9.29

1. (a) $\nabla f(x) = \begin{pmatrix} 4x_1 + 2x_2 + 2 \\ 5x_2 + 2x_1 - 2x_3 - 3 \\ 6x_3 - 2x_2 + 2 \end{pmatrix} = 0 \qquad \begin{cases} x_1 = -1 \\ x_2 = 1 \\ x_3 = 0 \end{cases}$

$\nabla^2 f(x) = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 5 & -2 \\ 0 & -2 & 6 \end{pmatrix}$, we have $4 > 0$, $\begin{vmatrix} 4 & 2 \\ 2 & 5 \end{vmatrix} = 16 > 0$, $|\nabla^2 f(x)| = 80 > 0$ is positive definite

$x = (-1, 1, 0)^T$, x is local minimum

(b) $\nabla f(x) = \begin{pmatrix} x_1 + 2x_2 \\ 2x_2 + 2x_1 - x_3 + 1 \\ -3x_3 - x_2 - 3 \end{pmatrix} = 0 \qquad \begin{cases} x_1 = -2.4 \\ x_2 = 1.2 \\ x_3 = -1.4 \end{cases}$

$\nabla^2 f(x) = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 2 & -1 \\ 0 & -1 & -3 \end{pmatrix}$, $1 > 0$, $\begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} < 0$, is not positive/negative semidefinite

x is neither

2. $3 > 0$ and $\begin{vmatrix} 3 & -1 \\ -1 & 1 \end{vmatrix} > 0$. $|A| = 6 - 2\alpha - 2\alpha - 4 - 2 - 3\alpha^2 \geq 0$

$\alpha \in [-\frac{4}{3}, 0]$

3. for $x, y \in f^{-1}(C)$ and $\theta \in [0,1]$

C is convex: $\theta f(x) + \bar{\theta} f(y) \in C$

$\theta(Ax + b) + \bar{\theta}(Ay + b) = A(\theta x + \bar{\theta} y) + b \in C$

$f(\theta x + \bar{\theta} y) \in C$, $\theta x + \bar{\theta} y \in f^{-1}(C)$

$f^{-1}(C)$ is convex

4. for $x_1, x_1' \in C_1$, $x_2, x_2' \in C_2$, $\theta_1, \theta_2 \in [0,1]$

$\theta_1 x_1 + \bar{\theta}_1 x_1' \in C_1$, $\theta_2 x_2 + \bar{\theta}_2 x_2' \in C_2$

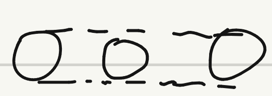
$x_1 + x_2 \in C_1 + C_2$, $x_1' + x_2' \in C_1 + C_2$, let $\theta_1 = \theta_2$

$\theta_1(x_1 + x_2) + \bar{\theta}_1(x_1' + x_2') = \theta_1 x_1 + \bar{\theta}_1 x_1' + \theta_2 x_2 + \bar{\theta}_2 x_2' \in C_1 + C_2$

$C_1 + C_2$ is convex

5. (a) $\forall x, y \in \operatorname{int} C$, $\exists r$, $B(x, r) \in \operatorname{int} C$, $B(y, r) \in \operatorname{int} C$ and are convex sets

C is convex, so for $\forall z \in B(\theta x + \bar{\theta} y, r)$, $z \in C$



since $\theta x + \bar{\theta} y$ is the centre of $B(\theta x + \bar{\theta} y, r)$ and $B(\theta x + \bar{\theta} y, r) \subset C$

(if exists)

$\theta x+\bar{\theta}y$ is not on the boundary of C, $\theta x+\bar{\theta}y \in \text{int } C$

so int C is convex

(b) \ref{wiki} \ref{pdf}

for x, y in $\bar{C}$, exists $\{x_n\}\{y_n\} \subset C$ s.t. $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$

C is convex. so $\{\theta x_n + \bar{\theta} y_n\} \subset C$, and $\lim_{n\to\infty} \theta x_n + \bar{\theta} y_n = \theta x + \bar{\theta} y$

since $\bar{C}$ is the closure of C, $\theta x+\bar{\theta}y \in \bar{C}$

$\bar{C}$ is convex.