5.1 QUADRATIC FUNCTIONS
(Adapted from "Precalculus" by Stewart et als.)

Graphing Quadratic Functions Using the Standard Form _ Maximum and Minimum Values of Quadratic Functions

A polynomial function is a function that is defined by a polynomial expression. So a **polynomial function of degree *n*** is a function of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

We have already studied polynomial functions of degree 0 and 1. These are functions of the form $P(x) = a_0$ and $P(x) = a_1 x + a_0$ respectively, whose graphs are lines. In this section we study polynomial functions of degree 2. These are called **quadratic functions**.

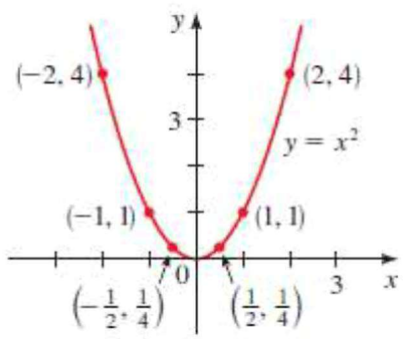
QUADRATIC FUNCTIONS

A **quadratic function** is a polynomial function of degree 2. So a quadratic function is a function of the form

$$f(x) = ax^2 + bx + c, \qquad a \neq 0$$

▼ Graphing Quadratic Functions Using the Standard Form

If we take $a = 1$ and $b = c = 0$ in the quadratic function $f(x) = ax^2 + bx + c$, we get the quadratic function, whose graph is the parabola graphed here.



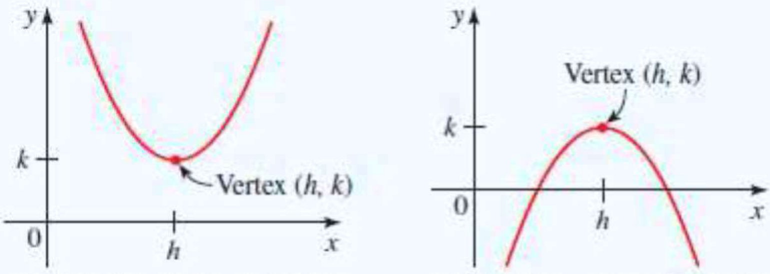
In fact, the graph of any quadratic function is a **parabola**; it can be obtained from the graph of $f(x) = x^2 + bx + c$ by the transformations discussed in Section 2.4.

STANDARD FORM OF A QUADRATIC FUNCTION

A quadratic function $f(x) = ax^2 + bx + c$ can be expressed in the **standard form**

$$f(x) = a(x - h)^2 + k$$

by completing the square. The graph of f is a parabola with **vertex** (h, k); the parabola opens upward if $a > 0$ or downward if $a < 0$.



$f(x) = a(x - h)^2 + k,\ a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k,\ a < 0$

E X A M P L E 1 | **Standard Form of a Quadratic Function**

Let . $f(x) = 2x^2 - 12x + 23$

(a) Express *f* in standard form. **(b)** Sketch the graph of *f*.

S O LU T I O N

(a) Since the coefficient of x^2 is not 1, we must factor this coefficient from the terms involving x before we complete the square.

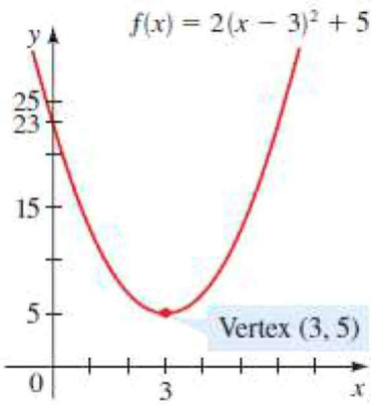
$$\begin{aligned} f(x) &= 2x^2 - 12x + 23 \\ &= 2(x^2 - 6x) + 23 && \text{Factor 2 from the } x\text{-terms} \\ &= 2(x^2 - 6x + 9) + 23 - 2\cdot 9 && \text{Complete the square: Add 9 inside parentheses, subtract } 2\cdot 9 \text{ outside} \\ &= 2(x - 3)^2 + 5 && \text{Factor and simplify} \end{aligned}$$

The standard form is $f(x) = 2(x - 3)^2 + 5$.

$f(x) = 2(x - 3)^2 + 5$

Vertex is $(3, 5)$

(b) The standard form tells us that we get the graph of *f* by taking the parabola $y = x^2$, shifting it to the right 3 units, stretching it by a factor of 2, and moving it upward 5 units. The vertex of the parabola is at $(3, 5)$ and the parabola opens upward. We sketch the graph in Figure 1 after noting that the y-intercept is $f(0) = 23$.



NOW TRY: A quadratic function is given. **(a)** Express the quadratic function in standard form. **(b)** Find its vertex and its x- and y-intercept(s). **(c)** Sketch its graph.

$$f(x) = x^2 + 4x + 3$$

▼ Maximum and Minimum Values of Quadratic Functions

If a quadratic function has vertex (h, k), then the function has a minimum value at the vertex if its graph opens upward and a maximum value at the vertex if its graph opens downward.

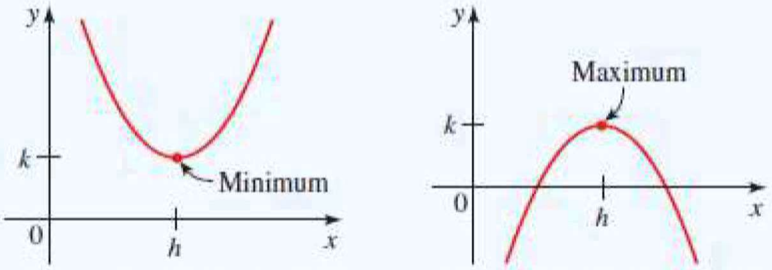
For example, the function graphed in Example 1 has minimum value 5 when $x = 3$, since the vertex $(3, 5)$ is the lowest point on the graph.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC FUNCTION

Let f be a quadratic function with standard form $f(x) = a(x - h)^2 + k$. The maximum or minimum value of f occurs at $x = h$.

If $a > 0$, then the **minimum value** of f is $f(h) = k$.

If $a < 0$, then the **maximum value** of f is $f(h) = k$.



$f(x) = a(x - h)^2 + k, a > 0$ $\qquad$ $f(x) = a(x - h)^2 + k, a < 0$

E X A M P L E 2 | Minimum Value of a Quadratic Function

Consider the quadratic function $f(x) = 5x^2 - 30x + 49$.

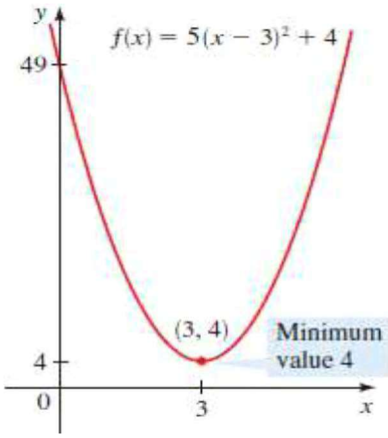
(a) Express f in standard form. **(b)** Sketch the graph of f. **(c)** Find the minimum value of f.

(a) To express this quadratic function in standard form, we complete the square.

$$\begin{aligned} f(x) &= 5x^2 - 30x + 49 \\ &= 5(x^2 - 6x) + 49 \\ &= 5(x^2 - 6x + 9) + 49 - 5\cdot 9 \\ &= 5(x - 3)^2 + 4 \end{aligned}$$

(c)
Since the coefficient of x^2 is positive, f has a minimum value.
The minimum value is $f(0) = 49$.

(b) The graph is a parabola that has its vertex at $(3, 4)$ and opens upward, as sketched here.



NOW TRY: A quadratic function is given. (a) Express the quadratic function in standard form. (b) Sketch its graph. (c) Find its maximum or minimum value.

$$f(x) = 3x^2 - 6x + 1$$

EXAMPLE 3 | Maximum Value of a Quadratic Function

Consider the quadratic function $f(x) = -x^2 + x + 2$.

(a) Express *f* in standard form.
(b) Sketch the graph of *f*.
(c) Find the maximum value of *f*.

SOLUTION

(a) To express this quadratic function in standard form, we complete the square.

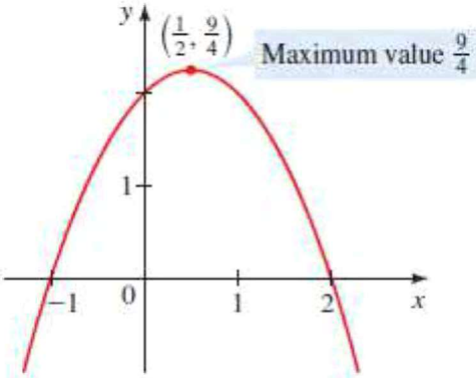
$$\begin{aligned} y &= -x^2 + x + 2 \\ &= -(x^2 - x) + 2 \\ &= -(x^2 - x + \tfrac{1}{4}) + 2 - (-1)\tfrac{1}{4} \\ &= -(x - \tfrac{1}{2})^2 + \tfrac{9}{4} \end{aligned}$$

(c)
Since the coefficient of x^2 is negative, f has a maximum value, which is $f(\tfrac{1}{2}) = \tfrac{9}{4}$.

(b) [From the standard form we see that]
The graph is a parabola that opens downward and has vertex $(\tfrac{1}{2}, \tfrac{9}{4})$.
As an aid to sketching the graph, we find the intercepts. The y-intercept is $f(0) = 2$. To find the x-intercepts, we set $f(x) = 0$ and factor the resulting equation.

$$-x^2 + x + 2 = 0$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$

Thus the x-intercepts are $x = 2$ and $x = -1$.
The graph of f is sketched here.



NOW TRY: A quadratic function is given. (a) Express the quadratic function in standard form. (b) Sketch its graph. (c) Find its maximum or minimum value.

$$f(x) = -x^2 - 3x + 3$$

Expressing a quadratic function in standard form helps us to sketch its graph as well as to find its maximum or minimum value. If we are interested only in finding the maximum or minimum value, then a formula is available for doing so. This formula is obtained by completing the square for the general quadratic function as follows:

$$
\begin{aligned}
f(x) &= ax^2 + bx + c \\
&= a\left(x^2 + \frac{b}{a}x\right) + c && \text{Factor } a \text{ from the } x\text{-terms} \\
&= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - a\left(\frac{b^2}{4a^2}\right) && \text{Complete the square: Add } \tfrac{b^2}{4a^2} \text{ inside parentheses, subtract } a\left(\tfrac{b^2}{4a^2}\right) \text{ outside} \\
&= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} && \text{Factor}
\end{aligned}
$$

This equation is in standard form with $h = -\dfrac{b}{2a}$ and $h = c - \dfrac{b^2}{4a}$. Since the maximum or minimum value occurs at $x = h$, we have the following result.

MAXIMUM OR MINIMUM VALUE OF A QUADRATIC FUNCTION

The maximum or minimum value of a quadratic function $f(x) = ax^2 + bx + c$ occurs at

$$x = -\frac{b}{2a}$$

If $a > 0$, then the **minimum value** is $f\left(-\dfrac{b}{2a}\right)$.

If $a < 0$, then the **maximum value** is $f\left(-\dfrac{b}{2a}\right)$.

EXAMPLE 4 Finding Maximum and Minimum Values of Quadratic Functions

Find the maximum or minimum value of each quadratic function.

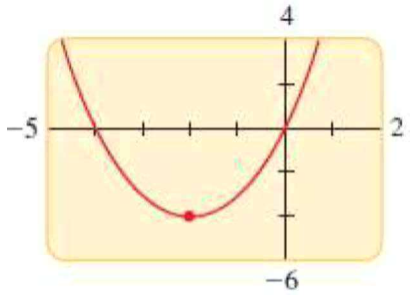
(a) $f(x) = x^2 + 4x$ (b) $g(x) = -2x^2 + 4x - 5$

SOLUTION

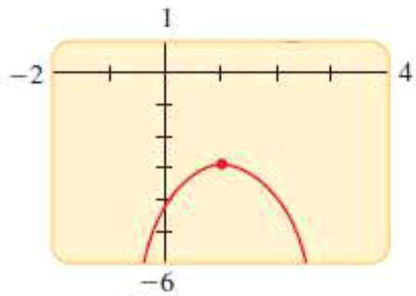
(a) This is a quadratic function with $a = 1$ and $b = 4$. Thus, the maximum or minimum value occurs at

$$x = -\frac{b}{2a} = -\frac{4}{2 \cdot 1} = -2$$

Since $a > 0$, the function has the *minimum* value $f(-2) = (-2)^2 + 4(-2) = -4$.



The minimum value occurs at $x = -2$.

(b) This is a quadratic function with $a = -2$ and $b = 4$. Thus, the maximum or minimum value occurs at $$x = -\frac{b}{2a} = -\frac{4}{2\cdot(-2)} = 1$$ Since $a < 0$, the function has the *maximum* value $g(1) = -2(1)^2 + 4(1) - 5 = -3$	 The maximum value occurs at $x = 1$.

NOW TRY: Find the maximum or minimum value of the function.

(b) $f(x) = x^2 + x + 1$ (b) $f(t) = 100 - 49t - 7t^2$

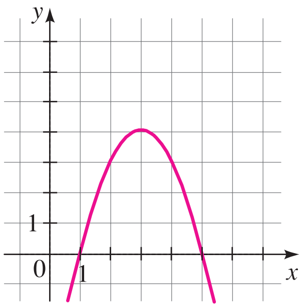
Exercise

1–4 ■ The graph of a quadratic function f is given.

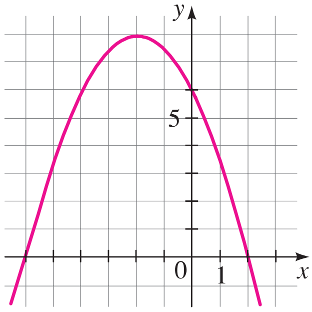
(a) Find the coordinates of the vertex.

(b) Find the maximum or minimum value of f.

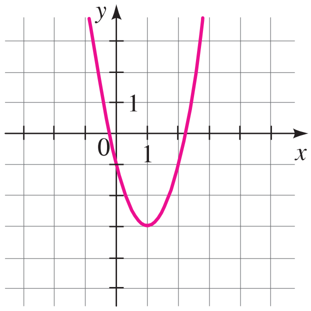
1. $f(x) = -x^2 + 6x - 5$



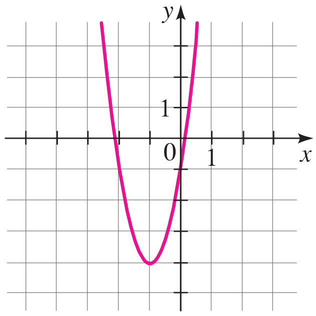
2. $f(x) = -\frac{1}{2}x^2 - 2x + 6$



3. $f(x) = 2x^2 - 4x - 1$



4. $f(x) = 3x^2 + 6x - 1$



5–18 ■ A quadratic function is given.

(a) Express the quadratic function in standard form.

(b) Find its vertex and its x- and y-intercept(s).

(c) Sketch its graph.

5. $f(x) = x^2 - 6x$ **6.** $f(x) = x^2 + 8x$

7. $f(x) = 2x^2 + 6x$ **8.** $f(x) = -x^2 + 10x$

9. $f(x) = x^2 + 4x + 3$ **10.** $f(x) = x^2 - 2x + 2$

11. $f(x) = -x^2 + 6x + 4$ **12.** $f(x) = -x^2 - 4x + 4$

13. $f(x) = 2x^2 + 4x + 3$ **14.** $f(x) = -3x^2 + 6x - 2$

15. $f(x) = 2x^2 - 20x + 57$ **16.** $f(x) = 2x^2 + x - 6$

17. $f(x) = -4x^2 - 16x + 3$ **18.** $f(x) = 6x^2 + 12x - 5$

19–28 ■ A quadratic function is given.

(a) Express the quadratic function in standard form.

(b) Sketch its graph.

(c) Find its maximum or minimum value.

19. $f(x) = 2x - x^2$ **20.** $f(x) = x + x^2$

21. $f(x) = x^2 + 2x - 1$ **22.** $f(x) = x^2 - 8x + 8$

23. $f(x) = -x^2 - 3x + 3$ **24.** $f(x) = 1 - 6x - x^2$

25. $g(x) = 3x^2 - 12x + 13$ **26.** $g(x) = 2x^2 + 8x + 11$

27. $h(x) = 1 - x - x^2$ **28.** $h(x) = 3 - 4x - 4x^2$

29–38 ■ Find the maximum or minimum value of the function.

29. $f(x) = x^2 + x + 1$ **30.** $f(x) = 1 + 3x - x^2$

31. $f(t) = 100 - 49t - 7t^2$ **32.** $f(t) = 10t^2 + 40t + 113$

33. $f(s) = s^2 - 1.2s + 16$ **34.** $g(x) = 100x^2 - 1500x$

35. $h(x) = \frac{1}{2}x^2 + 2x - 6$ **36.** $f(x) = -\dfrac{x^2}{3} + 2x + 7$

37. $f(x) = 3 - x - \frac{1}{2}x^2$ **38.** $g(x) = 2x(x - 4) + 7$

39. Find a function whose graph is a parabola with vertex $(1, -2)$ and that passes through the point $(4, 16)$.

40. Find a function whose graph is a parabola with vertex $(3, 4)$ and that passes through the point $(1, -8)$.

41–44 ■ Find the domain and range of the function.

41. $f(x) = -x^2 + 4x - 3$ **42.** $f(x) = x^2 - 2x - 3$

43. $f(x) = 2x^2 + 6x - 7$ **44.** $f(x) = -3x^2 + 6x + 4$

45–46 ■ A quadratic function is given.

(a) Use a graphing device to find the maximum or minimum value of the quadratic function f, correct to two decimal places.

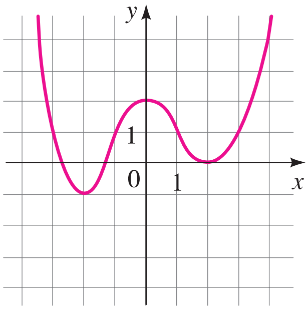
(b) Find the exact maximum or minimum value of f, and compare with your answer to part (a).

45. $f(x) = x^2 + 1.79x - 3.21$

46. $f(x) = 1 + x - \sqrt{2}x^2$

47–50 ■ Find all local maximum and minimum values of the function whose graph is shown.

47.



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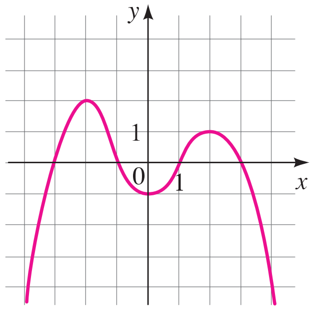

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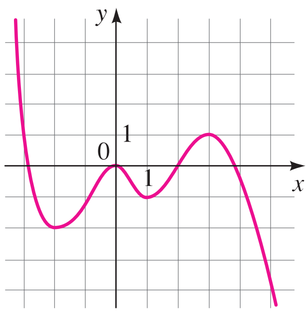

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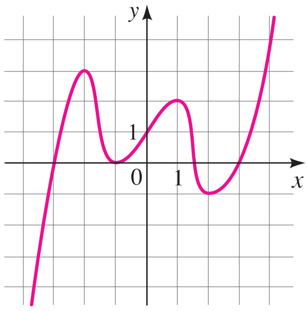


51–58 ■ Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer correct to two decimal places.

51. $f(x) = x^3 - x$ **52.** $f(x) = 3 + x + x^2 - x^3$

53. $g(x) = x^4 - 2x^3 - 11x^2$ **54.** $g(x) = x^5 - 8x^3 + 20x$

55. $U(x) = x\sqrt{6 - x}$ **56.** $U(x) = x\sqrt{x - x^2}$

57. $V(x) = \dfrac{1 - x^2}{x^3}$ **58.** $V(x) = \dfrac{1}{x^2 + x + 1}$

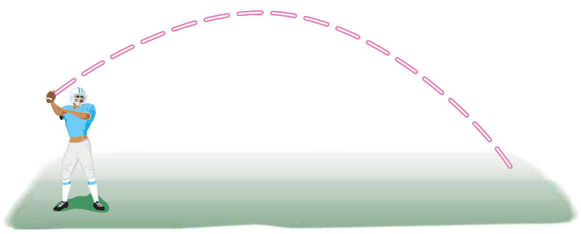
Applications

59. Height of a Ball If a ball is thrown directly upward with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. What is the maximum height attained by the ball?

60. Path of a Ball A ball is thrown across a playing field. Its path is given by the equation $y = -0.005x^2 + x + 5$, where x is the distance the ball has traveled horizontally, and y is its height above ground level, both measured in feet.

(a) What is the maximum height attained by the ball?

(b) How far has it traveled horizontally when it hits the ground?



61. Revenue A manufacturer finds that the revenue generated by selling x units of a certain commodity is given by the function $R(x) = 80x - 0.4x^2$, where the revenue $R(x)$ is measured in dollars. What is the maximum revenue, and how many units should be manufactured to obtain this maximum?

62. Sales A soft-drink vendor at a popular beach analyzes his sales records, and finds that if he sells x cans of soda pop in one day, his profit (in dollars) is given by

$$P(x) = -0.001x^2 + 3x - 1800$$

What is his maximum profit per day, and how many cans must he sell for maximum profit?

63. Advertising The effectiveness of a television commercial depends on how many times a viewer watches it. After some experiments an advertising agency found that if the effectiveness E is measured on a scale of 0 to 10, then

$$E(n) = \tfrac{2}{3}n - \tfrac{1}{90}n^2$$

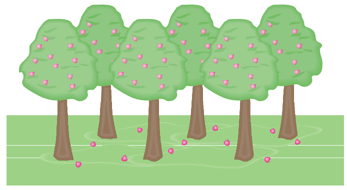
where n is the number of times a viewer watches a given commercial. For a commercial to have maximum effectiveness, how many times should a viewer watch it?

64. Pharmaceuticals When a certain drug is taken orally, the concentration of the drug in the patient's bloodstream after t minutes is given by $C(t) = 0.06t - 0.0002t^2$, where $0 \le t \le 240$ and the concentration is measured in mg/L. When is the maximum serum concentration reached, and what is that maximum concentration?

65. Agriculture The number of apples produced by each tree in an apple orchard depends on how densely the trees are planted. If n trees are planted on an acre of land, then each tree produces $900 - 9n$ apples. So the number of apples produced per acre is

$$A(n) = n(900 - 9n)$$

How many trees should be planted per acre in order to obtain the maximum yield of apples?



66. Migrating Fish A fish swims at a speed v relative to the water, against a current of 5 mi/h. Using a mathematical model of energy expenditure, it can be shown that the total energy E required to swim a distance of 10 mi is given by

$$E(v) = 2.73v^3\frac{10}{v - 5}$$

Biologists believe that migrating fish try to minimize the total energy required to swim a fixed distance. Find the value of v that minimizes energy required.

NOTE This result has been verified; migrating fish swim against a current at a speed 50% greater than the speed of the current.

67. Highway Engineering A highway engineer wants to estimate the maximum number of cars that can safely travel a particular highway at a given speed. She assumes that each car is 17 ft long, travels at a speed s, and follows the car in front of it at the "safe following distance" for that speed. She finds that the number N of cars that can pass a given point per minute is modeled by the function

$$N(s) = \frac{88s}{17 + 17\left(\dfrac{s}{20}\right)^2}$$

At what speed can the greatest number of cars travel the highway safely?

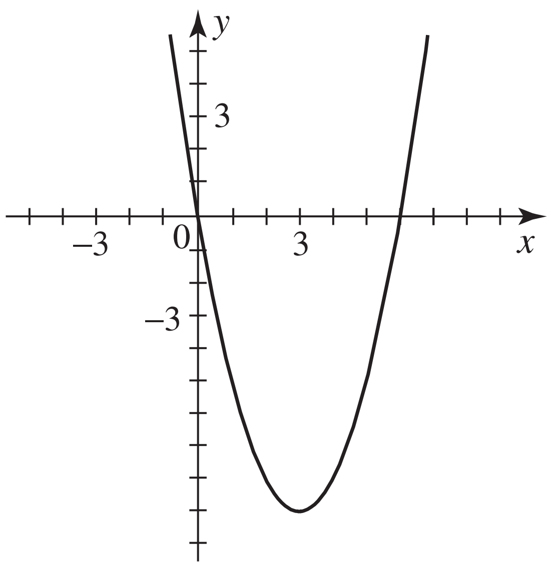
68. Volume of Water Between 0°C and 30°C, the volume V (in cubic centimeters) of 1 kg of water at a temperature T is given by the formula

$$V = 999.87 - 0.06426T + 0.0085043T^2 - 0.0000679T^3$$

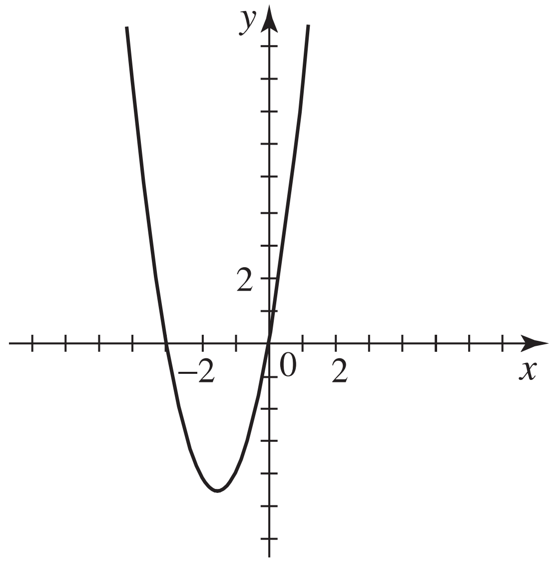
Find the temperature at which the volume of 1 kg of water is a minimum.

1. **(a)** $(3, 4)$ **(b)** 4 **3.** **(a)** $(1, -3)$ **(b)** -3

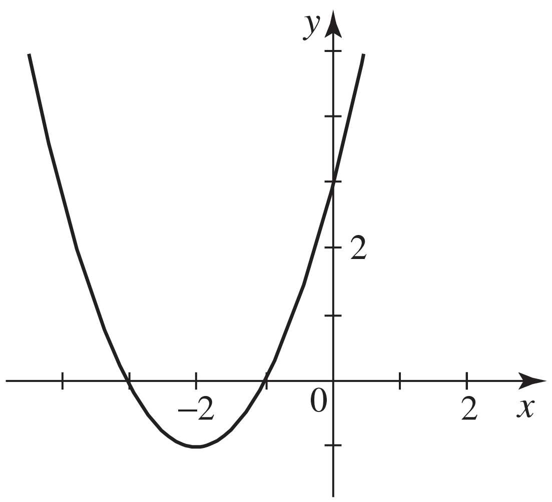
5. **(a)** $f(x) = (x - 3)^2 - 9$
(b) Vertex $(3, -9)$
x-intercepts 0, 6
y-intercept 0
(c)



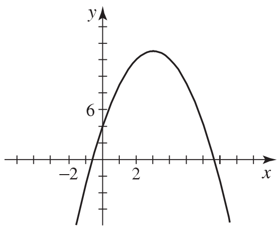
7. **(a)** $f(x) = 2\left(x + \frac{3}{2}\right)^2 - \frac{9}{2}$
(b) Vertex $\left(-\frac{3}{2}, -\frac{9}{2}\right)$
x-intercepts 0, -3,
y-intercept 0
(c)



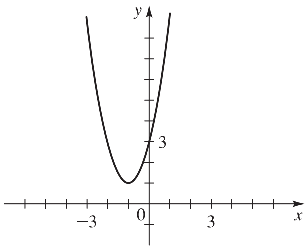
9. **(a)** $f(x) = (x + 2)^2 - 1$ **(b)** Vertex $(-2, -1)$
x-intercepts -1, -3, y-intercept 3
(c)



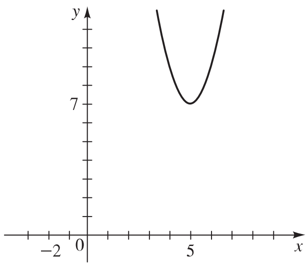
11. **(a)** $f(x) = -(x-3)^2 + 13$ **(b)** Vertex $(3, 13)$; x-intercepts $3 \pm \sqrt{13}$; y-intercept 4
(c)



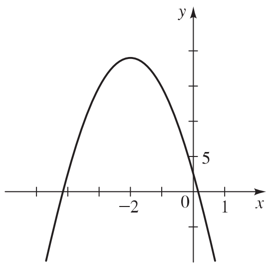
13. **(a)** $f(x) = 2(x+1)^2 + 1$ **(b)** Vertex $(-1, 1)$; no x-intercept; y-intercept 3
(c)



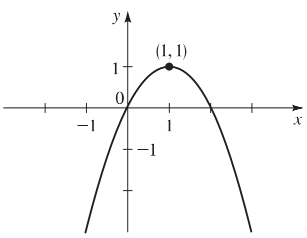
15. **(a)** $f(x) = 2(x-5)^2 + 7$ **(b)** Vertex $(5, 7)$; no x-intercept; y-intercept 57
(c)



17. **(a)** $f(x) = -4(x+2)^2 + 19$ **(b)** Vertex $(-2, 19)$; x-intercepts $-2 \pm \frac{1}{2}\sqrt{19}$; y-intercept 3
(c)

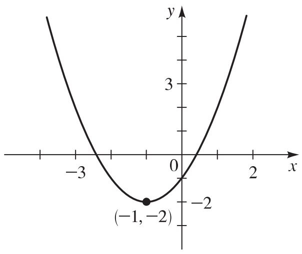


19. **(a)** $f(x) = -(x-1)^2 + 1$
(b)



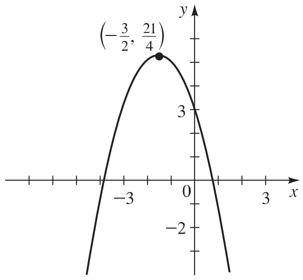
(c) Maximum $f(1) = 1$

21. **(a)** $f(x) = (x+1)^2 - 2$
(b)



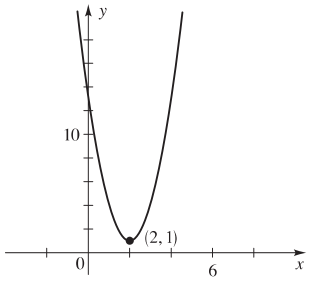
(c) Minimum $f(-1) = -2$

23. **(a)** $f(x) = -\left(x + \frac{3}{2}\right)^2 + \frac{21}{4}$
(b)



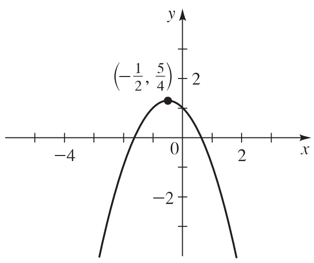
(c) Maximum $f\left(-\frac{3}{2}\right) = \frac{21}{4}$

25. **(a)** $g(x) = 3(x-2)^2 + 1$
(b)



(c) Minimum $g(2) = 1$

27. **(a)** $h(x) = -\left(x + \frac{1}{2}\right)^2 + \frac{5}{4}$
(b)



(c) Maximum $h\left(-\frac{1}{2}\right) = \frac{5}{4}$

29. Minimum $f\left(-\frac{1}{2}\right) = \frac{3}{4}$

31. Maximum $f(-3.5) = 185.75$

33. Minimum $f(0.6) = 15.64$

35. Minimum $h(-2) = -8$

37. Maximum $f(-1) = \frac{7}{2}$ **39.** $f(x) = 2x^2 - 4x$

41. $(-\infty, \infty), (-\infty, 1]$ **43.** $(-\infty, \infty), \left(-\frac{23}{2}, \infty\right)$

45. **(a)** -4.01 **(b)** -4.011025

47. Local maximum 2; local minimums -1, 0

49. Local maximums 0, 1; local minimums -2, -1

51. Local maximum ≈ 0.38 when $x \approx -0.58$; local minimum ≈ -0.38 when $x \approx 0.58$

53. Local maximum ≈ 0 when $x = 0$; local minimum ≈ -13.61 when $x \approx -1.71$; local minimum ≈ -73.32 when $x \approx 3.21$

55. Local maximum ≈ 5.66 when $x \approx 4.00$

57. Local maximum ≈ 0.38 when $x \approx -1.73$;

5.2 POLYNOMIAL FUNCTIONS AND THEIR GRAPHS

(Adapted from "Precalculus" by Stewart et als.)

Graphing Basic Polynomial Functions _ End Behavior and the Leading Term _ Using Zeros to Graph Polynomials _ Shape of the Graph Near a Zero _ Local Maxima and Minima of Polynomials

> **POLYNOMIAL FUNCTIONS**
>
> A **polynomial function of degree n** is a function of the form
>
> $$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$
>
> where n is a nonnegative integer and $a_n \neq 0$.
>
> The numbers $a_0, a_1, a_2, \ldots, a_n$ are called the **coefficients** of the polynomial.
>
> The number a_0 is the **constant coefficient** or **constant term**.
>
> The number a_n, the coefficient of the highest power, is the **leading coefficient**, and the term $a_n x^n$ is the **leading term**.

We often refer to polynomial functions simply as *polynomials*. The following polynomial has degree 5, leading coefficient 3, and constant term -6.

Leading coefficient 3   Degree 5   Constant term -6

$$3x^5 + 6x^4 - 2x^3 + x^2 + 7x - 6$$

Leading term $3x^5$ Coefficients 3, 6, -2, 1, 7, and -6

Here are some more examples of polynomials.

$P(x) = 3$	Degree 0
$Q(x) = 4x - 7$	Degree 1
$R(x) = x^2 + x$	Degree 2
$S(x) = 2x^3 - 6x^2 - 10$	Degree 3

If a polynomial consists of just a single term, then it is called a **monomial**. For example, and are monomials.

▼ Graphing Basic Polynomial Functions

The graphs of polynomials of degree 0 or 1 are lines, and the graphs of polynomials of degree 2 are parabolas. The greater the degree of a polynomial, the more complicated its graph can be. However, the graph of a polynomial function is **continuous**. This means that the graph has no breaks or holes (see Figure 1). Moreover, the graph of a polynomial function is a smooth curve; that is, it has no corners or sharp points (cusps) as shown here.

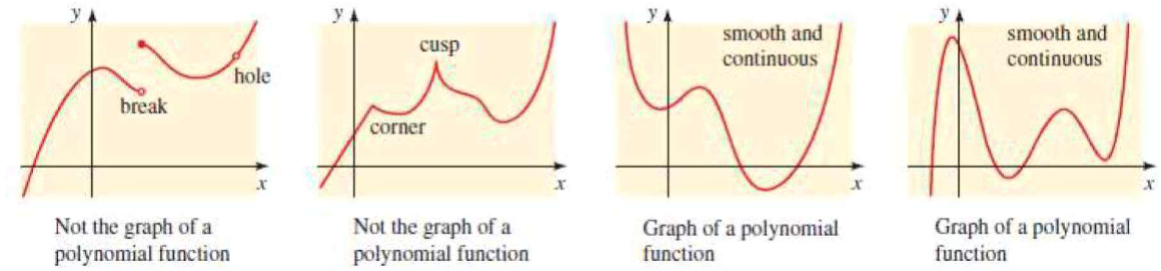
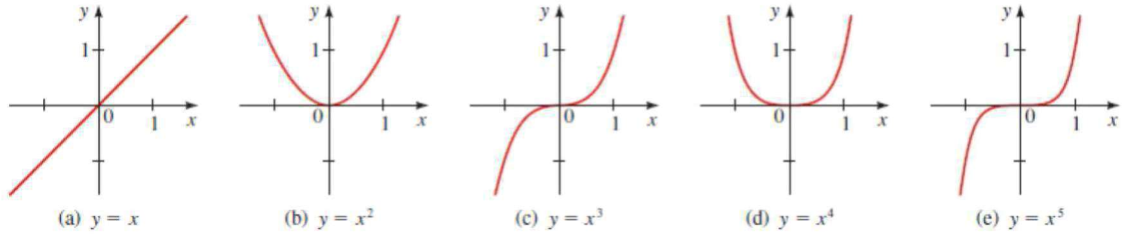


FIGURE 1

The simplest polynomial functions are the monomials $P(x) = x^n$, whose graphs are shown below. The graph of $P(x) = x^n$ has the same general shape as the graph of $y = x^2$ when n is even and the same general shape as the graph of $y = x^3$ when n is odd. However, as the degree n becomes larger, the graphs become flatter around the origin and steeper elsewhere.

Graphs of monomials (Figure2)



E X A M P L E 1 | Transformations of Monomials

Sketch the graphs of the following functions.

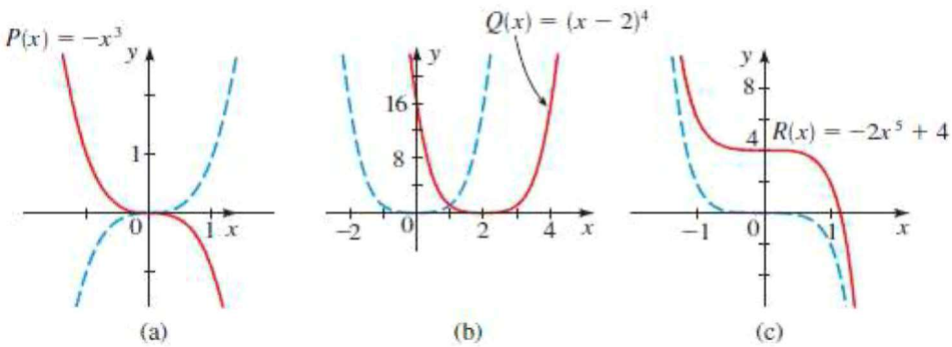
(a) $P(x) = -x^3$ **(b)** $Q(x) = (x-2)^4$ **(c)** $R(x) = -2x^5 + 4$

S O LU T I O N We use the graphs in Figure 2 and apply transformations.

(a) The graph of $P(x) = -x^3$ is the reflection of the graph of $y = x^3$ in the x-axis, as shown in the figure below.

(b) The graph of $Q(x) = (x-2)^4$ is the graph of $y = x^4$ shifted to the right 2 units, as shown in the figure below.

(c) We begin with the graph of $y = x^5$. The graph of $y = 2x^5$ is obtained by stretching the graph vertically and reflecting it in the x-axis (the dashed blue graph). Finally, the graph of $R(x)$ is obtained by shifting upward 4 units (the red graph).



NOW TRY: Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^n$ from Figure 2. Indicate all x- and y-intercepts on each graph.

(a) $P(x) = x^2 - 4$ (b) $Q(x) = (x-4)^2$ (c) $R(x) = 2x^2 - 2$ (d) $S(x) = 2(x-2)^2$

▼ End Behavior and the Leading Term

The **end behavior** of a polynomial is a description of what happens as x becomes large in the positive or negative direction. To describe end behavior, we use the following notation:

$x \to \infty$	means	"x becomes large in the positive direction"
$x \to -\infty$	means	"x becomes large in the negative direction"

For example, the monomial $y = x^2$ in Figure 2(b) has the following end behavior:

as $x \to -\infty, y \to \infty$ and as $x \to \infty, y \to \infty$

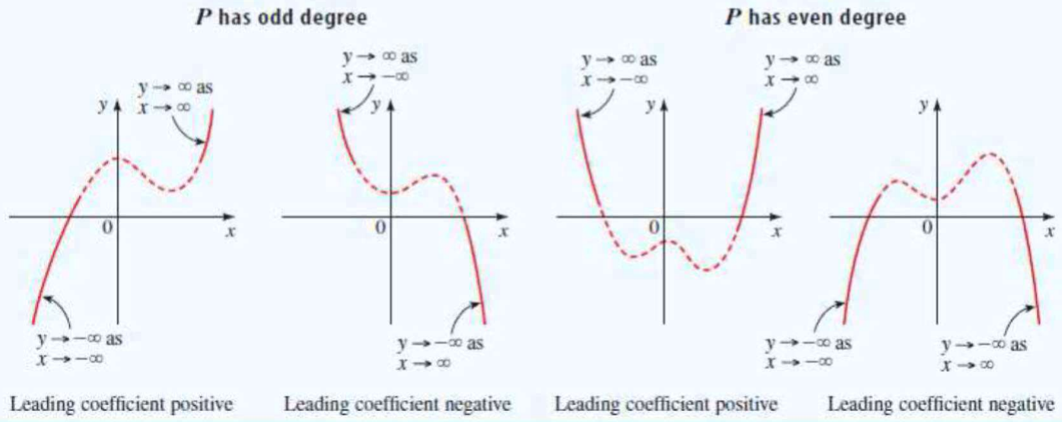
The monomial $y = x^3$ in Figure 2(c) has the following end behavior:

as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$

For any polynomial *the end behavior is determined by the term that contains the highest power of x,* because when x is large, the other terms are relatively insignificant in size. The following box shows the four possible types of end behavior, based on the highest power and the sign of its coefficient.

END BEHAVIOR OF POLYNOMIALS

The end behavior of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is determined by the degree n and the sign of the leading coefficient a_n, as indicated in the following graphs.

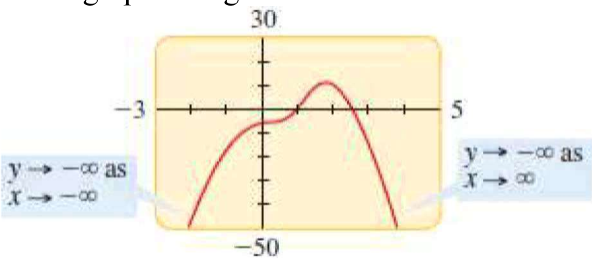


E X A M P L E 2 | End Behavior of a Polynomial

Determine the end behavior of the polynomial $P(x) = -2x^4 + 5x^3 + 4x - 7$

S O LU T I O N The polynomial P has degree 4 and leading coefficient -2.
Thus, P has *even* degree and *negative* leading coefficient,
so it has the following end behavior: as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to -\infty$

The graph in Figure 4 illustrates the end behavior of P.



NOW TRY EXERCISE **5-10**

E X A M P L E 3 | End Behavior of a Polynomial

(a) Determine the end behavior of the polynomial $P(x) = 3x^5 - 5x^3 + 2x$.

(b) Confirm that P and its leading term $Q(x) = 3x^5$ have the same end behavior by graphing them together. [Part (b) *requires the use of a graphing calculator or equivalent.*]

S O LU T I O N

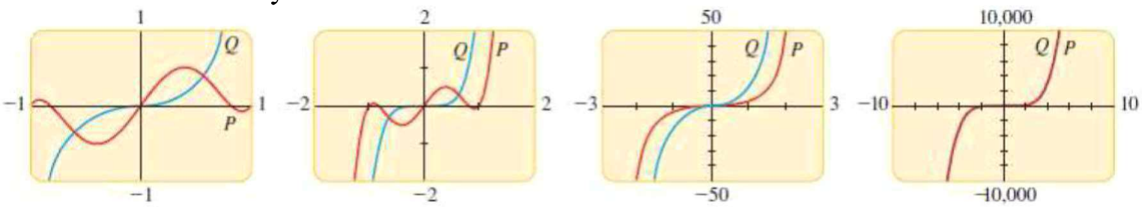
(a) Since P has odd degree and positive leading coefficient, it has the following end behaviour::

as $x \to -\infty, y \to -\infty$ and as $x \to \infty, y \to \infty$

(b) The figure below shows the graphs of P and Q in progressively larger viewing rectangles.

The larger the viewing rectangle, the more the graphs look alike.

This confirms that they have the same end behavior.



NOW TRYL Determine the end behavior of the polynomial $P(x) = 3x^3 - x^2 + 5x + 1$

To see algebraically why P and Q in Example 3 have the same end behavior, factor P as follows and compare with Q.

$$P(x) = 3x^5\left(1 - \frac{5}{3x^2} + \frac{2}{3x^4}\right) \qquad Q(x) = 3x^5$$

When x is large, the terms $\frac{5}{3x^2}$ and $\frac{2}{3x^4}$ are close to 0. So for large x, we have

$$P(x) \approx 3x^5(1 - 0 - 0) = 3x^5 = Q(x)$$

So when x is large, P and Q have approximately the same values. We can also see this numerically by making a table like the one shown below.

x	$P(x)$	$Q(x)$
15	2,261,280	2,278,125
30	72,765,060	72,900,000
50	936,875,100	937,500,000

By the same reasoning we can show that the end behavior of *any* polynomial is determined by its leading term.

▼ Using Zeros to Graph Polynomials

If P is a polynomial function, then c is called a **zero** of P if $P(c) = 0$.
In other words, the zeros of P are the solutions of the polynomial equation $P(x) = 0$.
Note that if $P(c) = 0$, then the graph of P has an x-intercept at $x = c$, so the x-intercepts of the graph are the zeros of the function.

REAL ZEROS OF POLYNOMIALS

If P is a polynomial and c is a real number, then the following are equivalent:

1. c is a zero of P.
2. $x = c$ is a solution of the equation $P(x) = 0$.
3. $x - c$ is a factor of $P(x)$.
4. c is an x-intercept of the graph of P.

To find the zeros of a polynomial P, we factor and then use the Zero-Product Property.
For example, to find the zeros of $P(x) = x^2 + x - 6$, we factor P to get

$$P(x) = (x-2)(x+3)$$

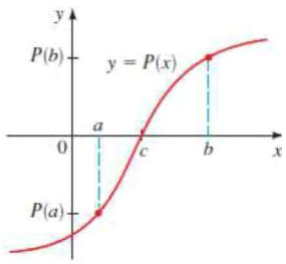
From this factored form we easily see that
1. 2 is a zero of P.
2. $x = 2$ is a solution of the equation $x^2 + x - 6 = 0$.
3. $x - 2$ is a factor of $x^2 + x - 6$.
4. 2 is an x-intercept of the graph of P.
The same facts are true for the other zero, -3.

The following theorem has many important consequences.

INTERMEDIATE VALUE THEOREM FOR POLYNOMIALS

If P is a polynomial function and $P(a)$ and $P(b)$ have opposite signs, then there exists at least one value c between a and b for which $P(c) = 0$.

Here we use it to help us graph polynomial functions.
We will not prove this theorem, but the following figure shows why it is intuitively plausible.



One important consequence of this theorem is that between any two successive zeros the values of a polynomial are either all positive or all negative. That is, between two successive zeros the graph of a polynomial lies *entirely above* or *entirely below* the x-axis. To see why, suppose c_1 and c_2 are successive zeros of P. If P has both positive and negative values between c_1 and c_2, then by the Intermediate Value Theorem P must have another zero between c_1 and c_2. But that's not possible because c_1 and c_2 are successive zeros. This observation allows us to use the following guidelines to graph polynomial functions.

GUIDELINES FOR GRAPHING POLYNOMIAL FUNCTIONS

1. **Zeros.** Factor the polynomial to find all its real zeros; these are the x-intercepts of the graph.
2. **Test Points.** Make a table of values for the polynomial. Include test points to determine whether the graph of the polynomial lies above or below the x-axis on the intervals determined by the zeros. Include the y-intercept in the table.
3. **End Behavior.** Determine the end behavior of the polynomial.
4. **Graph.** Plot the intercepts and other points you found in the table. Sketch a smooth curve that passes through these points and exhibits the required end behavior.

E X A M P L E 4 | Using Zeros to Graph a Polynomial Function

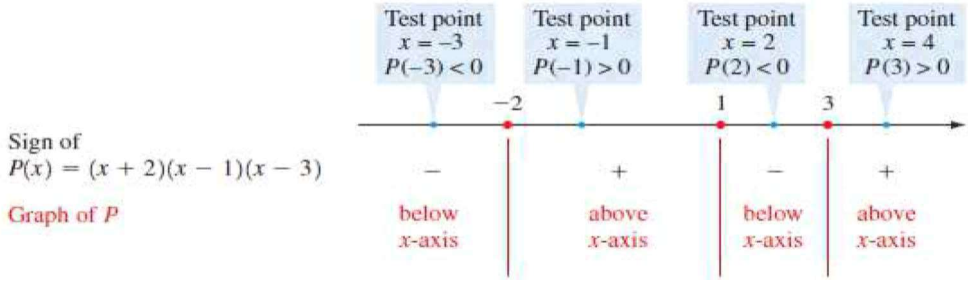
Sketch the graph of the polynomial function $P(x) = (x+2)(x-1)(x-3)$.

S O LU T I O N

The zeros are $x = -2$, 1, and 3.

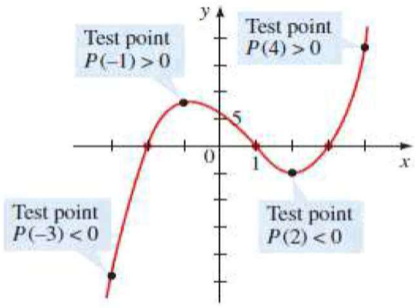
These determine the intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$, and $(3, \infty)$.

Using test points in these intervals, we get the information in the following sign diagram.



Plotting a few additional points and connecting them with a smooth curve helps us to complete the graph. $P(x) = (x+2)(x-1)(x-3)$

	x	$P(x)$
Test point →	−3	−24
	−2	0
Test point →	−1	8
	0	6
	1	0
Test point →	2	−4
	3	0
Test point →	4	18



NOW TRY: Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior. $P(x) = x(x-3)(x+2)$

E X A M P L E 5 | Finding Zeros and Graphing a Polynomial Function

Let $P(x) = x^3 - 2x^2 - 3x$

(a) Find the zeros of P. **(b)** Sketch a graph of P.

S O LU T I O N

(a) To find the zeros, we factor completely.

$$P(x) = x^3 - 2x^2 - 3x$$
$$= x(x^2 - 2x - 3)$$
$$= x(x-3)(x+1)$$

Thus, the zeros are $x = 0$, $x = 3$, and $x = -1$.

(b) The x-intercepts are $x = 0$, $x = 3$, and $x = -1$.

The y-intercept is $P(0) = 0$.

We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros.

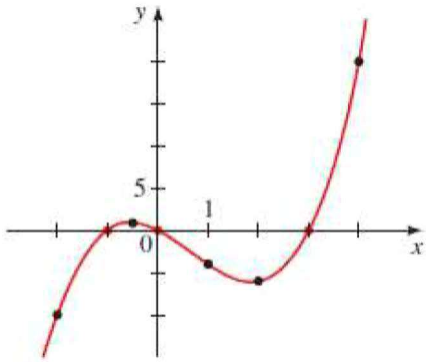
Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$y \to \infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$

We plot the points in the table and connect them by a smooth curve to complete the graph.

$$P(x) = x^3 - 2x^2 - 3x$$

	x	$P(x)$
Test point →	-2	-10
	-1	0
Test point →	$-\frac{1}{2}$	$\frac{7}{8}$
	0	0
Test point →	1	-4
	2	-6
	3	0
Test point →	4	20



NOW TRY: Factor the polynomial and use the factored form to find the zeros. Then sketch the graph. $P(x) = x^3 - x^2 - 6x$

E X A M P L E 6 | Finding Zeros and Graphing a Polynomial Function

Let $P(x) = -2x^4 - x^3 + 3x^2$

(a) Find the zeros of P. **(b)** Sketch a graph of P.

S O LU T I O N

(a) To find the zeros, we factor completely.

$$P(x) = -2x^4 - x^3 + 3x^2$$
$$= \qquad\qquad \text{Factor } -x^2$$
$$= -x^2(2x+3)(x-1) \qquad \text{Factor quadratic}$$

Thus, the zeros are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

(b) The x-intercepts are $x = 0$, $x = -\frac{3}{2}$, and $x = 1$.

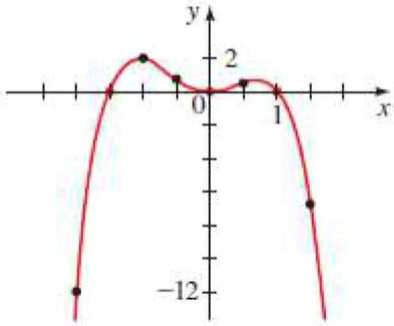
The y-intercept is $P(0) = 0$.

Since P is of even degree and its leading coefficient is negative, it has the following end behavior:

$y \to -\infty$ as $x \to \infty$ and $y \to -\infty$ as $x \to -\infty$

We make a table of values of $P(x)$, making sure that we choose test points between (and to the right and left of) successive zeros
We plot the points from the table and connect the points by a smooth curve to complete the graph.

x	$P(x)$
-2	-12
-1.5	0
-1	2
-0.5	0.75
0	0
0.5	0.5
1	0
1.5	-6.75



$P(x) = -2x^4 - x^3 + 3x^2$

The use of test points may not be necessary after we discuss the multiplicity of each zero. We shall make use of the multiplicity of each zero instead of the test points.

▼ Shape of the Graph Near a Zero

Although $x = 2$ is a zero of the polynomial in Example 7, the graph does not cross the x-axis at the x-intercept 2. This is because the factor corresponding to that zero is raised to an even power, so it doesn't change sign as we test points on either side of 2.

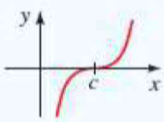
In general, if c is a zero of P, and the corresponding factor $x - c$ occurs exactly m times in the factorization of P, then we say that c is a **zero of multiplicity *m***. By considering test points on either side of the x-intercept c, we conclude that the graph crosses the x-axis at c if <u>the multiplicity m is odd</u> and does not cross the x-axis <u>if m is even</u>. Moreover, it can be shown by using calculus that near $x = c$ the graph has the same general shape as the graph of $y = A(x-c)^m$.

SHAPE OF THE GRAPH NEAR A ZERO OF MULTIPLICITY m

If c is a zero of P of multiplicity m, then the shape of the graph of P near c is as follows.

Multiplicity of c — **Shape of the graph of P near the x-intercept c**

m odd, $m > 1$



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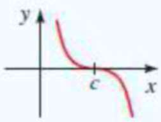
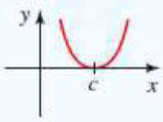


m even, $m > 1$



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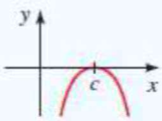
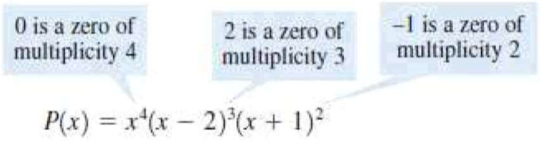


.

E X A M P L E 8 | Graphing a Polynomial Function Using Its Zeros
Graph the polynomial $P(x) = x^4(x-2)^3(x+1)^2$

SOLUTION

The zeros of P are -1, 0, and 2 with multiplicities 2, 4, and 3, respectively.



The zero 2 has *odd* multiplicity, so the graph crosses the x-axis at the x-intercept 2.
But the zeros 0 and -1 have *even* multiplicity, so the graph does not cross the x-axis at the x-intercepts 0 and -1.

Zero	Multiplicity	"touches" or "crosses"
-1	2	touches
0	4	touches
2	3	crosses

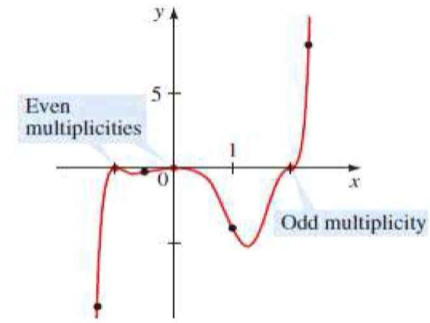
Since P is a polynomial of degree 9 and has positive leading coefficient, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

With this information and a table of values we sketch the graph.

x	$P(x)$
-1.3	-9.2
-1	0
-0.5	-3.9
0	0
1	-4
2	0
2.3	8.2

The test points are not really necessary.



$P(x) = x^4(x-2)^3(x+1)^2$

EXAMPLE 7 | Finding Zeros and Graphing a Polynomial Function

Let . $P(x) = x^3 - 2x^2 - 4x + 8$

(a) Find the zeros of P. **(b)** Sketch a graph of P.

SOLUTION

(a) To find the zeros, we factor completely.

$P(x) = x^3 - 2x^2 - 4x + 8$

$=$ Group and factor

$=$ Factor $x - 2$

$=$ Difference of squares

$= (x+2)(x-2)^2$ Simplify

Thus the zeros are $x = -2$ and $x = 2$.

Zero	Multiplicity	"touches" or "crosses"
-2	1	Crosses
2	2	Touches

(b) The x-intercepts are $x = -2$ and $x = 2$.

The y-intercept is $P(0) = 8$.

The table gives additional values of $P(x)$.

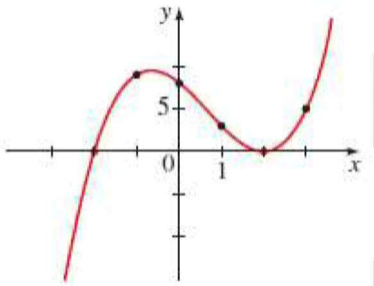
Since P is of odd degree and its leading coefficient is positive, it has the following end behavior:

$$y \to \infty \quad \text{as} \quad x \to \infty \qquad \text{and} \qquad y \to -\infty \quad \text{as} \quad x \to -\infty$$

We connect the points by a smooth curve to complete the graph.

x	$P(x)$
-3	-25
-2	0
-1	9
0	8
1	3
2	0
3	5

The test points are not really necessary.



$P(x) = x^3 - 2x^2 - 4x + 8$

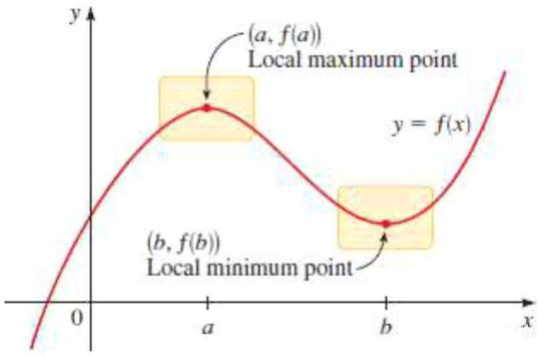
NOW TRY: Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior. $P(x) = x^4(x+2)(x-3)^2$

NOW TRY Factor the polynomial and use the factored form to find the zeros. Then sketch the graph.

(i) $P(x) = x^4 - 3x^3 + 2x^2$ (ii) $P(x) = x^3 + x^2 - x - 1$

▼ Local Maxima and Minima of Polynomials

Recall from Section 2.3 that if the point $(a, f(a))$ is the highest point on the graph of f within some viewing rectangle, then $f(a)$ is a local maximum value of f, and if $(b, f(b))$ is the lowest point on the graph of f within a viewing rectangle, then $f(b)$ is a local minimum value (see figure). We say that such a point $(a, f(a))$ is a **local maximum point** on the graph and that $(b, f(b))$ is a **local minimum point**. The local maximum and minimum points on the graph of a function are called its **local extrema**.



For a polynomial function the number of local extrema must be less than the degree, as the following principle indicates. (A proof of this principle requires calculus.)

LOCAL EXTREMA OF POLYNOMIALS

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n, then the graph of P has at most $n - 1$ local extrema.

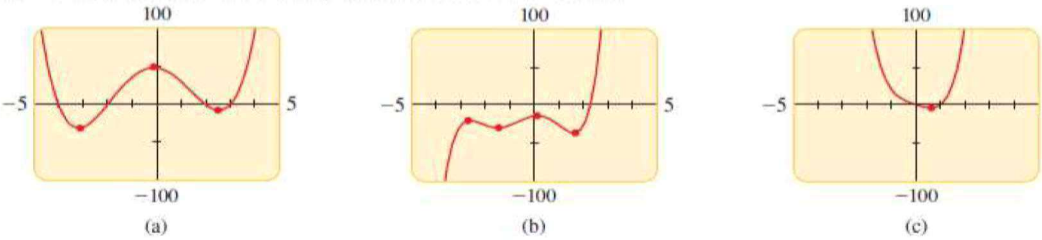
A polynomial of degree n may in fact have less than $n - 1$ local extrema. For example, $P(x) = x^5$ (graphed in Figure 2) has *no* local extrema, even though it is of degree 5. The preceding principle tells us only that a polynomial of degree n can have no more than $n - 1$ local extrema.

EXAMPLE 9 | The Number of Local Extrema – Some examples

(a) P_1 has two local minimum points and one local maximum point, for a total of three local extrema.

(b) P_2 has two local minimum points and two local maximum points, for a total of four local extrema.

(c) P_3 has just one local extremum, a local minimum.



(a) $P_1(x) = x^4 + x^3 - 16x^2 - 4x + 48$

(b) $P_2(x) = x^5 + 3x^4 - 5x^3 - 15x^2 + 4x - 15$

(c) $P_3(x) = 7x^4 + 3x^2 - 10x$

Exercise

1–4 ■ Sketch the graph of each function by transforming the graph of an appropriate function of the form $y = x^n$ from Figure 2. Indicate all x- and y-intercepts on each graph.

1. **(a)** $P(x) = x^2 - 4$ **(b)** $Q(x) = (x - 4)^2$
(c) $R(x) = 2x^2 - 2$ **(d)** $S(x) = 2(x - 2)^2$

2. **(a)** $P(x) = x^4 - 16$ **(b)** $Q(x) = (x + 2)^4$
(c) $R(x) = (x + 2)^4 - 16$ **(d)** $S(x) = -2(x + 2)^4$

3. **(a)** $P(x) = x^3 - 8$ **(b)** $Q(x) = -x^3 + 27$
(c) $R(x) = -(x + 2)^3$ **(d)** $S(x) = \frac{1}{2}(x - 1)^3 + 4$

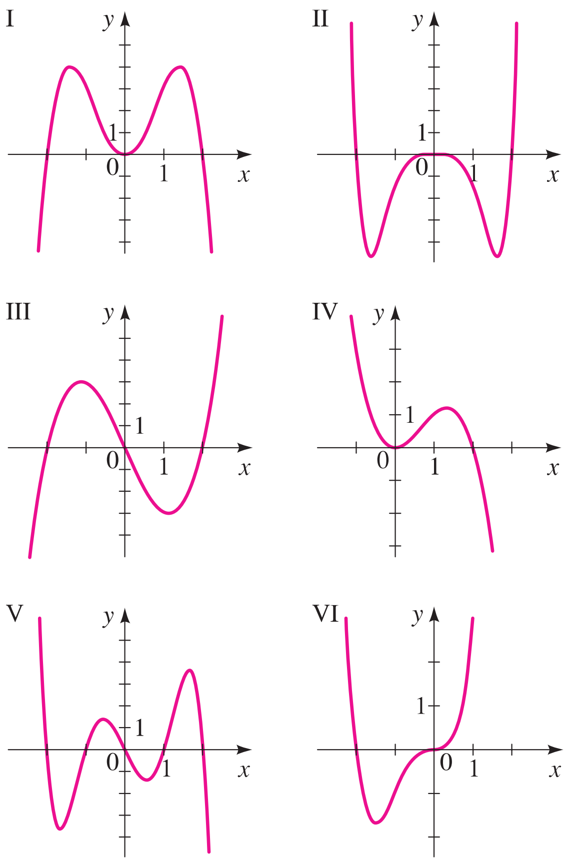
4. **(a)** $P(x) = (x + 3)^5$ **(b)** $Q(x) = 2(x + 3)^5 - 64$
(c) $R(x) = -\frac{1}{2}(x - 2)^5$ **(d)** $S(x) = -\frac{1}{2}(x - 2)^5 + 16$

5–10 ■ Match the polynomial function with one of the graphs I–VI. Give reasons for your choice.

5. $P(x) = x(x^2 - 4)$ **6.** $Q(x) = -x^2(x^2 - 4)$

7. $R(x) = -x^5 + 5x^3 - 4x$ **8.** $S(x) = \frac{1}{2}x^6 - 2x^4$

9. $T(x) = x^4 + 2x^3$ **10.** $U(x) = -x^3 + 2x^2$



11–22 ■ Sketch the graph of the polynomial function. Make sure your graph shows all intercepts and exhibits the proper end behavior.

11. $P(x) = (x - 1)(x + 2)$

12. $P(x) = (x - 1)(x + 1)(x - 2)$

13. $P(x) = x(x - 3)(x + 2)$

14. $P(x) = (2x - 1)(x + 1)(x + 3)$

15. $P(x) = (x - 3)(x + 2)(3x - 2)$

16. $P(x) = \frac{1}{5}x(x - 5)^2$

17. $P(x) = (x - 1)^2(x - 3)$

18. $P(x) = \frac{1}{4}(x + 1)^3(x - 3)$

19. $P(x) = \frac{1}{12}(x + 2)^2(x - 3)^2$

20. $P(x) = (x - 1)^2(x + 2)^3$

21. $P(x) = x^3(x + 2)(x - 3)^2$

22. $P(x) = (x - 3)^2(x + 1)^2$

23–36 ■ Factor the polynomial and use the factored form to find the zeros. Then sketch the graph.

23. $P(x) = x^3 - x^2 - 6x$

24. $P(x) = x^3 + 2x^2 - 8x$

25. $P(x) = -x^3 + x^2 + 12x$

26. $P(x) = -2x^3 - x^2 + x$

27. $P(x) = x^4 - 3x^3 + 2x^2$

28. $P(x) = x^5 - 9x^3$

29. $P(x) = x^3 + x^2 - x - 1$

30. $P(x) = x^3 + 3x^2 - 4x - 12$

31. $P(x) = 2x^3 - x^2 - 18x + 9$

32. $P(x) = \frac{1}{8}(2x^4 + 3x^3 - 16x - 24)^2$

33. $P(x) = x^4 - 2x^3 - 8x + 16$

34. $P(x) = x^4 - 2x^3 + 8x - 16$

35. $P(x) = x^4 - 3x^2 - 4$

36. $P(x) = x^6 - 2x^3 + 1$

37–42 ■ Determine the end behavior of P. Compare the graphs of P and Q on large and small viewing rectangles, as in Example 3(b).

37. $P(x) = 3x^3 - x^2 + 5x + 1$; $Q(x) = 3x^3$

38. $P(x) = -\frac{1}{8}x^3 + \frac{1}{4}x^2 + 12x$; $Q(x) = -\frac{1}{8}x^3$

39. $P(x) = x^4 - 7x^2 + 5x + 5$; $\quad Q(x) = x^4$

40. $P(x) = -x^5 + 2x^2 + x$; $\quad Q(x) = -x^5$

41. $P(x) = x^{11} - 9x^9$; $\quad Q(x) = x^{11}$

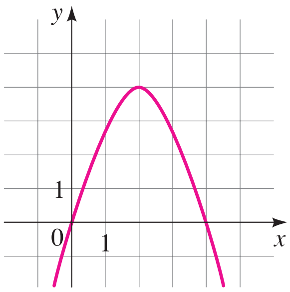
42. $P(x) = 2x^2 - x^{12}$; $\quad Q(x) = -x^{12}$

43–46 ■ The graph of a polynomial function is given. From the graph, find

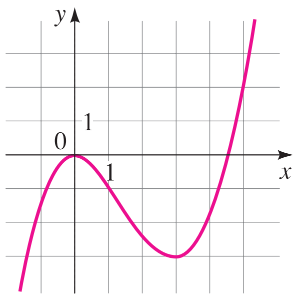
(a) the x- and y-intercepts

(b) the coordinates of all local extrema

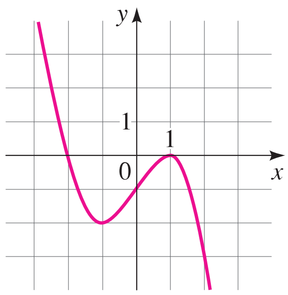
43. $P(x) = -x^2 + 4x$



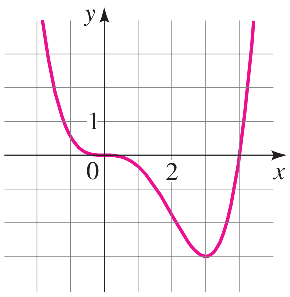
44. $P(x) = \frac{2}{9}x^3 - x^2$



45. $P(x) = -\frac{1}{2}x^3 + \frac{3}{2}x - 1$



46. $P(x) = \frac{1}{9}x^4 - \frac{4}{9}x^3$



47–54 ■ Graph the polynomial in the given viewing rectangle. Find the coordinates of all local extrema. State each answer correct to two decimal places.

47. $y = -x^2 + 8x$, $\quad [-4, 12]$ by $[-50, 30]$

48. $y = x^3 - 3x^2$, $\quad [-2, 5]$ by $[-10, 10]$

49. $y = x^3 - 12x + 9$, $\quad [-5, 5]$ by $[-30, 30]$

50. $y = 2x^3 - 3x^2 - 12x - 32$, $\quad [-5, 5]$ by $[-60, 30]$

51. $y = x^4 + 4x^3$, $\quad [-5, 5]$ by $[-30, 30]$

52. $y = x^4 - 18x^2 + 32$, $\quad [-5, 5]$ by $[-100, 100]$

53. $y = 3x^5 - 5x^3 + 3$, $\quad [-3, 3]$ by $[-5, 10]$

54. $y = x^5 - 5x^2 + 6$, $\quad [-3, 3]$ by $[-5, 10]$

55–64 ■ Graph the polynomial and determine how many local maxima and minima it has.

55. $y = -2x^2 + 3x + 5$

56. $y = x^3 + 12x$

57. $y = x^3 - x^2 - x$

58. $y = 6x^3 + 3x + 1$

59. $y = x^4 - 5x^2 + 4$

60. $y = 1.2x^5 + 3.75x^4 - 7x^3 - 15x^2 + 18x$

61. $y = (x - 2)^5 + 32$

62. $y = (x^2 - 2)^3$

63. $y = x^8 - 3x^4 + x$

64. $y = \frac{1}{3}x^7 - 17x^2 + 7$

65–70 ■ Graph the family of polynomials in the same viewing rectangle, using the given values of c. Explain how changing the value of c affects the graph.

65. $P(x) = cx^3$; $\quad c = 1, 2, 5, \frac{1}{2}$

66. $P(x) = (x - c)^4$; $\quad c = -1, 0, 1, 2$

67. $P(x) = x^4 + c$; $\quad c = -1, 0, 1, 2$

68. $P(x) = x^3 + cx$; $\quad c = 2, 0, -2, -4$

69. $P(x) = x^4 - cx$; $\quad c = 0, 1, 8, 27$

70. $P(x) = x^c$; $\quad c = 1, 3, 5, 7$

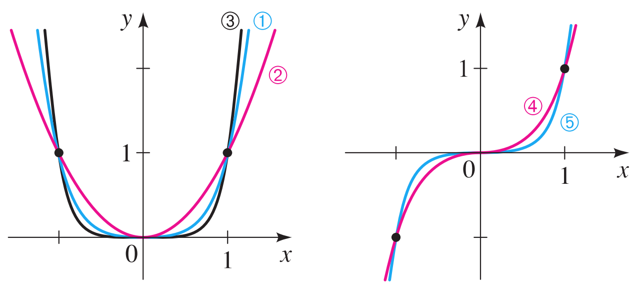
71. **(a)** On the same coordinate axes, sketch graphs (as accurately as possible) of the functions

$$y = x^3 - 2x^2 - x + 2 \qquad \text{and} \qquad y = -x^2 + 5x + 2$$

(b) Based on your sketch in part (a), at how many points do the two graphs appear to intersect?

(c) Find the coordinates of all intersection points.

72. Portions of the graphs of $y = x^2$, $y = x^3$, $y = x^4$, $y = x^5$, and $y = x^6$ are plotted in the figures. Determine which function belongs to each graph.



73. Recall that a function f is *odd* if $f(-x) = -f(x)$ or *even* if $f(-x) = f(x)$ for all real x.

(a) Show that a polynomial $P(x)$ that contains only odd powers of x is an odd function.

(b) Show that a polynomial $P(x)$ that contains only even powers of x is an even function.

(c) Show that if a polynomial $P(x)$ contains both odd and even powers of x, then it is neither an odd nor an even function.

(d) Express the function

$$P(x) = x^5 + 6x^3 - x^2 - 2x + 5$$

as the sum of an odd function and an even function.

74. (a) Graph the function $P(x) = (x - 1)(x - 3)(x - 4)$ and find all local extrema, correct to the nearest tenth.

(b) Graph the function

$$Q(x) = (x - 1)(x - 3)(x - 4) + 5$$

and use your answers to part (a) to find all local extrema, correct to the nearest tenth.

75. (a) Graph the function $P(x) = (x - 2)(x - 4)(x - 5)$ and determine how many local extrema it has.

(b) If $a < b < c$, explain why the function

$$P(x) = (x - a)(x - b)(x - c)$$

must have two local extrema.

76. (a) How many x-intercepts and how many local extrema does the polynomial $P(x) = x^3 - 4x$ have?

(b) How many x-intercepts and how many local extrema does the polynomial $Q(x) = x^3 + 4x$ have?

(c) If $a > 0$, how many x-intercepts and how many local extrema does each of the polynomials $P(x) = x^3 - ax$ and $Q(x) = x^3 + ax$ have? Explain your answer.

Applications

77. Market Research A market analyst working for a small-appliance manufacturer finds that if the firm produces and sells x blenders annually, the total profit (in dollars) is

$$P(x) = 8x + 0.3x^2 - 0.0013x^3 - 372$$

Graph the function P in an appropriate viewing rectangle and use the graph to answer the following questions.

(a) When just a few blenders are manufactured, the firm loses money (profit is negative). (For example, $P(10) = -263.3$, so the firm loses \$263.30 if it produces and sells only 10 blenders.) How many blenders must the firm produce to break even?

(b) Does profit increase indefinitely as more blenders are produced and sold? If not, what is the largest possible profit the firm could have?

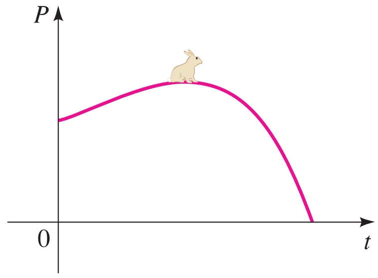
78. Population Change The rabbit population on a small island is observed to be given by the function

$$P(t) = 120t - 0.4t^4 + 1000$$

where t is the time (in months) since observations of the island began.

(a) When is the maximum population attained, and what is that maximum population?

(b) When does the rabbit population disappear from the island?

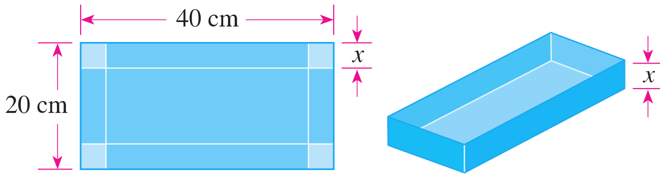


79. Volume of a Box An open box is to be constructed from a piece of cardboard 20 cm by 40 cm by cutting squares of side length x from each corner and folding up the sides, as shown in the figure.

(a) Express the volume V of the box as a function of x.

(b) What is the domain of V? (Use the fact that length and volume must be positive.)

(c) Draw a graph of the function V and use it to estimate the maximum volume for such a box.

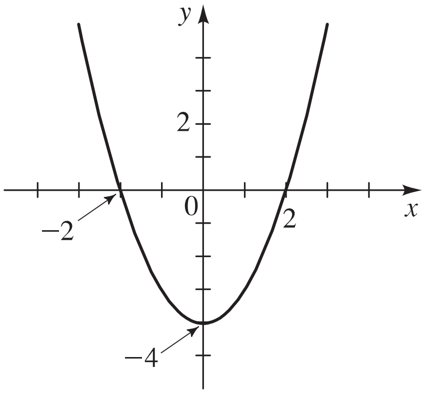


80. Volume of a Box A cardboard box has a square base, with each edge of the base having length x inches, as shown in the figure. The total length of all 12 edges of the box is 144 in.

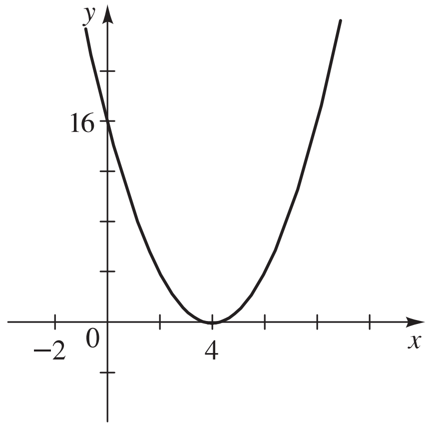
(a) Show that the volume of the box is given by the function $V(x) = 2x^2(18 - x)$.

Answer

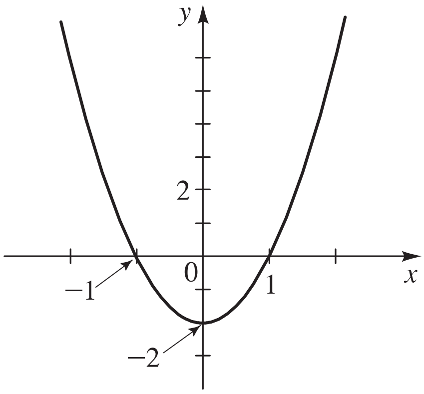
1. (a)



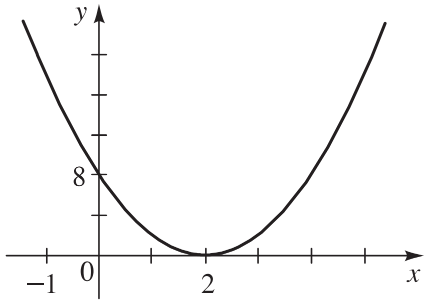
(b)



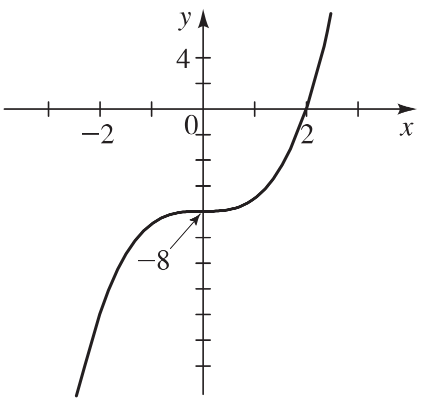
(c)



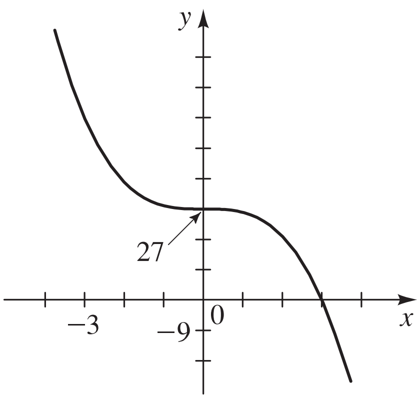
(d)



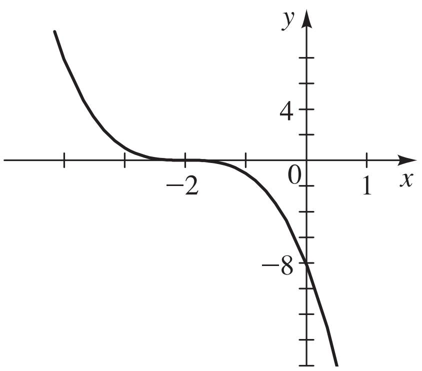
3. (a)



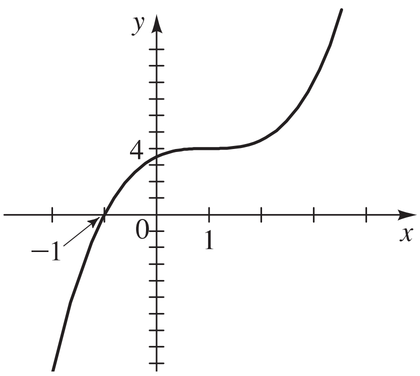
(b)



(c)

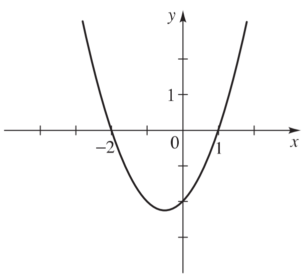


(d)



5. III **7.** V **9.** VI

11.



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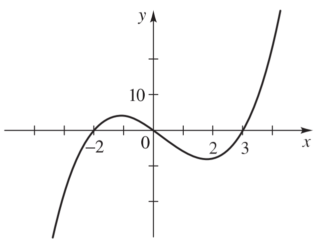

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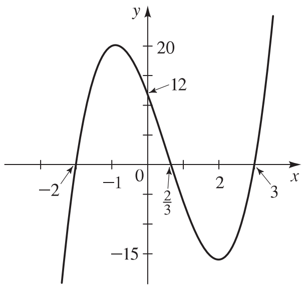

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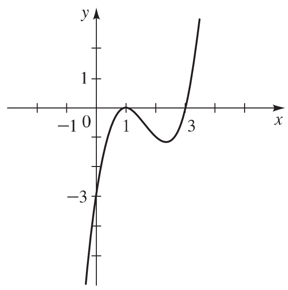

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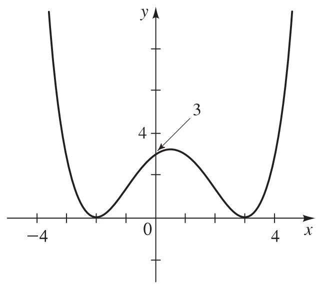

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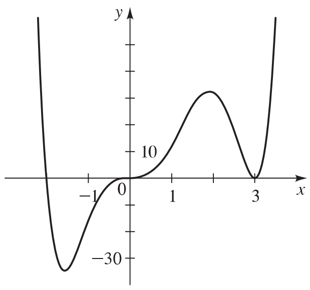
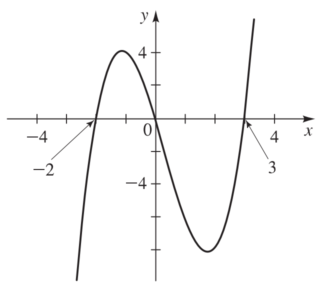
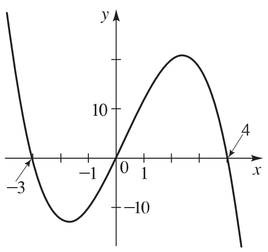


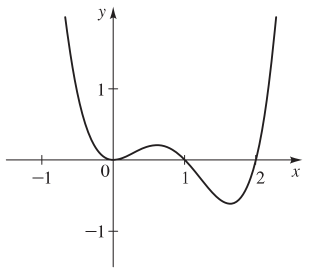
23. $P(x) = x(x + 2)(x - 3)$



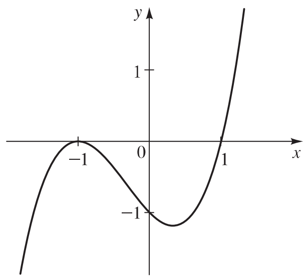
25. $P(x) = -x(x + 3)(x - 4)$



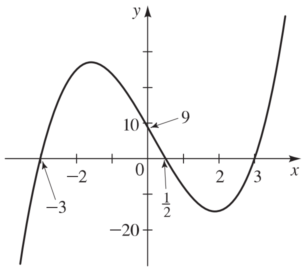
27. $P(x) = x^2(x - 1)(x - 2)$



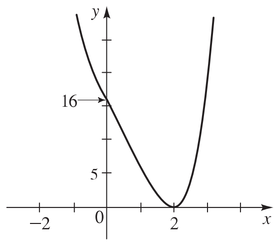
29. $P(x) = (x + 1)^2(x - 1)$



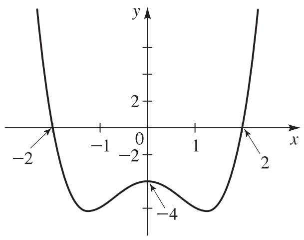
31. $P(x) = (2x - 1)(x + 3)(x - 3)$



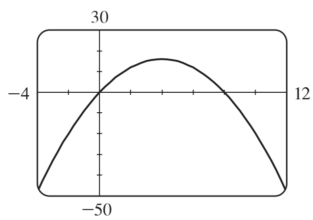
33. $P(x) = (x - 2)^2(x^2 + 2x + 4)$



35. $P(x) = (x^2 + 1)(x + 2)(x - 2)$

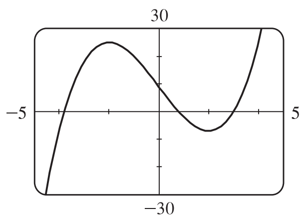


37. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
39. $y \to \infty$ as $x \to \pm\infty$
41. $y \to \infty$ as $x \to \infty$, $y \to -\infty$ as $x \to -\infty$
43. **(a)** x-intercepts 0, 4; y-intercept 0 **(b)** $(2, 4)$
45. **(a)** x-intercepts -2, 1; y-intercept -1
(b) $(-1, -2)$, $(1, 0)$
47.



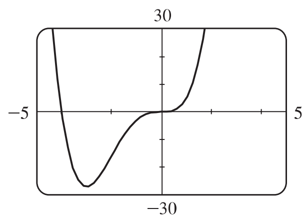
local maximum $(4, 16)$

49.



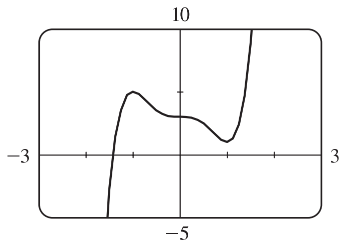
local maximum $(-2, 25)$,
local minimum $(2, -7)$

51.



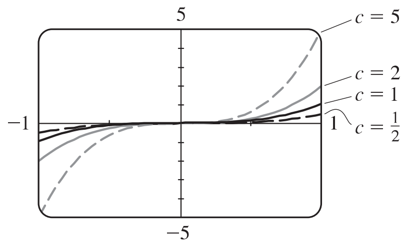
local minimum $(-3, -27)$

53.



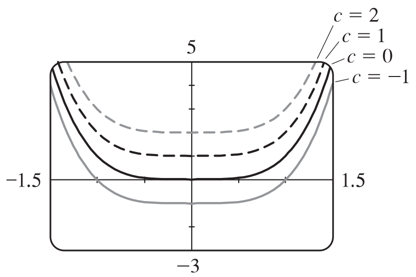
local maximum $(-1, 5)$,
local minimum $(1, 1)$

55. One local maximum, no local minimum
57. One local maximum, one local minimum
59. One local maximum, two local minima
61. No local extrema
63. One local maximum, two local minima
65.



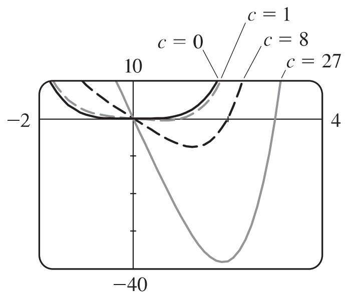
Increasing the value of c stretches the graph vertically.

67.



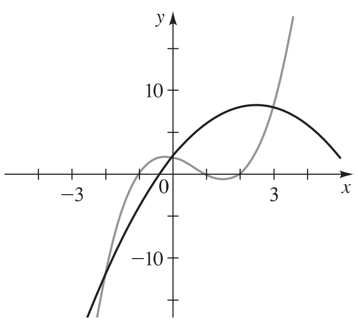
Increasing the value of c moves the graph up.

69.



Increasing the value of c causes a deeper dip in the graph in the fourth quadrant and moves the positive x-intercept to the right.

71. (a)



5.3 DIVIDING POLYNOMIALS

(Adapted from "Precalculus" by Stewart et als.)

Long Division of Polynomials _ Synthetic Division
_ The Remainder and Factor Theorems

So far in this chapter we have been studying polynomial functions *graphically*. In this section we begin to study polynomials *algebraically*. Most of our work will be concerned with factoring polynomials, and to factor, we need to know how to divide polynomials.

▼ Long Division of Polynomials

Dividing polynomials is much like the familiar process of dividing numbers. When we divide 38 by 7, the quotient is 5 and the remainder is 3. We write

Dividend

Remainder

$$\frac{38}{7} = 5 + \frac{3}{7}$$

Divisor

Quotient

To divide polynomials, we use long division, as follows.

DIVISION ALGORITHM

If $P(x)$ and $D(x)$ are polynomials, with $D(x) \neq 0$, then there exist unique polynomials $Q(x)$ and $R(x)$, where $R(x)$ is either 0 or of degree less than the degree of $D(x)$, such that

$$\underbrace{P(x)}_{\text{Dividend}} = \underbrace{D(x)}_{\text{Divisor}} \cdot \underbrace{Q(x)}_{\text{Quotient}} + \underbrace{R(x)}_{\text{Remainder}}$$

The polynomials $P(x)$ and $D(x)$ are called the **dividend** and **divisor**, respectively, $Q(x)$ is the **quotient**, and $R(x)$ is the **remainder**.

E X A M P L E 1 | Long Division of Polynomials

Divide $6x^2 - 26x + 12$ by $x - 4$

S O LU T I O N The *dividend* is $6x^2 - 26x + 12$ and the *divisor* is $x - 4$. We begin by arranging them as follows:

$$x - 4\overline{)6x^2 - 26x + 12}$$

Next we divide the <u>leading term in the dividend</u> by the <u>leading term in the divisor</u> to get the first term of the quotient: $\frac{6x^2}{x} = 6x$. Then we multiply the divisor by $6x$ and subtract the result from the dividend.

$$
\begin{array}{r|l}
\begin{array}{r}
6x \phantom{{}-26x+12} \\
x-4\overline{\smash{)}6x^2-26x+12} \\
\underline{6x^2-24x}\phantom{{}+12} \\
-2x+12
\end{array}
&
\begin{array}{l}
\text{Divide leading terms: } \dfrac{6x^2}{x}=6x \\
\\
\text{Multiply: } 6x(x-4)=6x^2-24x \\
\text{Subtract and "bring down" } 12
\end{array}
\end{array}
$$

We repeat the process using the last line $-2x+12$ as the dividend.

$$
\begin{array}{r|l}
\begin{array}{r}
6x-2 \phantom{{}-26x+12} \\
x-4\overline{\smash{)}6x^2-26x+12} \\
\underline{6x^2-24x}\phantom{{}+12} \\
-2x+12 \\
\underline{-2x+8} \\
4
\end{array}
&
\begin{array}{l}
\text{Divide leading terms: } \dfrac{-2x}{x}=-2 \\
\\
\\
\\
\text{Multiply: } -2(x-4)=-2x+8 \\
\text{Subtract}
\end{array}
\end{array}
$$

The division process ends when the last line is of lesser degree than the divisor. The last line then contains the *remainder*, and the top line contains the *quotient*. The result of the division can be interpreted in either of two ways.

$$\frac{\overbrace{6x^2-26x+12}^{\text{Dividend}}}{\underbrace{x-4}_{\text{Divisor}}}=\overbrace{6x-2}^{\text{Quotient}}+\frac{\overbrace{4}^{\text{Remainder}}}{x-4}$$

or

$$\underbrace{6x^2-26x+12}_{\text{Dividend}}=\underbrace{(x-4)}_{\text{Divisor}}\underbrace{(6x-2)}_{\text{Quotient}}+\overbrace{4}^{\text{Remainder}}$$

NOW TRY: Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $Q(x)$, and express P in the form $P(x)=D(x)\cdot Q(x)+R(x)$

$$P(x)=3x^2+5x-4,\quad D(x)=x+3$$

E X A M P L E 2 | Long Division of Polynomials

Let $P(x)=8x^4+6x^2-3x+1$ and $D(x)=2x^2-x+2$. Find polynomials $Q(x)$ and $R(x)$ such that $P(x)=D(x)\cdot Q(x)+R(x)$.

S O LU T I O N We use long division after first inserting the term $0x^3$ into the dividend to ensure that the columns line up correctly.

$$
\begin{array}{r|l}
\begin{array}{r}
4x^2+2x\phantom{{}+6x^2-3x+1} \\
2x^2-x+2\overline{\smash{)}8x^4+0x^3+6x^2-3x+1} \\
\underline{8x^4-4x^3+8x^2}\phantom{{}-3x+1} \\
4x^3-2x^2-3x\phantom{{}+1} \\
\underline{4x^3-2x^2+4x}\phantom{{}+1} \\
-7x+1
\end{array}
&
\begin{array}{l}
\\
\\
\text{Multiply divisor by } 4x^2 \\
\text{Subtract} \\
\text{Multiply divisor by } 2x \\
\text{Subtract}
\end{array}
\end{array}
$$

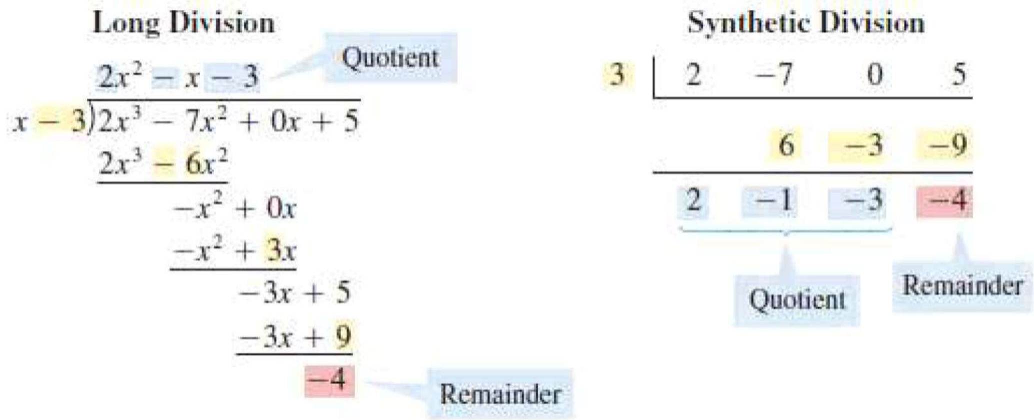
The process is complete at this point because $-7x+1$ is of lesser degree than the divisor $2x^2-x+2$. From the above long division we see that $Q(x)=4x^2+2x$ and $R(x)=-7x+1$, so $8x^4+6x^2-3x+1=(2x^2-x+2)(4x^2+2x)+(-7x+1)$

NOW TRY: Find the quotient and remainder using long division.

$$\frac{x^3+6x+3}{x^2-2x+2}$$

▼ Synthetic Division (Optional)

Synthetic division is a quick method of dividing polynomials; it can be used when the divisor is of the form $x-c$. In synthetic division we write only the essential parts of the long division. Compare the following long and synthetic divisions, in which we divide $2x^3-7x^2+5$ by $x-3$ (We'll explain how to perform the synthetic division in Example 3.)



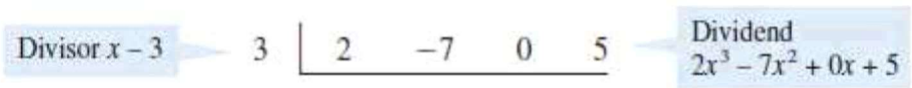
Note that in synthetic division we abbreviate $2x^3-7x^2+5$ by writing only the coefficients: 2 −7 0 5, and instead of $x-3$, we simply write 3. (Writing 3 instead of -3 allows us to add instead of subtract, but this changes the sign of all the numbers that appear in the gold boxes.)

The next example shows how synthetic division is performed.

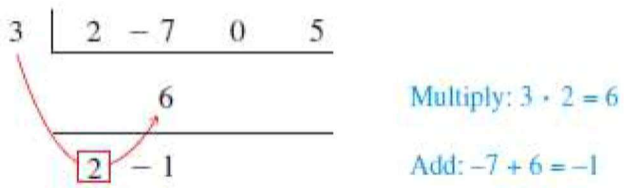
E X A M P L E 3 | Synthetic Division

Use synthetic division to divide $2x^3-7x^2+5$ by $x-3$.

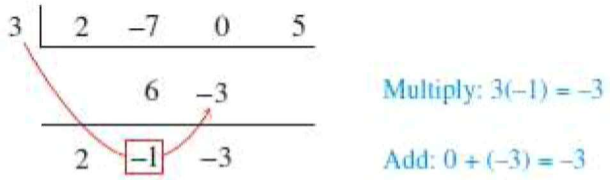
S O LU T I O N We begin by writing the appropriate coefficients to represent the divisor and the dividend.

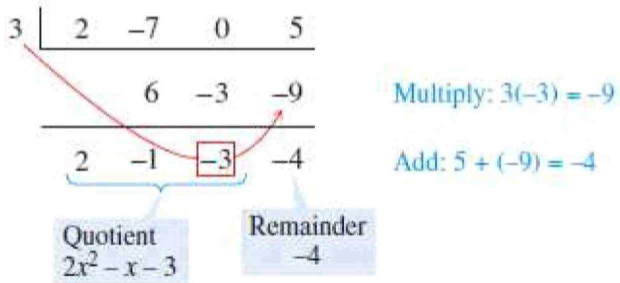


We bring down the 2, multiply $3\times 2=6$, and write the result in the middle row. Then we add.



We repeat this process of multiplying and then adding until the table is complete.





From the last line of the synthetic division we see that the quotient is $2x^2 - x - 3$ and the remainder is −4. Thus $2x^3 - 7x^2 + 5 = (x-3)(2x^2 - x - 3) + (-4)$

NOW TRY: : Find the quotient and remainder using synthetic division

$$\frac{x^3 - 8x + 2}{x+3}$$

Synthetic division, which is a 'shorter' equivalent of the long division, has been used for problems like those given here.
From past experience, most students would 'blindly' follow the procedure without understanding and would not remember learning much of anything after a break.
(To remember the procedure correctly for a longer period without understanding, one would need to do a lot of practice.)

Synthetic division works only for dividing a polynomial by a linear monomial $x + c$.

In contrast, the long division for polynomials is similar to long division for numbers learnt in primary schools; moreover it also works in more general situation: dividing a polynomial by another polynomial (not confined to just linear monomial $x + c$).
Examples : $(2x^3 - 7x^2 + 5) \div (2x+1)$, $(2x^3 - 7x^2 + 5) \div (x^2 + x + 1)$

Advice to students: Make sure you **know how to do long division**.
Every problem involving division of polynomials can be done using long division; but not so with synthetic division. It is **not necessary** to learn synthetic division.

▼ The Remainder and Factor Theorems

The next theorem shows how synthetic division can be used to evaluate polynomials easily.

> **REMAINDER THEOREM**
>
> If the polynomial $P(x)$ is divided by $x - c$, then the remainder is the value $P(c)$.

PROOF If the divisor in the Division Algorithm is of the form $x - c$ for some real number c, then the remainder must be a constant (since the degree of the remainder is less thanthe degree of the divisor). If we call this constant r, then

$$P(x) = (x-c)\cdot Q(x) + r$$

Replacing x by c in this equation, we get $P(c) = (c-c)\cdot Q(c = 0 + r = r) + r$, that is, $P(c)$ is the remainder r. ■

E X A M P L E 4 Using the Remainder Theorem to Find the Value of a Polynomial

Let $P(x) = 3x^5 + 5x^4 - 4x^3 + 7x + 3$.

(a) Find the quotient and remainder when $P(x)$ is divided by $x + 2$.

(b) Use the Remainder Theorem to find $P(-2)$

S O LU T I O N

(a) Carry out long division.

The quotient is $3x^4 - x^3 - 2x^2 + 4x - 1$, and the remainder is 5.

Make sure you know how to do the long division.

(b) By the Remainder Theorem, $3x^4 - x^3 - 2x^2 + 4x - 1$ is the remainder when $P(x)$ is divided by $x-(-2) = x+2$. From part (a) the remainder is 5, so . $P(-2) = 5$.

Wouldn't it be easier to **evaluate** $P(-2)$ **by mere substitution**

$P(-2) = 3(-2)^5 + 5(-2)^4 - 4(-2)^3 + 7(-2) + 3 = ... =$?

NOW TRY: Use (i) substitution , or (ii) synthetic division to evaluate $P(c)$.

$$P(x) = 4x^2 + 12x + 5, \quad c = -1 \blacksquare$$

The next theorem says that *zeros* of polynomials correspond to *factors*; we used this fact in Section 3.1 to graph polynomials.

> **FACTOR THEOREM**
>
> c is a zero of P if and only if $x - c$ is a factor of $P(x)$.

PROOF If $P(x)$ factors as $P(x) = (x - c) \cdot Q(x)$, then

$$P(c) = (c - c) \cdot Q(c) = 0 \cdot Q(c) = 0$$

Conversely, if $P(c) = 0$, then by the Remainder Theorem

$$P(x) = (x - c) \cdot Q(x) + 0 = (x - c) \cdot Q(x)$$

so $x - c$ is a factor of $P(x)$. ■

EXAMPLE 5 | Factoring a Polynomial Using the Factor Theorem

Let $P(x) = x^3 - 7x + 6$. Show that $P(1) = 0$, and use this fact to factor $P(x)$ completely.

SOLUTION Substituting, we see that $P(1) = 1^3 - 7 \cdot 1 + 6 = 0$.

By the Factor Theorem, $x - 1$ is a factor of $P(x)$.

Using long division (shown in the box), we see that

$P(x) = x^3 - 7x + 6 \ = (x-1)(x^2 + x - 6) = (x-1)(x-2)(x+3)$

$$\begin{array}{r} x^2 + x - 6 \\ x - 1 \overline{\big)\, x^3 + 0x^2 - 7x + 6} \\ \underline{x^3 - x^2 \qquad\qquad\quad} \\ x^2 - 7x \qquad \\ \underline{x^2 - x \qquad\;} \\ -6x + 6 \\ \underline{-6x + 6} \\ 0 \end{array}$$

Are there alternative ways?

NOW TRY EXERCISES **53** AND **57** ■

(a) Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

$$P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1 \ ■$$

(b) Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

$$P(x) = x^3 - x^2 - 11x + 15, \quad c = 3 \ ■$$

EXAMPLE 6 | Finding a Polynomial with Specified Zeros

Find a polynomial of degree 4 that has zeros −3, 0, 1, and 5.

SOLUTION By the Factor Theorem $x - (-3)$, $x - 0$, $x - 1$, and $x - 5$ must all be factors of the desired polynomial.

Let $P(x) = (x+3)x(x-1)(x-5)$

$$= x^4 - 3x^3 - 13x^2 + 15x$$

Since $P(x)$ is of degree 4, it is a solution of the problem. Any other solution of the problem must be a constant multiple of $P(x)$, since only multiplication by a constant does not change the degree.

NOW TRY EXERCISE 59 ■

Find a polynomial of the specified degree that has the given zeros.

Degree 3; zeros −1, 1, 3

Exercise

1–6 ■ Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$, and express P in the form $P(x) = D(x) \cdot Q(x) + R(x)$.

1. $P(x) = 3x^2 + 5x - 4, \quad D(x) = x + 3$

2. $P(x) = x^3 + 4x^2 - 6x + 1, \quad D(x) = x - 1$

3. $P(x) = 2x^3 - 3x^2 - 2x, \quad D(x) = 2x - 3$

4. $P(x) = 4x^3 + 7x + 9, \quad D(x) = 2x + 1$

5. $P(x) = x^4 - x^3 + 4x + 2, \quad D(x) = x^2 + 3$

6. $P(x) = 2x^5 + 4x^4 - 4x^3 - x - 3, \quad D(x) = x^2 - 2$

7–12 ■ Two polynomials P and D are given. Use either synthetic or long division to divide $P(x)$ by $D(x)$, and express the quotient $P(x)/D(x)$ in the form

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

7. $P(x) = x^2 + 4x - 8, \quad D(x) = x + 3$

8. $P(x) = x^3 + 6x + 5, \quad D(x) = x - 4$

9. $P(x) = 4x^2 - 3x - 7, \quad D(x) = 2x - 1$

10. $P(x) = 6x^3 + x^2 - 12x + 5, \quad D(x) = 3x - 4$

11. $P(x) = 2x^4 - x^3 + 9x^2, \quad D(x) = x^2 + 4$

12. $P(x) = x^5 + x^4 - 2x^3 + x + 1, \quad D(x) = x^2 + x - 1$

13–22 ■ Find the quotient and remainder using long division.

13. $\dfrac{x^2 - 6x - 8}{x - 4}$

14. $\dfrac{x^3 - x^2 - 2x + 6}{x - 2}$

15. $\dfrac{4x^3 + 2x^2 - 2x - 3}{2x + 1}$

16. $\dfrac{x^3 + 3x^2 + 4x + 3}{3x + 6}$

17. $\dfrac{x^3 + 6x + 3}{x^2 - 2x + 2}$

18. $\dfrac{3x^4 - 5x^3 - 20x - 5}{x^2 + x + 3}$

19. $\dfrac{6x^3 + 2x^2 + 22x}{2x^2 + 5}$

20. $\dfrac{9x^2 - x + 5}{3x^2 - 7x}$

21. $\dfrac{x^6 + x^4 + x^2 + 1}{x^2 + 1}$

22. $\dfrac{2x^5 - 7x^4 - 13}{4x^2 - 6x + 8}$

23–36 ■ Find the quotient and remainder using synthetic division.

23. $\dfrac{x^2 - 5x + 4}{x - 3}$

24. $\dfrac{x^2 - 5x + 4}{x - 1}$

25. $\dfrac{3x^2 + 5x}{x - 6}$

26. $\dfrac{4x^2 - 3}{x + 5}$

27. $\dfrac{x^3 + 2x^2 + 2x + 1}{x + 2}$

28. $\dfrac{3x^3 - 12x^2 - 9x + 1}{x - 5}$

29. $\dfrac{x^3 - 8x + 2}{x + 3}$

30. $\dfrac{x^4 - x^3 + x^2 - x + 2}{x - 2}$

31. $\dfrac{x^5 + 3x^3 - 6}{x - 1}$

32. $\dfrac{x^3 - 9x^2 + 27x - 27}{x - 3}$

33. $\dfrac{2x^3 + 3x^2 - 2x + 1}{x - \frac{1}{2}}$

34. $\dfrac{6x^4 + 10x^3 + 5x^2 + x + 1}{x + \frac{2}{3}}$

35. $\dfrac{x^3 - 27}{x - 3}$

36. $\dfrac{x^4 - 16}{x + 2}$

37–49 ■ Use synthetic division and the Remainder Theorem to evaluate $P(c)$.

37. $P(x) = 4x^2 + 12x + 5, \quad c = -1$

38. $P(x) = 2x^2 + 9x + 1, \quad c = \frac{1}{2}$

39. $P(x) = x^3 + 3x^2 - 7x + 6, \quad c = 2$

40. $P(x) = x^3 - x^2 + x + 5, \quad c = -1$

41. $P(x) = x^3 + 2x^2 - 7, \quad c = -2$

42. $P(x) = 2x^3 - 21x^2 + 9x - 200, \quad c = 11$

43. $P(x) = 5x^4 + 30x^3 - 40x^2 + 36x + 14, \quad c = -7$

44. $P(x) = 6x^5 + 10x^3 + x + 1, \quad c = -2$

45. $P(x) = x^7 - 3x^2 - 1, \quad c = 3$

46. $P(x) = -2x^6 + 7x^5 + 40x^4 - 7x^2 + 10x + 112, \quad c = -3$

47. $P(x) = 3x^3 + 4x^2 - 2x + 1, \quad c = \frac{2}{3}$

48. $P(x) = x^3 - x + 1, \quad c = \frac{1}{4}$

49. $P(x) = x^3 + 2x^2 - 3x - 8, \quad c = 0.1$

50. Let

$$P(x) = 6x^7 - 40x^6 + 16x^5 - 200x^4 - 60x^3 - 69x^2 + 13x - 139$$

Calculate $P(7)$ by **(a)** using synthetic division and **(b)** substituting $x = 7$ into the polynomial and evaluating directly.

51–54 ■ Use the Factor Theorem to show that $x - c$ is a factor of $P(x)$ for the given value(s) of c.

51. $P(x) = x^3 - 3x^2 + 3x - 1, \quad c = 1$

52. $P(x) = x^3 + 2x^2 - 3x - 10, \quad c = 2$

53. $P(x) = 2x^3 + 7x^2 + 6x - 5, \quad c = \frac{1}{2}$

54. $P(x) = x^4 + 3x^3 - 16x^2 - 27x + 63, \quad c = 3, -3$

55–56 ■ Show that the given value(s) of c are zeros of $P(x)$, and find all other zeros of $P(x)$.

55. $P(x) = x^3 - x^2 - 11x + 15, \quad c = 3$

56. $P(x) = 3x^4 - x^3 - 21x^2 - 11x + 6, \quad c = \frac{1}{3}, -2$

57–60 ■ Find a polynomial of the specified degree that has the given zeros.

57. Degree 3; zeros $-1, 1, 3$

58. Degree 4; zeros $-2, 0, 2, 4$

59. Degree 4; zeros $-1, 1, 3, 5$

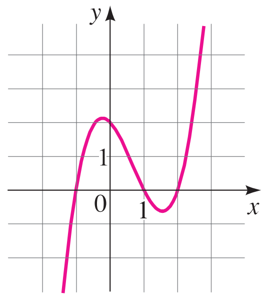
60. Degree 5; zeros $-2, -1, 0, 1, 2$

61. Find a polynomial of degree 3 that has zeros 1, -2, and 3, and in which the coefficient of x^2 is 3.

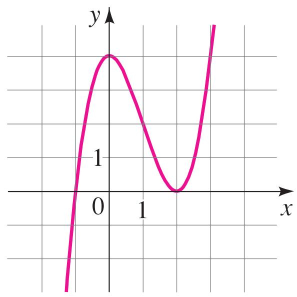
62. Find a polynomial of degree 4 that has integer coefficients and zeros 1, -1, 2, and $\frac{1}{2}$.

63–66 ■ Find the polynomial of the specified degree whose graph is shown.

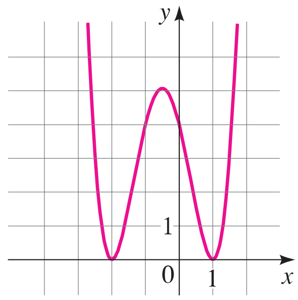
63. Degree 3



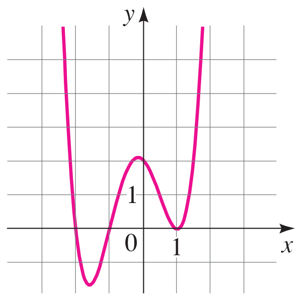
64. Degree 3



65. Degree 4



66. Degree 4



Discovery • Discussion

67. Impossible Division? Suppose you were asked to solve the following two problems on a test:

A. Find the remainder when $6x^{1000} - 17x^{562} + 12x + 26$ is divided by $x + 1$.

B. Is $x - 1$ a factor of $x^{567} - 3x^{400} + x^9 + 2$?

Obviously, it's impossible to solve these problems by dividing, because the polynomials are of such large degree. Use one or more of the theorems in this section to solve these problems *without* actually dividing.

Answer

1. $(x+3)(3x-4)+8$ **3.** $(2x-3)(x^2-1)-3$

5. $(x^2+3)(x^2-x-3)+(7x+11)$

7. $x+1+\dfrac{-11}{x+3}$

9. $2x-\frac{1}{2}+\dfrac{-\frac{15}{2}}{2x-1}$ **11.** $2x^2-x+1+\dfrac{4x-4}{x^2+4}$

In answers 13–36, the first polynomial given is the quotient and the second is the remainder.

13. $x-2, -16$ **15.** $2x^2-1, -2$ **17.** $x+2, 8x-1$
19. $3x+1, 7x-5$ **21.** $x^4+1, 0$ **23.** $x-2, -2$
25. $3x+23, 138$ **27.** $x^2+2, -3$ **29.** $x^2-3x+1, -1$
31. $x^4+x^3+4x^2+4x+4, -2$ **33.** $2x^2+4x, 1$
35. $x^2+3x+9, 0$ **37.** -3 **39.** 12 **41.** -7 **43.** -483
45. 2159 **47.** $\frac{7}{3}$ **49.** -8.279 **55.** $-1 \pm \sqrt{6}$
57. x^3-3x^2-x+3 **59.** $x^4-8x^3+14x^2+8x-15$
61. $-\frac{3}{2}x^3+3x^2+\frac{15}{2}x-9$ **63.** $(x+1)(x-1)(x-2)$
65. $(x+2)^2(x-1)^2$