Physics 410 – Mathematical Methods – Homework assignment 1

Due September 20-th, 2018, in class

For each question, explain what you are doing to solve the problem. Do not just write down the answer. Feel free to use a computer algebra program such as Mathematica, Python etc; when doing so, attach your computer code, the code output, and a human-readable explanation of what your code is doing.

Suggested reading: S. Lea, *Mathematics for Physicists*, ch. 2.

Dimensional analysis

(1pt) Consider the following generalized Gaussian integral:

$$I(a) = \int_0^\infty dx\; x^n\, e^{-ax^2}\,,$$

with $a > 0$, $n > 0$. Find how $I(a)$ depends on a. You don't need the exact answer for $I(a)$, just find the a-dependence.

(1pt) Black-body radiation is a gas of photons in thermal equilibrium at temperature T. For the black-body radiation in infinite volume, find how the pressure of the photon gas depends on temperature. You don't need the exact answer for $p(T)$, just find the T-dependence.

(1pt) The mean energy density for the black-body radiation is given by

$$u(\omega)\, d\omega = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3 d\omega}{e^{\frac{\hbar\omega}{kT}} - 1}\,,$$

where ω is the radiation frequency. This energy density has a maximum at a certain frequency $\omega_{\max}$. Find how $\omega_{\max}$ depends on temperature. You don't need the exact answer for $\omega_{\max}(T)$, just find the T-dependence.

Complex numbers basics

(2pt) Show that the equation

$$z = ae^{i\varphi} + be^{-i\varphi}$$

represents an ellipse in the complex plane. Here a and b are fixed *complex* constants, and the angle φ covers the range between 0 and 2π. Determine the semi-major and the semi-minor axes and the orientation of the ellipse.

(1pt) Find all solutions to the equation $\cos z = 100$. Plot the solutions in the complex z plane.

(1pt) Find all complex numbers z which satisfy $z = \ln(-5)$. Plot the values of z in the complex plane.

Analytic functions

(2pt) One of the two functions $u_1 = 2(x-y)^2$ and $u_2 = \frac{1}{3}x^3 - xy^2$ is the real part of an analytic function $w(z) = u + iv$. Which is it? Find the function $v(x, y)$ and write w as a function of z.

(1pt) Determine the (Taylor or Laurent) series for the function $f(z) = \frac{\cos z}{z-1}$ about $z = 1$.

(2pt) Determine all Taylor or Laurent series of the function $\frac{1}{z^2+1}$ about $z = i$.

Some integrals

Do the following integrals using the appropriate contours in the complex plane.

(2pt)

$$\int_0^{2\pi} \sin^{2n}\theta \; d\theta$$

(2pt)

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} dx$$

(2pt)

$$\mathcal{P} \int_{-\infty}^{\infty} \frac{x}{x^3 + 1} dx \,,$$

where $\mathcal{P}$ denotes the principal value.