Physics 410 – Mathematical Methods – Homework assignment 2

Due October 4-th, 2018, in class

For each question, explain what you are doing to solve the problem. Do not just write down the answer. Feel free to use a computer algebra program such as Mathematica, Python etc; when doing so, attach your computer code, the code output, and a human-readable explanation of what your code is doing.

Suggested reading to remind yourself of complex analysis: Dennery and Krzywicki, *Mathematics for Physicists*, ch. 1, or S. Lea, *Mathematics for Physicists*, ch. 2. For more in-depth applications to two-dimensional fluids (not needed for this assignment), take a look at Acheson, *Elementary Fluid Dynamics*, ch. 4.

Some integrals

(1pt) Show that

$$\int_0^\infty \sin x^2 \, dx = \int_0^\infty \cos x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \, .$$

Hint: Evaluate $\int_0^\infty e^{ix^2} dx$ by using a "pie-slice" shaped contour of angle $\pi/4$.

(2pt) Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^{2N}}$$

with integer $N \geqslant 1$ by integrating over a "pie-slice" shaped contour with sides at $\theta = 0$ and $\theta = \pi/N$.

(3pt) Evaluate the integral

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^{bx}} dx \, ,$$

where $0 < a < b$. Use a rectangular contour whose lower side is on the real axis, and the upper side is at $y = 2\pi/b$. Argue that the integral over the "short" sides of the rectangle does not contribute.

A potential flow [2pt]

Recall that a potential flow of a two-dimensional fluid can be described by the velocity potential φ, such that the fluid velocity is $\mathbf{v} = -\boldsymbol{\nabla}\varphi$. The potential φ is the real part of the complex analytic potential $\Phi(z)$, while the stream function ψ is the imaginary part of $\Phi(z)$. Find the fluid velocity $\mathbf{v} = (v_x, v_y)$ for the simplest potential flow inside a wedge of angle α, i.e. with hard walls at $\theta = 0$ and $\theta = \alpha$. Find the expression for the stream lines (the lines along which the fluid flows). Use a computer algebra program to plot several stream lines for $\alpha = \pi/3$.

Second-order ODEs [2pt]

Consider a homogeneous second-order ordinary differential equation (ODE)

$$y''(x) + f(x)y'(x) + g(x)y(x) = 0 \, .$$

Show that you can perform a transformation that eliminates the first derivative from the equation. In other words, show that you can find $u(x)$ [related to $y(x)$], such that the original ODE becomes

$$u''(x) + V(x)u(x) = 0\,.$$

In other words, the equation can be made to look like a time-independent Schrödinger equation. Find $V(x)$ in terms of $f(x)$ and $g(x)$.

Simple ODEs in Quantum Mechanics [5pt]

Consider the quantum-mechanical Hamiltonian $H(t) = H_0 + V(t)$, where H_0 does not depend on time and has orthonormal eigenstates $|n\rangle$ such that $H_0|n\rangle = E_n|n\rangle$. Any state can be represented as a superposition of the eigenstates of H_0, namely $|\psi,t\rangle = \sum_n c_n(t)e^{-iE_nt}|n\rangle$, with some coefficients $c_n(t)$.

a) Take the following Hamiltonian describing a two-state system:

$$H_0 = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2|\,, \qquad V(t) = \gamma e^{i\omega t}|1\rangle\langle 2| + \gamma e^{-i\omega t}|2\rangle\langle 1|\,,$$

where the parameters γ and ω are real and positive, and $E_2 > E_1$. This shows up in the description of spin-1/2 particles interacting with electromagnetic waves. Use the (time-dependent) Schrödinger equation to derive the differential equations satisfied by $c_1(t)$ and $c_2(t)$. Feel free to set $\hbar = 1$, and use $\omega_{21} \equiv E_2 - E_1$.

b) Solve your equations from part *a)* with the initial conditions $c_1(0)$=1, $c_2(0)$=0, i.e. the particle is in the lower-energy state $|1\rangle$ at t=0. Solve the equations by hand and find the probabilities $|c_1(t)|^2$, $|c_2(t)|^2$. Draw a box around your answer. Check that your answer satisfies $|c_1(t)|^2 + |c_2(t)|^2 = 1$.

Contour integrals in electromagnetism [3pt]

[Optional] For a point electric charge e undergoing a simple harmonic motion along the z axis, $z = a\cos\omega t$, the radiated power per unit solid angle is

$$\frac{dP(t)}{d\Omega} = \frac{e^2 c\beta^4}{4\pi a^2}\sin^2\theta\,\frac{\cos^2\omega t}{(1+\beta\cos\theta\sin\omega t)^5}\,,$$

where $\beta = a\omega/c$, and θ is the polar angle in spherical coordinates. Perform the time average over one period of oscillation to find the time-averaged $\langle dP/d\Omega\rangle$ as a function of θ. Use methods of contour integration in the complex plane.