $\varepsilon \to 0$ in (17) (complete spherical symmetry).

4. Conclusion

The result (16) may be considered as a two-step completed form of the relation (10). The first step is in observing the rotational dynamics of the Earth and the second is in considering the oblateness of it.

For the particular points at the equator ($\theta = 0, \pi$) and at the poles ($\theta = \pi/2, 3\pi/2$), both the results (12) and (16) have the same maximum and minimum magnitudes of $\left[\frac{-2GM_m R(\cos\omega t)}{d^3}\right]$ and 0 respectively. But of course, at the other points, there is a multiplicative factor of difference $\left[\sqrt{1-2\varepsilon\sin^2\theta+\varepsilon^2\sin^2\theta}\right]$ between the case of complete spherical symmetry (12) and that of oblate shape (17).

For example(s), for a point of latitude angle $\theta \approx 29°$ (somewhere near Bushehr beach on the Persian Gulf in Iran) there is a multiplicative factor of difference $\left[\sqrt{1-(2\varepsilon-\varepsilon^2)(0.235)}\right]$ and for a point of latitude angle $\theta \approx 45°$ (somewhere in New York), there is a multiplicative factor of difference $\left[\sqrt{1-(2\varepsilon-\varepsilon^2)(0.500)}\right]$.

In fact, the coefficient of oblateness for the Earth is small $\left(\varepsilon \approx \frac{1}{298}\right)$ [14] and the result (17) is only applicable to very accurate studies on the Earth–Moon system. The multiplicative factors of difference for the two above-noted examples are only about 0.998 and 0.997, respectively (i.e. the quantitative differences are of the order of 1/1000). Of course, the formulation used here can be easily applied to other systems of greater oblateness (e.g. the Jovian planets of our solar system that have ellipsoidal shapes with coefficients of oblateness greater than that of the Earth). For example, for Jupiter with a coefficient of oblateness $\varepsilon = 0.062$ [7] and at a point of latitude angle $\theta = 45°$ (considering the tidal effects of one of the moons of this planet), the multiplicative factor of difference between the two cases of complete spherical symmetry and the oblate shape is about 0.97 (i.e. the quantitative difference is of the order of 1/100).

We finish the work by mentioning a mathematical point. We know that the operator $\vec{\nabla}$ can be considered as a coordinate-free object (the general expansion/explicit form of this operator in generalized curvilinear systems of coordinates can be found in [15]); thus, the relation (7) is useful in solving the problems with different symmetries and coordinate systems. This may be considered as an advantage of the differential calculus approach of subsection 2.2 in comparison to the usual geometrical methods.

Appendix

Consider a three-dimensional function $f(\vec{x})$ with necessary conditions (e.g. differentiability) for having a Taylor expansion. We want to introduce a compact formula for the expansion of $f(\vec{x}+\vec{a})$ where $\vec{a}$ is an arbitrary vector of expansion. For simplicity, we work in Cartesian coordinates, but the final result is a coordinate-free formula and the generalization to other systems of coordinates is straightforward.

From the differential calculus of three-dimensional functions, we know

$$f(\vec{x}+\vec{a}) = f(x+a_x, y+a_y, z+a_z) = f(x,y,z) + \left[\left(a_x\frac{\partial}{\partial x}f\right) + \left(a_y\frac{\partial}{\partial y}f\right) + \left(a_z\frac{\partial}{\partial z}f\right)\right] + \frac{1}{2!}\left[\left(a_x\frac{\partial}{\partial x}f\right) + \left(a_y\frac{\partial}{\partial y}f\right) + \left(a_z\frac{\partial}{\partial z}f\right)\right]^2 + \text{higher order terms.} \tag{A.1}$$

Symbolically and in an operatorial form, we can write

$$\left(a_x\frac{\partial}{\partial x}f\right) + \left(a_y\frac{\partial}{\partial y}f\right) + \left(a_z\frac{\partial}{\partial z}f\right) = \left(a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z}\right) f = (\vec{a}\cdot\vec{\nabla})f \tag{A.2}$$

where $\vec{\nabla}$ is the gradient operator with Cartesian components $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Therefore, by means of (A.1), we find

$$\begin{aligned} f(\vec{x}+\vec{a}) &= f(x,y,z) + [(\vec{a}\cdot\vec{\nabla})]f + \left[\frac{1}{2!}(\vec{a}\cdot\vec{\nabla})^2\right] f + \text{higher order terms} \\ &= \left\{\hat{I} + (\vec{a}\cdot\vec{\nabla}) + \frac{1}{2!}(\vec{a}\cdot\vec{\nabla})^2 + \text{higher order terms}\right\} f = \left(\sum_{n=0}^{\infty}\frac{(\vec{a}\cdot\vec{\nabla})^n}{n!}\right) f \end{aligned} \tag{A.3}$$

where $\hat{I}$ is the unit operator.

Again symbolically and in an operatorial form, using the Maclurian expansion of the exponential function, we have

$$\left(\sum_{n=0}^{\infty}\frac{(\vec{a}\cdot\vec{\nabla})^n}{n!}\right) f = \mathrm{e}^{(\vec{a}\cdot\vec{\nabla})} f. \tag{A.4}$$

Thus, by means of (A.3) and (A.4), we find

$$f(\vec{x}+\vec{a}) = \mathrm{e}^{(\vec{a}\cdot\vec{\nabla})} f \tag{A.5}$$

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