Physics 500A – Quantum Mechanics – Homework assignment 2

Due October 9, 2019, in class

To be picked up by October 23, 2019

Textbook reading [5pt]

Read Sakurai-Napolitano *Modern Quantum Mechanics,* 2nd edition, sections 2.1 – 2.4, 2.6. Explicitly indicate which sections in the book you have read through.

Free particle in one dimension [5pt]

Consider a free particle in one dimension, $H = p^2/2m$.

a) [2pt] Find the transition amplitude $\langle x_2, t_2 | x_1, t_1 \rangle$.

b) [1pt] Your answer for the amplitude should depend on $x_2 - x_1$, rather than on x_2 and x_1 separately. Why is that? Give both a qualitative and a quantitative explanation.

c) [1pt] Your answer for the amplitude should depend on $t_2 - t_1$, rather than on t_2 and t_1 separately. Why is that? Give both a qualitative and a quantitative explanation.

d) [1pt] Find the probability density by squaring the amplitude. How does your probability density depend on $(x_2 - x_1)$? Give a qualitative explanation of that dependence.

Time-evolution operator [6pt]

a) [3pt] In class, I argued that the time-evolution operator for time-dependent Hamiltonians is

$$U(t) = e^{-i/\hbar \int_0^t dt' H(t')}$$

if $H(t')$ and $H(t'')$ commute, while $U(t)$ is given by the Dyson series

$$U(t) = 1 + \left(\frac{1}{i\hbar}\right) \int_0^t dt' H(t') + \left(\frac{1}{i\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') + \left(\frac{1}{i\hbar}\right)^3 \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' H(t') H(t'') H(t''') + \ldots$$

when $H(t')$ and $H(t'')$ do not necessarily commute. See Sakurai, ch.2. Show that when $H(t')$ and $H(t'')$ commute, the Dyson series reduces to the above exponential, in other words show that the two evolution operators are the same in that case.

b) [3pt] Show that the above Dyson series expression for $U(t)$ can be written as

$$U(t) = \mathcal{T} e^{-i/\hbar \int_0^t dt' H(t')} ,$$

where the $\mathcal{T}$ symbol denotes the time-ordering. Namely, when the exponential is expanded as a power series, the operators $H(t_1)$, $H(t_2)$ etc appearing in the multiple integral must be written in the order such that $H(t_i)$ appears to the left of $H(t_j)$ for $t_i \geqslant t_j$. For example, $\mathcal{T}(H(t_1)H(t_2))$ equals $H(t_1)H(t_2)$ if $t_1 > t_2$, and equals $H(t_2)H(t_1)$ if $t_2 > t_1$.

Transition amplitude for the oscillator [15pt]

The oscillator is described by the Hamiltonian

$$H(p,q) = \frac{p^2}{2m} + \frac{m}{2}\Omega^2 q^2 \,.$$

a) [1pt] Derive the Lagrangian $L(q,\dot{q})$ for the oscillator.

b) [2pt] Using the functional integral representation, show that the transition amplitude between two position eigenstates is given by

$$\langle q'', t'' | q', t' \rangle = F(t) \exp(iS_{\rm cl}) \tag{1}$$

where $F(t)$ is a function which depends only on $t = t'' - t'$ (but not on q'', q'), and $S_{\rm cl}$ is the action for the oscillator evaluated when q satisfies its equation of motion.

c) [2pt] Evaluate the classical action $S_{\rm cl}$ as a function of q'', q', t'', t'. Show that in the $\Omega \to 0$ limit $S_{\rm cl}$ reduces to the classical action for a free particle, evaluated in class.

d) [1pt] For a free particle, the amplitude (1) depends on $q'' - q'$ only, but for the oscillator it depends on q'' and q' sepatately. What is the physical reason for that?

e) [2pt] Apply the Schrödinger equation to the amplitude (1) and derive an ordinary differential equation for $F(t)$. Observe that all dependence on q'', q' drops out, providing a consistency check that (1) is the correct form of the amplitude. Solve the Schrödinger equation and find $F(t)$. Fix the integration constant so that the limit $\Omega \to 0$ reproduces $F(t)$ for a free particle, evaluated in class.

f) [1pt] Write down the answer for $\langle q'', t'' | q', t' \rangle$, and draw a box around it.

g) [1pt] Without using the explicit expression for the amplitude, show that if one takes time to be purely imginary, $t'' - t' = -iT$, then the amplitude takes the form

$$\langle q'', t'' | q', t' \rangle = \sum_n e^{-E_n T} \psi_n(q'') \psi_n(q')^* \,,$$

where $\psi_n(q)$ is the wave function of the n-th eigenstate of the Hamiltonian, and E_n its energy. The procedure of making time imaginary is called "Wick rotation", or "working in Euclidean time" (as opposed to "Minkowski time").

h) [1pt] Take your answer for the transition amplitude from part *f)* and perform the Wick rotation. Now take $T \to \infty$, and read off the ground state energy E_0 and the ground state wave function $\psi_0(q)$.

i) [2pt] Take the first subleading term in the large-T expansion of the amplitude, and read off the energy of the first excited state E_1 and its wave function $\psi_1(q)$. Does your result for E_1 and $\psi_1(q)$ agree with what you know from your undergraduate QM class?

j) [2pt] From the large-T limit of the amplitude, show that the energy of the n-th eigenstate is $E_n = \Omega(n + \frac{1}{2})$.