# Physics 500A – Quantum Mechanics – Homework assignment 6

Due December 4, 2019, in class

## Tritium beta decay [5pt]

The beta decay of the tritium nucleus is

$$^3\mathrm{H} \to {}^3\mathrm{He} + e^- + \bar{\nu}_e$$

where the energy of the emitted electron is of the order of several keV, and so you can ignore the recoil of the nucleus. As a result of the decay, the binding potential for the atomic electron changes from $-e^2/r$ to $-2e^2/r$. In this problem, take the potential change as instantaneous.

*a) [3pt]* For the tritium atom in the ground state ($n$=1) before the decay, find the probability $p_n$ that the atom is in the $n$-th excited state after the decay. You should find $p_1 = 512/729 \approx 0.7$. What do you find for $p_2$? What do you find for $p_3$?

*b) [2pt]* Now that you know $p_n$, evaluate numerically the sum of the probabilities $P = \sum_{n=1}^{\infty} p_n$. You should find that $P$ is less than 1 by several percent. Give a physical interpretation of the non-zero value of $1-P$.

## Gauge invariance [5pt]

In class, I claimed that under gauge transformations, the wave function gets multiplied by a phase. Show this. In other words, show that if $\psi(\mathbf{x}, t)$ satisfies the Schrödinger equation

$$i\hbar\, \partial_t \psi = \left( \frac{(\mathbf{p} - e\mathbf{A})^2}{2m} + V(\mathbf{x}) + e\varphi \right) \psi \,,$$

then the gauge-transformed wave function $\psi' = e^{ie\alpha/\hbar}\psi$ satisfies

$$i\hbar\, \partial_t \psi' = \left( \frac{(\mathbf{p} - e\mathbf{A}')^2}{2m} + V(\mathbf{x}) + e\varphi' \right) \psi' \,.$$

Here $\alpha = \alpha(\mathbf{x}, t)$ is the gauge function, and $\mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\alpha$ and $\varphi' = \varphi - \partial_t \alpha$ are the gauge transformed electromagnetic potentials. This tells you that the Schrödinger equation is invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu \alpha$, $\psi \to e^{ie\alpha/\hbar}\psi$.

## Lorentz force [3pt]

Recall that the time dependence of a Heisenberg-picture operator $\mathcal{O}$ is

$$\frac{d\mathcal{O}}{dt} = \frac{\partial \mathcal{O}}{\partial t} + \frac{i}{\hbar} [H, \mathcal{O}] \,.$$

Use this to detive the Lorentz force equation in Quantum Mechanics,

$$\frac{d}{dt}(m\mathbf{v}) = e\mathbf{E} + \frac{e}{2} (\mathbf{v} \times \mathbf{B} - \mathbf{B} \times \mathbf{v}) \,,$$

where $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity operator, and $\mathbf{E}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ are the electric and magnetic fields, which are functions of the operator $\mathbf{x}$.

## Aharonov-Bohm effect [10pt]

Consider a particle of charge $e$ moving between two very long concentric cylindrical shells with radii $\rho = a$ and $\rho = b > a$, in cylindrical coordinates. The particle can not penetrate the shells, hence its wave function $\psi(\mathbf{x})$ satisfies $\psi(\rho{=}a) = \psi(\rho{=}b) = 0$. Inside the inner shell there is a constant magnetic field $B \parallel \hat{z}$ with flux $\Phi$, while $B = 0$ everywhere outside the inner shell. In other words, the particle can only move in the region where there is no magnetic field. The motion along the $\mathbf{z}$ axis is just a plane wave, so let us look at $z$-independent states only.

*a) [4pt]* Because of the cylindrical symmetry, the wave function can be taken as $\psi(\mathbf{x}) = e^{i\ell\varphi}\psi_\ell(\rho)$. Explain why $\ell$ must be an integer. Solve the Schrödinger equation, and derive a transcendental equation which (implicitly) determines the energy levels for the particle in terms of its angular momentum $\ell$ and the flux $\Phi$. Express your answer in terms of the Bessel functions $J_\nu(z)$ and $N_\nu(z)$ [the Bessel function of the second kind $N_\nu(z)$ is often denoted as $Y_\nu(z)$, depending on the book].

*b) [2pt]* Show that the energy spectrum (i.e. the set of all possible energies) remains the same when $e\Phi/(2\pi\hbar)$ changes by an integer. In particular, this implies that the energy spectrum for $e\Phi/(2\pi\hbar)$ =integer is indistinguishable from the energy spectrum for $\Phi = 0$.

*c) [4pt]* Use a computer algebra program such as `Mathematica` to plot the energies of the first four lowest-energy states as a function of the flux $\Phi$, starting with $\Phi = 0$, up to $e\Phi/(2\pi\hbar) = 3$. You can pick some arbitrary values for the radii, e.g. $b = 3a$. How do the energies of the ground state, first excited state, second excited state etc depend on the flux $\Phi$? [Note: "the first four lowest-energy states" means "the first four lowest-energy states". It does *not* mean $\ell = 1, 2, 3, 4$.]

## Questions for Prof. Strickland [3pt]

[*Optional for extra credit*] Write down one question that you asked Prof. Donna Strickland on the morning of November 27-th, and give a brief version of her answer.