Physics 500A – Quantum Mechanics – Homework assignment 1

Due September 20, 2019, by 5pm in my office
(slide under the door if I'm not in)
To be picked up by October 7, 2019

Textbook reading [5pt]

Read all of chapter 1 in the Sakurai-Napolitano textbook (*Modern Quantum Mechanics,* 2nd edition). Explicitly indicate which sections in the book you have read through.

A shallow potential [3pt]

A particle of mass m in one dimension is bound by an attractive potential $V(x) \leqslant 0$ such that $V(x) = 0$ for $|x| > L$. For an even potential that is very shallow, find the energy of the bound state in terms of $V(x)$. Hint: replace $V(x)$ with $gV(x)$, such that $g \to 0$, and use your intuition for what the wave function is doing for very small g.

Bound states in one dimension [6pt]

The variational principle is based on a simple fact that for any state $|\psi\rangle$,

$$E_\psi \equiv \frac{\langle\psi|H|\psi\rangle}{\langle\psi|\psi\rangle} \geqslant E_0 \tag{1}$$

where H is the Hamiltonian, and E_0 is the ground state energy (the smallest eigenvalue of H). This relation means that the expectation value of energy in any state is greater than or equal to the ground state energy. The non-trivial part is to pick a suitable trial state $|\psi\rangle$ that depends on some parameter, and then minimize the energy with respect to that parameter. The answer then represents an upper bound on the ground state energy.

a) [1pt] Prove relation (1).

b) [2pt] Take the Hamiltonian to be $H = \frac{p^2}{2m} + V(x)$. Use the variational principle to prove that an attractive potential in one dimension always has a bound state. To do so, take the potential $V(x) \leqslant 0$ to be non-zero only in a bounded region $|x| < L$, and find a trial wave function that has $E_\psi < 0$.

c) [3pt] Show that the wave function $\psi(x)$ of the state with the smallest possible expectation value of energy E_ψ must satisfy the (time-independent) Schrödindger equation. Follow the same method you use to derive the Euler-Lagrange equations of motion in classical mechanics.

Quasi-bound states [9pt]

Consider a particle subject to the following potential in one dimension: $V(x) = \infty$ for $x < 0$, and $V(x) = \alpha\,\delta(x-L)$ for $x > 0$. Here $\alpha > 0$ determines the strength of the delta-function barrier at $x = L > 0$. Think about this potential as a crude model describing the decay of an unstable bound state.

a) [3pt] For $0 < x < L$, the solution to the Schrödinger equation is $\psi(x) = Ce^{ikx} + De^{-ikx}$. For $x > L$, the solution to the Schrödinger equation is $\psi(x) = Fe^{ikx} + Ge^{-ikx}$, where k is related to the energy by $E = \frac{\hbar^2}{2m}k^2$. Apply the appropriate boundary conditions at $x = 0$ and $x = L$, and express the coefficients C, D, and G in terms of F. In particular, you should find

$$C = \frac{F}{1 + \frac{2m\alpha}{\hbar^2 k} e^{-ikL} \sin(kL)} .$$

b) [2pt] Show that for real k, your result from part *a)* implies that $|F|^2 = |G|^2$. This means that the reflection coefficient $R = \frac{|F|^2}{|G|^2} = 1$. This makes sense: since the particles are not created or destroyed, everything that comes in from the right, bounces off to the right. These are the good old stationary scattering states.

c) [1pt] For finite α, the particle can tunnel through the delta function barrier, and will eventually escape to $x = \infty$. This is described through the boundary conditions $G = 0$, $F \neq 0$. It turns out that in this case the Schrödinger equation has solutions, provided k is *complex*, $k = k' + ik''$. These are the quasi-stationary states. What happens to k' as $\alpha \to \infty$? What happens to k'' as $\alpha \to \infty$? Do you expect k'' to be positive or negative?

d) [3pt] Show that your result from part *a)* indeed implies that the Schrödinger equation does have solutions with $G = 0$, but $F \neq 0$, provided k is complex. You can use a computer algebra program such as `Mathematica` or `Maple` to find these solutions numerically. Pick some large value of $2m\alpha L/\hbar^2$, and plot the first 10 values of k in the complex k plane.