# Physics 500A – Quantum Mechanics – Homework assignment 3

Due October 23, 2019, in class

To be picked up by November 6, 2019

## Normalization of number eigenstates [2pt]

In class, I claimed that for the oscillator number eigenstates $|n\rangle = c_n(a^\dagger)^n|0\rangle$, the normalization constant is $c_n = 1/\sqrt{n!}$ . Show this.

## Exponentials of operators [2pt]

In dealing with coherent states, exponentials of operators show up quite often. For two operators $A$ and $B$ which do not necessarily commute, derive the following formula:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}[A, [A, \ldots [A, B]]]$$

where the $n$-th term in the expansion has $n$ nested commutators. To do so, you may consider taking $A \to \alpha A$ for some constant $\alpha$, and then constructing a power expansion in $\alpha$.

## Squeezed states [8pt]

Let's introduce so-called "squeezed states" for the oscillator, which are a generalization of coherent states. Again, using the BCH formula in various forms will be helpful. To start with, take the following operator: $S(\phi) = e^{\frac{1}{2}(\phi aa - \phi^* a^\dagger a^\dagger)}$, where $\phi$ is a complex number, and $a$, $a^\dagger$ are the usual creation and annihilation operators for the oscillator. Squeezed states are defined as $|\phi, z\rangle = S(\phi)D(z)|0\rangle$. The state with $z = 0$ is also called "squeezed vacuum".

*a) [1pt]* Show that $S$ is unitary.

*b) [2pt]* Find $S^\dagger a S$ and $S^\dagger a^\dagger S$ in terms of $a$ and $a^\dagger$.

*c) [1pt]* Let $P \equiv p/\sqrt{m\omega}$, and $Q \equiv q\sqrt{m\omega}$. Note that the Hamiltonian is symmetric when expressed in terms of $P$ and $Q$. Show that $P$ and $Q$ have the same quantum uncertainty in coherent states, i.e. $\Delta P = \Delta Q$, in addition to $(\Delta P)(\Delta Q) = \hbar/2$.

*d) [2pt]* Now take $\phi$ real. Using your results from part *b)*, find the expectation values of $Q$ and $P$ in squeezed states.

*e) [2pt]* Using your result from part *d)*, express $\Delta Q$ and $\Delta P$ in the squeezed state $|\phi, z\rangle$ in terms of $\Delta Q$, $\Delta P$ in the corresponding coherent state $|z\rangle$. Show that squeezed states are also minimum-uncertainty states, just like coherent states, in other words show that $(\Delta Q)(\Delta P) = \hbar/2$. Note however that, unlike coherent states, the squeezed states have $\Delta P \neq \Delta Q$, which is exactly why they are called "squeezed".

## Quantized light [13pt]

*a) [3pt]* When thinking about quantized electromagnetic field in a cavity, the oscillator energy eigenstates $|n\rangle$ are photon number eigenstates with $n$ photons. Assume that the distribution of photons in a given cavity mode is the thermal Gibbs distribution (i.e. you are looking at the black-body radiation). What is the probability $P(n)$ to find $n$ photons in the thermal ensemble? Express $P(n)$ in terms of the average number of photons $\langle n \rangle$. Find an approximate expression for $P(n)$ when $\langle n \rangle$ is large (make sure your distribution is normalizable). Using a computer algebra program, plot $P(n)$ and your approximate expression for several values of $\langle n \rangle$.

*b) [2pt]* Now instead of the thermal ensemble, take your cavity mode to be in a coherent state $|z\rangle$. What is the probability $P(n)$ to find $n$ photons in the coherent state? Express $P(n)$ in terms of the average number of photons $\langle n \rangle$. What is the approximate form of the distribution when the average number of photons $\langle n \rangle$ is large? Using a computer algebra program, plot $P(n)$ and your approximate expression for several values of $\langle n \rangle$.

*c) [1pt]* Recall that for quantized electromagnetic field in a cavity, the vector potential of each mode is proportional to $(a\, e^{-i\omega_k t + i\mathbf{k}\cdot\mathbf{x}} + a^\dagger e^{i\omega_k t - i\mathbf{k}\cdot\mathbf{x}})$. Show that the electric field of each mode is proportional to $\mathcal{E} \equiv Q \cos\chi + P \sin\chi$, where $\chi = \omega_k t - \mathbf{k}\cdot\mathbf{x} + \pi/2$ is the phase of the wave.

*d) [2pt]* Find the average electric field $\bar{\mathcal{E}}$ in a coherent state $|z\rangle$, in a squeezed vacuum state $|\phi, z{=}0\rangle$, and in a general squeezed state $|\phi, z\rangle$, for real $\phi$.

*e) [3pt]* Find the uncertainty in the electric field $\Delta\mathcal{E}$ in a coherent state $|z\rangle$, in a squeezed vacuum state $|\phi, z{=}0\rangle$, and in a general squeezed state $|\phi, z{\neq}0\rangle$, for real $\phi$.

*f) [2pt]* When measuring electric field in a quantum state (such as laser light), you can think of $\bar{\mathcal{E}}$ as the signal, and of $\Delta\mathcal{E}$ as the quantum noise. Pick several coherent and squeezed states, and plot $\bar{\mathcal{E}}$ and $\bar{\mathcal{E}} \pm \Delta\mathcal{E}$ as a function of the phase $\chi$. Do your plots look like the measured signal (see attached photocopy)?

Of the various methods that have been proposed to reconstruct the quantum state numerically from the set of measured distributions $P_\theta$, two are employed here. The first method makes use of the fact that the distributions $P_\theta(x_\theta)$ are the marginals of the Wigner function $W(x, y)$ in rotated coordinates;

$$P_\theta(x_\theta) = \int_{-\infty}^{\infty} W(x_\theta \cos\theta - y_\theta \sin\theta, x_\theta \sin\theta + y_\theta \cos\theta)\, \mathrm{d}y_\theta \quad (1)$$

where $y_\theta = -x\sin\theta + y\cos\theta$. Therefore $W(x, y)$ can be obtained from the set $P_\theta$ by back-projection via the inverse Radon transform[2]. The second method furnishes the elements of the density matrix in the Fock basis via integration of the distributions $P_\theta$ over a set of pattern functions[3,4]. In contrast to the inverse Radon transform, this procedure does not involve any filtering of the experimental data and also allows an estimation of the propagation of statistical errors.

### The experiment

The experimental set-up is shown in Fig. 1. Central to the experiment is a monolithic standing-wave lithium-niobate optical parametric oscillator (OPA)[13,24], pumped by a frequency-doubled continuous-wave Nd:YAG laser (1,064 nm). The infrared laser wave is filtered by a high-finesse mode-cleaning cavity, which transmits 75% of the laser power. Its narrow linewidth of 170 kHz suppresses the high-frequency technical noise of the laser, yielding a shot-noise-limited local oscillator for light powers in the milliwatt range at frequencies $\geqslant$1 MHz (ref. 13). The pump wave $2\omega$ (power ~20–30 mW) for the OPA is generated by resonant second harmonic generation.

In the past OPAs have been frequently used as sources of non-classical light[10,13,25–28]. Operated below threshold, the OPA is a source of squeezed vacuum. We studied the field's spectral compnents around a frequency offset by $\Omega/2\pi = 1.5$ or 2.5 MHz from the optical frequency $\omega$, to avoid low-frequency laser excess noise. To generate bright light (that is, with non-vanishing average electric field at the frequencies $\omega \pm \Omega$), we employ the OPA in a dual port configuration[28]. A very weak wave split off the main laser beam is phase-modulated by an electro-optic modulator (EOM) at the frequency $\Omega$ (modulation index $\beta \ll 1$) and injected into the

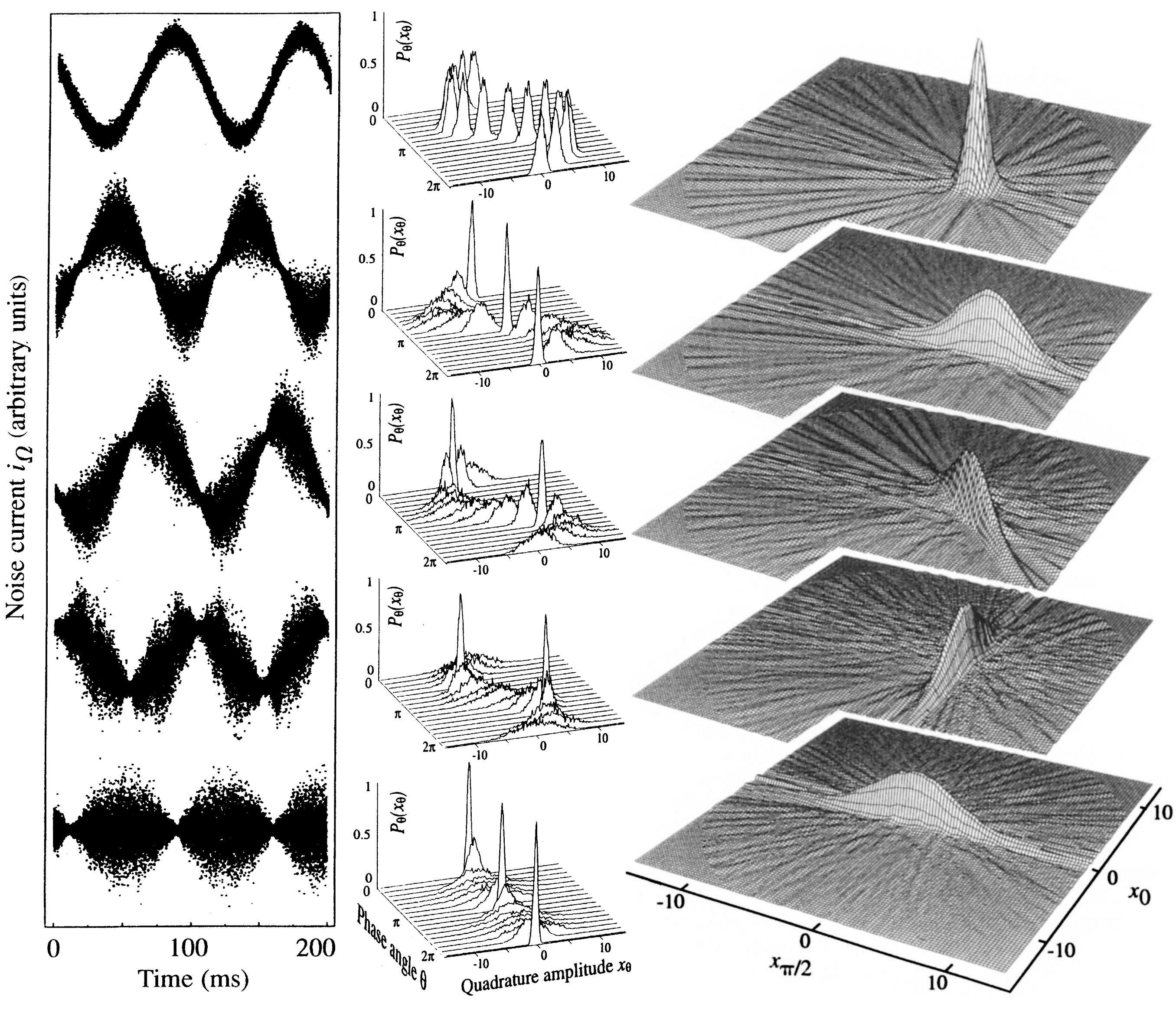


**Figure 2** Noise traces in $i_\Omega(t)$ (left), quadrature distributions $P_\theta(x_\theta)$ (centre), and reconstructed Wigner functions (right) of generated quantum states. From the top: Coherent state, phase-squeezed state, state squeezed in the $\phi = 48°$-quadrature, amplitude-squeezed state, squeezed vacuum state. The noise traces as a function of time show the electric fields' oscillation in a $4\pi$ interval for the upper four states, whereas for the squeezed vacuum (belonging to a different set of measurements) a $3\pi$ interval is shown. The quadrature distributions (centre) can be interpreted as the time evolution of wave packets (position probability densities) during one oscillation period. For the reconstruction of the quantum states a $\pi$ interval suffices.