discrete math

- A set is a collection of objects:

exp: $B = \{1, 2, 3, 4\}$

$\mathbb{Z}$ = integers = $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$

- Elements of a set:

exp: $3 \in \mathbb{Z}$ => 3 is an Element of integers set.

Element

$\pi \notin$ ~~in~~ $\mathbb{Z}$ => π is not an Element of integer set.

not an Element

- order and Repetition don't matter in a set.

exp: $\{1, 2, 3\} = \{3, 1, 2\} = \{1, 1, 2, 2, 3\}$

- subsets: A is a subset of B, if all elements in A are also in B

subset

exp: $\{2, 4\} \subseteq \{1, 2, 4\}$

$\{1,2,3\} \not\subseteq \{1,4,3,6\}$

not a sub set.

- set-Roster notation: means very large number of Elements in our set

exp: $\{0,2,4,\dots\}$

- set-Builder notation:

exp1: $\{ \underbrace{x}_{\text{variable}} \underbrace{|}_{\text{suchthat}} \underbrace{P(x)}_{\text{property is true}} \}$

exp2: Even integers = $\{x \mid x = \text{twice an integers}\}$

exp3: $\{x \mid \sqrt{x} \in \mathbb{Z}\}$

- Empty set = $\{\}$ or ϕ

- $\{\phi\}$ = a set with ~~just~~ only one Element (empty set)

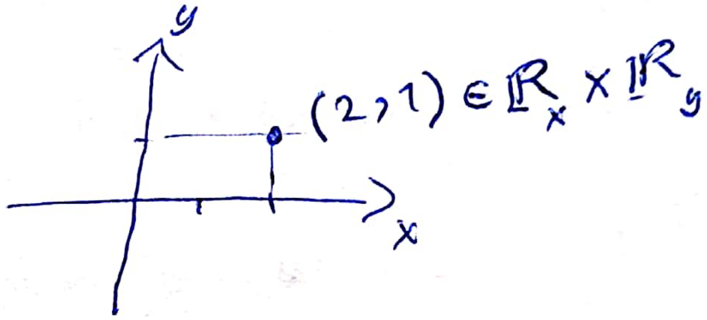
- is $\phi \subset \{1,2,3\}$? : True! → that is vacuously true!

- ordered Pairs (a, b)

- order matters $\Rightarrow$ $(a,b)=(c,d)$ if $a=c$ and $b=d$

- The <u>Cartesian Product</u> $A \times B$ is the set of all ordered Pairs (a,b) where $a \in A$ and $b \in B$

- cartesian plane:

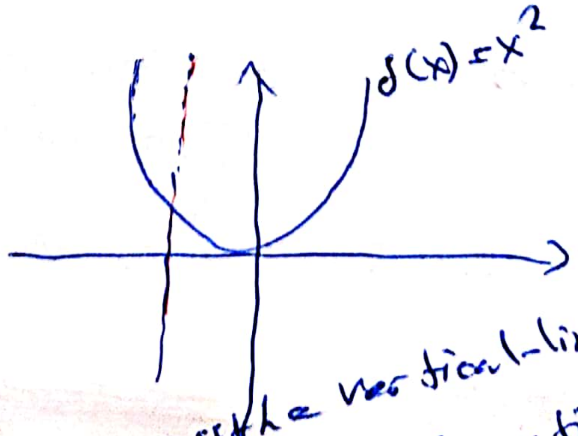


exp: $\underbrace{\{a,b\}}_{A} \times \underbrace{\{0,1\}}_{B} = \{(a,1),(a,0),(b,1),(b,0)\}$

Relations: A Relation R between A and B is a subset of $A \times B$. ie: ordered Pairs $(a,b) \in A \times B$.

- functions:

exp: $f(x)=x^2$



it pass the vertical-line-Test, so it is a function.

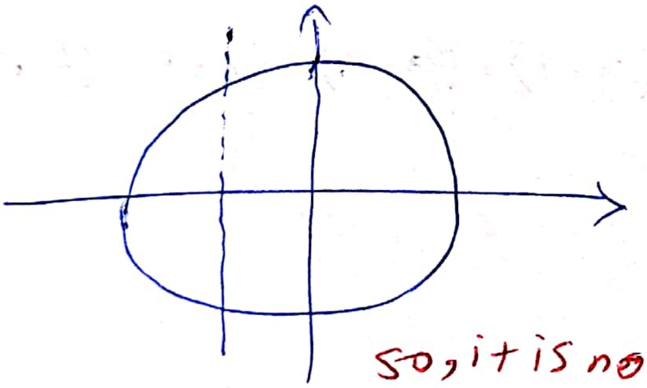
- vertical-line Test: each input has only one output.

- A function f between A and B is a relation between A and B such that:

subset $A \times B$

- for every element $x \in A$ there is an element $y \in B$ such that $(x, y) \in f$ or $f(x) = y$
- if $(x, y) \in f$ and $(x, z) \in f$ then $y = z$ (vertical-line-test)

exp: $x^2 + y^2 = 1$ is a function?



1) for every single there is an output ✓

2) vertical-line-Test ✗

so, it is not a function

- Statement

exp: "$\overbrace{\text{my shirt is gray}}^{p}$ but $\underbrace{\text{my shorts are not}}_{\sim q}$"

- $\sim p$ => not p

p = "my shirt is gray"

- $p \wedge q$ => p and q

q = "my shirts are gray"

- $p \vee q$ => p or q

~~$p \wedge q$~~

$\boxed{p \wedge \sim q}$

Truth Tables

p	$\sim p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

- $p \equiv \sim(\sim p)$

$\equiv$: logically equivalent

- A tautology t is a statement is always True.

exp: a dog is not a monkey is always true (a tautology statement)

- A contradiction c is a statement that is always False.

exp: a dog is a monkey!

- $\sim(p \vee q) \equiv (\sim p) \wedge (\sim q)$
- $\sim(p \wedge q) \equiv (\sim p) \vee (\sim q)$
- $\sim(\sim p) = p$
- $p \vee c \equiv p$
- $p \wedge t \equiv p$

$p \vee t \equiv t$

$p \wedge c \equiv c$

5

- The universal Quantifier "$\forall$" means "for all"

exp: $\forall x \in D, P(x)$

for all x in the domain, P(x) is true.

- every dog is a mammal.

- The Existential Quantifier $\exists$ means "there exists"

exp: $\exists x \in D, P(x)$

there exists x in the domain, such that P(x) is True.

- some person is the oldest in the world.

$\exists \in D = \{\text{People in the world}\}$; $P(x) = x$ is the ~~oldest~~ oldest.

"$\forall x \in \mathbb{Z}^+, x > 3$" negate

$\exists x \in \mathbb{Z}^+, x \not> 3$

or

$\exists x \in \mathbb{Z}^+, x \leq 3$

Negating a universal:

$\sim(\forall x \in D, P(x)) \equiv \exists x \in D, \sim P(x)$.

Negating an existential!

$\sim(\exists x \in D, P(x)) \equiv \forall x \in D, \sim P(x)$

Every integer has a larger integer!

~~$\forall x \in \mathbb{Z}$~~

$$\forall x \in \mathbb{Z}, \underbrace{\exists y \in \mathbb{Z}, y > x}_{P(x)}$$ → a true statement.

True: choose $y = x+1 \in \mathbb{Z}$

Negate: $\exists x \in \mathbb{Z}, \sim P(x)$

$\boxed{\Rightarrow \exists x \in \mathbb{Z}, \forall y \in \mathbb{Z}, y \leq x}$ ← this statement is false, so the first statement is true

---

"Some number in D is ~~two~~ the largest"

$\exists x \in D, P(x)$

$\Rightarrow \exists x \in D, \forall y \in D, x \geq y$

Negate: $\forall x \in D, \exists y \in D, x < y$

$$A(x) \Rightarrow B(x)$$

$A(x)$ is a sufficient condition for $B(x)$

Being a square is a sufficient condition for being a rectangle.

$B(x)$ is a necessary condition for $A(x)$

having a rectangle is a necessary condition for being a square.

- precisely define even and odd integers.

- n is an even integer if n can be written as twice an integer. (if $\exists k \in \mathbb{Z}$ such that $n = 2k$)

$\Downarrow$

n is an even integer,

- n is an odd integer, if n is an integer that is not even.

- n is an odd integer if $\exists k \in \mathbb{Z}$ such that $n = 2k+1$

$\boxed{\text{Theorem}}$: an even integer plus an odd integer is another odd integer.

$\boxed{\text{Proof}}$: suppose $\begin{cases} m \text{ is even} \\ n \text{ is odd} \end{cases}$

$\begin{cases} -\exists k_1 \in \mathbb{Z} \\ -\exists k_2 \in \mathbb{Z} \end{cases}$ so $\begin{cases} m = 2k_1 \\ n = 2k_2 + 1 \end{cases}$

then $\begin{cases} m+n = (2k_1) + (2k_2+1) \\ = 2(k_1+k_2)+1 \end{cases}$

let $k_3 = k_1 + k_2$ & $k_3 \in \mathbb{Z}$

Hence:

$\exists k_3 \in \mathbb{Z}$ so that

$\boxed{m+n = 2k_3 + 1}$

Thus m+n is odd

Direct proofs of: $\forall x \in D, P(x) \Longrightarrow Q(x)$

1) State the Assumptions.

2) formally Define the Assumptions.

3) Manipulate.

4) Arrive at Definition of conclusion.

5) state the conclution.

$\boxed{\text{Theorem:}}$ an even integer times an even integer is another even integer.

~~Proof~~ → for n an integers: n is even $\Longleftrightarrow \exists k \in \mathbb{Z}$ such that $n = 2k$

① $\forall m, n \in \mathbb{Z}$, if m, n are even,

② then $m \times n$ is even.

③ $4 \times 8 = 32$

$2(2) \quad 2(4) \quad 2(2) \cdot 2(4) = 2(2 \cdot 2 \cdot 4)$

$\boxed{\text{Proof}}$

(1) suppose m and n are even integers.

(2) $\exists r$ so that $m = 2r$

$\exists s$ so that $n = 2s$

3. $mn = (2r)(2s)$

$= 2(2rs)$

3. let $t = 2rs$, and note it is an integer.

4. Hence, $\exists t \in \mathbb{Z}$ so that $mn = 2t$

5. thus mn is even $\blacksquare$

Rational numbers ($\mathbb{Q}$)

- n is a rational number if it is a fraction
~~(R)~~ ($\mathbb{Q}$)

exp: $\frac{3}{7}$

- n is a Rational number if $\exists p \in \mathbb{Z}, \exists q \in \mathbb{Z} / \{0\}$ such that $n = \frac{p}{q}$

q ~~must~~ must not be Zero.

~~Proof:~~
theorem: The sum of two rational number is another rational number.

Proof: suppose m and n are rational.

$\exists p_1, p_2 \in \mathbb{Z}$ and $\exists q_1, q_2 \in \mathbb{Z} / \{0\}$ so that $m = \frac{p_1}{q_1}$ and $n = \frac{p_2}{q_2}$

then, $m+n = \frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}$

let $p_3 = \cancel{q_1 q_2 +}\ p_1 q_2 + p_2 q_1$ and $q_3 = q_1 q_2$

Hence, $\exists p_3 \in \mathbb{Z}, \exists q_3 \in \mathbb{Z} \setminus \{0\}$ so $m+n = \frac{p_3}{q_3}$

thus $m+n$ is rational.

Divisibility

- 12 is divisible by 3 $\Longrightarrow \frac{12}{3} = 4 \longrightarrow 4 \in \mathbb{Z}$
- $\cancel{\frac{12}{5}}$ 12 is not divisible by 5 $\Rightarrow \frac{12}{5} \notin \mathbb{Z}$
- Definition: for n and d integers, $d \neq 0$,

$d \mid n \Longleftrightarrow$ if $\exists k \in \mathbb{Z}$ such that $n = dk$

it means n is divisible by d $\left(\frac{n}{d}\right)$

or

d divides n

or

n is a multiple of d

d is a factor of n

theorem

if a is divisible by b.
and b is divisible by c.
then a is divisible by c.

exp: $4 | 12$
$2 | 4$
$2 | 12$

proof

let: $b | a$ and $c | b$

$\underbrace{\exists s, t}_{\in \mathbb{Z}}$, $a = sb$, $b = tc$

want: $a = cu$, $u \in \mathbb{Z}$
$\parallel$
$sb = stc = c(st)$

then: $a = sb = s(tc)$
$= c(st)$, $st \in \mathbb{Z}$

$\Rightarrow c | a$ ■

- for $a, b \in \mathbb{Z}$, $a^2 > b^2$ implies $a > b$

$a^2 > b^2 \Rightarrow \pm\sqrt{a^2} > \pm\sqrt{b^2} \not\Rightarrow a > b$!?

exp: $(-4)^2 > 3^2 \Rightarrow 16 > 9 \Rightarrow$ $\boxed{-4 \not> 3 \text{ or } -4 < 3}$?

$\longrightarrow$ so false by counterexample

method of counter example

- to prove $P(x) \Rightarrow Q(x)$ is false.

$\sim(\forall x \in D, P(x) \longrightarrow Q(x))$

$\exists x \in D, \sim(P(x) \longrightarrow Q(x))$ (Domain)

- method of counterexample: find one $a \in D$ where $P(a) \wedge \sim Q(a)$

Division into Cases

Theorem: The square of an integer has the same parity.

~~Case 1~~: Assume n is even $(\exists k_1 \in \mathbb{Z}, n = 2k_1)$

so $n^2 = (2k_1)^2 = 2(2k_1^2)$

let $k_2 = 2k_1^2$.

so $n^2 = 2k_2$

thus n^2 is even

Case 2: ~~the square of an int~~ assume n is odd ?

Division into cases:

to prove $P(x) \vee Q(x) \Rightarrow R(x)$

case 1: $P(x) \Rightarrow R(x)$

case 2: $Q(x) \Rightarrow R(x)$

Proof by contradiction

- contradiction is an argument of the form: $\sim p \rightarrow c$ therefor, p.

1) suppose $\sim p$ is True
2) Get a contradiction like $0=1$
3) therefore, p is True. (!)

exp: ~~no integer~~

$\boxed{\text{Theorem}}$: no integer is both even and odd.

- $\forall n \in \mathbb{Z}, \sim(n \text{ is even } \& \text{ odd})$

$\boxed{\text{Proof}}$ Assuming false $\cancel{\sim}(\forall n \in \mathbb{Z}, \cancel{\sim}(n \text{ is even } \& \text{ odd}))$

so: $\exists n \in \mathbb{Z}$, n is even $\&$ odd

so $\exists k_1, k_2 \in \mathbb{Z}$, $n = 2k_1$, $n = 2k_2 + 1$

$2k_1 = 2k_2 + 1 \Rightarrow 2(k_1 - k_2) = 1 \Rightarrow k_1 - k_2 = \frac{1}{2}$

$k_1 - k_2$ must be integer but $(\frac{1}{2})$ is not.

$k_1 - k_2 \in \mathbb{Z}$ but $k_1 - k_2 \notin \mathbb{Z}$

$\Rightarrow\Leftarrow$

so no integer, both even and odd

Proof by contrapositive

1) Goal: prove $P(x) \Longrightarrow Q(x)$

2) instead, prove the contrapositive:

$$\sim Q(x) \Longrightarrow \sim P(x)$$

exp: if n^2 is even, then n is even.

$\sqrt{n^2} = \sqrt{2k}$?

- if n is not even, n^2 is not even.

Assume n is odd (not even)

$\exists k_1 \in \mathbb{Z}, n = 2k_1 + 1$

so $n^2 = (2k_1+1)^2 = 4k_1^2 + 4k_1 + 1 = 2(2k_1^2 + 2k_1) + 1$

let $k_2 = 2k_1^2 + 2k_1$

so $n^2 = 2k_2 + 1$

Thus n^2 is odd.

So if n^2 is even, then n is even ✓

Quotient - Remainder - Theorem

- For all $n \in \mathbb{Z}$, $d \in \mathbb{Z}^+$, there exists $q, r \in \mathbb{Z}$ so that $n = dq + r$ with $0 \leq r < d$

exp,

$$\begin{array}{r|l} 26 & 3 \\ \hline -24 & 8 \\ \hline 02 & \end{array}$$

$$\underset{n}{26} = \underset{d}{3} \times \underset{q}{8} + \underset{r}{2}$$

- Today's Friday. what day the week is it in 50 days?

$$\begin{array}{r|l} 50 & 7 \rightarrow \text{a week} \\ \hline -49 & 7 \\ \hline 01 \rightarrow r & \end{array}$$

so, friday + 1 day = Saturday

Are there infinitely many Primes?

prime numbers: p is prime if it is a positive integer whose only factors are 1 and p

- Composite Number: c is composite if it is an integer >1 that is not not prime.

Theorem: every composite number can be written as a product of prime numbers.

Proof: prove with contradiction

- Assume there are finitely many primes.
- let's call them $P_1, P_2, \dots, P_n$.
- consider $P = P_1 P_2 \cdots P_n + 1$.
- Thus P is number bigger than all primes.

  i.e. P is compositive.

- $$\frac{P_1 P_2 \cdots P_n + 1}{P_1} = P_2 \cdots P_n + \frac{1}{P_1} \longrightarrow \text{not} \in \mathbb{Z}$$

- no prime P_i are factors as they all have reminders

so P is prime.

- so P is prime and not prime! Contradiction!
- therefore, there are infinitely many primes.

Sequences

Ex: 1, 2, 3, ...

Ex: 1, 0, 1, 0, ...

(informal)

Sequence: An infinite ordered list of objects.

- a_k is the k^{th} term.

Ex: $a_k = k$ so: 1, 2, 3, ...

Ex: $b_k = (-1)^k (3k)$ so: $-3, 6, -9, 12, \dots$

sequence (formal): A sequence is a function

$$f : \mathbb{Z}^+ \longrightarrow C$$

Sumation notation

$$a_m + a_{m+1} + \dots + a_n = \sum_{k=m}^{n} a_k$$

Product notation

$$a_m \cdot a_{m+1} \cdots a_n = \prod_{k=m}^{n} a_k$$

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factorial notation

$$1.2.3.4.\ldots.n = \prod_{k=1}^{n} k = n!$$

Ex: $4! = 1.2.3.4 = 24$

Ex: $0! = 1$ (!)

mathematical Induction

$\boxed{\text{Prove}}$ $1+2+3+\ldots+n = \frac{n(n+1)}{2}, \forall n \geq 1$

Ex: $n=5$: $1+2+3+4+5 = 15 = \frac{5(5+1)}{2}$

Proves $P(n) = \forall n \geq 1$

$\boxed{\text{step 1:}}$ prove that $P(a)$ is true $\boxed{\text{basis step}}$

$\boxed{\text{step 2:}}$ Assume $P(k)$ is true, $\boxed{\text{Induction step}}$

Prove $P(k+1)$ is true.

(for $k \geq a$)

Basis step: $1 = \frac{1(1+1)}{2}$ ✓

Induction step) Assume $P(k) = 1+2+\ldots+k = \frac{k(k+1)}{2}$

consider: $1+2+\ldots+k+(k+1) = \frac{k(k+1)}{2} + (k+1)$

$$= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

therefore, $P(k+1)$?!!!

Carl Gauss (1777–1855)

summed from 1 to 100

at approx. age 8

$$\overbrace{50+51}^{101}$$

$$1+2+3+\dots+50+51+\dots+98+99+100$$

101

101

101

So, $\sum_{k=1}^{100} k = 50(101) = \frac{100(101)}{2}$

Induction with Inequalities

$\boxed{\text{Prove:}}$ $2^n > n \quad \forall n \geq 0$

$\boxed{\text{Basis Step:}}$ $2^0 > 0 \Rightarrow 1 > 0$ ✓

$\boxed{\text{Induction step}}$

Induction Hypothesis: $2^k > k, \; k \geq 0$

$2^{k+1} > k+1$?

$\Rightarrow 2 \cdot 2^k > k+1$

$> 2 \cdot k$ $\quad > k+1 \quad$ for $k > 1$

$= k+k$

Strong Induction

Goal: $P(n)$, $n \geq a$

(basis)
step 1: $P(a)$, $P(a+1)$, ..., $P(b)$ ← prove these

step 2 (Induction): Assume $P(i)$ for $a \leq i \leq k$

Prove: $P(k+1)$

Ex: $a_1 = 1$, $a_2 = 3$, $a_k = a_{k-2} + 2a_{k-1}$

So: 1, 3, 7, 17 ...

claim: a_k is odd always!

$\boxed{\text{Proof}}$

step 1: $a_1 = 1$ & $a_2 = 3$ are odd ✓

Step 2: Assume that a_i is odd

for $1 \leq i \leq k$

then. $a_{k+1} = \underbrace{a_{k-1}}_{\text{odd}} + 2\underbrace{a_k}_{\text{odd}}$ ✓

$= 2_{p+1} + 2(2g+1)$ for $p, g \in \mathbb{Z}$

$= 2(p+2g+1)+1$

So a_{k+1} is odd ✓

Recursive Sequences

Ex: $a_1 = 1$

$a_k = a_{k-1} + 3, \; k > 1$

$a_1 = 1$ → Zero three

$a_2 = 1 + 3 = 4$ → 1 three

$a_3 = (1+3) + 3 = 7$ → zero three

$a_4 = (1+3+3+\cancel{3}) + 3 = 10$ → 3 three

...

$a_k = \cancel{(1+3)_{(k-1)}}$ $\; 1 + 3(k-1)$ → k-1 three

Fibonacci sequence

$f(1) = 1$

$f(2) = 1$

$f(k) = f(k-1) + f(k-2)$

subset: $A \subseteq B$

means: if $x \in A$, then $x \in B$

or: $\forall x \in A, \; x \in B$

Suppose that:

$A = \{ n \in \mathbb{Z} \mid n = 4p,\ p \in \mathbb{Z} \}$

$B = \{ m \in \mathbb{Z} \mid m = 2q,\ q \in \mathbb{Z} \}$

Prove: $A \subseteq B$

$\boxed{\text{Proof}}$ let $x \in A$, $x = 4p$, some $p \in \mathbb{Z}$

$x = 2(2p)$

let $q = 2p$.

then $x = 2q \in \mathbb{Z}$

~~A~~ so $x \in B$ $\blacksquare$

so $A \subseteq B$

~~AU~~

union

$\boxed{\text{fact: } A \cup B = B \cup A}$

or

$\boxed{\text{Proof}}$ let $\boxed{x \in A \cup B} \iff x \in A \lor x \in B$

$\iff x \in B \lor x \in A$

$\iff \boxed{x \in B \cup A \ \blacksquare}$

Union: $A \cup B = \{x \mid x \in A \vee x \in B\}$

suppose that: $A = \{n \in \mathbb{Z} \mid n = 2p, p \in \mathbb{Z}\}$

$B = \{m \in \mathbb{Z} \mid n = 2q + 1, q \in \mathbb{Z}\}$

compute: $A \cup B = ?$

$A = \{\ldots, -4, -2, 0, 2, 4, \ldots\}$ even

$B = \{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$ odd

So: $A \cup B = \{-2, -1, 0, 1, 2, \ldots\} = \mathbb{Z}$

Proof! $n \in A \cup B \iff n \in A \vee n \in B$

$\iff$ n is even or n is odd

$\iff$ n is an integer ∎

so $A \cup B = \mathbb{Z}$

---

intersection: $A \cap B = \{x \mid x \in A \overset{\text{and}}{\wedge} x \in B\}$

Suppose that:

$A = \{ n \in \mathbb{Z} \mid n = 2p , p \in \mathbb{Z} \}$

$B = \{ m \in \mathbb{Z} \mid n = 3q , q \in \mathbb{Z} \}$

Compute: $A \cap B$?

$A = \{ \ldots, -4, -2, 0, 2, 4, \ldots \}$

$B = \{ \ldots, -6, -3, 0, 3, 6, \ldots \}$

$A \cap B = \{ \ldots, -12, -6, 0, 6, 12, \ldots \}$

Proof let $n \in A \cap B \iff \boxed{n = 2p = 3q} , p, q \in \mathbb{Z}$

$\iff n = 2 \cdot 3 r , r \in \mathbb{Z}$

$\iff n = 6r , r \in \mathbb{Z}$

$\iff n \in \{ x \in \mathbb{Z} \mid x = 6r , r \in \mathbb{Z} \}$ ∎

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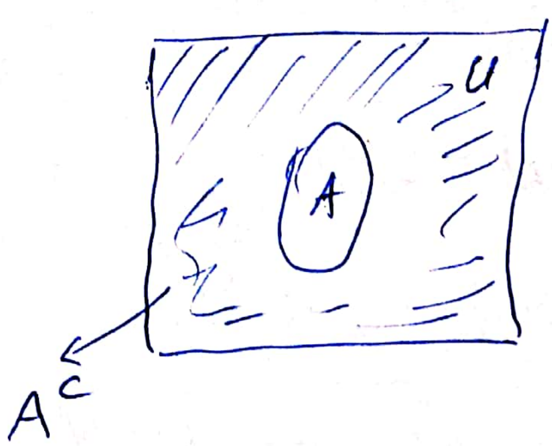
complement

$A^c = \{ x \in U \mid x \notin A \}$

universe



Prove: $B^c \subseteq A^c$ if $A \subseteq B \rightarrow$ if $x \in A$ then $x \in B$

Proof:

$x \in B^c$

$\Rightarrow x \notin B$

$\Rightarrow x \notin A$

$\downarrow$

$x \in A^c$ ■

- The power set $P(A)$ is the set of all subsets of A.

Ex: $A = \{1, 2, 3\}$

$$P(A) = \left\{ \begin{array}{l} \{1\}, \{2\}, \{3\}, \\ \{1,2\}, \{1,3\}, \{2,3\}, \\ \{1,2,3\}, \underset{\text{empty set}}{\emptyset} \end{array} \right\}$$

$\{\} = \emptyset$ ü

$P(\emptyset) = \{\emptyset\}$

$P(P(\emptyset)) = \{\emptyset, \{\emptyset\}\}$

For $|A| = n$

$|P(A)| = 2^n$

let $A = \{1, 2, 3, \ldots\} = \mathbb{N}$

$|P(\mathbb{N})| = ?$

$f: P(\mathbb{N}) \rightarrow$ Binary numbers

Ex: $\{1, 3\} \rightarrow 0.101 = \frac{1}{2^1} + \frac{0}{2^2} + \frac{1}{2^3}$

1th 3th

Ex: $\{2, 4, 5, 7\} \rightarrow 0.0101101$

2th 4th 5th 7th

$f: P(\mathbb{N}) \rightarrow [0, 1]$ is onto.

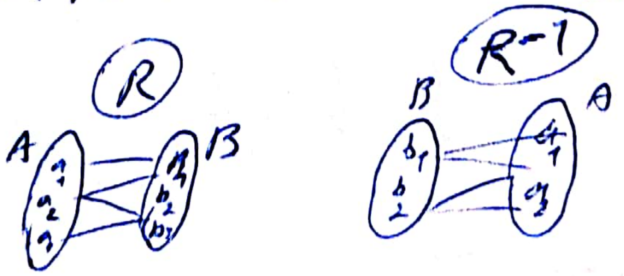
Big Idea: $P(\mathbb{N})$ is uncountably infinite. !

Big Idea: $P(A)$ has greater cardinality than A.

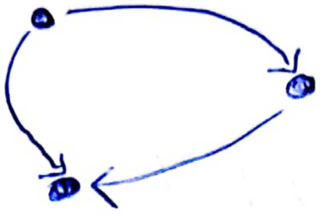
- Relation: A relation R from A to B is a subset of $A \times B$

- An element in R is written aRb.

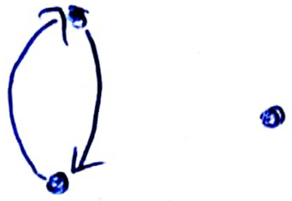
- the inverse relation to R is denoted R^{-1} and is a subset of $B \times A$.



- **Transitive**: every 2-step Path has a 1-step Path.



- **Symetric**: Every Path has a reverse path

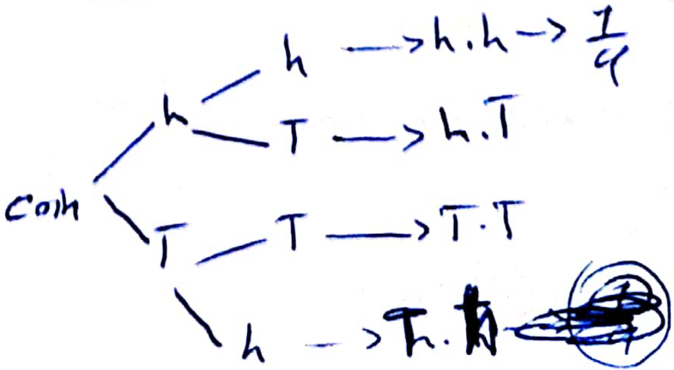


- **Reflexive**: every Path has a path to itself.

Probability

- what is the probability of fliping $\boxed{\text{heads}}$ two times in a Row?

$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$



- Sample Space (S) is some set

Ex: all pairs of coins.

- An event E is a subset of S.

Ex: The pair T-T or H-H or H-T or T-H

- The probability that E occurs is denoted P(E).

Ex: P(heads-heads) = 0.25

$$P(E) = \frac{\text{size of } E}{\text{total size of } S} = \frac{N(E)}{N(S)}$$

Ex: $N(h\text{-}h) = 1$

$N(S) = 4$

$P(h\text{-}h) = \frac{1}{4}$.

- what is the probability of flipping one h and one T in any order?

$P(1h, 1T) = \frac{2}{4} = 0.5$

- what is the probability of selecting an even integer between 3 and 14, inclusive?

3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14

$$P(E) = \frac{N(E)}{N(S)} = \frac{6}{12} = \frac{1}{2} = 0.5 \text{ or } 50\%$$

- what is the probability of guessing 4 digit pin code?

(~~it is an~~ independent events)
they are

for independent events: multiply the probability for each level.

$$\underline{0.1}\ \underline{0.1}\ \underline{0.1}\ \underline{0.1} = (0.1)^4 = 0.0001 \text{ or } \frac{1}{10000}$$

Ex: How many ways to reorder the letters of the word "FROM"? (~~they are dependent events~~)

(note: no repeats)
but they are independent.

$$\underset{4}{\underline{\tfrac{1}{4}}}\ \underset{3}{\underline{\tfrac{1}{3}}}\ \underset{2}{\underline{\tfrac{1}{2}}}\ \underset{1}{\underline{1}} \qquad \frac{1}{24}$$

The number all the different possibility = 4! or

$4 \times 3 \times 2 \times 1$

$\boxed{= 24}$

Ex: How many ways to order four letters from the word "FORMULA"? (without repeating letters).

r

$\underline{7}\ \underline{6}\ \underline{5}\ \underline{4} = 840$ different possibility

n

$$N(S) = \boxed{\frac{7!}{3!}} = \frac{7\times6\times5\times4\times3\times2\times1}{3\times2\times1}$$

$n-r$

- The number of permutations that pick $\boxed{r \text{ items}}$ from $\boxed{n \text{ possibilities}}$ without repeats in order is

$$\boxed{P(n,r) = \frac{n!}{(n-r)!}}$$

Ex: How many 1, 2 or 3 letter passwords are there, using 26 letters only?

$\boxed{S = S_1 \sqcup S_2 \sqcup S_3}$

1 letter: $N(S_1) = 26^1$

2 letter: $N(S_2) = 26^2$

3 letter: $N(S_3) = 26^3$

total number: $26 + 26^2 + 26^3 = 18278$

intersection

when $A \cap B = \emptyset$, the union is called the disjoint union and is denoted

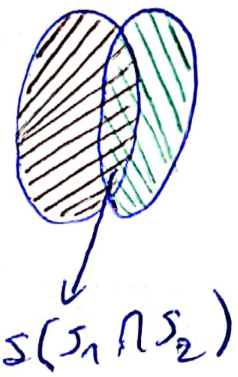
$$A \sqcup B$$

- when the sample space is a disjoint union $S = S_1 \sqcup S_2$,

$N(S) = N(S_1) + N(S_2)$

- when the Sample space is a union $S = S_1 \cup S_2$,

$\cancel{S(x)}$ $N(S) = N(S_1) + N(S_2) - N(S_1 \cap S_2)$



Ex: How many numbers between 1 and 100 are multiples of 4 or 6? $\boxed{S_4 \cap S_6 \neq \emptyset}$

$S = S_4 \cup S_6$

$$\begin{cases} N(S_4) = 25 \\ N(S_6) = 16 \\ N(S_4 \cap S_6) = 8 \end{cases}$$

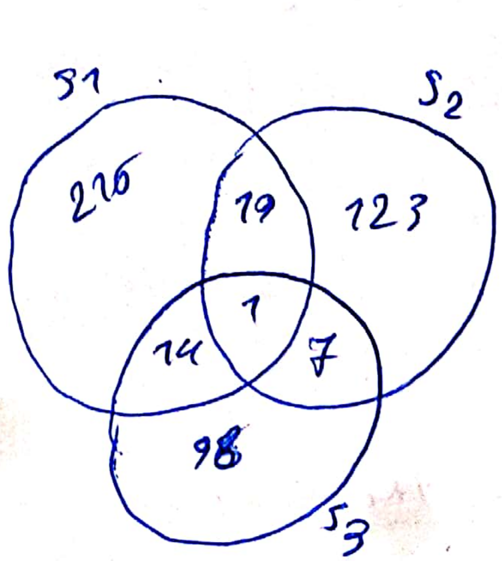
$N(S) = 25 + 16 - 8 = \underline{33}$

"counting with triple Intersections"

$S_1 = 250$

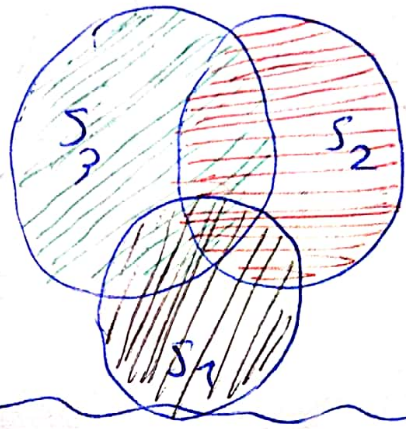
$S_2 = 150$

$S_3 = 120$



$(S_1 \cup S_2 \cup S_3) = 1 + 7 + 14 + 98 + 19 + 216 + 123$
$= \underline{478}$

$$N(S_1 \cup S_2 \cup S_3) = N(S_1) + N(S_2) + N(S_3) - N(S_1 \cap S_2)$$
$$- N(S_1 \cap S_3) - N(S_2 \cap S_3) + N(S_1 \cap S_2 \cap S_3)$$



EX: How many ways can I choose a 3 person team from 5 possible choices?

- If i did care about order: $\underline{5}\ \underline{4}\ \underline{3}$

$$P(5,3) = \frac{5!}{(5-3)!} = \frac{5 \times 4 \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times \cancel{1}}$$

- if i didn't care about order:

$$\binom{n}{r} = \frac{P(n,r)}{r!} = \frac{1}{r!}\left[\frac{n!}{(n-r)!}\right] = \frac{n!}{r!\,(n-r)!}$$

Ex: How many rearrangments of "MISSISSIPPI" are there?

S I S M I S S P I P I

$$\binom{11}{4} \cdot \binom{7}{4} \cdot \binom{4}{2} \cdot \binom{1}{1}$$

choose 4 spots from 11 total for "S" — I — P — M

$$= \left[\frac{11!}{4!\,(11-4)!}\right] \cdot \left[\frac{7!}{4!\,3!}\right] \cdot \left[\frac{4!}{2!\,2!}\right] \cdot \left[\frac{1!}{1!\,0!}\right]$$

$\overline{} \to = 1$

Ex: compute $\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = \frac{5 \times 4 \times 3!}{2 \times 3!} = \frac{20}{2} = 10$

Ex: How many ways are there to reorder the letters in the word "CINCINNATI"?

$$\binom{10}{2} \cdot \binom{8}{3} \cdot \binom{5}{3} \cdot \binom{2}{1} \cdot \binom{1}{1} = \frac{10!}{2!\,(10-2)!} \cdot \frac{8!}{3!\,(8-3)!} \cdot$$

C — I — N — A — T

$$\frac{5!}{3!\,(5-3)!} \cdot \frac{2!}{1!\,(2-1)!} \cdot \frac{1!}{1!\,(1-1)!}$$

EX: a coin is tossed 10 times and the results recorded.

A) How many Total possible outcomes are there?

$$\underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2}\ \underline{2} = 2^{10} = 1024$$

B) How many outcomes have exactly 5 heads?

$$\binom{10}{5} = \frac{10!}{5!\,(10-5)!} = \frac{10 \times 9 \times \cancel{8} \times 7 \times \cancel{6} \times \cancel{5!}}{\cancel{5!} \times 5 \times 4 \times \cancel{3 \times 2}}$$

$$= \frac{90 \times 56}{20} = 252$$

C) How many outcomes have at least 7 heads?

$$\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}$$

D) How many outcomes have at most 2 ~~head~~ heads?

$$\binom{10}{0} + \binom{10}{1} + \binom{10}{2}$$

EX: An Instructor gives an exam with fourteen questions. Students are allowed to choose any 10 to answer.

A) How many different choices of 10 questions are there?

$$\binom{14}{10} = \frac{14!}{10!\,(14-10)!}$$

B) suppose six questions require proof and eight do not.

i) How many groups of 10 questions contain four that require proof and six that do not?

$$\binom{6}{4} \cdot \binom{8}{6}$$

ii) How many groups of ten questions contain at least one that requires proof?

All of them $\binom{14}{10}$

iii) How many groups of ten contain at most three that require proof?

$$\binom{6}{2}\binom{8}{8} + \binom{6}{3}\binom{8}{7}$$

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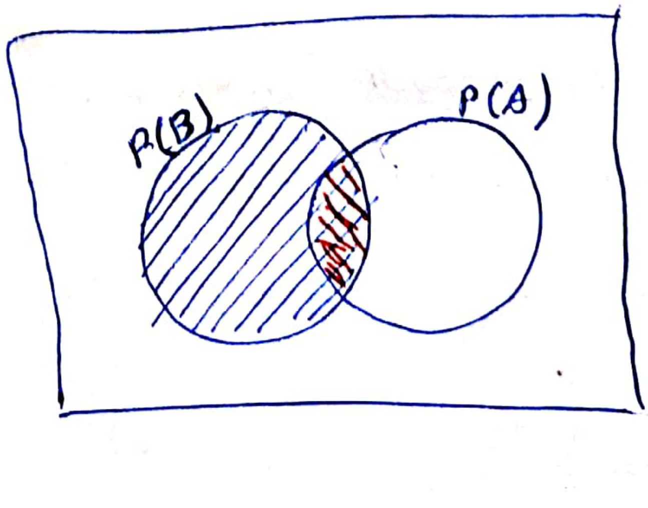
conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



Ex: the % of adults who are men & alcoholics is 2.25%. what is the probability of being an alcoholic, given being a man?

A = alcoholic

B = man

$P(A \cap B) = 0.0225$

? $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.0225}{0.5} = 0.045$

Ex: what is the probability of two children being girls if we are told at least one is a girl?

$P(2 \text{ girls} | \text{atleast one girls})$

$= \frac{P(2G \cap \text{atleast } 1G)}{P(\text{at least } 1G)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{4}{12}$

$\boxed{= \frac{1}{3}}$

Bayes' Theorem

previously: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

or

$P(B|A) = \frac{P(B \cap A)}{P(A)}$

$\Rightarrow$ Bayes' Theorem

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Ex: what is the probability of two girls given at least one girls?

$$P(2G \mid \text{atleast } 1G)$$

$$= \frac{P(1G \mid 2G) \cdot P(2G)}{P(1G)} = 1 \cdot \tfrac{1}{4} \Big/ \tfrac{3}{4} = \tfrac{1}{4} \times \tfrac{4}{3} = \boxed{\tfrac{1}{3}}$$

Ex: You test positive in a test with a 5% false positive Rate.

- what is the chance you have the disease?
(suppos 1 in 100 have the illness.)
(the test also has a false negative rate of 10%)

A = Have the disease
B = Test positive

have the disease and test positive.

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B)}$$

Have the disease and test positive (or) don't have the disease and test positive.

$$P(A \mid B) = \frac{P(B \mid A)\, P(A)}{P(B \mid A)\, P(A) + P(B \mid \sim A)\, P(\sim A)}$$

$$= \frac{0.9 \times 0.07}{0.9 \times 0.07 + 0.05 \times 0.99} = 0.159 = 15.9\%$$

the correct hait rate

A = have the disease

B = Test positive Twice

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B|A)\,P(A) + P(B|\sim A)\,P(\sim A)}$$

$$= \frac{0.87 \times 0.07}{0.87 \times 0.07 + 0.0025 \times 0.99} = 0.77 = \boxed{77\%}$$

the correct hait rate after the second Test. arithe The Same resalt

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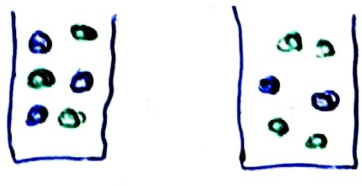
Ex: If we randomly draw a blue ball, what is the probability of being in 1st backet?

A: Select a blue ball.



B_1 B_2

$P(A|B_1) = \frac{1}{2}$

$P(A|B_2) = \frac{1}{3}$

$P(B_1) = P(B_2) = \frac{1}{2}$

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)}}$$

$$P(A) = P(A \cap B_1) + P(A \cap B_2)$$

$$= P(A|B_1) \cdot P(B) + P(A|B_2) \cdot P(B_2)$$

$$= \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{4} + \frac{1}{6} = \frac{5}{12}$$

$$P(B_1|A) = \frac{P(A|B_1) \cdot P(B_1)}{P(A)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{2}}{5/12} = \frac{3}{5}$$

Frequentist

$$P(A) = \lim_{n \to \infty} \frac{\#\ \text{outcomes that are } A}{\#\ \text{outcomes Total}}$$

Bayesian

Prior Belief: $P(A)$

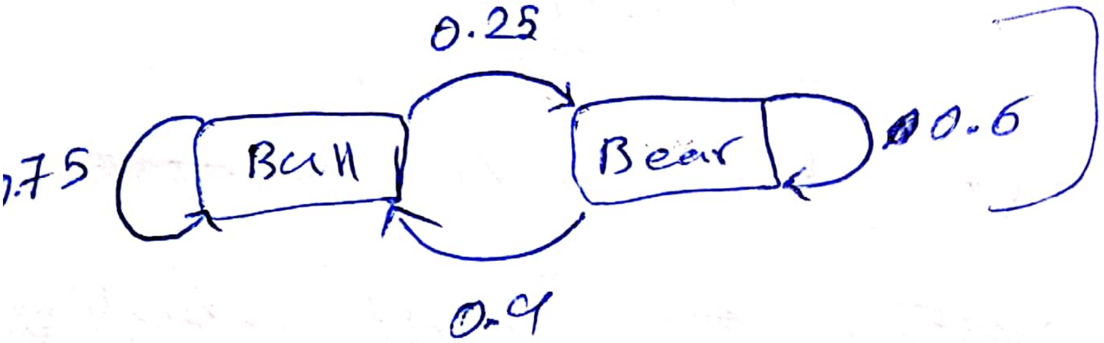
updated Belief: $P(A|B)$

Markov chains

- A markov chain is a sequence of events where the probabilities of the future only depend on the present.

Ex: Stock Market

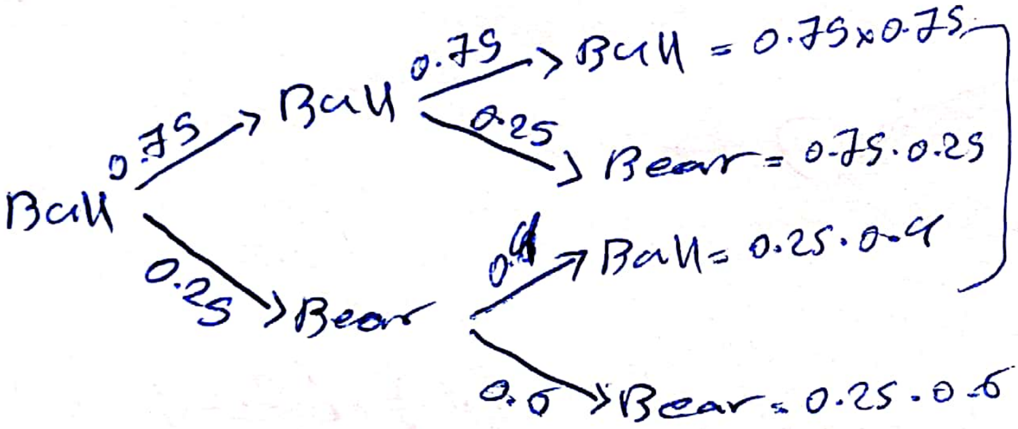
- 75% a bull market followed by a bull market.
- 60% a bear market followed by a bear market.



transition diagram

- if is a bull market this week.

what are the probabilities in two weeks?



tree diagram

$P(\text{Bull}) = (0.75 \cdot 0.75) + (0.25 \cdot 0.4) = 0.66$

$P(\text{Bear}) = (0.75 \cdot 0.25) + (0.25 \cdot 0.6) = ?\ 0.34$

$$\begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} = \begin{pmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad S_1 = P(S_0)$$

$$\begin{pmatrix} 0.66 \\ 0.34 \end{pmatrix} = \begin{pmatrix} 0.75 & 0.4 \\ 0.25 & 0.6 \end{pmatrix} \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} \quad S_2 = P(S_1)$$

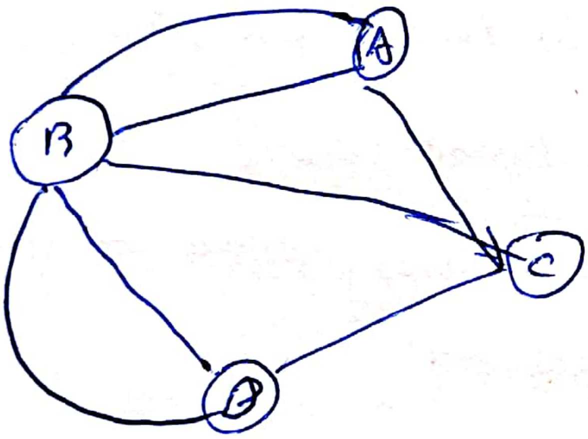
$S_2 = P S_1 = P^2 S_0$

$$\boxed{S_n = P^n S_0}$$

Graph Theory

Definition: A graph (V, E) has a set "V" called ~~vector~~ "vertices" and a set E called "Edges" consisting of two-element subsets of "V".

Ex:

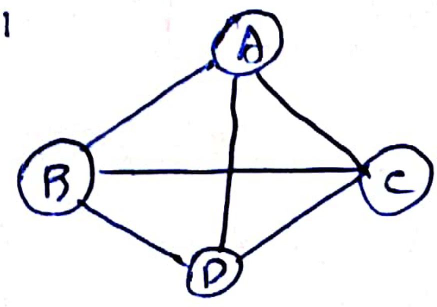


$E = \{\{A,B\}, \{A,B\}, \{B,D\}, \{B,D\}, \{B,C\}, \{C,D\}, \cancel{\{C,D\}}, \{A,C\}\}$

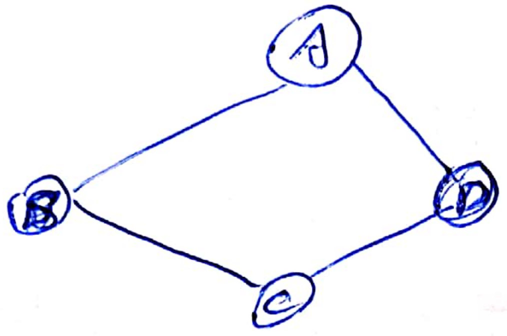
$V = \{A, B, C, D\}$

- K_n is the complete graph on n vertices, i.e. one of every possible edge

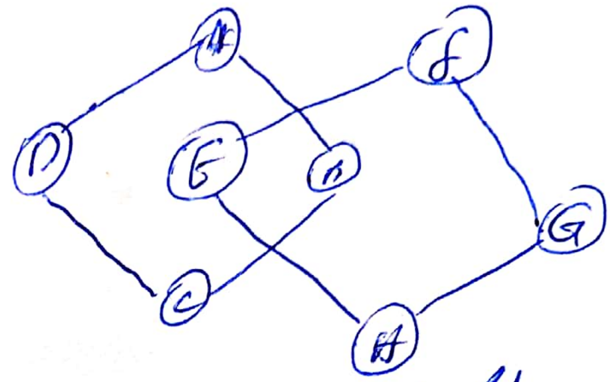
K_4:



- A graph is connected if you can get from any vertex to any other via edges.

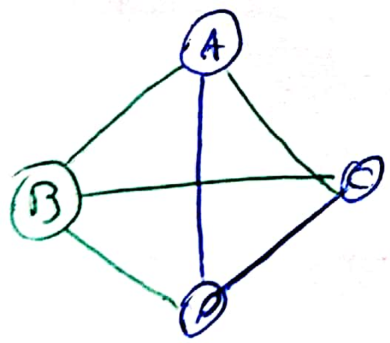


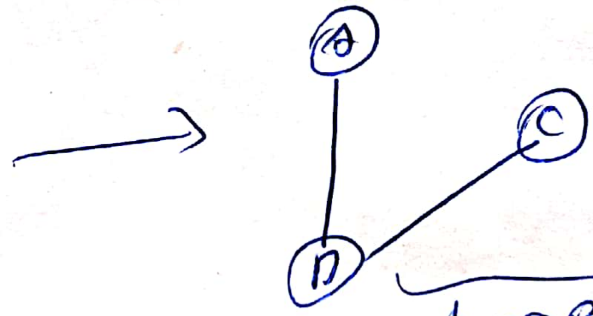
connected!



not connected!

- (V_1, E_1) is a subgraph of (V_2, E_2) if it is a graph where $V_1 \subseteq V_2$ and $E_1 \subset E_2$.





a subgraph