Quantum Mechanics – Homework assignment 5

Due November 20, 2019, in class

Born approximation and the optical theorem [6pt]

For a spherically symmetric potential, prove the optical theorem to lowest order in the Born approximation. Namely, do the following.

a) [3pt] Using the first-order Born approximation, show that the total cross section is

$$\sigma_{\text{tot}} = \frac{m^2}{\pi\hbar^4}\int d^3x\, d^3x'\, V(r)V(r')\frac{\sin^2(k|\mathbf{x}-\mathbf{x}'|)}{k^2|\mathbf{x}-\mathbf{x}'|^2}\,.$$

b) [3pt] Find the imaginary part of the forward scattering amplitude to the lowest non-vanishing order in the expansion of $f(\theta)$ in powers of V, and show that $\mathrm{Im} f(\theta{=}0)$ is equal to $\frac{k}{4\pi}\sigma_{\text{tot}}$ where σ_{tot} is given by the expression above.

Scattering by a hard sphere [5pt]

Scattering by a hard sphere is a simple toy model for scattering in three dimensions (like the infinite square well is a toy model for bound states). The potential for the hard sphere is $V = \infty$ for $r < R$, and $V = 0$ for $r > R$. The phase shifts δ_ℓ for the hard sphere scattering are determined by

$$\tan\delta_\ell = \frac{j_\ell(kR)}{n_\ell(kR)}\,.$$

a) [2pt] Plot the total (integrated over all angles) cross section σ_{tot} as a function of kR. Give a physical interpretation to your plot.

b) [3pt] Plot the differential cross section $d\sigma/d\Omega$ as a function of the scattering angle θ for several different energies, from low $kR \ll 1$ to high $kR \gg 1$. Produce plots for $kR = 0.1, 0.5, 1, 2, 10$. For $kR = 10$, produce a semi-logarithmic plot. Compare your high-energy cross section with the high-energy cross section for scattering of classical electromagnetic waves by a conducting sphere (Jackson's E&M, Figure 10.16), and give a physical interpretation to your plot of $d\sigma/d\Omega$.

Attractive potential: One-dimensional scattering [6pt]

Consider scattering in one dimension. A particle with energy E in incident from $x = -\infty$ upon the potential well of depth V_0, namely $V(x) = -V_0 < 0$ for $|x| < a$, and $V(x) = 0$ for $|x| > a$.

a) [2pt] Find the transmission probability T as a function V_0, a, and E.

b) [2pt] For some fixed value of V_0, plot your transmission probability as a function of E (or rather of $2ma^2E/\hbar^2$). Which values of E give $T{=}1$ (perfect transmission)? For a fixed small value of E, plot your transmission probability as a function of V_0 (or rather of $2ma^2V_0/\hbar^2$). Your plot $T(V_0)$ should have peaks. Determine the locations of these peaks and comment on their physical interpretation.

c) [2pt] Write your wave function for $|x| > a$ as $\psi(x) = e^{ikx} + f(\theta)e^{ikr}$, where $r \equiv |x|$, and $\theta = 0$ corresponds to $x > 0$, while $\theta = \pi$ corresponds to $x < 0$. The analogue of the total cross section in one dimension is $\sigma_{\text{tot}} = |f(0)|^2 + |f(\pi)|^2$. Pick a value of V_0, and plot σ_{tot} as a function of E (or rather of $2ma^2E/\hbar^2$). Pick a value of E, and plot σ_{tot} as a function of V_0 (or rather of $2ma^2V_0/\hbar^2$).

Attractive potential: three-dimensional scattering [7pt]

Consider now scattering by the attractive potential well in three dimensions, $V = -V_0 < 0$ for $r < R$, and $V = 0$ for $r > R$. Recall that both the differential cross section $d\sigma/d\Omega$ and the total cross section σ_{tot} can be expressed in terms of phase shifts δ_ℓ.

a) [2pt] Derive the expression for $\tan\delta_\ell$ in terms of V_0, R, and kR.

b) [2pt] For some fixed value of V_0, plot the total (integrated over all angles) cross section σ_{tot} as a function of kR. Give a physical interpretation to your plot. Comment on how the energy dependence of the total cross section changes as you vary V_0.

c) [3pt] Now consider low-energy scattering, i.e. small kR. Find the s-wave phase shift δ_0 for small kR. For a fixed small value of kR, plot the total cross-section σ_{tot} as a function of V_0 (or rather of $2mR^2V_0/\hbar^2$). Your plot $\sigma_{\text{tot}}(V_0)$ should have peaks. Find the locations of these peaks and comment on their physical interpretation.