Physics 410 – Mathematical Methods – Homework assignment 5

Due December 5-th, 2018, by 6pm in my office

(slide under the door if I'm not in)

For each question, explain what you are doing to solve the problem. Do not just write down the answer. Feel free to use a computer algebra program such as Mathematica, Python etc; when doing so, attach your computer code, the code output, and a human-readable explanation of what your code is doing.

Exponentials of operators [4pt]

a) For two $N \times N$ matrices (or two operators) A and B which do not necessarily commute, derive the following formula:

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \dots$$

To do so, you may consider taking $A \to \alpha A$ for some constant α, and then constructing a power expansion in α. Convince yourself that the n-th term indeed has $1/n!$, and n nested commutators.

b) Use your formula from part *a)* to show that for a rotation implemented by a 3×3 orthogonal matrix R, the expectation values of the angular momentum operator $\mathbf{J}$ transform as a vector, i.e. that the components transform as

$$\langle\psi|J_k|\psi\rangle \to {}_R\langle\psi|J_k|\psi\rangle_R = R_{kl}\langle\psi|J_l|\psi\rangle\,.$$

You can consider just the case of the rotation by ϕ about the z-axis and show that $\langle J_x\rangle \to \langle J_x\rangle \cos\phi - \langle J_y\rangle \sin\phi$. Here $\langle \dots \rangle = \langle\psi| \dots |\psi\rangle$, as usual. Recall that the rotation operator is $\mathcal{D}(R) = \exp(-i\,\mathbf{J}\cdot\mathbf{n}\,\phi)$ (you can set $\hbar = 1$). Do not assume that you have a spin-1/2 particle.

Rotation algebra [5pt]

The algebra of rotations is defined by the commutation relations

$$[J_k, J_l] = i\epsilon_{klm}J_m\,, \tag{1}$$

where ϵ_{klm} is the totally antisymmetric Levi-Civita tensor, with $\epsilon_{123} = 1$. In Quantum Mechanics, J_k represent angular momentum operators, whose eigenstates are denoted by $|j, m\rangle$, with

$$\mathbf{J}^2|j, m\rangle = j(j+1)|j, m\rangle\,, \qquad J_z|j, m\rangle = m|j, m\rangle\,.$$

The parameter j (called "spin") can be one of 0, 1/2, 1, 3/2, etc, and for a given value of j, the parameter m ranges from $-j$ to j, in integer steps.

a) The raising and lowering operators are defined by $J_\pm \equiv J_x \pm iJ_y$. Their action changes the value of m by one unit, $J_\pm|j,m\rangle = C_\pm|j,m\pm 1\rangle$, where $C_\pm$ are some normalization constants. Calculate $C_\pm$ by demanding that the states are normalized, $\langle j,m|j',m'\rangle = \delta_{jj'}\delta_{mm'}$.

b) Using your result from part *a)*, write a computer algebra program that for a given j finds the matrix representation of the operators J_k satisfying (1) as $(2j+1)\times(2j+1)$ square matrices. Check that for $j = 1/2$ your matrices become $J_k = \frac{1}{2}\sigma_k$, where σ_k are the Pauli matrices. What is the matrix representation of J_k for $j = 5/2$?

c) In the Euclidean three-dimensional space, rotations of vectors can be implemented by three $SO(3)$ matrices $R_a(\theta_a)$. Each matrix describes a rotation about the axis x_a by the angle θ_a, for example

$$R_3(\theta_3) = \begin{pmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and similarly for $R_1(\theta_1)$, $R_2(\theta_2)$. A completely general rotation matrix parametrized by the above three angles θ_a can be written as $R(\boldsymbol{\theta}) = \exp(-i\theta_a T^a)$, with an implicit summation over a. The matrices T^a are called the *generators* of the group $SO(3)$. Find the three generators T^a (as 3×3 matrices). Check that your generators satisfy Eq. (1).

SU(2) and SO(3) [5pt]

Your generators T^a of $SO(3)$ satisfy Eq. (1) and give $\exp(-i\theta_a T^a) \in SO(3)$. On the other hand, the matrices $\tau^a = \frac{1}{2}\sigma^a$ (where σ^a are the Pauli matrices) also satisfy (1), and give $\exp(-i\theta_a\tau^a) \in SU(2)$. Thus τ^a are the generators of $SU(2)$, and you see that the generators of both $SU(2)$ and $SO(3)$ obey the same Lie algebra (1). You might think this implies that $SO(3)$ is isomorphic to $SU(2)$, but that's not quite the case.

a) For every real vector $\mathbf{x}$ in the three-dimensional Euclidean space, we can associate a Hermitian traceless 2×2 matrix $X \equiv x_i\sigma^i$, where σ^i are the Pauli matrices (implicit summation over i here). For vectors $\mathbf{x}$ and $\mathbf{y}$, show that $\mathbf{x}^2 = -\det(X)$, and $\mathbf{x}\cdot\mathbf{y} = -\frac{1}{2}(\det(X+Y) - \det(X) - \det(Y))$.

b) Show that a transformation $X \to X' = AXA^\dagger$, where A is a 2×2 unitary matrix with determinant 1, *i)* leaves X Hermitian and traceless, and *ii)* leaves $\mathbf{x}\cdot\mathbf{y}$ invariant. Thus, the dot product is preserved, and the transformation $X \to X'$ corresponds to a rotation of the vector $\mathbf{x}$.

c) You have seen that a rotation of the position vector $\mathbf{x}$ can be implemented either by $x_i \to R_{ij}x_j$ with $R \in SO(3)$, or by $X \to AXA^\dagger$, with $A \in SU(2)$. Find the explicit relation between R and A, expressing the components R_{ij} in terms of A. Your relationship should have the property that for every element R of $SO(3)$

there are *two* corresponding elements A of $SU(2)$. Because of this, $SU(2)$ is called a "double cover" of $SO(3)$, which is written as $SO(3) = SU(2)/\mathbf{Z}_2$. Hint: use $\mathrm{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$.

A representation of Z_3 [2pt]

Recall that the cyclic group Z_3 is a group of three elements e, a, b (e is the identity element) which obey the multiplication rule $aa = b$, $bb = a$, $ab = ba = e$. Consider the following three-dimensional representation of the group: $D_3(e) = R_3(\theta{=}0)$, $D_3(a) = R_3(\theta{=}\frac{2\pi}{3})$, $D_3(b) = R_3(\theta{=}\frac{4\pi}{3})$, where R_3 is the rotation matrix in the xy-plane. Show that this representation is reducible, and is equivalent to a direct sum of three one-dimensional representations. In other words, show that there is a matrix S such that $D_3'(g) = SD_3(g)S^{-1}$ is diagonal for all $g = e, a, b$, and that the diagonal elements of D_3' form one-dimensional representations of Z_3. Find a matrix S that does that.