The pendulum problem and elliptic integrals

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1 Introduction

This paper is a collection of notes based on an interesting video I watched with my friend, and a series of seminars given by my cal 3 teacher in college. I will reveal the connections between pendulums in physics, elliptic integrals, functional equations, the arithmetic-geometric mean, and more.

I will present these topics in the way that I discovered them, starting with the example of finding the period of a pendulum, then moving on to functional equations and elliptic integrals.

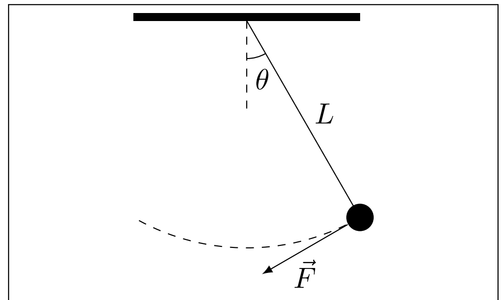
2 The pendulum problem

2.1 The problem

This first part is based on the video my friend and I found titled: The complete elliptic integral of the first kind. The problem: find the period of a pendulum.

We've all learned in mechanics class that the period of a pendulum can be found using the equation

$$T = 2\pi\sqrt{\frac{L}{g}}$$



A simple mechanical pendulum

However this formula is only accurate for small values of θ, so it's only natural to wonder at what the real period formula looks like.

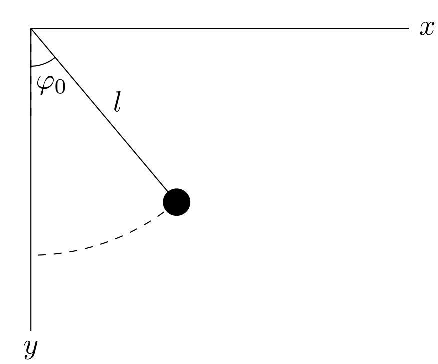
2.2 Setting up the integral

The guy starts off the video by showing the titular **complete elliptic integral of the first kind**, which looks like

$$K(n) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - n^2 \sin^2\theta}}$$

He claims that this integral appears as a solution for certain physics problems such as a pendulum problem. Sounds believable enough. He also mentions that he will be looking at some interesting values of this function.

The setup is as follows:



A pendulum of mass m, attached to a string of length l, oscillating with an amplitude of φ_0

We look at the energy of the pendulum at an angle φ. The x-axis is set as the level of 0 gravitational potential energy.

The kinetic energy $K = \frac{1}{2}mv^2$ and the potential energy $U = -mgl\cos\varphi$. If energy is conserved then the total energy of the pendulum at any point is

$$E = -mgl\cos\varphi_0 = \frac{1}{2}mv^2 - mgl\cos\varphi$$

Where the velocity v is defined as the derivative of the arc length

$$v = \frac{ds}{dt} = \frac{ld\varphi}{dt}$$

So we have the differential equation

$$E = -mgl\cos\varphi_0 = \frac{1}{2}ml^2\dot{\varphi}^2 - mgl\cos\varphi$$

And we can rearange to get

$$\begin{aligned}
\frac{1}{2}ml^2\dot{\varphi}^2 &= mgl(\cos\varphi - \cos\varphi_0) \\
\frac{1}{2}l\dot{\varphi}^2 &= g(\cos\varphi - \cos\varphi_0) \\
\dot{\varphi}^2 &= \frac{2g}{l}(\cos\varphi - \cos\varphi_0) \\
\frac{d\varphi}{dt} &= \sqrt{\frac{2g}{l}(\cos\varphi - \cos\varphi_0)}
\end{aligned}$$

So far pretty simple, only some basic knowledge in mechanics and calculus required.

Since we are trying to find the period, we isolate dt and integrate.

$$\int dt = \sqrt{\frac{l}{2g}} \int \frac{d\varphi}{\sqrt{(\cos\varphi - cos\varphi_0)}}$$

The pendulum takes a quarter period to move from 0 to φ_0 so we get

$$\frac{T}{4} = \sqrt{\frac{l}{2g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{(\cos\varphi - cos\varphi_0)}}$$

And so the true formula for the period of the pendulum is

$$T = 2\sqrt{2}\sqrt{\frac{l}{g}} \int_0^{\varphi_0} \frac{d\varphi}{\sqrt{(\cos\varphi - cos\varphi_0)}}$$

2.3 Start substituting

But wait, I thought the solution was going to look something like our elliptic integral $\int \frac{d\theta}{\sqrt{1-n^2\sin^2\theta}}$?

We are going to have to make a few substitutions.
First, the identity $\cos(2\varphi) = 1 - 2\sin^2\varphi \longrightarrow \cos\varphi = 1 - 2\sin^2\frac{\varphi}{2}$

$$T = 2\sqrt{2}\sqrt{\frac{l}{g}}\int_0^{\varphi_0} \frac{d\varphi}{\sqrt{(1 - 2\sin^2\frac{\varphi}{2} - cos\varphi_0)}}$$

3 Functional Equations

In order to understand the solution above, we must talk about functional equations. Functional equations are similar to differential equations in that their solutions are functions themselves. Consider the functional equation

$$f(xy) = f(x) + f(y)$$

You might know a function that solves this equation, but for the moment, pretend we don't know what that function is, what can we say about this function?

If we take $x = 0, y = 0$ then we see that $f(0) = f(0) + f(0) \Rightarrow f(0) = 0$

References

[1] *Title*, Website Title `real`