Algorithm Design and Analysis (Fall 2022)
Final Exam (B)

1. (25 points) Given a set of n points $x_1, \ldots, x_n$ in $\mathbb{R}^1$, the objective is to use a minimum number of unit intervals (closed intervals with length 1, $[t, t+1]$) to cover all the n points. A point x is covered by the interval $[t, t+1]$ if $x \in [t, t+1]$. Design a polynomial time algorithm for deciding the minimum number of unit intervals needed to cover all the n points. Prove the correctness of your algorithm, and analyze its time complexity.

2. (25 points) Given an undirected edge-weighted graph $G = (V, E)$, two vertices $s, t \in V$, and $\theta \in \mathbb{Z}^+$, design a polynomial time algorithm to decide if G contains a minimum spanning tree such that s and t are connected by the tree edges with weights at most θ. Prove the correctness of your algorithm, and analyze its time complexity. You can assume the edge weights are positive integers.

3. (25 points) Given a directed edge-weighted graph $G = (V, E)$ (where the weights are integers and can be negative), two vertices s and t, and an integer k, the problem is to decide if there is a simple s-t path (an s-t path that does not visit a vertex more than once) with length exactly k.

   (a) (10 points) Prove that this problem is NP-complete.

   (b) (15 points) Suppose G is known to be a directed acyclic graph. Is this problem in P or still NP-complete? Prove your answer.

4. (25 points) Given a ground set $U = \{1, \ldots, n\}$ and a collection of k subsets $\mathcal{A} = \{A_1, \ldots, A_k\}$, a *system of distinct representatives* of $\mathcal{A}$ is a "representative" collection T of distinct elements from the sets in $\mathcal{A}$. Specifically, we have $|T| = k$, and the k *distinct* elements in T can be ordered as $u_1, \ldots, u_k$ such that $u_i \in A_i$ for each $i = 1, \ldots, k$. For example, $\{A_1 = \{2, 8\}, A_2 = \{8\}, A_3 = \{4, 5\}, A_4 = \{2, 4, 8\}\}$ has a system of distinct representatives $\{2, 4, 5, 8\}$ where $2 \in A_1, 4 \in A_4, 5 \in A_3, 8 \in A_2$, while $\{A_1 = \{2, 8\}, A_2 = \{8\}, A_3 = \{4, 8\}, A_4 = \{2, 4, 8\}\}$ does not have a system of distinct representatives.

   (a) (10 points) Design a polynomial time algorithm to decide if $\mathcal{A}$ has a system of distinct representatives.

   (b) (15 points) Given a ground set $U = \{1, \ldots, n\}$ and two collections of k subsets $\mathcal{A} = \{A_1, \ldots, A_k\}$ and $\mathcal{B} = \{B_1, \ldots, B_k\}$, a *common system of distinct representatives* is a collection T of k elements that is a system of distinct representatives of both $\mathcal{A}$ and $\mathcal{B}$. Design a polynomial time algorithm to decide if $\mathcal{A}$ and $\mathcal{B}$ have a common system of distinct representatives.

   For each part, prove the correctness of your algorithm, and analyze its time complexity.