Discrete Math (Honor) 2021-Fall Homework-6

Instructor: Xiang YIN
Due: 2021.11.26 Friday in Class
(Please use A4 paper. Do not use exercise book!)

Problem 1. *(5 Points)*
Write $\langle a, \langle b, c\rangle, d\rangle$ by set representation, e.g., $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$.

Answer: $< a, < b, c >, d >$
$= \{\{< a, < b, c >>\}, \{< a, < b, c >>, d\}\}$
$= \{\{\{\{a\}, \{a, < b, c >\}\}\}, \{\{\{a\}, \{a, < b, c >\}\}, d\}\}$
$= \{\{\{\{a\}, \{a, \{\{b\}, \{b, c\}\}\}\}\}, \{\{\{a\}, \{a, \{\{b\}, \{b, c\}\}\}\}, d\}\}$

Problem 2. *(8 Points)*
Let $R = \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\}$ be a relation. Determine each of the followings.

1. $R \circ R$ 2. R^{-1} 3. $R\restriction\emptyset$ 4. $R\restriction\{\emptyset\}$ 5. $R\restriction\{\emptyset, \{\emptyset\}\}$ 6. $R[\emptyset]$ 7. $R[\{\emptyset\}]$ 8. $R[\{\emptyset, \{\emptyset\}\}]$

Answer:
1. $= \{< \{\emptyset\}, \{\emptyset, \{\emptyset\}\} >\}$
2. $= \{\langle\{\emptyset, \{\emptyset\}\}, \emptyset, \rangle, \langle\emptyset, \{\emptyset\}\rangle\}$
3. $= \emptyset$
4. $= \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle\}$
5. $= R, \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\}$
6. $= \emptyset$
7. $= \{\{\emptyset, \{\emptyset\}\}\}$
8. $= \{\{\emptyset, \{\emptyset\}\}, \emptyset\}$

Problem 3. *(8 Points)*
Let $A = \{a, b, c, d\}$, $R_1 = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, d\rangle\} \subseteq A \times A$ and $R_2 = \{\langle a, d\rangle, \langle b, c\rangle, \langle b, d\rangle, \langle c, b\rangle\} \subseteq A \times A$.
Draw the relation graph for each of the following relations.

1.$R_1 \circ R_2$ 2.$R_2 \circ R_1$ 3.$R_1 \circ R_1$ 4.$R_2 \circ R_2$

Answer: Just draw...
1. $= \{< c, d >\}$
2. $= \{< a, d >, < a, c >\}$
3. $= \{< a, a >, < a, b >, < a, d >\}$
4. $= \{< b, b >, < c, c >, < c, d >\}$

Problem 4. *(8 Points)*
Let $A = \{a, b, c\}$ be a set.

1. Provide a relation $R \subseteq A \times A$ that is symmetric, anti-symmetric and transitive.
2. Provide a relation $R \subseteq A \times A$ that is not symmetric, not anti-symmetric and transitive.

Answer:
1. R = $\{< a, a >\}$
2. R = $\{< a, a >, < a, b >, < b, a >, < b, b >< b, c >, < a, c >\}$

Problem 5. *(12 Points)*
Let $A = \{1, 2, \ldots, 10\}$ be a set. Determine if each of the following relations is reflexive, inreflexive, symmetric, anti-symmetric or transitive. (You do not need to write down each relation explicitly)

1. $R_1 = \{\langle x, y\rangle : x + y = 10\}$
2. $R_2 = \{\langle x, y\rangle : x + y \text{ is odd}\}$
3. $R_3 = \{\langle x, y\rangle : x + y \text{ is even}\}$

Answer:
1. not reflexive, not in-reflexive, symmetric, not anti-symmetric, not transitive
2. not reflexive, in-reflexive, symmetric, not anti-symmetric, not transitive
3. reflexive, not in-reflexive, symmetric, not anti-symmetric, transitive

Problem 6. *(15 Points)*
For any relation $R \subseteq A \times A$, $R^n \subseteq A \times A$ is a new relation on A defined inductively by

$$R^0 := I_A, \text{ and for any } k \geq 0, \text{ we have } R^{k+1} := R^k \circ R$$

Then answer the following questions:

1. Let $A = \{a, b, c, d\}$ and $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, d\rangle\} \subseteq A \times A$. Compute R^2, R^3 and R^4. (You can show your result by relation graph)
2. Prove that $R^m \circ R^n = R^{m+n}$ for any n and m.
3. Prove that, if R is symmetric, then R^n is also symmetric for any n.

Answer:
1.
$R^2 = \{< a, c >, < b, b >, < b, d >, < c, c >\}$
$R^3 = \{< a, b >, < a, d >, < b, c >, < c, b >, < c, d >\}$
$R^4 = \{< a, c >, < b, b >, < b, d >, < c, c >\}$

2.
You should use mathematical induction Induction on n:
i. Base Step(n=0):
$R^m \circ R^0 = R^m$ (details omitted)
ii. Induction Step (suppose it was found when n = k, consider n = k+ 1):
$R^m \circ R^{k+1} = R^m \circ (R^k \circ R^1) = (R^m \circ R^k) \circ R^1 = (R^{m+k}) \circ R^1 = R^{m+k+1}$

3.
You should use mathematical induction Induction on n:
i. Base Step (n=0):
R^0 is symmetric (obviously) ii. Induction Step (n=k+1):
For any $< x, y > \in R^{k+1} = R^k \circ R$,
$\exists z, < x, z > \in R^k \wedge < z, y > \in R$
Then $\exists z, < z, x > \in R^k \wedge < y, z > \in R$
Then $< y, x > \in R \circ R^k = R^{1+k} = R^{k+1}$

Problem 7. *(15 Points)*
Let $R = A \times A$ be a relation on A. Prove the followings formally:

1. R is reflexive if and only if $I_A \subseteq R$.
2. R is inreflexive if and only if $I_A \cap R = \emptyset$.
3. R is transitive if and only if $R \circ R \subseteq R$.

Answer:

1. if R is reflexive, then for any $< x, x > \in I_A, < x, x > \in R$;
if $I_A \subseteq R$, then for any $x \in A, < x, x > \in I_A$, so $< x, x > \in R$

2. if R is in-reflexive, then suppose any $< x, x > \in I_A \cap R$, then $< x, x > \in R$, which is impossible;
if $I_A \cap R = \emptyset$, then suppose $< x, x > \in R$, since $< x, x > \in I_A$, $< x, x > \in I_A \cap R$, which is impossible;

3. if R is transitive, then for any $< x, y > \in R \circ R$, $\exists z, < x, z > \in R \wedge < z, y > \in R$
Then $< x, y > \in R$;
if $R \circ R \subseteq R$, then for any $< x, y >, < y, z > \in R$, $< x, z > \in R \circ R$. Therefore, $< x, z > \in R$