王凯录 521030910356

2021级数学分析Ⅱ(荣誉)大作业

1. $xy''-3y'=x^2$, $y(1)=0$, $y'(1)=0$

$x^2y''-3xy'=x^3 \overset{x=e^t}{\Rightarrow} (D(D-1)-3D)y=e^{3t}$

特征方程 $r^2-4r=0$, $r=0$ 或 4

设特解 $y^*=c_3x^3$, $y=c_1+c_2x^4+c_3x^3$

$$\begin{cases} y(1)=c_1+c_2+c_3=0 \\ y'(1)=4c_2+3c_3=0 \\ x(12c_2x^2+6c_3x)-3(4c_2x^3+3c_3x^2)=x^2 \end{cases}$$ 解得 $$\begin{cases} c_1=\frac{1}{12} \\ c_2=\frac{1}{4} \\ c_3=-\frac{1}{3} \end{cases}$$

故特解 $y=\frac{1}{12}+\frac{1}{4}x^4-\frac{1}{3}x^3$

2. $y''-2y'-3y=5e^{-x}+\cos x$

$y''-2y'-3y=0 \Rightarrow r^2-2r-3=0$

$r=3$ 或 -1, 通解 $y=c_1e^{-x}+c_2e^{3x}$

设特解 $y_1^*=axe^{-x}$

$-4ae^{-x}=5e^{-x}$, $a=-\frac{5}{4}$

由 $\cos x$ 知 $\alpha=0$, $\beta=1$, 设 $y_2^*=c_3\cos x+c_4\sin x$

$(-4c_3-2c_4)\cos x+(2c_3-4c_4)\sin x=\cos x$

$c_3=-\frac{1}{5}$, $c_4=-\frac{1}{10}$

故解为 $y=(c_1-\frac{5}{4}x)e^{-x}+c_2e^{3x}-\frac{1}{5}\cos x-\frac{1}{10}\sin x$

3. 点 $M_1(0,1,-12)\in L_1$, $M_2(-7,5,9)\in L_2$

$\vec{s_1}=\begin{vmatrix} i & j & k \\ 2 & -2 & -1 \\ 1 & -1 & -1 \end{vmatrix}=i+j$, $\vec{s_2}=(3,1,-4)$

$\vec{s_1}\times\vec{s_2}=\begin{vmatrix} i & j & k \\ 1 & 1 & 0 \\ 3 & 1 & -4 \end{vmatrix}=-4i+4j-2k$

$d=\frac{|[\overrightarrow{M_1M_2},\vec{s_1},\vec{s_2}]|}{\|\vec{s_1}\times\vec{s_2}\|}=\frac{2}{\sqrt{36}}=\frac{1}{3}$

公垂线方向 $\vec{s_3}=\vec{s_1}\times\vec{s_2}=(-4,4,-2)$

L_1: $x=m$, $y=m+1$, $z=-12$, L_2: $x=3n-7$, $y=n+5$, $z=-4n+9$

$\frac{m-3n+7}{-4}=\frac{m+1-n-5}{4}=\frac{-12+4n-9}{-2} \Rightarrow m=\frac{163}{18}$, $n=\frac{95}{18}$

公垂线 $\frac{x-\frac{163}{18}}{4}=\frac{y-\frac{181}{18}}{-4}=\frac{z+12}{2}$ 或 $\begin{cases} x-y-4z-47=0 \\ 7x+11y+8z-78=0 \end{cases}$

4. 设点 $C\cdot(a-1,-2a+7,a)$，$\vec{CA}(a-1,-2a+5,a+4)$

$\vec{S}_{AB}(1,2,-2)$，L_{AB}: $\frac{x}{1}=\frac{y-2}{2}=\frac{z+x}{-2}$

$d=\frac{\|\vec{CA}\times\vec{S}\|}{\|\vec{S}\|}=\frac{(a-2)^2+261}{\|\vec{S}\|}$，$a=2$

故点 $C(1,3,2)$ 使面积最小

5. $\lim\limits_{(x,y)\to(0,5)}(y^2\ln(1+x)+e^x)^{\frac{1}{xy}}$

$=e^{\lim\limits_{(x,y)\to(0,5)}\frac{1}{xy}\ln(e^x(\frac{y^2}{e^x}\ln(1+x)+1))}$

$=e^{\lim\limits_{(x,y)\to(0,5)}\frac{1}{xy}\cdot(x+\ln(\frac{y^2}{e^x}\ln(1+x)+1))}$

$=e^{\lim\limits_{(x,y)\to(0,5)}\frac{1}{y}+\frac{1}{xy}\ln(\frac{y^2}{e^x}\ln(1+x)+1)}$

$=e^{\frac{1}{5}}\cdot e^{\lim\limits_{(x,y)\to(0,5)}\frac{1}{xy}\cdot\frac{y^2}{e^x}\ln(1+x)}$

$=e^{\frac{1}{5}}\cdot e^{\lim\limits_{(x,y)\to(0,5)}\frac{y}{x}e^x\cdot x}$

$=e^{\frac{1}{5}}\cdot e^{\lim\limits_{(x,y)\to(0,5)}ye^x}$

$=e^{\frac{1}{5}+5}=e^{\frac{26}{5}}$

6. 令 $r=x^2+y^2$，$\lim\limits_{r^2\to+\infty}f(x,y)=+\infty \Rightarrow$

$\forall M>0,\exists R\in\mathbb{R},\forall r>R,f_r(x,y)>M$. 不妨取 $M>f(0,0)$

$U(0,M)$ 为有界闭域，由最值定理，$f(x,y)$ 在 $P_0\in U(0,M)$ 处有最小值

且 $f(P_0)\le f(0,0)<M$，故 $f(P_0)$ 是 $\mathbb{R}^2$ 上的最小值

7. $\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)}{x^2+y^2}$ 存在 $\Rightarrow\lim\limits_{P\to(0,0)}f(P)=0$

由 $f(x,y)$ 连续，$f(0,0)=0$

故 $\lim\limits_{\substack{x\to0\\y\to0}}\frac{f(x,y)-f(0,0)}{\sqrt{x^2+y^2}}=\lim\limits_{\substack{x\to0\\y\to0}}C\sqrt{x^2+y^2}=0$.

即 $f(\Delta x,\Delta y)-f(0,0)=o(\rho)$，$f(x,y)$ 在 $(0,0)$ 处可微

8. $F(x,y)=e^{xy}-xy-2=0$

$\frac{dy}{dx}=-\frac{F_x}{F_y}=-\frac{ye^{xy}-y}{xe^{xy}-x}=-\frac{y}{x}$

$G(x,z)=e^x-\int_0^{x-z}\frac{\sin t}{t}dt$

$\frac{dz}{dx}=-\frac{G_x}{G_z}=-\frac{e^x-\frac{\sin(x-z)}{x-z}}{\frac{\sin(x-z)}{x-z}}=1-\frac{(x-z)e^x}{\sin(x-z)}$

$\frac{du}{dx}=\frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}\cdot\frac{y}{x}+\frac{\partial f}{\partial z}\left(1-\frac{(x-z)e^x}{\sin(x-z)}\right)$

9. $F(x,y,z) = xyz - \lambda$

$\vec{n} = (y_0 z_0, z_0 x_0, x_0 y_0)$

$G(x,y,z) = x^2+y^2+z^2-a^2$

$\vec{m} = (2x_0, 2y_0, 2z_0)$, $x_0 . y_0, z_0 \neq 0$

$\frac{y_0 z_0}{x_0} = \frac{z_0 x_0}{y_0} = \frac{x_0 y_0}{z_0} \Rightarrow x_0 = y_0 = z_0 = \sqrt{\frac{a^2}{3}}$

$F(\sqrt{\frac{a^2}{3}}, \sqrt{\frac{a^2}{3}}, \sqrt{\frac{a^2}{3}}) = 0$, $\lambda = \frac{a^2}{3}\sqrt{\frac{a^2}{3}} = \frac{\sqrt{3}}{9}a^3$

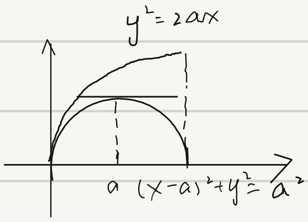
10. 设切点 (x_0, y_0, z_0). $\vec{n} = (\frac{1}{z_0}F_2 - \frac{y_0}{x_0^2}F_3, \frac{1}{x_0}F_3 - \frac{z_0}{y_0^2}F_1, \frac{1}{y_0}F_1 - \frac{x_0}{z_0^2}F_2)$

切平面 $(\frac{1}{z_0}F_2 - \frac{y_0}{x_0^2}F_3)(x-x_0) + (\frac{1}{x_0}F_3 - \frac{z_0}{y_0^2}F_1)(y-y_0) + (\frac{1}{y_0}F_1 - \frac{x_0}{z_0^2}F_2)(z-z_0) = 0$

化简得 $(\frac{1}{z_0}F_2 - \frac{y_0}{x_0^2}F_3)x + (\frac{1}{x_0}F_3 - \frac{z_0}{y_0^2}F_1)y + (\frac{1}{y_0}F_1 - \frac{x_0}{z_0^2}F_2)z = 0$

显然. 过 $(0,0,0)$ 原点.

11. $\int_0^{2a} dx \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy$



$= \int_0^a dy \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx$

$+ \int_0^a dy \int_{\sqrt{a^2-y^2}+a}^{2a} f(x,y)\,dx + \int_a^{2a} dy \int_{\frac{y^2}{2a}}^{2a} f(x,y)\,dx$

12. $\iint_D e^{\frac{x-2y}{x+2y}}\,dx\,dy$

令 $x-2y = u$, $x+2y = v$

原积分 $= \iint_{D'} e^{\frac{u}{v}} \cdot \frac{1}{4}\,du\,dv$

$= \int_0^4 dv \int_{-v}^{v} e^{\frac{u}{v}} \cdot \frac{1}{4}\,du$

$= \int_0^4 \frac{v}{4}(e-\frac{1}{e})\,du$

$= \frac{v^2}{8}(e-\frac{1}{e}) = 2(e-\frac{1}{e})$

13. $\iiint_\Omega x^2+5xy^2\sin\sqrt{x^2+y^2}\,dx\,dy\,dz$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

$= \int_1^4 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{2z}} r^3\cos^2\theta + 5r^4\cos\theta\sin^2\theta\sin r\,dr$

$= \int_1^4 dz \int_0^{2\pi} d\theta \int_0^{\sqrt{2z}} r^3\cos^2\theta\,dr = \int_1^4 dz \int_0^{2\pi} \frac{4z^2}{4}\cos^2\theta\,d\theta$

$= \int_1^4 \pi z^2\,dz = 21\pi$

14. $\iiint\limits_{\Omega} z\,dx\,dy\,dz = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{2\pi} d\theta \int_0^{2R\cos\varphi} \rho\cos\varphi \cdot \rho^2 \sin\varphi \, d\rho$

$= 2\pi \int_0^{\frac{\pi}{4}} \frac{\sin\varphi}{4} \cdot 16R^4 \cos^5\varphi \, d\varphi$

$= 2\pi \cdot \int_0^{\frac{\pi}{4}} 4R^4 \cos^5\varphi \, d\cos\varphi$

$= 8\pi R^4 \cdot \frac{7}{48}$

$= \frac{7}{6}\pi R^4$

15. (1) $F(0) = f(0)g(0) = 0$

$F'(x) = f'(x)g(x) + g'(x)f(x)$

$= f^2(x) + g^2(x)$

$= [f(x)+g(x)]^2 - 2f(x)g(x)$

$= 4e^{2x} - 2F(x)$

$F'(x) + 2F(x) = 4e^{2x}$

(2) $r+2=0$, $r=-2$

$F'(x) + 2F(x) = 0$ 通解为 $C_1 e^{-2x}$

设特解 $F^*(x) = ae^{2x}$

$(2a+2)e^{2x} = 4e^{2x}$, $a=1$, $F(0)=0 \Rightarrow C_1 = -1$

$F(x) = e^{2x} - e^{-2x}$

16. $F(x,y) = S(\sqrt{x^2+y^2}) = S(r)$

$F_x = S'(r)\frac{x}{r} = f'(x)$

$F_y = S'(r) \cdot \frac{y}{r} = g'(y)$

$\frac{x}{y} = \frac{f'(x)}{g'(y)}$

$\frac{f'(x)}{x} = \frac{g'(y)}{y}$, $\frac{f'(x)}{x} = \frac{g'(y_0)}{y_0} = \frac{f'(x_0)}{x_0} = \frac{g'(y)}{y} = C_1$

故 $f(x) = C_1x^2 + C_2$, $g(y) = C_1y^2 + C_3$

$F(x,y) = C_1(x^2+y^2) + C_2$

17. 令 $x = r\cos\theta$, $y = r\sin\theta$, $r \neq 0$

$\frac{\partial f}{\partial r} = \frac{\partial}{\partial r} f(r\cos\theta, r\sin\theta) = f_1 \cdot \cos\theta + f_2 \cdot \sin\theta$

$= \frac{1}{r}(f_1 x + f_2 y) = 0$

故 $f(x,y)=f(0)$. 而 $f(x,y)$ 连续

$\lim\limits_{r\to 0} f(r,0)=f(0,0)$，故 $f(x,y)=f(0,0)$

18.(1) 令 $F(x,y)=f(x,y)-g(x,y)$

由 $\lim\limits_{x^2+y^2\to 1} f(x,y)=+\infty$，$g(x,y)$ 有界

$\lim\limits_{x^2+y^2\to 1} F(x,y)=+\infty$，又 $F(x,y)$ 连续

$\forall M>\min f(x,y)$，$\exists\delta>0$，$\forall x^2+y^2>1-\delta$，$F(x,y)>M$

且在有界闭域 $U(0,1-\delta)$ 中，$F(x,y)$ 存在最小值，记点 (x_0,y_0) P_0

P_0 在 D 内部，必为极值点，即有

$$H(P_0)=\begin{pmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy}\end{pmatrix}$$ 正定，$F_{xx}>0$，$F_{yy}>0$，而

$$F_{xx}+F_{yy}=f_{xx}+f_{yy}-g_{xx}-g_{yy}$$
$$\le e^{f(P_0)}-e^{g(P_0)}<0$$，矛盾

于是必有 $F(x,y)\ge 0$，即 $f(x,y)\ge g(x,y)$

19. $\iint\limits_{D_\varepsilon} \frac{x}{x^2+y^2}\frac{\partial f}{\partial x}+\frac{y}{x^2+y^2}\frac{\partial f}{\partial y}\,dxdy$，由于 $\frac{\partial\frac{x}{x^2+y^2}}{\partial x}+\frac{\partial\frac{y}{x^2+y^2}}{\partial y}=0$，

$$=\iint\limits_{D_\varepsilon} \frac{\partial}{\partial x}\left(\frac{xf}{x^2+y^2}\right)+\frac{\partial}{\partial y}\left(\frac{yf}{x^2+y^2}\right)dxdy$$

$$\overset{\text{Green}}{=}\oint_{\partial D^+} -\frac{yf}{x^2+y^2}dx+\frac{xf}{x^2+y^2}dy-\frac{1}{x^2+y^2}\oint_{\partial D_\varepsilon^+} -yf\,dx+xf\,dy$$

$$=0+\frac{1}{\varepsilon^2}\oint_{\partial(D-D_\varepsilon)^+} yf\,dx-xf\,dy$$

$$=\frac{1}{\varepsilon^2}\iint_{\partial(D-D_\varepsilon)^+}\left(-\frac{\partial xf}{\partial x}-\frac{\partial yf}{\partial y}\right)dxdy$$

$$=-\frac{1}{\varepsilon^2}\iint_{\partial(D-D_\varepsilon)^+}(2f+xf_x+yf_y)\,dxdy$$

$$=-\pi\left(2f(P_\varepsilon)+\alpha f(P_\alpha)+\beta f(P_\beta)\right)$$，由积分中值定理

其中，当 $\varepsilon\to 0^+$ 时，$\alpha,\beta\to 0^+$

$P_\varepsilon, P_\alpha, P_\beta\to(0,0)$

故原式 $=-2\pi f(0,0)$

20. (1) 正定性 $P_0(\vec{x})=0 \Rightarrow P_0^2(x) = x_1^2+x_2^2+\cdots+x_n^2=0$，$\vec{x}\in\mathbb{R}_n$

故 $x_1=x_2=\cdots=x_n=0$，$\vec{x}=\vec{0}$

正齐次性：$P_0(a\vec{x}) = \sqrt{(ax_1)^2+(ax_2)^2+\cdots+(ax_n)^2}$

$= \sqrt{a^2(x_1^2+x_2^2+\cdots+x_n^2)}$

$= |a|\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$

$= |a|\,P_0(\vec{x})$

三角不等式：要证 $P(\vec{x}+\vec{y}) \le P(\vec{x})+P(\vec{y})$

即 $0 \le \sqrt{(x_1+y_1)^2+\cdots+(x_n+y_n)^2} \le \sqrt{x_1^2+x_2^2+\cdots+x_n^2}+\sqrt{y_1^2+y_2^2+\cdots+y_n^2}$

两边平方，消去平方项：$2x_1y_1+2x_2y_2+\cdots+2x_ny_n \le 2\sqrt{(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)}$

由柯西不等式：$\sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2 \ge (\sum_{i=1}^n x_iy_i)^2$

即 $|\sum_{i=1}^n x_iy_i| \le \sqrt{\sum_{i=1}^n x_i^2 \cdot \sum_{i=1}^n y_i^2}$

提 $x_1y_1+x_2y_2+\cdots+x_ny_n \le |\sum_{i=1}^n x_iy_i| \le \sqrt{(x_1^2+\cdots+x_n^2)(y_1^2+\cdots+y_n^2)}$

故 $P_0(\vec{x})$ 为范数

(2) 记 $\mathbb{R}^n$ 上任一固定点 $\vec{x}_0(a_1,a_2\cdots a_n)$，

$\forall \varepsilon>0$，$\exists \vec{x}_1=(x_1,x_2,\cdots x_n)$ 满足 $P_0(\vec{x}_1-\vec{x}_0)<\delta=\frac{\varepsilon}{P(\vec{E})}$，其中 $\vec{E}=(1,1\cdots 1)$

由三角不等式，$P(\vec{x}_1) \le P(\vec{x}_1-\vec{x}_0)+P(\vec{x}_0)$

$P(\vec{x}_1)-P(\vec{x}_0) \le P(\vec{x}_1-\vec{x}_0)$　　其中 $e_i=(0,0\cdots,\overset{\text{第}i\text{个}}{1},\cdots,0)$

$= P((x_1-a_1)e_1+(x_2-a_2)e_2+\cdots+(x_n-a_n)e_n)$

三角不等式 $\le P((x_1-a_1)e_1)+P((x_2-a_2)e_2)+\cdots+P((x_n-a_n)e_n)$

由 $P_0(\vec{x}_1-\vec{x}_0)<\delta$，$|x_i-a_i|<\delta$，故

上式 $<\delta P(\vec{E})<\varepsilon$

(3) 要证任意两个范数等价，不妨证都与 $\|\cdot\|$ 等价

取集合 $D=\{x\in\mathbb{R}^n : \|x\|=1\}$ 为有界闭集，由 $P(\vec{x})$ 连续

故任意 $P(\vec{x})$ 在 D 上有最值 $M\ge m>0$（由正定性）

$\forall \vec{x}\in\mathbb{R}^n$，取 $k=\max\{M,\frac{1}{m}\}+1>1$，

显然，$\frac{1}{k}P_0(\frac{\vec{x}}{\|\vec{x}\|}) \le P(\frac{\vec{x}}{\|\vec{x}\|}) \le k\,P_0(\frac{\vec{x}}{\|\vec{x}\|})$，$\frac{\vec{x}}{\|\vec{x}\|}\in D$

而由正齐次性，$\frac{1}{k}P_0(\vec{x}) \le P(\vec{x}) \le kP_0(\vec{x})$

故任意 $P(\vec{x})$ 与 $P_0(\vec{x})$ 等价，证毕

5.7 P40

1.(1) 收敛. 正确

若$\{f_n(x)\}$在(a,b)收敛, 则$\forall x\in D$, $f_n(x)\to f(x)$

$\forall [a,b]\subset D$, $\forall x\in[a,b]$, $f_n(x)\to f(x)$ $\Rightarrow$ 内闭收敛

内闭收敛, $\forall x\in D$, $\exists a,b\in D$ 且 $x\in(a,b)$. $f_n(x)\to f(x)$

由x任意性, $\{f_n(x)\}$在D上收敛

一致收敛. 不正确

$f_n(x)=x^n$, $\forall [a,b]\in(0,1)$. $\forall \varepsilon>0$, $\exists N=[\log_b\varepsilon+1]$

$0<a^n\le x^n\le b^n<\varepsilon\to 0$, $f_n(x)$内闭一致收敛

取点列 $1-\frac{1}{n}$, $\lim\limits_{n\to\infty}|f_n(x_n)-f(x_n)|=|\frac{1}{e}-1|\ne 0$

$f_n(x)$ 不一致收敛

(2) 正确.

若$\{f_n(x)\}$在$D_1\cup D_2$上一致收敛, 则 $\forall x\in D_1\cup D_2$, $f_n(x)\Rightarrow f(x)$

$\forall x\in D_1$, $x\in D_1\cup D_2$, $f_n(x)\overset{D_1}{\Rightarrow}f(x)$, 同理 $f_n(x)\overset{D_2}{\Rightarrow}f(x)$

若$\{f_n(x)\}$在D_1和D_2上一致连续, $\forall x\in D_1\cup D_2$, $x\in D_1$或$x\in D_2$, 故$f_n(x)\Rightarrow f(x)$

于是 $f_n(x)\overset{D_1\cup D_2}{\Rightarrow}f(x)$

(3) 错误, 考虑 $f_n(x)=x$, $g_n(x)=\frac{1}{n}$, $D=R$

显然 $f_n(x)\overset{D}{\Rightarrow}x$, $g_n(x)\overset{D}{\Rightarrow}0$

而 $\lim\limits_{n\to\infty}f_n(x)g_n(x)=\lim\limits_{n\to\infty}\frac{x}{n}=F(x)$ (假定收敛)

$F(n)=1\ne f_n(n)g_n(n)=0$

(4) 错误. 考虑 $u_n(x)=\frac{x^n}{n}$, $D=[0,1)$

$\lim\limits_{n\to\infty}u_n(x)=0$, $\forall\varepsilon>0$, $\exists N=[\frac{1}{\varepsilon}]+1$. $\forall n>N$. $f_n(x_n)-f(x_n)<\varepsilon$

而 $\lim\limits_{k\to\infty}\sum u_n(x)=0$. 取点列 $x_n=1-\frac{1}{n}$

$\sum u_n(x_n)>\frac{1}{4}\sum\frac{1}{n}$ 发散

2.(1) $1+n^5x^2\ge 2n^{\frac{5}{2}}|x|\ge 2n^{\frac{5}{2}}x$,

$\left|\frac{nx}{1+n^5x^2}\right| \le \left|\frac{1}{2n^{\frac{3}{2}}}\right|$，由P判别法.

$\lim\limits_{n\to\infty} n^{\frac{3}{2}}\cdot\frac{1}{2n^{\frac{3}{2}}}=1$. $\sum\frac{1}{2n^{\frac{3}{2}}}$ 收敛

故原级数一致收敛

(4) $f(x)=x\ln x$, $f'(x)=1+\ln x$.

$f(x)$ 在 $(0,\frac{1}{e})$ 减. $(\frac{1}{e},1)$ 增

提 $0<x\le 1$ 时. $\lim\limits_{x\to 0} f(x)=0$, $f(1)=0$

$f(\frac{1}{e})\le f(x)\le 0$, $f(\frac{1}{e})=-\frac{1}{e}>-\frac{1}{2}$

故 $\sum f(x)^n < \sum\left|-\frac{1}{2}\right|^n \overset{(0,1]}{\Rightarrow} 1$

故原级数一致收敛

3(2) $\frac{1}{n+\sin x}>0$, $\frac{1}{n+\sin x}-\frac{1}{n+1+\sin x}=\frac{1}{(n+\sin x)(n+1+\sin x)}>0$

$\lim\limits_{n\to\infty}\frac{1}{n+\sin x}=0$. 且 $\sum\limits_{n=1}^{\infty}(-1)^n$ 有界

故由D判别法，一致收敛

(4) $\exists\varepsilon=1$, $\forall N\in\mathbb{N}$, $\exists n=N+1$, $p=1$. $x_n=\frac{2}{3^{N+2}\pi}$

$f_{n+p}(x)-f_n(x) = x_{n+1}=2^{N+2}\cdot\sin\frac{\pi}{2} = 2^{N+2}>1$

故不一致收敛(Cauthy)

(6) (i) 若一致收敛. 由Cauthy, $\forall\varepsilon>0$, $\exists N\in\mathbb{N}$, $\forall n,p,x$:

$|u_{n+1}(x)+\cdots+u_{n+p}(x)|<\varepsilon$

现取 $N'=2N+2$, $n_0=N+1$, 对相应p.

$|u_{n+1}(x)+\cdots+u_{N'+1}(x)|<\varepsilon$ $\sin x+\sin 2x$

取 $x=\frac{\pi}{2N'}$, $u_{n_i}(x)=\frac{\sin n_i x}{n_i}$

$\sin n_i x > \sin\frac{(N+1)\pi}{2N'} > \sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

$\frac{\sin n_i x}{n_i} > \frac{\sqrt{2}}{2}\cdot\frac{1}{N'+1}$.

$|u_{n+1}(x)+\cdots+u_{N'+1}(x)| > \frac{\sqrt{2}}{2}=\varepsilon=\frac{1}{2}$，非一致收敛

(ii) $x\ge a$, $\left|\sum\limits_{n=1}^{\infty}\sin nx\right|\le\frac{1}{|\sin\frac{x}{2}|}\le\frac{1}{\sin\frac{a}{2}}$ 有界 且 $\frac{1}{n}$单调趋于0

由D判别法. 一致收敛

其中，$\sum \sin nx = \sin x + \sin 2x + \cdots + \sin nx$

$-2\sin\frac{x}{2}\sum \sin nx = \cos(x+\frac{x}{2}) - \cos(x-\frac{x}{2}) + \cos(2x+\frac{x}{2}) - \cos(2x-\frac{x}{2}) \cdots$

$= \cos(nx+\frac{x}{2}) - \cos(x-\frac{x}{2})$

绝对值，有 $|\sum \sin nx| \le \left|\frac{1}{\sin\frac{x}{2}}\right|$

6(1) $x \le b$，$\lim\limits_{n\to\infty} \ln(1+\frac{x}{n\ln^2 n}) = \frac{x}{n\ln^2 n} + o(\frac{1}{n^2})$

$\le \frac{b}{n\ln^2 n} + o(\frac{1}{n^2})$

$\sum\limits_{n=1} \frac{b}{n\ln^2 n}$ 收敛，由积分判别法 $\int \frac{1}{(1-p)\ln^{1-p}x}$

故一致收敛

P45 1.(1) $\sum\limits_{n=1}^{\infty} x^n \ln x = \begin{cases} \ln x \, \frac{1-x^n}{1-x} x & x \ne 1 \\ 0, & x=1 \end{cases}$

$\lim\limits_{x\to 1} \ln x \cdot \frac{1-x^n}{1-x} x = \lim\limits_{x\to 1} \ln x(1-x^n) + 1 - x^n - n x^n \ln x$

$= \lim\limits_{x\to 1} -n x^n \ln x$

$= 1 \ne 0$

若一致连续，与连续性定理矛盾

故不一致连续

2. $0 < \frac{1}{n^x} \le 1$ 且单调，且 $\sum\limits_{n=1}^{\infty} a_n$ 收敛

故 $\sum\limits_{n=1}^{\infty} \frac{a_n}{n^x}$ 一致收敛

于是知 $\sum\limits_{n=1}^{\infty} \frac{a_n}{n^x}$ 在 $(0, \varepsilon)$ 连续，

$\lim\limits_{x\to 0^+} \sum\limits_{n=1}^{\infty} \frac{a_n}{n^x} = \sum\limits_{n=1}^{\infty} \frac{a_n}{n^0} = \sum\limits_{n=1}^{\infty} a_n$

5.9

P45 1(2) $\sum_{n=1}^{\infty} x^n(\ln x)^2$, $x\in[0,1]$，x在0处无定义

$= \begin{cases} (\ln x)^2 \cdot \frac{1-x^n}{1-x} \times \xrightarrow[x\neq 1]{n\to\infty} (\ln x)^2 \frac{x}{1-x} \\ 0 \quad x=1 \end{cases}$

$\lim_{x\to 1^-} (\ln x)^2 \frac{x}{1-x} = \lim_{x\to 1^-} \frac{(\ln x)^2 + 2\ln x}{-1} = 0$，$f(x)$在$[0,1]$连续

$x^n(\ln x)^2$ 在$[0,1]$连续，且 $x^n(\ln x)^2 > 0$

由Dini定理知原级数在$[0,1]$一致连续

3. 1° $\sum_{n=1}^{\infty} x^n(1-x)$，$S_n(x) = 1-x^{n+1} \xrightarrow{n\to\infty} \begin{cases} 0, & x=1 \\ 1, & 0\le x<1 \end{cases}$ 绝对收敛

2° 由 $\sum_{n=1}^{\infty}(-1)^n \in \{-1,0,\}$ 有界，$f_n(x) = x^n - x^{n+1}$，$f_n'(x) = x^{n-1}(n-(n+1)x)$

$f_n(x) \le \left(\frac{n}{n+1}\right)^n\left(1-\frac{n}{n+1}\right) < \frac{1}{n+1}$，$\forall \varepsilon>0$，$\exists N = \left[\frac{1}{\varepsilon}\right]$，$\forall n, x$

$f_n(x) < \varepsilon$，$f_n(x)$ 一致收敛于0

由D判别法，可知一致收敛

3° 对 $\sum_{n=1}^{\infty} x^n(1-x)$，$S_n(x) = 1-x^{n+1}$，只需取点列 $x = \frac{1}{\sqrt[n+1]{2}}$，

$S_n(x_n) - S(x_n) = \frac{1}{2} > \varepsilon$ 取 $\frac{1}{3}$

故绝对收敛 一致收敛但不绝对一致收敛

4(1) 1° $\lim_{n\to\infty} \sqrt[n]{(x+\frac{1}{n})^n} = \lim_{n\to\infty} |x+\frac{1}{n}| < 1$，由根值判别，原级数收敛

2° 由于 $(x+\frac{1}{n})^n \in ((-1,1))$，而 $u(1) = e$，

若级数在$(-1,1)$一致连续，可知在 $x=1$处也连续（由10.1例5）

而 $u(1) \not\to 0$，矛盾，故$(0,1)$不一致连续

3° $\forall [a,b] \in (-1,1)$，$|u(x)| \le u(|a|>|b|?|a|:|b|)$ $(x\in[a,b])$

$u(a) = u(b) = 0$，由M判别法，$\sum_{n=1}^{\infty}(x+\frac{1}{n})^n$ 在$[a,b]$一致收敛

可知：由连续性定理，$f(x)$在$[a,b]$连续

由 a,b 任意性，$f(x)$在$(-1,1)$连续

5.(1) $\lim_{n\to\infty} n^\alpha x e^{-nx} = \begin{cases} 0 & x=0 \\ \lim_{n\to\infty} x\cdot \frac{n^\alpha}{e^{nx}}, & x>1 \end{cases}$

$\forall x>1$，$\lim_{n\to\infty} \frac{n^\alpha}{e^{xn}} = 0$，提 $\forall \alpha\in R$，均有 $f_n(x)$ 收敛

(2) 取点列 $x_n = \frac{1}{n}$

$|f_n(x_n) - f(x_n)| = n^{\alpha-1}e^{-1}$,

$\alpha < 1$ 时 $n^{\alpha-1}e^{-1} \to 0$,

$\alpha \geq 1$ 时 $n^{\alpha-1}e^{-1} \not\to 0$. 不一致收敛

下证 $\alpha < 1$ 时一致收敛.

$f_n'(x) = n^{\alpha}(1-nx)e^{-nx}$, $x = \frac{1}{n}$ 时 $f_n(x)$ 取最大值

$\lim\limits_{n\to\infty} \sup|f_n(x) - f(x)| = \lim\limits_{n\to\infty} n^{\alpha-1}e^{-1} = 0$, 一致收敛.

故 $\alpha < 1$

(3) 即证 $\lim\limits_{n\to\infty}\int_0^1 f_n(x)\,dx = \int_0^1 \lim\limits_{n\to\infty} f_n(x)\,dx$.

$\int_0^1 \lim\limits_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$

$\lim\limits_{n\to\infty}\int_0^1 f_n(x)\,dx = \lim\limits_{n\to\infty}\int_0^1 n^{\alpha}xe^{-nx} = \lim\limits_{n\to\infty} n^{\alpha}\left(\frac{1}{n^2} - \frac{1}{n}e^{-n} - \frac{1}{n^2}e^{-n}\right)$

$= \lim\limits_{n\to\infty} n^{\alpha-2}$,

$\alpha < 2$ 时 $\lim\limits_{n\to\infty} n^{\alpha-2} = 0$. $\alpha \geq 2$ 时 $\lim\limits_{n\to\infty} n^{\alpha-2} \neq 0$

故 $\alpha < 2$

8 对 $f_n(x)$, $\forall \varepsilon > 0$, $\exists \delta > 0$, $\forall |x'-x''| < \delta$, $|f_n(x') - f_n(x'')| < \frac{\varepsilon}{3}$

由 $f_n(x)$ 一致收敛于 $f(x)$, 可知

$\lim\limits_{n\to\infty}|f_n(x) - f(x)| \leq \lim\limits_{n\to\infty} \sup|f_n(x) - f(x)| < \frac{\varepsilon}{3}$

故有 $|f_n(x') - f(x')| < \frac{\varepsilon}{3}$, $|f(x'') - f_n(x'')| < \frac{\varepsilon}{3}$

于是 $|f(x'') - f(x')| \leq |f(x'') - f_n(x'')| + |f_n(x'') - f_n(x')|$

$+ |f_n(x') - f(x')| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$

故 $f(x)$ 在 R 上一致连续

10 (1) $\lim\limits_{n\to\infty}\sqrt{n}\,xe^{-nx^2} = 0$

$\lim\limits_{n\to\infty}\frac{\sqrt{n}\,xe^{-nx^2}}{n^2} = 0$. 由比较判别法, 级数收敛

取 $x_n = \frac{1}{\sqrt{n}}$, $u_n(x_n) = e^{-1}$.

$|u_n(x_n) - u(x_n)| = e^{-1} > \varepsilon$ 取 0.1

故不一致收敛

(2) $\forall [a,b] \subset (0,+\infty)$, $u_n'(x) = \sqrt{n}(1-2nx^2)e^{-nx^2}$, $x = \sqrt{\frac{1}{2n}}$ 时取最大值

$\lim\limits_{n\to\infty}\sqrt{\frac{1}{2n}}<a$，$\lim\limits_{n\to\infty}\sup|u_n(x)-u(x)|=\lim\limits_{n\to\infty}|u_n(a)-u_n(a)|$

$u_n(x)\in C[a,b]$ $=0$

由连续性定理，$[a,b]$ 上 $f(x)$ 连续

由 a,b 任意性 $f(x)\in C(0,+\infty)$

(3) $\sum\limits_{n=1}^{\infty}(\sqrt{n}xe^{-nx^2})'=\sum\limits_{n=1}^{\infty}\sqrt{n}(1-2nx^2)e^{-nx^2}$

$v_n'(x)=\sqrt{n}(4n^2x^3-6nx)e^{-nx^2}$

$\lim\limits_{n\to\infty}\sqrt{\frac{3}{2n}}<a$，$\lim\limits_{n\to\infty}\sup|v_n(x)-v(x)|$

$=\lim\limits_{n\to\infty}|v_n(a)-v(a)|$

$=0$，内闭一致连续

提导函数和 $g(x)$ 连续，而由逐项求导定理

$f'(x)=g(x)$ 且 $g(x)$ 连续

故可逐项求导，且连续

5.11 P46

4(2). $\sum\limits_{n=1}^{\infty}\left(\frac{x}{x^2+n^2}+\frac{n(-1)^n}{x^2+n^2}\right)$

其中 $\frac{x}{x^2+n^2}<\frac{x}{n^2}$，对 $\forall x$，$x\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ 收敛，知 $\frac{x}{x^2+n^2}$ 在 R 上点态收敛

而对于 $\frac{n(-1)^n}{x^2+n^2}$，$\sum\limits_{n=1}^{\infty}(-1)^n<2$ 有界 且 $\frac{n}{x^2+n^2}$ 单调 且 $\lim\limits_{n\to\infty}\frac{n}{x^2+n^2}<\lim\limits_{n\to\infty}\frac{1}{n}=0$

由D判别法，$\frac{n(-1)^n}{x^2+n^2}$ 在 R 上一致收敛

故原级数在 R 上收敛，而 $\forall N\in\mathbb{N}$，$\exists\varepsilon=\frac{1}{10}$，$\exists n=N+1$，$p=n$，$x=n$

$\left|\frac{n}{n^2+(n+1)^2}+\frac{n}{n^2+(n+2)^2}+\cdots+\frac{n}{n^2+(2n)^2}\right|\geqslant\left|\frac{n^2}{n^2+(2n)^2}\right|=\frac{1}{5}>\varepsilon$

故 $\frac{x}{x^2+n^2}$ 不一致收敛，即原级数不一致收敛

$\forall a>0$，由 $|x\sum\limits_{n=1}^{\infty}|\leqslant a\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$，$x\in[-a,a]$ 可知，$\frac{x}{x^2+n^2}$ 在 R 上内闭一致收敛

由连续性定理，和函数在 R 上连续

11. (1) 由连续性定理，$f(x)\in C[a,b]$，且由一致收敛

$\forall\varepsilon>0$，$\exists N\in\mathbb{N}$，$\forall n>N$，$\sup\limits_{x\in[a,b]}|f_n(x)-f(x)|<\varepsilon$

不妨记 $m=\inf\limits_{x\in[a,b]}|f(x)|$，取 $\varepsilon=\frac{m}{2}$，则 $\exists N_0$，

$\forall n>N_0$，$|f_n(x)-0|>m-\varepsilon=\frac{m}{2}>0$

$f_n(x)$ 无零点.

(2) $\forall \varepsilon>0$, 取 $\delta=\frac{m^2}{2}\varepsilon$,

由一致连续, $\exists N_0\in\mathbb{N}$, $\forall n>N$,

$$\sup_{x\in D}|f_n(x)-f(x)|<\frac{m^2}{2}\varepsilon=\delta$$

$$\sup_{x\in D}\left|\frac{1}{f_n(x)}-\frac{1}{f(x)}\right|=\sup_{x\in D}\frac{|f_n(x)-f(x)|}{|f_n(x)|\cdot|f(x)|}<\frac{\frac{m^2}{2}\varepsilon}{m\cdot\frac{m}{2}}=\varepsilon$$

故 $\frac{1}{f_n(x)}$ 一致收敛于 $\frac{1}{f(x)}$

12. 由 $|f_n(x)-f_n(y)|\leq K|x-y|$,

$\forall\varepsilon>0$, $\exists\delta=\frac{\varepsilon}{K}$. $\forall x,y$, $|x-y|<\delta$.

$|f_n(x)-f_n(y)|<\varepsilon$. $f_n(x)$ 一致连续

$f_n(x)\xrightarrow{n\to\infty}f(x)$, 则 $f(x)$ 一致连续

于是有 $\forall\varepsilon>0$, $\exists\delta=\frac{\varepsilon}{3K}$. $\forall|x-x'|<\delta$,

$|f_n(x)-f_n(x')|<\frac{\varepsilon}{3}$, $|f(x)-f(x')|<\frac{\varepsilon}{3}$

$$\forall x\in D,\ |f_n(x)-f(x)|\leq|f_n(x)-f_n(x')|+|f_n(x')-f(x')|+|f(x')-f(x)|$$
$$<\frac{2}{3}\varepsilon+|f_n(x')-f(x')|$$

由 $f_n(x)\xrightarrow{n\to\infty}f(x)$, 对于 x', $\exists N\in\mathbb{N}$ $\forall n>N$, $|f_n(x')-f(x')|<\frac{\varepsilon}{3}$

则对同一个 N. $|f_n(x)-f(x)|<\varepsilon$, 则 $\exists N_0\in\mathbb{N}$.

$\forall n>N_0$. $|f_n(x)-f(x)|<\varepsilon$, 一致收敛

14. $\forall\varepsilon>0$, $\exists N$, $\forall n.p>0$, $x\in D$.

$$|f_{n+p}(x)-f_n(x)|\leq|f^{(n+p)}(x)-f^{(n+p-1)}(x)|+\cdots+|f^{(n+1)}(x)-f^{(n)}(x)|$$
$$\leq\frac{1}{(n+p)^2}+\cdots+\frac{1}{(n+1)^2}$$

由 $\sum_{n=1}^{\infty}\frac{1}{n^2}$ 一致收敛与 Cauchy 准则, 知 $\frac{1}{(n+p)^2}+\cdots+\frac{1}{(n+1)^2}<\varepsilon$.

于是 $\{f^{(n)}(x)\}$ 一致收敛, 记收敛于 $f(x)$

又知 $\lim_{n\to\infty}|f^{(n)}(x)-f^{(n-1)}(x)|=0$,

$$\lim_{n\to\infty}\frac{df_n(x)}{dx}=\lim_{n\to\infty}f_n(x)$$

$f'(x)=f(x)$,

解之得 $f(x)=Ce^x$

P54

1.(5) $r=\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=1$，易知收敛半径为1

$|x|=1$时 由于$\sum\limits_{n=1}^{\infty}\frac{1}{n}$发散，知收敛域$(-1,1)$

(7) $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=\frac{1}{4}$，收敛半径为$\frac{1}{4}$，

$x=-\frac{1}{4}$，则 $\sum\frac{4^n+(-3)^n}{n}\cdot(-1)^n\frac{1}{4^n}$

$=\sum\frac{(-1)^n}{n}+\sum\frac{1}{n}\cdot(\frac{3}{4})^n$

其中，$\sum(\frac{-1}{n})^n=\sum\limits_{n=2k}\frac{1}{n(n-1)}$ 收敛

$\sum\frac{1}{n}(\frac{3}{4})^n<\sum(\frac{3}{4})^n$ 收敛

$x=-\frac{3}{4}$，$\sum\frac{1}{n}+\sum\frac{1}{n}(-\frac{3}{4})^n$

其中 $\sum\frac{1}{n}$发散

故收敛域 $[-\frac{3}{4},\frac{3}{4})$

(8) $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim\limits_{n\to\infty}\frac{\frac{n!}{n^n}}{\frac{(n+1)!}{(n+1)^{n+1}}}=(\frac{n+1}{n})^n=e$

收敛半径为e

$|x|=e$，$x^n\to\infty$

故收敛域$(-e,e)$

(12) 令$t=(x(1+x))^3$，$\sum\limits_{n=1}^{\infty}t^n$收敛，

则$|t|<1$，

$(x(1+x))^3<1$，

$x\in(-\frac{\sqrt5+1}{2},\frac{\sqrt5-1}{2})$，即收敛域

2(1) 若$a=-b$，则$R\to+\infty$

由Abel第一定理，不妨$r_1<r_2$

则$|x|<r_1$时$\sum a_nx^n$收敛

且$|x|<r_1<r_2$时，b_nx^n也收敛

故 $R\geq\min\{r_1,r_2\}$

5.16

P55

4(2) 记 $f(x)=\sum\limits_{n=1}^{\infty}\frac{x^{2n-1}}{n(2n-1)}$

$f'(x)=\sum\limits_{n=1}^{\infty}\frac{x^{2n-2}}{n}$

令 $g(x)=\sum\limits_{n=1}^{\infty}\frac{x^{2n}}{n}$

$g'(x)=2\sum\limits_{n=1}^{\infty}x^{2n-1}$

$=2\cdot\frac{x}{1-x^2},\ x\neq\pm1$

$g(x)=\int\frac{1}{1-x}-\frac{1}{1+x}$

$=-\ln(1-x)-\ln(1+x)$

$f'(x)=-\frac{\ln(1-x)+\ln(1+x)}{x^2}$

$f(x)=\frac{\ln(1-x^2)}{x}+\ln\frac{1+x}{1-x}$

$\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim\limits_{n\to\infty}\frac{(n+1)(2n+1)}{n(2n-1)}=1$

且 $x=\pm1$ 时，由 $\sum\frac{1}{n(2n-1)}$ 收敛知原级数收敛

$$f(x)=\begin{cases}\frac{\ln(1-x^2)}{x}+\ln\frac{1+x}{1-x}, & x\neq0\\ 0, & x=0\end{cases}$$，定义域 $[-1,1]$

(5) $\overline{\lim\limits_{n\to\infty}}\sqrt[n]{n^2}=1$，且 $x=\pm1$ 时发散，故定义域显然为 $(-1,1)$

令 $f(x)=\sum\limits_{n=1}^{\infty}n^2x^{n-1}$，$\sum\limits_{n=1}^{\infty}n^2x^n(-1)^{n-1}=xf(-x)$

$\int_0^x f(x)=n\int_0^x\sum\limits_{n=1}^{\infty}nx^{n-1}=\sum\limits_{n=1}^{\infty}nx^n$

令 $g(x)=\sum\limits_{n=1}^{\infty}nx^{n-1}$，$\int_0^x g(x)=\sum\limits_{n=1}^{\infty}x^n=\frac{x}{1-x}$

$g(x)=\frac{1}{(1-x)^2}$，$f(x)=\left(\frac{x}{(1-x)^2}\right)'$

$=\frac{1+x}{(1-x)^3}$

$S(x)=x\frac{1-x}{(1+x)^3}$，$D=(-1,1)$

(6) $\lim\limits_{n\to\infty}\frac{a_n}{a_{n+1}}=\lim\limits_{n\to\infty}\frac{(2n+1)(n+1)!}{(2n+3)(n)!}=+\infty$，故 $D=R$

令 $f(x)=\sum\limits_{n=1}^{\infty}\frac{2n+1}{n!}x^{2n}$

$\int_0^x f(x)=\sum\limits_{n=1}^{\infty}\frac{1}{n!}x^{2n+1}=x\sum\limits_{n=1}^{\infty}\frac{(x^2)^n}{n!}$

$=x(e^{x^2}-1)$

$f(x)=\left(x(e^{x^2}-1)\right)'=e^{x^2}-1+2x\cdot xe^{x^2}$

$=(2x^2+1)e^{x^2}-1$，$D=R$

(12) 记 $f(x)=\sum_{n=1}^{\infty}\frac{nx^{n-1}}{2^n}$. $\lim_{n\to\infty}\frac{n2^{n+1}}{(n+1)2^n}=2$，收敛半径 $(-2,2)$

$\int_0^x f(x)=\sum_{n=1}^{\infty}\frac{x^n}{2^n}$　　$x=\pm 2$ 显然发散

$=\sum_{n=1}^{\infty}(\frac{x}{2})^n$,

$=\frac{\frac{x}{2}}{1-\frac{x}{2}}$

$=\frac{x}{2-x}$

$f(x)=\frac{2}{(2-x)^2}$

$f(-1)=\frac{2}{9}$

(14) 记 $f(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n}$, $\overline{\lim_{n\to\infty}}:\sqrt[n]{\frac{1}{2n+1}}=1$

$g(x)=xf(x)$. $g'(x)=\sum_{n=0}^{\infty}(-1)^n x^{2n}$

$=\sum_{n=0}^{\infty}(-x^2)^n$

$=\frac{1}{1+x^2}$

$g(x)=\arctan x$

$f(x)=\frac{\arctan x}{x}$

$f(\frac{1}{\sqrt{3}})=\frac{\sqrt{3}}{6}\pi$

6. $\forall[a,b]\ (-\infty,+\infty)$. 由 a_nx^n 连续

由连续性定理，$f(x)$ 在 $[a,b]$ 上连续，且一致连续

$\forall\varepsilon>0, \exists\delta>0, \forall|x'-x''|<\delta$.

$|f(x')-f(x'')|<\varepsilon$

又：$f_n(x)$ 在 $[a,b]$ 上一致收敛

对 δ，$\exists N\in\mathbb{N}, \forall n>N$. $\sup_{x\in D}|f_n(x)-f(x)|<\delta$

于是 $\sup|f(f_n(x))-f(f(x))|<\varepsilon$

证毕

5.18

P60 1.

(1)
$$f(x) = x^2 \cdot \frac{1}{(x-\frac{3+\sqrt5}{2})(x-\frac{3-\sqrt5}{2})}$$
$$= x^2 \cdot \frac{1}{\sqrt5}\left(\frac{1}{x-\frac{3+\sqrt5}{2}} - \frac{1}{x-\frac{3-\sqrt5}{2}}\right)$$
$$= -x^2 \frac{1}{\sqrt5}\left(\frac{2}{3+\sqrt5}\cdot\frac{1}{1-\frac{2}{3+\sqrt5}x} - \frac{2}{3-\sqrt5}\frac{1}{1-\frac{2}{3-\sqrt5}x}\right)$$
$$= \frac{1}{\sqrt5}x^2\left(\frac{2}{3-\sqrt5}\sum_{n=0}^{\infty}\left(\frac{2}{3-\sqrt5}\right)^n x^n - \frac{2}{3+\sqrt5}\sum_{n=0}^{\infty}\left(\frac{2}{3+\sqrt5}\right)^n x^n\right)$$
$$= \frac{1}{\sqrt5}\left(\sum_{n=0}^{\infty}\left(\frac{2}{3-\sqrt5}\right)^{n+1}x^{n+2} - \sum_{n=0}^{\infty}\left(\frac{2}{3+\sqrt5}\right)^{n+1}x^{n+2}\right)$$

(3)
$$f^{(n)}(x) = (-1)^n (n+1)! \frac{1}{x^{n+2}}$$
$$f^{(n)}(1) = (-1)^n (n+1)!$$
$$f(x) = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)!}{n!}(x-1)^n$$
$$= \sum_{n=0}^{\infty}(-1)^n(n+1)(x-1)^n$$

(5)
$$f^{(n)}(x) = (-1)^{n-1}(n-1)!\, x^{-n}$$
$$f^{(n)}(2) = (-1)^{n-1}(n-1)!\, 2^{-n}$$
$$f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(n-1)!}{n!}2^{-n}(x-2)^n + \ln 2$$
$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n2^n}(x-2)^n + \ln 2$$

3.
$$f'(x) = \frac{1}{1+(\frac{2x}{1-x^2})^2}\cdot\frac{2+2x^2}{(1-x^2)^2}$$
$$= \frac{2}{1+x^2}$$
$$= 2\sum_{n=0}^{\infty}(-x^2)^n$$
$$f(x) = 2\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$$
故
$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \frac12\arctan 1 = \frac{\pi}{8}$$

4.
$$f(x) = \frac{-1}{x^2+x-1} = \frac{1}{\sqrt5}\left(\frac{1}{x-\frac{-1-\sqrt5}{2}} - \frac{1}{x-\frac{-1+\sqrt5}{2}}\right)$$
$$= \frac{1}{\sqrt5}\left(\frac{2}{-1+\sqrt5}\cdot\frac{1}{1-\frac{2}{-1+\sqrt5}x} - \frac{2}{-1-\sqrt5}\cdot\frac{1}{1-\frac{2}{-1-\sqrt5}x}\right)$$
$$= \frac{1}{\sqrt5}\sum_{n=0}^{\infty}\left(\left(\frac{2}{-1+\sqrt5}\right)^{n+1} - \left(\frac{2}{-1-\sqrt5}\right)^{n+1}\right)x^n = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n$$
故
$$f^{(n)}(0) = \frac{n!}{\sqrt5}\left(\left(\frac{2}{-1+\sqrt5}\right)^{n+1} - \left(\frac{2}{-1-\sqrt5}\right)^{n+1}\right)$$
$$\sum_{n=0}^{\infty}\frac{n!}{f^{(n)}(0)} = \sum_{n=0}^{\infty}\frac{\sqrt5}{\left(\frac{2}{-1+\sqrt5}\right)^{n+1} - \left(\frac{2}{-1-\sqrt5}\right)^{n+1}}$$
，其中，$\frac{2}{-1+\sqrt5} > 1$，$\frac{2}{-1-\sqrt5} < 1$

由根值判别法，收敛

5. (1) 令 $f(x)=\frac{1}{\sqrt{1-x^2}}=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^{2n}$, $-1<x<1$

$\int_0^x f(t)dt=\arcsin x=x+\sum_{n=1}^{\infty}\frac{1}{2n+1}\cdot\frac{(2n-1)!!}{(2n)!!}x^{2n+1}$, $-1<x<1$

$x\in[0,\frac{\pi}{2})$, $\sin x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{\sin^{2n+1}x}{2n+1}\xrightarrow{D}x$

当 $x=\frac{\pi}{2}$ 时, $1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\sim\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)}$ 显然收敛

故由 Abel 定理, $[0,\frac{\pi}{2}]$ 上 $\sin x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{\sin^{2n+1}x}{2n+1}=x$

(2) $\int_0^{\frac{\pi}{2}}\sin x+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\sin^{2n+1}x\,dx=\int_0^{\frac{\pi}{2}}x\,dx$

$1+\sum_{n=1}^{\infty}\int_0^{\frac{\pi}{2}}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\sin^{2n+1}x\,dx=\frac{\pi^2}{8}$

$1+\sum_{n=1}^{\infty}\frac{1}{2n+1}\frac{(2n-1)!!}{(2n)!!}\frac{(2n)!!}{(2n+1)!!}=\frac{\pi^2}{8}$

$1+\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$

即 $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$, $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$

$\sum_{n=1}^{\infty}\frac{1}{n^2}=\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}+\sum_{n=1}^{\infty}\frac{1}{(2n)^2}$

故 $\frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{8}$

$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

5.23 P65 1.

(1) $f(x)=x$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx = 0$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$=\frac{1}{\pi}\quad\frac{1}{n}\left(x\sin nx+\frac{\cos nx}{n}\right)\Big|_{-\pi}^{\pi}$

$=0$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx$

$=\frac{1}{\pi}\cdot\frac{1}{n}\left(\frac{\sin nx}{n}-x\cos nx\right)\Big|_{-\pi}^{\pi}$

$=\frac{(-1)^{n-1}}{n\pi}\cdot 2\pi$

$f(x)\sim\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}2\sin nx$

(3) $f(x)=\frac{x^2}{2}-\pi$

$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\,dx$

$=\frac{1}{\pi}\left(\frac{\pi^3}{3}-2\pi^2\right)$

$=\frac{\pi^2}{3}-2\pi$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx$

$=\frac{1}{\pi}\left(\int_{-\pi}^{\pi}\frac{x^2}{2}\cos nx\,dx+\frac{\pi\sin nx}{n}\Big|_{-\pi}^{\pi}\right)$

$=\frac{1}{n\pi}\left(\frac{x^2}{2}\sin nx+\frac{x\cos nx}{n}-\frac{\sin nx}{n^2}\right)\Big|_{-\pi}^{\pi}$

$=\frac{(-1)^n\cdot 2}{n}$

$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0$

$f(x)\sim\frac{\pi^2}{6}-\pi+\sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\cos nx$

5.25 P65

2.(2) 正弦级数:

$b_n=\frac{2}{\pi}\int_0^{\pi}e^{ax}\sin nx\,dx$

$=\frac{2}{\pi}\int_0^{\pi}\frac{\sin nx}{a}\,de^{ax}$

$=\frac{2}{\pi}\left(e^{ax}\frac{\sin nx}{a}\Big|_0^{\pi}-\int_0^{\pi}\frac{e^{ax}}{a}\,d\sin nx\right)$

$=-\frac{2}{\pi}\int_0^{\pi}\frac{n}{a}\cos nx\,e^{ax}\,dx$

$=-\frac{2}{\pi}\int_0^{\pi}\frac{n}{a^2}\cos nx\,de^{ax}$

$=-\frac{2}{\pi}\left(\frac{n}{a^2}\cos nx\cdot e^{ax}\Big|_0^{\pi}+\int_0^{\pi}\frac{n^2}{a^2}\sin nx\,e^{ax}\,dx\right)$

$$\frac{2}{\pi}\left(1+\frac{n^2}{a^2}\right)\int_0^\pi e^{ax}\sin nx\,dx = -\frac{2}{\pi}\frac{n}{a^2}\cdot\left((-1)^n e^{a\pi}-1\right)$$

$$\frac{2}{\pi}\int_0^\pi e^{ax}\sin nx\,dx = b_n = -\frac{n}{a^2+n^2}\,\frac{2}{\pi}\left[(-1)^n e^{a\pi}-1\right]$$

$$= \frac{n}{a^2+n^2}\left((-1)^{n+1}e^{a\pi}+1\right)$$

$$e^{ax}\sim \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{n}{a^2+n^2}\left[(-1)^{n+1}e^{a\pi}+1\right]\sin nx$$

余弦级数

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^\pi e^{ax}\,dx = \frac{e^{a\pi}-1}{a\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi e^{ax}\cos x\,dx$$

$$= -\frac{a^2}{n}b_n$$

$$e^{ax}\sim \frac{e^{a\pi}-1}{a\pi}+\sum_{n=1}^{\infty}\frac{a^2}{a^2+n^2}\left((-1)^n e^{a\pi}-1\right)\cos nx$$

(3) 正弦级数

$$\frac{2}{\pi}\int_0^\pi \sin ax\sin nx\,dx = \frac{1}{\pi}\int_0^\pi \cos(n-a)x - \cos(n+a)x\,dx$$

$$= \frac{1}{\pi}\left(\frac{\sin(n-a)\pi}{n-a} - \frac{\sin(n+a)\pi}{n+a}\right)$$

$$= \frac{1}{\pi}\frac{1}{n^2-a^2}\left((n+a)(-\cos n\pi\sin a\pi) - (n-a)\cos n\pi\sin a\pi\right)$$

$$= \frac{1}{\pi}\frac{1}{n^2-a^2}\,2n\sin a\pi\cdot(-1)^{n-1}$$

$$\sin ax\sim \frac{2\sin a\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{n^2-a^2}$$

余弦级数

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^\pi \sin ax\,dx = \frac{1}{a\pi}\int_0^\pi \sin ax\,dax = -\frac{1}{a\pi}\cos x\Big|_0^{a\pi} = \frac{1-\cos a\pi}{a\pi}$$

$$a_n = \frac{2}{\pi}\int_0^\pi \sin ax\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^\pi \sin(nx+ax) - \sin(nx-ax)\,dx$$

$$= \frac{1}{\pi}\,-\left(\frac{\cos(n+a)x}{n+a} - \frac{\cos(n-a)x}{n-a} - \frac{1}{n+a}+\frac{1}{n-a}\right)$$

$$= -\frac{1}{\pi}\frac{1}{n^2-a^2}\left((n-a)\cos n\pi\cos a\pi - (n+a)\cos n\pi\cos a\pi + 2a\right)$$

$$= \frac{1}{\pi}\frac{2}{n^2-a^2}\left(a\cos a\pi\cdot(-1)^n - a\right)$$

$$\sin ax\sim \frac{1-\cos ax}{a\pi} + \frac{2a}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n\cos a\pi-1}{n^2-a^2}\cos nx$$

P72/4 分段连续，知 $f(x)$ 分段可积，即 $f(x)$ 可积

$$\lim_{u\to 0}\frac{f(u)-f(-u)}{2\sin\frac{u}{2}} = \lim_{u\to 0}\frac{f(u)-f(0)+f(0)-f(-u)}{u} = f'_+(0) - f'_-(0)$$

故 $\frac{f(u)-f(-u)}{2\sin\frac{u}{2}}$ 在 $[-\pi,\pi]$ 可积和绝对可积。

故由 Riemann 引理，$\lim_{p\to\infty}\frac{1}{2}\int_0^\pi f(u)\,f(-u)\,\frac{\cos px}{\sin\frac{u}{2}}du=0$

故 $\lim\limits_{p\to+\infty}\int_{-\pi}^{\pi} f(x)\dfrac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}dt = \frac{1}{2}\lim\limits_{p\to+\infty}\int_0^{\pi}(f(x)-f(-x))\cdot\dfrac{\cos\frac{t}{2}-\cos pt}{2\sin\frac{t}{2}}$

$= \frac{1}{2}\int_0^{\pi}(f(x)-f(-x))\cdot\cot\frac{t}{2}\,dt$

5.30

P72

5. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{e^{\pi}-e^{-\pi}}{\pi}$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\cos nx\,dx$

$= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos nx\,de^x$

$= \frac{1}{\pi}\left(e^x\cos nx\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} e^x n\sin nx\,dx\right)$

$= \frac{1}{\pi}\left((-1)^n e^{\pi} - (-1)^n e^{-\pi} + e^x n\sin nx\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^x n^2\cos nx\,dx\right)$

$a_n = \frac{(-1)^n}{(n^2+1)\pi}(e^{\pi}-e^{-\pi})$

$nb_n = a_n - \frac{(-1)^n}{\pi}(e^{\pi}-e^{-\pi})$

$b_n = -\frac{n(-1)^n}{(n^2+1)\pi}(e^{\pi}-e^{-\pi})$

$f(x) \sim \frac{e^{\pi}-e^{-\pi}}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2+1}(\cos nx - n\sin x)\right]$

令 $\cos nx - n\sin x = (-1)^n$, $x=\pi$

$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{e^{\pi}-e^{-\pi}}{\pi}\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{1}{n^2+1}\right)$

$\sum_{n=1}^{\infty}\frac{1}{n^2+1} = \frac{\pi(e^{\pi}+e^{-\pi})-1}{2(e^{\pi}-e^{-\pi})}$

6. $a_0 = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}dx = 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\cos nx\,dx$

$= \frac{1}{2\pi}\int_0^{2\pi}\cos nx\,dx - \frac{1}{2\pi}\int_0^{2\pi} x\cos nx\,dx$

$= \frac{1}{2}\frac{\sin nx}{n}\Big|_0^{2\pi} - \frac{1}{2\pi}\left(\frac{x\sin nx}{n}+\frac{\cos nx}{n^2}\right)\Big|_0^{2\pi} = 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{\pi-x}{2}\sin nx\,dx$

$= \frac{1}{\pi}\left(-\frac{\cos nx}{2n}\Big|_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} x\sin x\,dx\right)$

$= \left(\frac{x\cos nx}{2n\pi} - \frac{\sin nx}{2n^2\pi}\right)\Big|_0^{2\pi}$

$= \frac{1}{n}$

$f(x) \sim \sum_{n=1}^{\infty}\frac{1}{n}\sin nx$

$\frac{\pi}{4} = \sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}$

7. (1) 奇函数，则 $a_n=0$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{0}-\frac{\pi}{4}\sin nx\,dx+\frac{1}{\pi}\int_{0}^{\pi}\frac{\pi}{4}\sin nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin nx\,dx$$

$$=\frac{2}{\pi}\cdot\frac{\pi}{4}\cdot-\frac{\cos nx}{n}\Big|_0^{\pi}$$

$$=\begin{cases}\frac{1}{n}, & n=2k-1\\ 0, & n=2k\end{cases}$$

$$f(x)\sim\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{2n-1}$$

① 令 $x=\frac{\pi}{2}$，$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}\cdots$

② $\frac{\pi}{12}=\frac{1}{3}-\frac{1}{9}+\frac{1}{15}\cdots$，与①相加

$\frac{\pi}{3}=1+\frac{1}{5}-\frac{1}{7}-\frac{1}{11}+\cdots$

③ 令 $x=\frac{\pi}{3}$，$\frac{\sqrt{3}}{6}\pi=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}\cdots$

P78 4. (1) $\int_0^x t\,dt=2\sum_{n=1}^{\infty}\int_0^{x}(-1)^{n+1}\frac{\sin nt}{n}dt$

$$\frac{x^2}{2}=2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^x\sin nt\,dt$$

$$x^2=4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(-\frac{\cos nt}{n}\Big|_0^x\right)$$

$$=4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$$

$$=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$\int_0^x t^2dt=\int_0^x\frac{\pi^2}{3}dt+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^x\cos nt\,dt$$

$$\frac{x^3}{3}=\frac{\pi^2}{3}x+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot\frac{\sin nt}{n}\Big|_0^x$$

$$=\frac{\pi^2}{3}2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}+4\sum_{n=1}^{\infty}\frac{(-1)^n\sin nx}{n^3}$$

$$=\sum_{n=1}^{\infty}(-1)^{n+1}\left(\frac{2\pi^2}{3n}-\frac{4}{n^3}\right)\sin nx$$

$$x^3=\sum_{n=1}^{\infty}(-1)^n\left(\frac{12}{n^3}-\frac{2\pi^2}{n}\right)\sin nx \qquad \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}=\frac{\pi^2}{12}$$

$$\int_0^x t^3dt=2\sum_{n=1}^{\infty}(-1)^n\left(\frac{6}{n^3}-\frac{\pi^2}{n}\right)\int_0^x\sin nt\,dt \qquad \sum_{n=1}^{\infty}(-1)^n\frac{1}{n^4}=-\frac{7}{720}\pi^4$$

$$\frac{x^4}{4}=2\sum_{n=1}^{\infty}(-1)^n\left(\frac{6}{n^3}-\frac{\pi^2}{n}\right)\left(-\frac{\cos nx}{n}+\frac{1}{n}\right)$$

$$=2\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi^2}{n^2}-\frac{6}{n^4}\right)\cos nx+\sum_{n=1}^{\infty}(-1)^n\frac{6}{n^4}+\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\pi^2}{n^2}$$

$$x^4=\frac{\pi^4}{5}+8\sum_{n=1}^{\infty}(-1)^n\left(\frac{\pi^2}{n^2}-\frac{6}{n^4}\right)\cos nx$$

(2) 上有 $\frac{x^3}{3}=\frac{\pi^2}{3}x+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^3}\sin nx$

积分之，$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}(\cos x-1)=\frac{2\pi^2x^2-x^4}{48}$

令 $x=\pi$，$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}((-1)^n-1)=\frac{\pi^4}{48}$

化简有 $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^4} = \frac{\pi^4}{96}$

而 $\sum_{n=1}^{\infty} \frac{1}{(2n)^4} + \frac{\pi^4}{96} = \sum_{n=1}^{\infty} \frac{1}{n^4}$,

$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$, $\sum_{n=1}^{\infty} \frac{1}{(2n)^4} = \frac{\pi^4}{2^4 90}$

故 $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^4} = \frac{7}{720}\pi^4$