Phys 515 Assignment #4 Due: Nov 21, 1 pm

In this assignment, you analyze an experiment with statistical and systematic uncertainty. Using a Bayesian approach, you find credible intervals for parameters using a Markov Chain Monte Carlo.

The experiment is setup to measure the position, width, and brightness of a distant object. To simplify the problem we work with a 1D image. Your telescope optics projects photons onto a sensor with 0.5 mm wide pixels. Each pixel counts the number of photons that strike that pixel. Due to thermal effects, additional counts can appear in each pixel, known as dark noise. When operated in a dark room, each pixel has equal probability to have a dark noise count per unit time. The photons from the object are scattered by the atmosphere, resulting in Gaussian smearing. The object has uniform brightness across its surface. The datafile has the counts in the 20 pixels from $x = 0$ to $x = 10$mm.

We model this experiment by first working out the photon density from the object on the sensor. Convolving a uniform distribution centered at x_0 with width w with a Gaussian having standard deviation σ, yields,

$$f_s(x|s, x_0, w, \sigma) = \frac{s}{2w}\left[\mathrm{erf}\left(\frac{x - \left(x_0 - \frac{1}{2}w\right)}{\sqrt{2}\sigma}\right) - \mathrm{erf}\left(\frac{x - \left(x_0 + \frac{1}{2}w\right)}{\sqrt{2}\sigma}\right)\right] \tag{1}$$

where s is the total number of photons arriving at the sensor. In addition, there are dark counts uniformly distributed across the sensor, so the density of both signal and noise counts is given by

$$f_c(x|s, x_0, w, \sigma, d) = f_s(x|s, x_0, w, \sigma) + d \tag{2}$$

The expected number of counts in pixel i spanning (a_i, b_i) is therefore

$$\nu_i(s, x_0, w, \sigma, d) = \int_{a_i}^{b_i} f_c(x|s, x_0, w, \sigma, d)\, dx \tag{3}$$

The smearing parameter, σ , is not perfectly known. As a result of separate studies, your prior degree of belief for σ is represented by a Gaussian pdf with mean σ_0 and standard deviation σ_σ. Your prior degree of belief for the other parameters follow uniform distributions. For all parameters, negative values are not allowed (in other words, the prior distributions have zero density below zero).

1. Using Bayes theorem, **show** that the log of the posterior probability density for all parameters is given by:

$$\ln P(s, x_0, w, \sigma, d|\vec{n}) = c + \sum_i [n_i \ln \nu_i(s, x_0, w, \sigma, d) - \nu_i(s, x_0, w, \sigma, d)] - \frac{1}{2}\frac{(\sigma - \sigma_0)^2}{\sigma_\sigma^2}$$

   where n_i is the number of counts in pixel i and c is a constant (independent of the parameters and not necessary to evaluate), provided $s > 0$, $x_0 > 0$, $w > 0$, $\sigma > 0$, and $d > 0$. Otherwise, $P = 0$.

2. Write a program that implements the Markov Chain Monte Carlo algorithm to produce points in the parameter space $\vec{\theta} = (s, x_0, w, \sigma, d)$ distributed with density proportional to the probability density, $P(s, x_0, w, \sigma, d|\vec{n})$. (See pages 133-135). Start the chain at the nominal values shown in the table below, and use steps drawn <u>uniformly</u> from the hypercube:

   $\Delta s \in [-d_s, +d_s]$ $\Delta x_0 \in [-d_x, +d_x]$ $\Delta w \in [-d_w, +d_w]$ $\Delta \sigma \in [-d_\sigma, +d_\sigma]$ $\Delta d \in [-d_d, +d_d]$

   Parameter table:

parameter	symbol	nominal value	related parameter(s)
photons from source	s	200.	$d_s = 10.$
centre of object	x_0	4.5	$d_d = 0.05$
width of object	w	2.5	$d_w = 0.1$
smearing parameter	σ	$\sigma_0 = 0.4$	$d_\sigma = 0.05,\ \sigma_\sigma = 0.2$
dark noise density	d	12.	$d_d = 1.$

   Your program should define variables with simple names and assign these values together in one location. The rest of the program should only use variables – to make your program more robust and easier to spot errors. To avoid underflow, your program should not evaluate $P(\vec{\theta})$ directly; only evaluate logarithms and ratios of the posterior probability density. So, if a proposed step from $\vec{\theta}$ to $\vec{\theta}'$ has $P(\vec{\theta}') > 0$ and $\ln P(\vec{\theta}') < \ln P(\vec{\theta})$, to decide if the step is accepted, you should calculate the ratio of the probability densities as follows:

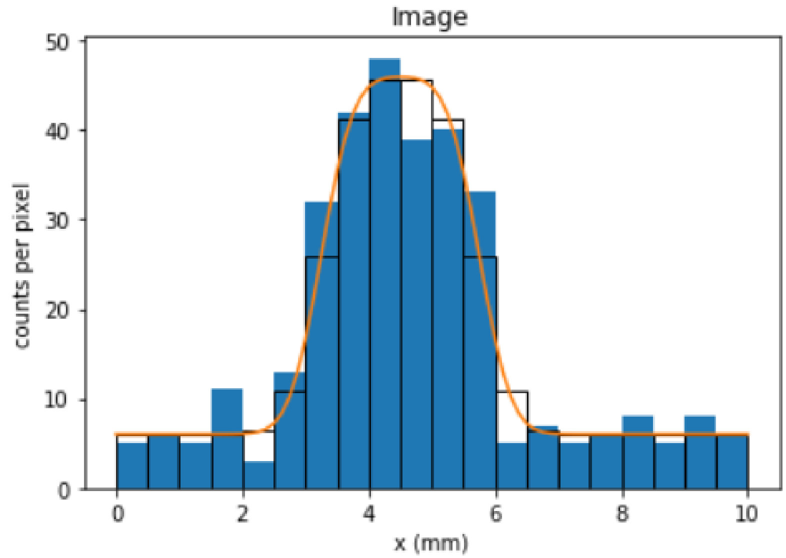
$$\frac{P(\vec{\theta}')}{P(\vec{\theta})} = \exp\left[\ln P(\vec{\theta}') - \ln P(\vec{\theta})\right]$$

   Note: It is typical to discard the first several thousand points (known as "burn-in") because the first points may not be representative (especially if first point is far away from the highest density region). For this exercise, it is not necessary.

3. After testing your program with short chains, run your program to produce a MCMC chain with 5×10^4 points and the data sample provided. The fraction of accepted steps should be about 66%. Show scatter plots of all pairs of the 5 parameters and comment on any pair showing strong correlation (explain why the correlation exists). In python, the scatter_matrix method from pandas allows you to do this quite efficiently. Report the median value and 90% central credible intervals for each parameter. The numpy.median and numpy.percentile methods make this easy to do. Comment on whether your knowledge about the systematic parameter, σ, is significantly improved as a result of this analysis.

<u>**Hints:**</u>

- To evaluate the integral in equation (3) use, $\int \operatorname{erf} x \ dx = x \operatorname{erf} x + e^{-x^2}/\sqrt{\pi} + c$
- As a cross check, you may compare you calculations with the figure on the following page.



The orange curve shows the count density (equation 2), using the values for the parameters in the table above. The bars outlined with black lines show the expected counts in each pixel (equation 3). The blue histogram shows the counts in each pixel that appears in the datafile.

Submit your .ipynb file to the course website.