Discrete Math (Honor) 2021-Fall Homework-4

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Due: 2021.10.29 Friday in Class

(Please use A4 paper. Do not use exercise book!)

Problem 1. *(5 Points)*
Provide a predicate formula that is saftifiable for domain of discourse $D_1 = \{1, 2, 3\}$ but is not saftifiable for domain of discourse $D_2 = \{1, 2\}$ or $D_2 = \{2, 3\}$.

Answer:

Note: unsatisfiable under domain D means unsatisfiable whatever the interpretation of a relation R (i.e. $I(R)$) is, including interpreting symbol "=" to "unequal", or interpreting symbol "<" to "larger than"

An example:
$(\forall x)(R(x, x) \to \mathbf{F}) \to (\forall x)(\forall y)(R(x, y) \to R(y, x) \to F) \to (\forall x)(\exists y)(\exists z)(R(x, y) \wedge R(y, z))$

Problem 2. *(12 Points)*
Determine if each of the following statements is correct; justify your answer.

1. $(\forall x)P(x) = \mathsf{F}$, if and only if, for any $x_0 \in D$, we have $P(x_0) = \mathsf{F}$.
2. $(\forall x)(P(x) \wedge Q(x)) = \mathsf{F}$, if and only if, for any $x_0 \in D$, we have $P(x_0) = \mathsf{F}$ and $Q(x_0) = \mathsf{F}$.
3. $P(a) \to Q(b) \Rightarrow (\exists x)(P(x) \to Q(x))$
4. $P(a) \to Q(b) \Rightarrow (\exists x)(\exists y)(P(x) \to Q(y))$

Answer

1. wrong
2. wrong
3. correct (discuss $P(a)$ and $P(b)$ respectively) (need excluded-middle axiom if using inference rules)
4. correct

Problem 3. *(12 Points)*
Write the prenex normal form for each of the following formulas.

1. $(\forall x)(\forall y)((\exists z)P(x, y, z) \wedge ((\exists u)Q(x, u) \to (\exists v)Q(y, v)))$
2. $(\neg(\exists x)F(x) \vee (\forall y)G(y)) \wedge (F(a) \to (\forall z)H(z))$

Answer:

1. $(\forall x)(\forall y)((\exists z)P(x, y, z) \wedge (\neg(\exists u)Q(x, u) \vee (\exists v)Q(y, v)))$
   $(\forall x)(\forall y)(\exists z)(P(x, y, z) \wedge (\neg(\exists u)Q(x, u) \vee (\exists v)Q(y, v)))$
   $(\forall x)(\forall y)(\exists z)(P(x, y, z) \wedge ((\forall u)\neg Q(x, u) \vee (\exists v)Q(y, v)))$
   $(\forall x)(\forall y)(\exists z)(P(x, y, z) \wedge (\forall u)(\neg Q(x, u) \vee (\exists v)Q(y, v)))$
   $(\forall x)(\forall y)(\exists z)(P(x, y, z) \wedge (\forall u)(\exists v)(\neg Q(x, u) \vee Q(y, v)))$
   $(\forall x)(\forall y)(\exists z)(\forall u)(\exists v)(P(x, y, z) \wedge (\neg Q(x, u) \vee Q(y, v)))$

2. $(\neg(\exists x)F(x) \vee (\forall y)G(y)) \wedge (F(a) \to (\forall z)H(z))$
   $((\forall x)\neg F(x) \vee (\forall y)G(y)) \wedge (\neg F(a) \vee (\forall z)H(z))$
   $(\forall x)(\forall y)(\neg F(x) \vee G(y)) \wedge (\forall z)(\neg F(a) \vee H(z))$
   $(\forall x)(\forall y)((\neg F(x) \vee G(y)) \wedge (\forall z)(\neg F(a) \vee H(z)))$
   $(\forall x)(\forall y)(\forall z)((\neg F(x) \vee G(y)) \wedge (\neg F(a) \vee H(z)))$

Problem 4. *(12 Points)*
Prove each of the following equivalences.

1. $(\forall x)P(x) \to q = (\exists x)(P(x) \to q)$
2. $(\forall y)(\exists x)((P(x) \to q) \vee S(y)) = ((\forall x)P(x) \to q) \vee (\forall y)S(y)$

Answer

1. $(\forall x)P(x) \to q$
   $= \neg(\forall x)P(x) \vee q$
   $= (\exists x)\neg P(x) \vee q$
   $= (\exists x)(\neg P(x) \vee q)$
   $= (\exists x)(P(x) \to q)$

2. $(\forall y)(\exists x)((P(x) \to q) \vee S(y))$
   $= (\forall y)((\exists x)(P(x) \to q) \vee S(y))$
   $= (\exists x)(P(x) \to q) \vee (\forall y)S(y)$
   $= (\exists x)(\neg P(x) \vee q) \vee (\forall y)S(y)$
   $= ((\exists x)(\neg P(x)) \vee q) \vee (\forall y)S(y)$
   $= (\neg(\forall x)P(x) \vee q) \vee (\forall y)S(y)$
   $= ((\forall x)P(x) \to q) \vee (\forall y)S(y)$

Problem 5. *(8 Points)*
Formalize the following inference and prove it.

Students in Discrete Math class either like studying or like playing games. Bob does not like playing games. Therefore, if Bob is a student in Discrete Math class, then he likes studying.

Answer Formalize:
$S(x)$: x is a student
$DM(x)$: x is in Discrete Math class
$LS(x)$: x likes studying
$LPG(x)$: x likes playing games
c_i: Bob

Conclusion:
$(\forall x)((S(x) \wedge DM(x)) \to (LS(x) \oplus LPG(x))) \wedge \neg LPG(c_i) \wedge S(c_i) \wedge DM(c_i) \vdash LS(c_i)$