Physics 410 – Mathematical Methods – Homework assignment 3

Due October 18-th, 2018, in class

For each question, explain what you are doing to solve the problem. Do not just write down the answer. Feel free to use a computer algebra program such as Mathematica, Python etc; when doing so, attach your computer code, the code output, and a human-readable explanation of what your code is doing.

Some ODEs

[2pt] Find two independent solutions of the Airy equation

$$y'' = xy,$$

by using the series expansion about $x = 0$ and finding the coefficients of the series. (You don't need to epxress the solutions in terms of $Ai(x)$ and $Bi(x)$.)

[2pt] Solve the equation

$$4x^2y'' + 4xy' + (4x^2 - 1)y = 0$$

as a Frobenius series in powers of x. Sum the series to obtain closed-form expressions for the two solutions.

[2pt] Consider the equation

$$(1 - x^2)y'' - 2xy' = 0\,.$$

Clearly, one solution is just a constant, $y(x) = C$. Use the Frobenius method to find the second solution by expanding about $x = 1$. Sum the series to obtain a closed-form expression for the second solution.

Double-well potential [3pt]

Recall that in one-dimensional quantum mechanics, the ground state energy of a particle bound by the potential of two delta functions, $V(x) = -\frac{\hbar^2}{2m}v[\delta(x-a) + \delta(x+a)]$ (with positive v and a) is determined by the equation

$$z = 1 + e^{-Az}\,, \tag{1}$$

where $A \equiv va$, $z \equiv 2\kappa/v$, and the energy is $E = -\hbar^2\kappa^2/2m$. Equation (1) needs to be solved to find $z(A)$, which in turn gives the ground state energy $E(v, a)$. For two widely separated delta functions (large A), we clearly have $z = 1 + \ldots$, where the "…" terms are small when A is large. Let's find how small. I'll write the solution to the above equation as

$$z = 1 + \frac{1}{A}f(A)\,.$$

[2pt] Convince yourself that $f(A)$ can be written as $f(A) = \sum_{n=1}^{\infty} c_n(Ae^{-A})^n$, in other words the small parameter is e^{-A}, not $1/A$. Find the coefficients c_1, c_2, c_3.

[1pt] Note that when A is large, $g \equiv 1/A$ is small. Plot the function $e^{-A} = e^{-1/g}$, as a function of g. Find the Taylor series expansion of this function for small g, and write down this Taylor series next to your plot.

[2pt] [Optional] Use a computer program to calculate c_n for n up to 100. What does your result indicate about the convergence of the series you found in part *a)*?

Funny perturbation theory [9pt]

a) [2pt] Consider the following function

$$Z(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\varphi\, e^{-\frac{1}{2}\varphi^2 - \lambda\varphi^4} \,. \tag{2}$$

By assuming that λ is small and expanding the exponential in powers of λ, you generate the expansion $Z(\lambda) = \sum_n c_n \lambda^n$. Such a procedure is called "perturbation theory", and is widely used in all fields of physics. Find c_n.

b) [1pt] In practice, one keeps only a finite number of terms in the perturbative expansion, i.e. one approximates $Z(\lambda) \approx Z_m(\lambda)$, where $Z_m(\lambda) = \sum_{n=0}^{m} c_n \lambda^n$ is the result of the perturbation theory to m-th order. Plot $Z_m(\lambda)$ as a function of λ for $m = 0, 1, 2, 3, 4$. Write down an explicit formula for $Z(\lambda)$ to 4-th order.

c) [1pt] We can also evaluate $Z(\lambda)$ explicitly (e.g. numerically), without using any expansions. Plot the exact result for $Z(\lambda)$ as a function of λ. Comment on how your exact result compares with the perturbation theory result from part *b)*. Do you get better approximations to the exact answer by going to higher and higher order in perturbation theory?

d) [2pt] Find the approximate behavior of the perturbation theory coefficients c_n at large n. What is the radius of convergence of the perturbation theory series for $Z(\lambda)$?

e) [2pt] For a given small fixed value of λ, estimate at which order (as a function of λ) in the expansion the perturbation theory stops making sense in this problem.

f) [1pt] Write a paragraph explaining what lesson you learned from this problem.