Physics 500A – Quantum Mechanics – Homework assignment 4

Due November 6, 2019, in class

To be picked up by November 20, 2019

Time-dependent perturbation theory [2pt]

Consider the Hamiltonian $H(t) = H_0 + V(t)$, where H_0 does not depend on time and has eigenstates $|n\rangle$ such that $H_0|n\rangle = E_n|n\rangle$. In class, we derived the basic differential equations for the coefficients $c_n(t)$ in the expansion $|\psi, t\rangle = \sum_n c_n(t)e^{-iE_nt}|n\rangle$, see Sakurai eq.(5.5.15). Derive the expansion of $c_n(t)$ in powers of V, i.e. derive Sakurai (5.6.17) [1-st edition] or (5.7.17) [2-nd edition]. This expansion is called the time-dependent perturbation theory. Sakurai's book uses the so-called "interaction picture", but you don't have to use it.

Spin magnetic resonance [10pt]

As promised some time ago, here is the problem to work out the two-state problem with a periodic time-dependent potential (see Sakurai, section 5.5). The Hamiltonian is $H = H_0 + V(t)$, where $H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|$, and $V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|$. The parameters γ and ω are real and positive, and we can take $E_2 > E_1$.

a) [3pt] Recall that transition amplitudes satisfy $i\hbar\dot{c}_n(t) = \sum_m V_{nm}(t)\exp(i\omega_{nm}t)c_m(t)$. Take the system to be in the lower-energy state $|1\rangle$ at t=0. Find the probability $P(1\to 1)$ that the system is still in state $|1\rangle$ at $t > 0$. Find the probability $P(1\to 2)$ that the system is in state $|2\rangle$ at $t > 0$. (I.e., derive the answer that I wrote down in class.) Solve the equations by hand, do not use any computer programs. Check that your answers for c_1 and c_2 satisfy $|c_1|^2 + |c_2|^2 = 1$.

b) [3pt] Now use time-dependent perturbation theory to lowest non-vanishing order to find the same probabilities $P(1\to 1)$ and $P(1\to 2)$. [You will have to go to 1-st order for c_2 and to 2-nd order for c_1.] Compare your answer with the exact result of part *a)* for small γ.

c) [2pt] Take the frequency of the external perturbation to be far from the resonance, $\hbar|\omega - \omega_{21}| \gg \gamma$. When is the perturbation theory answer of part *b)* reliable? Now take the frequency of the external perturbation to be close to the resonance, $\hbar|\omega - \omega_{21}| \ll \gamma$. When is the perturbation theory answer of part *b)* reliable?

d) [2pt] Write a short (one page) essay on how you understand the phenomenon and applications of the nuclear magnetic resonance, and what this problem has to do with NMR.

Neutron interference in magnetic field [4pt]

Read the papers by Werner *et al.* in *Phys. Rev. Lett.* **35**, 1053 (1975), and by Rauch *et al.* in *Phys. Lett. A* **54**, 425 (1975), and the comments on them in Sakurai. Write a 1–2 page summary of how you understand the papers, and what the results are. [In particular, explain how the phase difference comes about, and what the experiment has to do with 2π or 4π rotations.]

Orbital angular momentum [8pt]

The orbital angular mometum operator in three dimensions is $\mathbf{L} = \mathbf{x}\times\mathbf{p}$, or in components $L_i = \epsilon_{ijk}\, x_j\, p_k$, where small Latin indices label x, y, and z. It is easy to write down how the L_i operator acts on position-space wave functions in Cartesian coordinates $\psi(x, y, z)$, because you know how p_i acts on $\psi(x, y, z)$. Study Sections 3.5 and 3.6 in Sakurai if you haven't yet. For the calculations below, you can do them by hand, but also feel free to use a computer algebra program such as `Mathematica`.

a) [2pt] Transform from Cartesian to spherical coordinates, and derive how the orbital angular momentum operator L_i acts on the position-space wave function $\psi(r, \theta, \varphi)$ in spherical coordinates. In other words, derive equations (3.6.9), (3.6.11), (3.6.12) in Sakurai.

b) [2pt] Find how the operator $\mathbf{L}^2$ acts on the position-space wave function $\psi(r, \theta, \varphi)$ in spherical coordinates. In other words, derive equation (3.6.15) in Sakurai.

c) [4pt] At the beginnig of Section 3.6 in Sakurai, there is an argument that $\mathbf{L}$ is a generator of rotations, based on the action of $\mathbf{L}$ on position eigenstates. Extend this argument to finite rotations for both position and momentum eigenstates. In other words, show that

$$e^{-\frac{i\phi L_z}{\hbar}} |\mathbf{x}\rangle = |\mathbf{x}'\rangle \,,$$
$$e^{-\frac{i\phi L_z}{\hbar}} |\mathbf{p}\rangle = |\mathbf{p}'\rangle \,,$$

where $x'_i = R_{ij}\, x_j$, $p'_i = R_{ij}\, p_j$, and R is the rotation matrix, see Eq.(3.1.3) in Sakurai. You may it helpful to show first that the position operator is a generator of momentum translations.