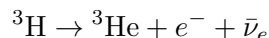


Physics 500A – Quantum Mechanics – Homework assignment 6

Due December 4, 2019, in class

Tritium beta decay [5pt]

The beta decay of the tritium nucleus is



where the energy of the emitted electron is of the order of several keV, and so you can ignore the recoil of the nucleus. As a result of the decay, the binding potential for the atomic electron changes from $-e^2/r$ to $-2e^2/r$. In this problem, take the potential change as instantaneous.

- a) [3pt] For the tritium atom in the ground state ($n=1$) before the decay, find the probability p_n that the atom is in the n -th excited state after the decay. You should find $p_1 = 512/729 \approx 0.7$. What do you find for p_2 ? What do you find for p_3 ?
- b) [2pt] Now that you know p_n , evaluate numerically the sum of the probabilities $P = \sum_{n=1}^{\infty} p_n$. You should find that P is less than 1 by several percent. Give a physical interpretation of the non-zero value of $1-P$.

Gauge invariance [5pt]

In class, I claimed that under gauge transformations, the wave function gets multiplied by a phase. Show this. In other words, show that if $\psi(\mathbf{x}, t)$ satisfies the Schrödinger equation

$$i\hbar \partial_t \psi = \left(\frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + V(\mathbf{x}) + e\varphi \right) \psi,$$

then the gauge-transformed wave function $\psi' = e^{ie\alpha/\hbar} \psi$ satisfies

$$i\hbar \partial_t \psi' = \left(\frac{(\mathbf{p} - e\mathbf{A}')^2}{2m} + V(\mathbf{x}) + e\varphi' \right) \psi'.$$

Here $\alpha = \alpha(\mathbf{x}, t)$ is the gauge function, and $\mathbf{A}' = \mathbf{A} + \nabla\alpha$ and $\varphi' = \varphi - \partial_t\alpha$ are the gauge transformed electromagnetic potentials. This tells you that the Schrödinger equation is invariant under the gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu\alpha$, $\psi \rightarrow e^{ie\alpha/\hbar}\psi$.

Lorentz force [3pt]

Recall that the time dependence of a Heisenberg-picture operator \mathcal{O} is

$$\frac{d\mathcal{O}}{dt} = \frac{\partial\mathcal{O}}{\partial t} + \frac{i}{\hbar} [H, \mathcal{O}].$$

Use this to derive the Lorentz force equation in Quantum Mechanics,

$$\frac{d}{dt} (m\mathbf{v}) = e\mathbf{E} + \frac{e}{2} (\mathbf{v} \times \mathbf{B} - \mathbf{B} \times \mathbf{v}),$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity operator, and $\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ are the electric and magnetic fields, which are functions of the operator \mathbf{x} .

Aharonov-Bohm effect [10pt]

Consider a particle of charge e moving between two very long concentric cylindrical shells with radii $\rho = a$ and $\rho = b > a$, in cylindrical coordinates. The particle can not penetrate the shells, hence its wave function $\psi(\mathbf{x})$ satisfies $\psi(\rho=a) = \psi(\rho=b) = 0$. Inside the inner shell there is a constant magnetic field $B \parallel \hat{z}$ with flux Φ , while $B = 0$ everywhere outside the inner shell. In other words, the particle can only move in the region where there is no magnetic field. The motion along the \mathbf{z} axis is just a plane wave, so let us look at z -independent states only.

- a) [4pt] Because of the cylindrical symmetry, the wave function can be taken as $\psi(\mathbf{x}) = e^{i\ell\varphi}\psi_\ell(\rho)$. Explain why ℓ must be an integer. Solve the Schrödinger equation, and derive a transcendental equation which (implicitly) determines the energy levels for the particle in terms of its angular momentum ℓ and the flux Φ . Express your answer in terms of the Bessel functions $J_\nu(z)$ and $N_\nu(z)$ [the Bessel function of the second kind $N_\nu(z)$ is often denoted as $Y_\nu(z)$, depending on the book].
- b) [2pt] Show that the energy spectrum (i.e. the set of all possible energies) remains the same when $e\Phi/(2\pi\hbar)$ changes by an integer. In particular, this implies that the energy spectrum for $e\Phi/(2\pi\hbar) = \text{integer}$ is indistinguishable from the energy spectrum for $\Phi = 0$.
- c) [4pt] Use a computer algebra program such as **Mathematica** to plot the energies of the first four lowest-energy states as a function of the flux Φ , starting with $\Phi = 0$, up to $e\Phi/(2\pi\hbar) = 3$. You can pick some arbitrary values for the radii, e.g. $b = 3a$. How do the energies of the ground state, first excited state, second excited state etc depend on the flux Φ ? [Note: “the first four lowest-energy states” means “the first four lowest-energy states”. It does *not* mean $\ell = 1, 2, 3, 4$.]

Questions for Prof. Strickland [3pt]

[Optional for extra credit] Write down one question that you asked Prof. Donna Strickland on the morning of November 27-th, and give a brief version of her answer.