Physics 410 – Mathematical Methods – Homework assignment 4

(Assignment 3.5 is your midterm)

Due November 15-th, 2018, in class

For each question, explain what you are doing to solve the problem. Do not just write down the answer. Feel free to use a computer algebra program such as Mathematica, Python etc; when doing so, attach your computer code, the code output, and a human-readable explanation of what your code is doing.

Special relativity [6pt]

In class, I defined Lorentz transformations as isometries of the Minkowski spacetime, i.e. the transformations $x^{\mu'} = \Lambda^{\mu'}{}_{\mu}\, x^{\mu}$, $x^{\mu} = \underline{\Lambda}^{\mu}{}_{\mu'}\, x^{\mu'}$ whose transformation matrices satisfy

$$\eta_{\mu'\nu'} = \eta_{\mu\nu}\, \underline{\Lambda}^{\mu}{}_{\mu'}\, \underline{\Lambda}^{\nu}{}_{\nu'}\,, \tag{1}$$

where <u>both</u> $\eta_{\mu\nu}$ and $\eta_{\mu'\nu'}$ are diagonal matrices with $-1, 1, 1, 1$ on the diagonal. If we define a matrix ℓ with components $\ell_{\mu\mu'} \equiv \underline{\Lambda}^{\mu}{}_{\mu'}$, then Eq. (1) can be written as $\eta = \ell^T \eta \ell$, which is a matrix-form definition of a Lorentz transformation.

Consider the transformation

$$t' = \gamma(t - vx)\,, \qquad x' = \gamma(-vt + x)\,, \qquad y' = y\,, \qquad z' = z\,, \tag{2}$$

with $\gamma = 1/\sqrt{1-v^2}$. This transformation is called a *boost*, and relates the space and time experienced by an "unprimed" and a "primed" observer which move with constant velocity v with respect to each other. Here the speed of light is $c = 1$.

a) Find the matrices Λ and $\underline{\Lambda}$ corresponding to the transformation (2).

b) Show that your matrices satisfy (1), i.e. show that (2) is a Lorentz transformation.

c) Show that if the matrices ℓ_1 and ℓ_2 each describe a Lorentz transformation, so does $\ell_1 \ell_2$.

d) Show that $\det \ell = \pm 1$ for Lorentz transformations. Therefore, the matrices ℓ are invertible. Show that if ℓ describes a Lorentz transformation, so does ℓ^{-1}. Show that the identity matrix also describes a Lorentz transformation. You've just shown that Lorentz transformations form a group, called the Lorentz group.

e) For the transformation (2), give a physical interpretation to the matrix ℓ^{-1}.

Polar coordinates [3pt]

a) Show that the Jacobian of the coordinate transformation from Cartesian to polar coordinates vanishes at $r = 0$, thus the polar coordinates are singular at the origin. Explain why that happens.

b) The basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and the corresponding basis 1-forms of the "standard" vector calculus are defined by

$$\hat{\mathbf{r}} = \hat{e}_{(r)}\,, \qquad \hat{\boldsymbol{\theta}} = \frac{1}{r}\hat{e}_{(\theta)}\,, \qquad \tilde{\boldsymbol{\omega}}^{\hat{r}} = \tilde{d}r\,, \qquad \tilde{\boldsymbol{\omega}}^{\hat{\theta}} = r\tilde{d}\theta\,.$$

Show that there are no coordinates ξ, η such that the above are coordinate-basis vectors and 1-forms. To do so, show that with

$$\tilde{d}\xi = (\partial\xi/\partial x)\tilde{d}x + (\partial\xi/\partial y)\tilde{d}y\,, \qquad \tilde{d}\eta = (\partial\eta/\partial x)\tilde{d}x + (\partial\eta/\partial y)\tilde{d}y\,,$$

there are no coordinates ξ, η such that $\tilde{d}\xi = \tilde{\boldsymbol{\omega}}^{\hat{r}}$, $\tilde{d}\eta = \tilde{\boldsymbol{\omega}}^{\hat{\theta}}$. [Use the equality of mixed derivatives.]

Basic covariant derivatives [5pt]

Recall that the covariant derivative of a vector V is a (1,1) tensor whose components in any coordinate basis are given by

$$(\nabla V)^{\alpha}{}_{\beta} = \nabla_\beta V^\alpha = \frac{\partial V^\alpha}{\partial x^\beta} + V^\mu\, \Gamma^\alpha_{\mu\beta}\,, \tag{3}$$

where $\Gamma^\alpha_{\mu\beta}$ are the Christoffel symbols (also called "connection coefficients"). The Christoffel symbols are defined by

$$\frac{\partial}{\partial x^\beta}\hat{e}_{(\alpha)} = \Gamma^\mu_{\alpha\beta}\,\hat{e}_{(\mu)}\,.$$

Consider a vector field V in two dimensions. Use only coordinate bases in this problem.

a) In Cartesian coordinates, $V = V(x,y)$. Compute the Christoffel symbols in Cartesian coordinates, and use Eq. (3) to find all the components $\nabla_\beta V^\alpha$ in terms of x, y derivatives.

b) In polar coordinates, $V = V(r,\theta)$. Compute the Christoffel symbols in polar coordinates, and use Eq. (3) to find all the components $\nabla_\beta V^\alpha$ in terms of r, θ derivatives.

c) Use the transformation matrices $\Lambda^{\alpha'}{}_{\beta}$ and the inverse $\underline{\Lambda}^{\beta}{}_{\alpha'}$ between Cartesian and polar coordinates to verify that your answers in parts *a)* and *b)* are related by the transformation law of the components of a (1,1) tensor.

Spherical coordinates [8pt]

The spherical coordinates in three dimensions are defined by $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, $z = r\cos\theta$, with $0 \leqslant \theta < \pi$, $0 \leqslant \varphi < 2\pi$, $0 \leqslant r < \infty$. The space is Euclidean, i.e. the metric components in Cartesian coordinates form the unit matrix. Use only coordinate bases in this problem.

a) Express the coordinate basis vectors $\hat{e}_{(r)}$, $\hat{e}_{(\theta)}$, $\hat{e}_{(\varphi)}$ in terms of $\hat{e}_{(x)}$, $\hat{e}_{(y)}$, $\hat{e}_{(z)}$.

b) Find the metric components $g_{\alpha\beta}$ in spherical coordinates.

c) Using the definition

$$\frac{\partial}{\partial x^\beta}\,\hat{e}_{(\alpha)} = \Gamma^\mu_{\alpha\beta}\,\hat{e}_{(\beta)}\,,$$

compute the Christoffel symbols in spherical coordinates.

d) Using your metric from part *b)*, compute the Christoffel symbols in spherical coordinates by

$$\Gamma^\mu_{\alpha\beta} = \tfrac{1}{2} g^{\mu\lambda}\left(\frac{\partial g_{\lambda\beta}}{\partial x_\alpha} + \frac{\partial g_{\alpha\lambda}}{\partial x_\beta} - \frac{\partial g_{\alpha\beta}}{\partial x_\lambda}\right)\,.$$

Did you get the same answer for $\Gamma^\mu_{\alpha\beta}$?

e) Using your expressions for $\Gamma^\mu_{\alpha\beta}$, compute all the components of

$$\nabla_\mu\, g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} - g_{\nu\beta}\Gamma^\nu_{\alpha\mu} - g_{\alpha\nu}\Gamma^\nu_{\beta\mu}\,,$$

where ∂_μ stands for $\partial/\partial x^\mu$. Is your (0,3) tensor ∇g zero or non-zero?