From $\lambda = \lambda_o$, $u = (1 - 1.2500z^{-1})(1 - 1.8388z^{-1})$: antistable,

$N = 4.9709 - 5.7883z^{-1} + 1.0298z^{-2}$ and the controller is:

$$C_{on} = 4.4869 - 5.5108z^{-1} + 1.4864z^{-2}$$

$$C_{od} = 2.2985 - 5.7369z^{-1} + 3.7160z^{-2}.$$

The roots of the closed-loop polynomial are $z = 0.8$, 0.5438, 0.4. This is an optimal answer but it is not the only one. For $\lambda = -\lambda_o$, $u = (1 + 4.6036z^{-1})(1 - 1.25z^{-1})$: antistable

$$N = 9.7626 - 13.7602z^{-1} + 2.5783z^{-2}$$

and the controller is

$$C_{on} = 4.1530 - 10.9645z^{-1} + 3.7213z^{-2}$$

$$C_{od} = -5.7545 - 5.4030z^{-1} + 9.3034z^{-2}.$$

The roots of the closed-loop polynomial are

$$z = 0.8, 0.4, -0.2172.$$

Note that in this second solution there is a c.l. pole on the negative side of the imaginary axis. From all examples encountered so far it seems that in the nongeneric case the solution for the negative $\lambda$ has a negative c.l. pole.

## VI. Conclusions

A solution to the single-input single-output mixed sensitivity $H_\infty$ optimal-control problem has been obtained in a discrete-time setting. Both generic and nongeneric solutions where considered.

A Wiener type of argument was used to demonstrate, how, in the generic case, the solution loses degree and becomes the same as that of [3] where the generic case was solved using a link with the LQG problem. A set of equations equivalent to that of [3] was derived.

It was also shown that by correctly choosing (increasing) the degree of the solution, Grimble's equations can be used in the nongeneric case. However, the alternative system of equations presented here [(5)–(7), (20), (21)] is considered to have some advantage as far as computational effort is concerned. It is necessary to iterate on a single diophantine equation, rather than a coupled pair of equations.

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# A Simple PD Controller for Robots with Elastic Joints

Patrizio Tomei

*Abstract*—**This note is concerned with the point to point control of manipulators having elastic joints. It is shown that a simple PD controller, similar to that used for rigid robots, suffices to globally stabilize the elastic joint robots about a reference position. A robustness analysis is also given with respect to uncertainties on the robot parameters. This note is completed by some simulations tests referred to a manipulator with three revolute elastic joints.**

## I. Introduction

Experimental results have shown that the joint elasticity should be taken into account in the modeling of robotic manipulators [1]. The elasticity in the joints may be caused by the *harmonic drives*, that are special type of gear mechanisms having high transmission ratio, low weight, and small size. As a counterpart, these gear-boxes introduce no negligible elasticity, due to their mechanical structure. Joint flexibility can be also used to obtain approximate models of robots having elastic links.

The introduction of joint flexibility in the robot modeling complicates considerably the equations of motion. In particular, the order of the related dynamics becomes twice that of rigid robots [2]. Moreover, the property owned by rigid robots of being linearizable by static-state feedback is lost, in general [3].

Consequently, the control laws proposed for elastic joint robots are more complex than those valid for rigid robots. One approach is that based on feedback linearization by dynamic state feedback [4]. Other used approaches are based on the singular perturbation theory [2] and on the concept of integral manifold [5], [6]. In [7] it is shown that linearization by static-state feedback can be still obtained if one simplificative hypothesis is made in the derivation of the dynamic model. An adaptive controller, based on this simplified model and on the assumption of weak elasticity, has been also proposed [8]. Basically, the aforementioned assumption consists of considering the motion of the actuator rotors as pure rotations with respect to an inertial frame. Unfortunately, this hypothesis may not be reasonable for some operating conditions [9]. Hence, throughout this note we do not make such an assumption.

As is well known, a simple PD controller suffices to stabilize any kind of rigid manipulator about a reference position, provided that the gravitational forces are compensated [10]. Although these rigid robots are exactly linearizable by static-state feedback [11], the stability of the PD controller does not rely on this property, but it is

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rather related to the intrinsic passivity properties of robotic manipulators. This remark suggests that a similar control algorithm could be derived also for elastic joint robots.

In this note, taking benefit of the passivity properties and of the particular structure of the dynamic equations, we prove that a simple PD controller globally stabilizes, about a reference position, robots having elastic joints. The PD controller is exactly the same used to control robots with rigid joint. Indeed, it consists of linear constant feedback from the position and the velocity of the motor rotors, plus a term to compensate the gravity forces. Therefore, the proposed controller can be implemented by using the sensors which are usually mounted on current industrial robots; that is, one position and one velocity sensor on the motor shaft of each joint. Thus, even if the state of each elastic joint is completely defined by four state variables (for instance, position and velocity of the link and of the motor rotor) only half of them are needed to implement this PD control law.

The organization of this note is as follows. In Section II we recall the dynamic model of elastic joint robots and derive some useful structural properties. Section III is devoted to present the PD control algorithm and to demonstrate its global asymptotic stability. Section IV presents the robustness analysis with respect to uncertainties on the gravity terms and on the elastic constants. We prove that the PD control law is still stable but the equilibrium point differs, in general, from the desired one. Finally, simulation results referred to a three-revolute jointed robot are given in Section V and some conclusions are drawn in Section VI.

## II. Modeling Issues

We refer to a robot having $n+1$ rigid links, interconnected by $n$ elastic revolute joints. The following quite general assumptions are made about the mechanical structure.

*A1:* The elasticity in the joint can be modeled as a linear torsional spring, as schematically represented in Fig. 1.

*A2:* The rotors of the actuators can be modeled as uniform bodies of revolution.

Let $q_1$ represent the $n \times 1$ vector of the position of the links, and let $q_2$ represent the $n \times 1$ vector of the actuator positions. The external torque applied to the $i$th joint is denoted by $u_i$. The kinetic energy of the whole structure is given by

$$T = \frac{1}{2}\dot{q}^T B(q)\dot{q} \tag{1}$$

where $q^T = [q_1^T, q_2^T]$ and $B(q)$ is the inertia matrix, which is symmetric positive definite and bounded for any $q$. Owing to assumption A2, $B(q)$ is structured as follows [2]

$$B(q) = B(q_1) = \begin{bmatrix} B_1(q_1) & B_2(q_1) \\ B_2^T(q_1) & B_3 \end{bmatrix} \tag{2}$$

in which $B_1$ and $B_2$ are $n \times n$ matrices, and $B_3$ is a constant diagonal matrix depending on the actuator inertias and gear ratios.

The potential energy $U$ is given by the sum of two terms. The first one is the gravitational term that, under assumption A2, takes the form

$$U_1 = U_1(q_1) \tag{3}$$

and is bounded for any $q_1$. The second one, arising from joint elastic torques, can be written as

$$U_2 = \frac{1}{2}(q_1 - q_2)^T K_e (q_1 - q_2) \tag{4}$$

in which $K_e = \mathrm{diag}[k_1, \cdots, k_n]$, $k_i$ being the elastic constant of joint $i$. Defining the matrix

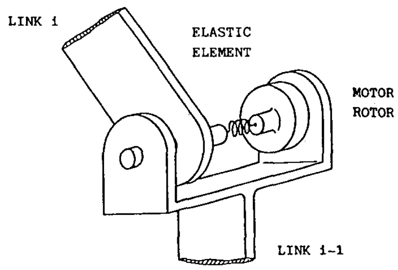


Fig. 1. Elastic joint.

$$K_E = \begin{bmatrix} K_e & -K_e \\ -K_e & K_e \end{bmatrix}$$

(4) can be rewritten as

$$U_2 = \frac{1}{2}q^T K_E q. \tag{5}$$

Following the Lagrangian formulation, the dynamic equations of the robot motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = m \tag{6}$$

where $L(q, \dot{q}) = T(q, \dot{q}) - U(q)$ is the Lagrangian function and $m = [0, \cdots, 0, u_1, \cdots, u_n]^T$. Substituting (1), (3), and (5) into (6), we get

$$B(q_1)\ddot{q} + C(q, \dot{q})\dot{q} + K_E q + e(q_1) = m \tag{7}$$

in which

$$C(q, \dot{q})\dot{q} = \dot{B}(q_1)\dot{q} - \frac{1}{2}\frac{\partial \dot{q}^T B(q_1)\dot{q}}{\partial q}$$

$$e(q_1) = \frac{\partial U_1(q_1)}{\partial q} = \begin{bmatrix} e_1(q_1) \\ 0 \end{bmatrix} \tag{8}$$

with $e_1 = \partial U_1 / \partial q_1$.

Frictional forces can be taken into account by modifying (7) as follows:

$$B(q_1)\ddot{q} + C(q, \dot{q})\dot{q} + K_E q + e(q_1) + f(q, \dot{q}) = m \tag{9}$$

where $f(q, \dot{q})$ is such that

$$f(q, 0) = 0 \tag{10}$$

$$\dot{q}^T f(q, \dot{q}) \geq 0 \tag{11}$$

$$\dot{q}^T f(q, \dot{q}) = 0 \Rightarrow f(q, \dot{q}) = 0. \tag{12}$$

For instance, if we deal with viscous frictional forces $f(q, \dot{q}) = F\dot{q}$, where $F$ is a symmetric positive semidefinite matrix.

Referring to the model (9), the following useful structural properties can be derived.

*P1:* If the elements of $C(q, \dot{q})$ are defined as

$$C_{ij}(q, \dot{q}) = \frac{1}{2}\left[\dot{q}^T \frac{\partial B_{ij}}{\partial q} + \sum_{k=1}^{2n}\left(\frac{\partial B_{ik}}{\partial q_j} - \frac{\partial B_{jk}}{\partial q_i}\right)\dot{q}_k\right] \qquad i, j = 1, \cdots, 2n \tag{13}$$

with $q_i$ denoting the $i$th element of vector $q$, then matrix $\dot{B} - 2C$ is skew-symmetric [12].

*P2:* If $C(q, \dot{q})$ is defined according to (13), then it can be decomposed as

$$C(q, \dot{q}) = C_A(q_1, \dot{q}_2) + C_B(q_1, \dot{q}_1) \tag{14}$$

with

$$C_A(q_1, \dot{q}_2) = \begin{bmatrix} C_{A1}(q_1, \dot{q}_2) & 0 \\ 0 & 0 \end{bmatrix},$$

$$C_B(q_1, \dot{q}_1) = \begin{bmatrix} C_{B1}(q_1, \dot{q}_1) & C_{B2}(q_1, \dot{q}_1) \\ C_{B3}(q_1, \dot{q}_1) & 0 \end{bmatrix}$$

where the elements of the $n \times n$ matrices $C_{A1}$, $C_{B1}$, $C_{B2}$, and $C_{B3}$ are

$$C_{A1,ij}(q_1, \dot{q}_2) = \frac{1}{2}\left(\frac{\partial B_2^i}{\partial q_{1,j}} - \frac{\partial B_2^j}{\partial q_{1,i}}\right)\dot{q}_2$$

$$C_{B1,ij}(q_1, \dot{q}_1) = \frac{1}{2}\left[\dot{q}_1^T \frac{\partial B_{1,ij}}{\partial q_1} + \left(\frac{\partial B_1^i}{\partial q_{1,j}} - \frac{\partial B_1^j}{\partial q_{1,i}}\right)\dot{q}_1\right]$$

$$C_{B2,ij}(q_1, \dot{q}_1) = \frac{1}{2}\left[\dot{q}_1^T \frac{\partial B_{2,ij}}{\partial q_1} - \frac{\partial (B_2^T)^j}{\partial q_{1,i}}\dot{q}_1\right]$$

$$C_{B3,ij}(q_1, \dot{q}_1) = \frac{1}{2}\left[\dot{q}_1^T \frac{\partial B_{2,ji}}{\partial q_1} + \frac{\partial (B_2^T)^i}{\partial q_{1,j}}\dot{q}_1\right] \qquad (15)$$

with $B^i$ denoting the $i$th row of matrix $B$.

This is a direct consequence of (2) and (13).

*P3:* Matrix $B_2(q_1)$ owns the structure

$$B_2(q_1) = \begin{bmatrix} 0 & b_{12}(q_{1,1}) & b_{13}(q_{1,1}, q_{1,2}) & \cdots & b_{1n}(q_{1,1}, \cdots, q_{1,n-1}) \\ 0 & 0 & b_{23}(q_{1,2}) & \cdots & b_{2n}(q_{1,2}, \cdots, q_{1,n-1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \qquad (16)$$

where $b_{ij}$ are suitable scalar functions. Indeed, the kinetic energy of the robot is given by the sum of the kinetic energies of the links (including the motor stators) and of the motor rotors. Since the kinetic energy of each link depends only on the links variables, it does not affect matrix $B_2$. Now, define for each rotor a frame solidal to the corresponding stator, having origin in the center of mass of the rotor. The kinetic energy of rotor $i$ is then given by

$$T_{R,i} = \frac{1}{2} m_{R,i} v_{2,i}^T v_{2,i} + \frac{1}{2}\omega_{2,i}^T I_{R,i}\omega_{2,i} \qquad (17)$$

where $v_{2,i}$ and $\omega_{2,i}$ are the absolute linear and angular velocity vectors of the rotor expressed in the above cited frame, $m_{R,i}$ and $I_{R,i}$ are the mass and the inertia tensor of the rotor. Given that $v_{2,i}$ only depends on links variables, the first term on the right side of (17) does not affect $B_2$. The angular velocity $\omega_{2,i}$ can be calculated by the recursive formula

$$\omega_{2,i} = R_{R,i}[\omega_{1,i-1} + g_{2,i}\dot{q}_{2,i}]$$
$$\omega_{1,i} = R_i(q_{1,i})[\omega_{1,i-1} + g_{1,i}\dot{q}_{1,i}] \qquad (18)$$

in which $\omega_{1,i}$ is the absolute angular velocity vector of link $i$ in a reference solidal to the link itself, $g_{1,i}$ denotes the constant unit vector of the angular velocity of link $i$ relative to link $i-1$, $g_{2,i}$ is the constant unit vector of the angular velocity of rotor $i$ relative to link $i-1$, $R_i$ is the $3 \times 3$ transformation matrix from reference of link $i-1$ to reference of link $i$, $R_{R,i}$ is the constant $3 \times 3$ transformation matrix from reference of link $i-1$ to reference of rotor $i$.

The equations (17) and (18) imply (16).

*P4:* A positive constant $\alpha$ exists such that

$$\left\| \frac{\partial e_1(q_1)}{\partial q_1} \right\| \le \alpha, \qquad \forall q_1 \in R^n. \qquad (19)$$

This property relies on the fact that $e_1(q_1)$ is formed by trigonometric functions of the links variables $q_{1,i}$. The inequality (19) implies, by the mean value theorem

$$\| e_1(q_1) - e_1(\bar{q}_1) \| \le \alpha \| q_1 - \bar{q}_1 \|, \qquad \forall q_1, \bar{q}_1 \in R^n. \quad (20)$$

## III. PD Controller

In this section, we will show how a simple PD controller globally stabilizes, about the desired reference position, the robot system (9). We adopt the following notations and definitions.

Given a symmetric positive definite bounded matrix $A(x)$, we indicate by $A_M$ and $A_m$ its maximum and minimum eigenvalues, respectively, for any $x \in R^n$. The norm of an $n \times 1$ vector $x$ is defined as

$$\| x \| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

and the norm of a matrix $A$ is defined as the corresponding induced norm

$$\| A \| = \sqrt{\max_{\text{eigenvalue}} A^T A}\,.$$

Consider the control law

$$u = -K_P(q_2 - q_{2_0}) - K_D\dot{q}_2 + e_1(q_{1_0}) \qquad (21)$$

where $K_P$ and $K_D$ are $n \times n$ symmetric positive definite matrices, $q_{1_0}$ is the desired position of the links and $q_{2_0}$ is related to $q_{1_0}$ by

$$q_{2_0} = q_{1_0} + K_e^{-1} e_1(q_{1_0}). \qquad (22)$$

Define the matrix

$$K = \begin{bmatrix} K_e & -K_e \\ -K_e & K_e + K_P \end{bmatrix}.$$

The following theorem holds.

*Theorem 1:* Consider the system (9) with the control law (21). If $K_m > \alpha$, then $q_1 = q_{1_0}$, $q_2 = q_{2_0}$, $\dot{q} = 0$ is the unique equilibrium point. Moreover, this equilibrium is globally asymptotically stable.

*Proof:* The equilibrium positions of (9), (21) are the solutions of

$$K_e(q_1 - q_2) + e_1(q_1) = 0 \qquad (23)$$

$$K_e(q_1 - q_2) - K_P(q_2 - q_{2_0}) + e_1(q_{1_0}) = 0. \qquad (24)$$

Recalling (22), we can add to (23) and (24) the term $K_e(q_{2_0} - q_{1_0}) - e_1(q_{1_0})$. We obtain

$$K_e(q_1 - q_{1_0}) - K_e(q_2 - q_{2_0}) + e_1(q_1) - e_1(q_{1_0}) = 0$$
$$K_e(q_1 - q_{1_0}) - (K_e + K_P)(q_2 - q_{2_0}) = 0$$

that can be rewritten in matrix form as

$$K(q - q_0) = e(q_{1_0}) - e(q_1) \qquad (25)$$

where $q_0^T = [q_{1_0}^T, q_{2_0}^T]$. The inequality (20) enables us to write

$$\|K(q - q_0)\| \geq K_m\|q - q_0\| > \alpha\|q - q_0\| \geq \|e(q_{1_0}) - e(q_1)\|, \qquad \forall q \neq q_0.$$

Hence, (25) has the unique solution $q = q_0$.

Now, define the function

$$P_1(q) = \frac{1}{2}(q - q_0)^T K(q - q_0) + U(q_1) - q^T e(q_{1_0}).$$

The stationary points of $P_1(q)$ are given by the solutions of

$$\frac{\partial P_1(q)}{\partial q} = 0$$

that coincides with (25). Therefore, $P_1(q)$ owns the unique stationary point $q = q_0$. Moreover

$$\frac{\partial^2 P_1(q)}{\partial q^2} = K + \frac{\partial e(q_1)}{\partial q}. \tag{26}$$

By virtue of (19) and of the assumption $K_m > \alpha$, the matrix on the right side of (26) is positive definite. As a consequence, $q = q_0$ is an absolute minimum point for $P_1(q)$. Consider the candidate Lyapunov function

$$v(q, \dot{q}) = \frac{1}{2}\dot{q}^T B(q_1)\dot{q} + P_1(q) - P_1(q_0) \tag{27}$$

that is positive definite with respect to $q = q_0$, $\dot{q} = 0$. The time derivative of (27), along (9), (21), is given by

$$\begin{aligned}\dot{v}(q, \dot{q}) = &\frac{1}{2}\dot{q}^T \dot{B}(q_1)\dot{q} + \dot{q}^T[-C(q, \dot{q})\dot{q} - K_E q \\ &-f(q, \dot{q}) - e(q_1)] - \dot{q}_2^T K_P(q_2 - q_{2_0}) \\ &- \dot{q}_2^T K_D \dot{q}_2 + \dot{q}_2^T e_1(q_{1_0}) + \dot{q}^T K(q - q_0) \\ &+ \dot{q}^T \frac{\partial U_1(q_1)}{\partial q} - \dot{q}^T e(q_{1_0}).\end{aligned} \tag{28}$$

Recalling property P1 and (8), (28) reduces to

$$\begin{aligned}\dot{v}(q, \dot{q}) = &-\dot{q}_2^T K_D \dot{q}_2 - \dot{q}^T f(q, \dot{q}) \\ &+ (\dot{q}_1 - \dot{q}_2)^T [K_e(q_{2_0} - q_{1_0}) - e_1(q_{1_0})]\end{aligned}$$

which, in turn, by virtue of (22) becomes

$$\dot{v}(q, \dot{q}) = -\dot{q}_2^T K_D \dot{q}_2 - \dot{q}^T f(q, \dot{q}).$$

Equations (11) and (12) imply that $\dot{v}$ is negative semidefinite and vanishes if and only if $\dot{q}_2 = 0$, $f(q, \dot{q}) = 0$. Substituting these conditions into (9) and recalling (14) and (15), we get

$$B_1(q_1)\ddot{q}_1 + C_{B1}(q_1, \dot{q}_1)\dot{q}_1 + K_e q_1 + e_1(q_1) = K_e q_2 = \text{constant} \tag{29}$$

$$\begin{aligned}B_2^T(q_1)\ddot{q}_1 &+ C_{B3}(q_1, \dot{q}_1)\dot{q}_1 - K_e q_1 \\ &= -K_e q_2 - K_P(q_2 - q_{2_0}) + e_1(q_{1_0}) = \text{constant}.\end{aligned} \tag{30}$$

Taking (16) and the last of (15) into account, the first equation of (30) becomes

$$q_{1,1} = \text{constant} \tag{31}$$

Substitution of (31) into the second equation of (30) yields $q_{1,2}$ = constant. Proceeding in the same way, we finally obtain

$$q_1 = \text{constant}$$

that, substituted in (29) and (30), leads to

$$\begin{aligned}K_e(q_1 - q_2) + e_1(q_1) &= 0 \\ K_e(q_2 - q_1) + K_P(q_2 - q_{2_0}) - e_1(q_{1_0}) &= 0.\end{aligned} \tag{32}$$

Since, as previously shown, (32) owns the unique solution $q = q_0$, we conclude that $q = q_0$, $\dot{q} = 0$ is the largest invariant subset of the set $\dot{v} = 0$. By applying the Lasalle theorem [13, p. 108] the thesis is proved. △

*Remark:* The hypothesis $K_m > \alpha$ of Theorem 1 is not restrictive, since by increasing in norm $K_P$ it can be satisfied for all applications.

## IV. Analysis of Robustness

The PD control law (21) is robust with respect to some model uncertainties. In particular, the asymptotic stability is guaranteed even if the inertial and frictional parameters of the robot are not known. Conversely, uncertainties on the gravitational and elastic parameters may affect the stability of this controller, since they appear explicitly in the control law (see (21) and (22)).

In this section we will show that the PD controller, subject to uncertainties on the aforementioned parameters, is still stable but the equilibrium point is, in general, different from the desired one.

Suppose that the gravity vector $e_1(q_1)$ and the matrix of the elastic constants $K_e$ are not perfectly known. Let $\hat{e}_1(q_1)$ and $\hat{K}_e$ be, respectively, the available estimates. The control law (21) becomes

$$u = -K_P(q_2 - \hat{q}_{2_0}) - K_D\dot{q}_2 + \hat{e}_1(q_{1_0}) \tag{33}$$

where $\hat{q}_{2_0} = q_{1_0} + \hat{K}_e^{-1}\hat{e}_1(q_{1_0})$.

*Theorem 2:* If $K_m > \alpha$, then system (9), (33) owns only one equilibrium point, denoted by $q = \bar{q}_0$, $\dot{q} = 0$. Moreover, this equilibrium point is globally asymptotically stable.

*Proof:* The equilibrium positions of (9), (33) are given by the solutions of

$$\begin{aligned}K_e(q_1 - q_2) + e_1(q_1) &= 0 \\ K_e(q_1 - q_2) - K_P(q_2 - \hat{q}_{2_0}) + \hat{e}_1(q_{1_0}) &= 0\end{aligned}$$

that can be rewritten as

$$K_e(q_1 - q_2) + e_1(q_1) = 0 \tag{34}$$

$$\begin{aligned}K_e(q_1 - q_2) - K_P(q_2 - q_{2_0}) + e_1(q_{1_0}) + K_P(\hat{q}_{2_0} - q_{2_0}) \\ + \hat{e}_1(q_{1_0}) - e_1(q_{1_0}) = 0.\end{aligned} \tag{35}$$

By adding to (34) and (35) the term $K_e(q_{2_0} - q_{1_0}) - e_1(q_{1_0})$, which is null by (22), we obtain

$$K(q - q_0) = \begin{bmatrix} e_1(q_{1_0}) - e_1(q_1) \\ K_P(\hat{q}_{2_0} - q_{2_0}) + \hat{e}_1(q_{1_0}) - e_1(q_{1_0}) \end{bmatrix}. \tag{36}$$

Consider the function

$$\begin{aligned}P_2(q) &= \frac{1}{2}(q - q_0)^T K(q - q_0) \\ &\quad - (q - q_0)^T \begin{bmatrix} e_1(q_{1_0}) \\ K_P(\hat{q}_{2_0} - q_{2_0}) + \hat{e}_1(q_{1_0}) - e_1(q_{1_0}) \end{bmatrix} \\ &\quad + U(q_1) \\ &= P_c(q) + U(q_1).\end{aligned}$$

It is easy to see that $P_c(q)$ is a convex function, whose absolute minimum is assumed for $q_1 = q_{1_0} + \hat{q}_{2_0} - q_{2_0} + K_e^{-1}e_1(q_{1_0}) + K_P^{-1}\hat{e}_1(q_{1_0})$, $q_2 = \hat{q}_{2_0} + K_P^{-1}\hat{e}_1(q_{1_0})$. Recalling that $U(q_1)$ is a bounded function, it follows that $P_2(q)$ owns absolute minimum for a finite value of $q$. Therefore, the equation

$$\frac{\partial P_2(q)}{\partial q} = 0 \tag{37}$$

which gives the stationary points of $P_2(q)$, has at least one solution. Let $q = \bar{q}_0$ be this solution. Since (37) coincides with (36), we can write

$$K(\bar{q}_0 - q_0) = \begin{bmatrix} e_1(q_{1_0}) - e_1(\bar{q}_{1_0}) \\ K_P(\hat{q}_{2_0} - q_{2_0}) + \hat{e}_1(q_{1_0}) - e_1(q_{1_0}) \end{bmatrix}. \tag{38}$$

Subtracting (38) from (36), we obtain

$$K(q - \bar{q}_0) = e(\bar{q}_{1_0}) - e(q_1). \tag{39}$$

As is shown in the proof of Theorem 1, the hypothesis $K_m > \alpha$ implies that (39) owns the unique solution $q = \bar{q}_0$. Hence, the first part of this theorem has been proved.

The global asymptotic stability of the equilibrium point $q = \bar{q}_0$, $\dot{q} = 0$ can be proved by considering the Lyapunov function

$$v(q, \dot{q}) = \frac{1}{2}\dot{q}^T B(q_1)\dot{q} + P_2(q) - P_2(\bar{q}_0)$$

and proceeding as in the proof of Theorem 1. △

*Remark:* By comparing (25) to (38) we can see that if $K_P(\hat{q}_{2_0} - q_{2_0}) + \hat{e}_1(q_{1_0}) - e_1(q_{1_0}) \neq 0$, then the actual equilibrium position $\bar{q}_0$ differs from the desired one $q_0$. An estimate of this difference can be derived from (38), yielding

$$\|\bar{q}_0 - q_0\| \leq \frac{1}{K_m - \alpha} \cdot \left(K_{P_M}\|\hat{q}_{2_0} - q_{2_0}\| + \|\hat{e}_1(q_{1_0}) - e_1(q_{1_0})\|\right).$$

## V. Simulation Results

In order to test the dynamic performances of the PD controller presented in the previous sections, numerical simulations were carried out on a three link manipulator. The considered robot consists of three rigid bodies (shoulder, upper arm, and forearm) interconnected by three revolute elastic joints. Frictional phenomena were neglected. According to the notations of (9), the nonzero elements of $B$, $e$, $K_e$, and $C$ are given by

$$B_{11} = a_1 + a_2 \cos q_{1,2} + a_3 \cos(q_{1,2} + q_{1,3}) + a_4 \cos q_{1,2} \cos(q_{1,2} + q_{1,3})$$

$$B_{22} = a_5 + a_4 \cos q_{1,3}, \quad B_{23} = a_8 + a_7 \cos q_{1,3}, \quad B_{26} = a_9$$

$$B_{32} = B_{23}, \quad B_{62} = B_{26}, \quad B_{33} = a_8$$

$$B_{44} = a_{10}, \quad B_{55} = a_{11}, \quad B_{66} = a_{10}$$

$$e_2 = a_{12} \cos q_{1,2} + a_{13} \cos(q_{1,2} + q_{1,3}),$$

$$e_3 = a_{13} \cos(q_{1,2} + q_{1,3})$$

$$k_1 = 14\,210, \quad k_2 = 29\,800, \quad k_3 = 14\,210$$

$$C_{11} = -\frac{1}{2}\left(r(q_1)\dot{q}_{1,2} + s(q_1)\dot{q}_{1,3}\right), \quad C_{12} = -\frac{1}{2}r(q_1)\dot{q}_{1,1}$$

$$C_{13} = -\frac{1}{2}s(q_1)\dot{q}_{1,1}, \quad C_{21} = \frac{1}{2}r(q_1)\dot{q}_{1,1}$$

$$C_{22} = -\frac{1}{2}t(q_1)\dot{q}_{1,3}, \quad C_{23} = -\frac{1}{2}t(q_1)(\dot{q}_{1,2} + \dot{q}_{1,3})$$

$$C_{31} = \frac{1}{2}s(q_1)\dot{q}_{1,1}, \quad C_{32} = \frac{1}{2}t(q_1)\dot{q}_{1,2}$$

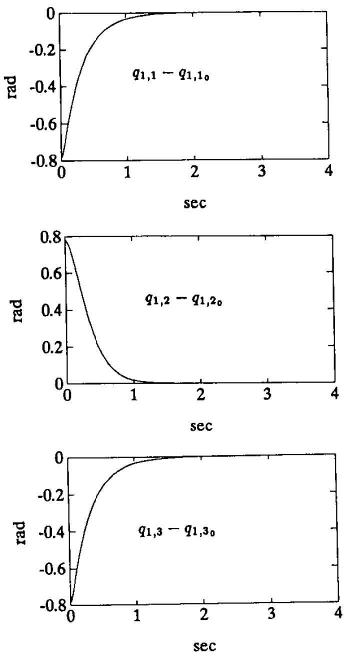


Fig. 2. Link errors.

TABLE I
Robot Parameters for Different Payloads

| $m_p = 0$ kg | $m_p = 5$ kg |
|---|---|
| $a_1 = 0.0840$ | $a_1 = 0.0840$ |
| $a_2 = 9.2063$ | $a_2 = 10.456$ |
| $a_3 = 2.4515$ | $a_3 = 3.7015$ |
| $a_4 = 5.4000$ | $a_4 = 7.9000$ |
| $a_5 = 11.743$ | $a_5 = 14.243$ |
| $a_6 = 2.6274$ | $a_6 = 3.8774$ |
| $a_7 = 2.7000$ | $a_7 = 3.9500$ |
| $a_8 = 2.4817$ | $a_8 = 3.7317$ |
| $a_9 = 0.1456$ | $a_9 = 0.1456$ |
| $a_{10} = 23.296$ | $a_{10} = 23.296$ |
| $a_{11} = 70.656$ | $a_{11} = 70.656$ |
| $a_{12} = 189.17$ | $a_{12} = 213.67$ |
| $a_{13} = 52.928$ | $a_{13} = 77.432$ |

where

$$r(q_1) = 2a_2 \sin q_{1,2} \cos q_{1,2} + a_3 \sin 2(q_{1,2} + q_{1,3}) + a_4 \sin(2q_{1,2} + q_{1,3})$$

$$s(q_1) = a_3 \sin 2(q_{1,2} + q_{1,3}) + a_4 \cos q_{1,2} \sin(q_{1,2} + q_{1,3})$$

$$t(q_1) = a_4 \sin q_{1,3}.$$

The values of the parameters $a_i$, corresponding to the nominal payload $m_p = 5$ kg and to a different payload $m_p = 0$ kg, are listed in Table I. All parameters are in *SI* units.

The considered problem is that of regulation about the reference position

$$q_{1,1_0} = q_{1,2_0} = q_{1,3_0} = \pi/4$$

starting from the following initial conditions

$$q_1(0) = q_2(0) = [0, \pi/2, 0]^T, \quad \dot{q}_1(0) = \dot{q}_2(0) = 0.$$

The proportional and derivative gain matrices were chosen as

$$K_P = \text{diag}[3000], \quad K_D = \text{diag}[1000].$$

In the first set of simulation runs we assumed that the actual payload and the elastic constants were exactly known. The corresponding results are reported in Fig. 2., where are drawn the errors between the desired and the actual position of the links. The rotor position errors were substantially the same.

In order to verify the robustness of the PD controller, the test was repeated by using an actual payload $m_p = 0$ and actual elastic constants that are 10% greater than the nominal ones. The adopted controller was the same of the first simulation; that is, the controller designed with the nominal parameters. The resulting errors practically coincided with those of the first case. However, according to Theorem 2, steady-state errors appeared of the order of $10^{-4}$ rad.

## VI. Concluding Remarks

If high performances are to be achieved, the joint elasticity introduced by the actuators should be taken into consideration in the modeling and control of robotic manipulators. The incorporation of the joint flexibility into the robot dynamic model complicates remarkably the equations of motion. In particular, the property owned by rigid joint robots of being linearizable by static-state feedback is lost, in general.

However, some structural properties, such as the passivity between the generalized $2n$ vector $m$ of the joint torques and the $2n$ vector $\dot{q}$ of the velocities, are preserved. Hence, it could be reasonable that control laws based on this passivity property are still valid when the elasticity in the joints is taken into account.

In this note, we have shown that the previous conjecture is true for the PD control algorithm. More in details, we have proved that proportional plus derivative control algorithms which stabilize rigid robots keep their stabilizing action also for elastic joint manipulators. This is a very nice property which allows us to control manipulators having elastic joints by using the same sensors needed to control rigid robots.

The dynamic behavior of the proposed PD controller has been pointed out by means of simulation tests referred to a robot having three revolute elastic joints.

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# Parameter Estimation for Linear Discrete-Time Models with Random Coefficients

Michal Horváth

*Abstract*—**This note deals with the problem of parameter estimation in linear discrete-time systems with random coefficients. In particular, the maximum *likelihood* (ML) *estimators* and their consistency for defined structure of the model are derived. The estimators have similar structure as the least square estimators for the linear discrete-time system with constant coefficients.**

## Introduction

The linear discrete-time single-input single-output system with random coefficients can be expressed in the following form [2], [6]:

$$y(t) = A_1(t)y(t-1) + \cdots + A_n(t)y(t-n) + B_1(t)u(t-1) + \cdots + B_n(t)u(t-n) + v(t)$$

where $u(t)$ is the input, $y(t)$ is the output, and $A_1(t),\cdots, A_n(t)$, $B_1(t),\cdots, B_n(t)$, $v(t)$ are Gaussian random processes. In [1], the model is considered without the random process $v(t)$ and in [3], the coefficients $B_1,\cdots, B_n$ are constants.

The problem of estimation of the model parameters is treated only in few papers. Bohlin [2] shows that ML estimates, in general, may be found only by means of an optimization method that maximizes the likelihood function. The same conclusion is obtained by Nicholls and Quinn [5] for the statistical model which does not include the input $u(t)$. They also pointed out that the numerical procedure is rather difficult. Chen and Caines [3] used in their adaptation algorithm modified least squares method which provided the estimates of the constant mean values of $A_1(t),\cdots, A_n(t)$.

In this note, we consider the model which contains a smaller number of parameters that enables us to derive suitable estimators. We adopt the assumption that the variances of coefficients disturbances are equal and we do not consider the additive noise $v(t)$. Such a model has the following form:

$$\begin{aligned} y(t) = {} & [a_1 + \alpha_1(t)]y(t-1) + \cdots \\ & + [a_n + \alpha_n(t)]y(t-n) \\ & + [b_1 + \beta_1(t)]u(t-1) + \cdots \\ & + [b_n + \beta_n(t)]u(t-n) \end{aligned} \tag{1}$$

where $u(t)$ is the stochastic input, $y(t)$ is the output, $a_1,\cdots, a_n$, $b_1,\cdots, b_n$ are real coefficients, $\alpha_1(t),\cdots,\alpha_n(t)$, $\beta_1(t),\cdots,\beta_n(t)$ are mutually independent Gaussian white noises with zero mean and the finite variance $r > 0$, $u(t)$ is furthermore independent with $\alpha_1(t),\cdots,\alpha_n(t)$, $\beta_1(t),\cdots,\beta_n(t)$.

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