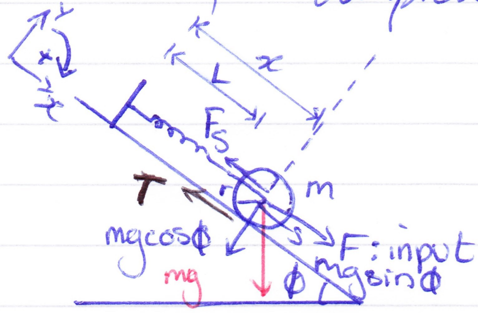
ELE 2024 Complete example: modelling to simulations.

MODELLING



Nonlinear spring

$$F_S = -k_1(x-L) - k_2(x-L)^3$$

$$k_1, k_2 > 0$$

No-slip condition, T causes rotation

First sketch all forces applied

x-axis forces:

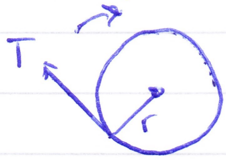
F_S, F, $mg\sin\phi$, T

→ Total force along the x-axis:

$$F + mg\sin\phi - \underbrace{T}_{\text{unknown.}} - k_1(x-L) - k_2(x-L)^3 = m\ddot{x} \quad (1)$$

→ Total force along the y-axis:

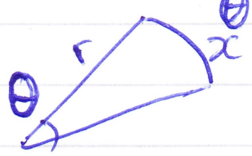
$$N - mg\cos\phi = 0$$



$$M_T^{(0)} = +T\cdot r \iff I\ddot{\theta} = T\cdot r \iff \frac{1}{2} m r^2 \ddot{\theta} = T r$$

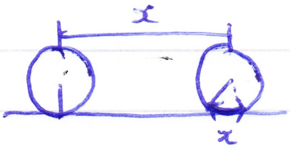
$$\Rightarrow \boxed{\frac{1}{2} m r \ddot{\theta} = T}$$

relationship between θ and x



$$x = r\theta$$
$$\dot{x} = r\dot{\theta}$$
$$\ddot{x} = r\ddot{\theta}$$

$$\Rightarrow \frac{1}{2} m\ddot{x} = T$$



(1)

$$F + mg\sin\phi - \frac{1}{2} m\ddot{x} - k_1(x-L) - k_2(x-L)^3 = m\ddot{x}$$

State: (x_1, x_2)
output: x
input: F

$$\Rightarrow \boxed{F + mg\sin\phi - \cancel{\tfrac{1}{2} m\ddot{x}}\, k_1(x-L) - k_2(x-L)^3 = \frac{3}{2} m\ddot{x}}$$

$$F \cancel{+} mg\sin\phi - k_1(x-L) - k_2(x-L)^3 = \frac{3}{2}m\ddot{x} \quad (3)$$

EQUILIBRIUM

multiply (3) by $\frac{2}{3m}$

$x_1 = x, \quad x_2 = \dot{x}_1 = \dot{x}$

$\dot{x}_2 = \ddot{x}_1 = \ddot{x}$

$$\Rightarrow \boxed{\dot{x}_2 = \frac{2}{3m}F + \frac{2}{3}g\sin\phi - \frac{2k_1}{3m}(x_1 - L) - \frac{2k_2}{3m}(x_1 - L)^3}$$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \frac{2}{3m}F - \frac{2}{3}g\sin\phi - \frac{2k_1}{3m}(x_1 - L) - \frac{2k_2}{3m}(x_1 - L)^3 \end{cases}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

SSR

$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{2}{3m}F - \frac{2}{3}g\sin\phi - \frac{2k_1}{3m}(x_1 - L) - \frac{2k_2}{3m}(x_1 - L)^3 \end{bmatrix}$$

$$\dot{x} = f(x, F)$$

→ A pair (x^{eq}, F^{eq}) is an EQUILIBRIUM POINT of the system if and only if:

$$f(x^e, F^e) = 0$$

so, evaluate at equilibrium point:

x_2 is linear velocity so this makes sense

$$\boxed{\begin{bmatrix} x_2^{eq} = 0 \\ \frac{2}{3m}F^{eq} - \frac{2}{3}g\sin\phi - \frac{2k_1}{3m}(x_1^{eq} - L) - \frac{2k_2}{3m}(x_1^{eq} - L)^3 = 0 \end{bmatrix}}$$

under these conditions, $(x_1^{eq}, x_2^{eq}, F^{eq})$ is an equilibrium point

$$\dot{x}_1 = x_2$$

constant

$$\dot{x}_2 = \frac{2}{3m}F - \boxed{\frac{2}{3}g\sin\phi} - \frac{2k_1}{3m}(x_1 - L) - \frac{2k_2}{3m}(x_1 - L)^3$$

non-linear

LINEARISATION

$$g(x_1) = (x_1 - L)^3$$

LINEARISE

$$g'(x_1) = 3(x_1 - L)^2$$

$$g'(x_1^{eq}) = 3(x_1^{eq} - L)^2$$

$$g(x_1^{eq}) = (x_1^{eq} - L)^3$$

Taylor's approximation theorem on g at x_1^{eq}

$$g(x_1) \approx g(x_1^{eq}) + g'(x_1^{eq}).(x_1 - x_1^{eq})$$

for x_1 close to x_1^{eq}

$$(x_1 - L)^3 = \underbrace{(x_1^{eq} - L)^3}_{\text{constant}} + \underbrace{3(x_1^{eq} - L)^2}_{\text{constant}}(x_1 - x_1^{eq})$$

$$\iff (x_1 - L)^3 - (x_1^{eq} - L)^3 = 3(x_1^{eq} - L)^2(x_1 - x_1^{eq})$$

OBSERVE THAT THIS IS A DEVIATION VARIABLE $\overline{x_1}$

$$\dot{x}_2 = \frac{2}{3m}F - \frac{2}{3}g\sin\phi - \frac{2k_1}{3m}(x_1 - L) - \frac{2k_2}{3m}(x_1 - L)^3$$

$$(-) \quad 0 = \frac{2}{3m}F^{eq} - \frac{2}{3}g\sin\phi - \frac{2k_1}{3m}(x_1^{eq} - L) - \frac{2k_2}{3m}(x_1^{eq} - L)^3$$

$$\dot{x}_2 = \frac{2}{3m}(F - F^{eq}) - \frac{2k_1}{3m}(x_1 - x_1^{eq}) - \frac{2k_2}{3m}\left[(x_1 - L)^3 - (x_1^{eq} - L)^3\right]$$

DEVIATION VARIABLE $\overline{F}$

ANOTHER DEVIATION VARIABLE: $\overline{x_1}$

$$\Rightarrow \dot{x}_2 = \frac{2}{3m}\overline{F} - \frac{2k_1}{3m}\overline{x_1} - \frac{2k_2}{3m}.3(x_1^{eq} - L)^3\cancel{(x_1 - x_1^{eq})}.\overline{x_1}$$

$$\Rightarrow \dot{x}_2 = \frac{2}{3m}\overline{F} - \frac{2}{3m}\left(k_1 - k_2 3(x_1^{eq} - L)^3\right).\overline{x_1}$$

$$\dot{\bar{x}}_2 = \underbrace{\boxed{\frac{2}{3m}}}_{a}\bar{F} - \underbrace{\boxed{\frac{2}{3m}\left(k_1 + 3k_2(x_1^{eq} - L)^3\right)}}_{b} \cdot \bar{x}_1$$

(to make it simpler)

TRANSFER FUNCTION

$$\boxed{\dot{\bar{x}}_2 = a\bar{F} - b\bar{x}_1}$$

$\dot{\bar{x}}_1 = ?$

$$\dot{x}_1 = x_2 \quad (-)$$
$$0 = x_2^{eq}$$

$$\dot{\bar{x}}_1 = x_2 - x_2^{eq}$$
$$\Rightarrow \dot{\bar{x}}_1 = \bar{x}_2$$

LINEARISED ↓

$$\boxed{\begin{aligned} \dot{\bar{x}}_1 &= \bar{x}_2 \\ \dot{\bar{x}}_2 &= a\bar{F} - b\bar{x}_1 \end{aligned}}$$

$$\bar{F} \longrightarrow \boxed{G(s)} \longrightarrow \bar{x} = \bar{x}_1$$

$$G(s) = \frac{\bar{x}(s)}{\bar{F}(s)}$$

Apply laplace transform:

$$\left.\begin{aligned} s\bar{x}_1 &= \bar{x}_2 \\ s\bar{x}_2 &= a\bar{F} - b\bar{x}_1 \end{aligned}\right]$$

* plug in $s\bar{x}_1$

$$s^2\bar{x}_1 = a\bar{F} - b\bar{x}_1 \iff (s^2 + b)\bar{x}_1 = a\bar{F}$$

in $G(s) = \frac{\bar{x}(s)}{\bar{F}(s)}$ form:

$$\boxed{\frac{\bar{x}_1}{\bar{F}} = \frac{a}{s^2 + b}} \leftarrow \text{TRANSFER FUNCTION.}$$

↓

(impulse) kick (when $\bar{F}(t) = \delta(t)$)

(step) push (when $\bar{F}(t) = A$)

(frequency) shake (when $\bar{F}(t) = A\sin(\omega t)$

$t \geq 0$

or if a little more general, include 'A'

(Applications of inverse Laplace transform)

SIMULATIONS

- Impulse response

if $\bar{F}(t) = \delta(t) \Rightarrow \bar{F}(s) = 1 \Rightarrow \bar{x}_1(s) = G(s) \cdot \bar{F}(s)$

$$\Rightarrow \bar{x}_1(t) = \mathcal{L}^{-1}\left\{\frac{a}{s^2+b}\right\} = \mathcal{L}^{-1}\left\{\frac{a}{s^2+\sqrt{b}^2}\right\}$$

$$= \frac{a}{\sqrt{b}} \mathcal{L}^{-1}\left\{\frac{\sqrt{b}}{s^2+\sqrt{b}^2}\right\}$$

$$= \frac{a}{\sqrt{b}} \sin(\sqrt{b} \cdot t), \quad t \geq 0$$

$$\bar{x}_1(t) = \frac{a}{\sqrt{b}} \cdot \sin(\sqrt{b} \cdot t)$$

$$\parallel$$

$$x_1(t) - x_1^{eq} = \frac{a}{\sqrt{b}} \sin(\sqrt{b}\, t)$$

$$\Rightarrow x_1(t) = x_1^{eq} + \frac{a}{\sqrt{b}} \cdot \sin(\sqrt{b} \cdot t).$$

- Step response

$$G(s) = \frac{a}{s^2+b}, \quad \bar{F}(t) = 1 \Rightarrow \bar{F}(s) = \frac{1}{s}$$

$$\bar{x}_1(s) = G(s) \cdot \bar{F}(s) = \frac{a}{(s^2+b)(s)}$$

$$\bar{x}_1(t) = \mathcal{L}^{-1}\left\{\boxed{\frac{a}{(s^2+b)s}}\right\}$$

poles: $0, \pm\sqrt{b} \cdot j$

PARTIAL FRACTIONS!!

$$\frac{a}{(s^2+b)s} = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+b}$$

$$\therefore A_1 = \frac{a}{b}, \quad A_2 = \frac{-a}{b}, \quad A_3 = 0$$

$$\Rightarrow \frac{a/b}{s} + \frac{-a/b \cdot s}{s^2+b}$$

$$\mathcal{L}^{-1}\left\{\frac{a/b}{s}\right\} = \frac{a}{b}, \quad \mathcal{L}^{-1}\left\{\frac{\boxed{-\frac{a}{b}} s}{s^2+b}\right\} = -\frac{a}{b}\mathcal{L}^{-1}\left\{\frac{s}{s^2+b}\right\}$$

$$= -\frac{a}{b} \cdot \cos(\sqrt{b} \cdot t)$$

$$\bar{x}_1(t) = \mathcal{L}^{-1}\{\bar{x}_1(s)\}$$

$$= \frac{a}{b} - \frac{a}{b} \cdot \cos(\sqrt{b} \cdot t)$$

$$= \frac{a}{b}\left[1 - \cos(\sqrt{b} \cdot t)\right], \quad t \geq 0$$

- Frequency Response

$$\bar{F}(t) = \sin(\omega t), \quad \omega \in \mathbb{R}$$

$$\left.\begin{aligned} \bar{F}(s) &= \frac{\omega}{s^2+\omega^2} \\ G(s) &= \frac{a}{s^2+b} \end{aligned}\right] \quad \bar{x}_1(s) = \frac{\omega}{s^2+\omega^2} \cdot \frac{a}{s^2+b}$$

$$\downarrow \mathcal{L}^{-1}$$

- Partial fractions expansion (PFE)
- Inverse Laplace tranform (ILT)

Careful!!

We have to treat the case $w^2 = b$ separately, because if $w^2 = b$:

$$\bar{x}_1(s) = \frac{\omega a}{(s^2+b)^2}$$

$\uparrow$ Double pair of imaginary poles.

$\bar{x}_1(t) = \ldots$ approximation for large t

$\downarrow$