Replication Project Empirical Finance

Mayur Choudhary

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Summary and Main Ideas: Lewellen, Nagel and Shanken (2010)

The authors take a *skeptical approach* to evaluating the validity of linear factor models in asset pricing. I provide here a brief description of the common methodology in these papers which is based on Fama and MacBeth (1973). A more detailed version is in Appendix: Basic Setup. These models rely on conducting 2 stages of regression,

1. **Time Series Regressions:** The first stage to determine the loadings on the proposed factors. In this step we estimate the loadings on a given factor for a particular asset. A measure of how the asset prices are correlated with the risk factors the marginal investor cares about.

2. **Cross Section Regressions:** The second stage regression is used to recover the price of risk of the proposed factors. The fit of the model in the second stage also serves as a test of the validity of the model. The idea is if the proposed linear SDF of the marginal investor is indeed valid then all that should matter is how the stocks comove with the factors (β), and the price of risk of each of these factors λ in explaining the cross-sectional dispersion of average expected returns of assets.

A number of models in this class such as the CAPM, consumption-CAPM (CCAPM), Lettau and Ludvigson (2001), Fama and French (1992) etc. justify their validity by reporting a high R^2 in the second stage. Lewellen, Nagel and Shanken (2010)'s main idea is to demonstrate how, just obtaining a high in sample R^2 is not a good enough hurdle to meet in order to claim that these models successfully explain the dispersion in average returns of assets in the cross-section.

If the data truly has a factor structure,

$$\mathbf{R}_{T\times N} = \mathbf{B}_{T\times K}\mathbf{F}_{K\times N} + \mathbf{e}_{T\times N}$$

such that $Cov(\mathbf{F}, \mathbf{e}) = 0$. Then the authors make the following claims about finding alternative time series variables $\mathbf{P}$ (proposed factors) that may seem like they *explain* the cross-sectional variation in average returns $E[\mathbf{R}] = \boldsymbol{\mu}_{N\times 1}$.

Issue 1: Any set of random factors $\mathbf{P}$ even mildly correlated with $\mathbf{F}$ ($det[Cov(\mathbf{P}, \mathbf{F})] \neq 0$) and $Cov(\mathbf{P}, \mathbf{e}) = 0$ would have the same R^2 in the cross section regression as that in the true model.

In order to understand this, project $\mathbf{P}$ on $\mathbf{F}$. Let the coefficient of regression be $\mathbf{Q}$ which is non-singular as there is some correlation between the 2 sets of factors. Taking expectation of the original regression with the true data generating process

$$E[\mathbf{R}] = \boldsymbol{\mu} = \mathbf{B}E[\mathbf{F}] + \mathbf{0}$$

where $\mathbf{B} = Cov(\mathbf{R}, \mathbf{F})V(\mathbf{F})^{-1}$. In a similar way by using $Cov(\mathbf{P}, \mathbf{e}) = 0$ and the definition of $\mathbf{C}$,

$$\mathbf{C} = Cov(\mathbf{R}, \mathbf{P})V(\mathbf{P})^{-1} = \mathbf{B}Cov(P, F)V(\mathbf{F})^{-1} = \mathbf{BQ}$$

Hence,

$$\boldsymbol{\mu} = \mathbf{B}E[\mathbf{F}] = \mathbf{BQQ}^{-1}E[\mathbf{F}] = \mathbf{CQ}^{-1}E[\mathbf{F}]$$

Here, $\mathbf{Q}^{-1}E[\mathbf{F}]$ will be the price of risk. This will have a high value if there is a small correlation between F, P.

We can construct any random ($Cov(P, e) = 0$) set of factors which are the same in number as the original set of factors and mildly correlated with them $Cov(F, P) \neq 0$, and we can end up with a model that performs very well in terms of the fit, in the second stage. The authors employ this idea in Figure (1a) below, where they artificially cook up random factors which are correlated with the Fama-French 3 factors (Mkt, SMB, HML) and show that even with 2 such factors we end up with a very high sample R^2. One reason for this is that since we are performing the tests on the FF 25 Size by B/M sorted portfolios, where the $\beta_{Mkt} \approx 1$, hence in the cross-section it is effectively a 2- factor model. The results are replicated in the original sample, exactly and the patterns do not change even in the current data sample from $1963Q1 - 2018Q4$.

Issue 2: If we assume a strict factor structure,

$$V[\mathbf{e}] = diag(\sigma_1^2 \dots \sigma_N^2)$$

Then forming a set of factors from the test assets $K < N$, we can still end up with a high R^2 for the remaining $N - K$ assets. This follows from observation 1 as the strict factor structure ensures $Cov(P, e_{N-k}) = 0$. The results to this observation are in Figure(1a) where the authors make up factors by selecting at random factors from the test assets and show how we still end up with a high R^2 for 2 and 3 factors. This exercise is repeated by making $E[P] = 0$ in Figure(1a) and we still end up with similar results. This signifies the importance of the economic interpretation of $\boldsymbol{\lambda} = E[\mathbf{F}]$. The results are replicated exactly as in the original paper and the patterns persist in the current data as well. Please see figures (1b,1b,1b).

Issue 3: Although the true R^2 may be high or low in the population, since we do not have a huge sample size (≤ 500 in most papers) there may be sampling error in the R^2 reported in the proposed models. The authors create these confidence intervals by simulations in which they first assume a true population R^2, then cook up factors P that would have that true R^2. Randomly sample from the data to get the simulated returns data treating the proposed model as true and then re-running the 2 pass regressions to obtain the sample R^2. The authors find that we are likely to end up getting a high R^2 in sample even when the true population R^2 might be nearly 0. The results are in figures (1.)

Prescription 1: One clear prescription to avoid the above pitfalls is to use test assets that do not have such strong factor structure. The authors suggest using the FF25 along with the 30 Industries sorted portfolios availabe at `https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. The results are in figure (1). Clearly the OLS R^2 is smaller with a wider set of test assets.

Prescription 2: Impose the theoretical and economically meaningful restriction on the coefficient of the second stage regression that it must be equal to the risk premium (if the factors are traded), or the average returns on factor mimicking portfolios. This can be done by using GLS instead of OLS. GLS R^2 also have the added economic significance in the sense that they represent the distance of the 'best' combination of the factors to the mean variance frontier. Moreover, although GLS R^2 still suffers from Issues 1 and 2, it still provides a stringent hurdle for the models to pass as the values are generally lower than OLS R^2. The results are in figures(3a,3a,3a). The results are again exactly replicated except that the GLS R^2 seem to be shifted up by 0.1 but still quite low compared to the OLS R^2. Otherwise the patterns are exactly the same as the original paper.

Prescription 3: Report the confidence intervals of R^2. There may be closed form solutions to this, but almost always we can address this issue by way of bootstrapping and showing the confidence interval for the in-sample observed R^2. That is the true R^2 for which the observed R^2 value is in the 90% confidence interval.

Prescription 4: Report the confidence intervals for weighted squared sum of pricing errors a la Shanken (1985) T^2 statistic. This statistic is a quadratic form of the pricing errors with the weight matrix being the variance matrix of the estimator. The authors show that the asymptotic distribution of this statistic is χ^2 with centrality parameter q which is a measure of the distance between the factor-mimicking portfolio from the mean-variance frontier.

This statistic and GLS also allow us to test if the mean-variance frontier is spanned by our factors. Which should be the case if we have posited the correct SDF. Both tests rely on imposing restrictions on the interpretation of the coefficient of the second stage as the average expected return on the factor mimiicking portfolio or the risk premium of traded factors. This allows us to avoid the pitfall shown in Figure (1a).

Testing Empirical Models: Ultimately the authors take this set of improved asset pricing tests and deploy them to examine 8 asset pricing models. They find that the issues that they highlight in fact are true in practice. I highlight these findings in the context of the Fama and French (1992)and Lettau and Ludvigson (2001) models.

For example, the FF 3 factor model does a good job of explaining the cross-sectional R^2 in OLS (≈ 0.8), but fails to do so in GLS. The confidence interval of the OLS R^2 is still tight in this case, but as soon as we include the 30 Industries portfolios as test assets, the OLS R^2 goes down. Similarly, Lettau and Ludvigson (2001) also has a high cross-section OLS R^2 (≈ 0.6), however the confidence interval for this is $(0.4, 1)$, i.e. even with a low true cross-sectional R^2 we might end up with a high observed in sample R^2. Again, the GLS R^2 is significantly lower and even the OLS R^2 is quite lower when we expand our test assets to include 30 Industries portfolios. This final fact highlights that the *cay* and Δc series may be correlated with the (SMB, HML) factors, and as a result produce a decent fit in the OLS second stage regression.

In what follows we go sequentially in replicating the figures and tables as they appear in the paper. I will briefly describe the method used in the paper and any slight modifications that I may have made to the approach mentioned in the paper. In replicating the R^2 confidence intervals for different empirical models, the authors are not very clear about their methodology. In this case I employ a methodology as described in HW2Q3. I however had to make a small adjustment because I think there was an issue with the method suggested in the HW assignment. I replicate the 3 standard models $CAPM$, Consumption $CAPM$, Fama French 3 factor model along with Lettau and Ludvigson (2001). The in-sample results for these tests seem to match the results in the original paper except for a few GLS R^2 confidence intervals. I could not replicate the results in Lettau and Ludvigson (2001) for the current data, probably because I am not using the correct consumption growth and *cay* series (more details in the data section). I don't replicate the other 4 models because I feel that the main idea and results of this paper become apparent in the models already replicated. It may not be hard (but it is time consuming) to replicate the other papers, all we need is to simply change the data inputs for factors.

Data

Every monthly series is converted to quarterly as follows, I take the three months $\{m_1, m_2, m_3\}$ (in %) in a quarter and compute the quarterly figures as follows (in %)

$$R_q = 100 \cdot (\prod_{i=1}^{3}(1 + R_{m_i}) - 1)$$

Portfolio Data

FF25 : I obtain the monthly returns on the 25 Size by B/M sorted portfolios from Kenneth French's website (KF, hereafter) https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html. I convert this data to monthly.

FF25 + 30 Ind : Again I supplement my data with the 30 Industry sorted portfolios from KF. I make the returns quarterly following the procedure outlined above.

Factors: CAPM + FF3

1. I obtain the monthly FF 3 factor returns data from KF. Again, I make the data quarterly using the procedure described earlier.

2. I use the RF here and subtract it from the portfolio data to obtain the excess returns.

LL

1. For the original sample I could not find the data on the authors' website, so I obtained the data from the dataset shared in class as part of HW1. That data series is from $1963Q3 - 1998Q3$

2. For the current data I got the cay series data from Martin Lettau's website, https://drive.google.com/file/d/1nrtJOK4N2KfN2m-LdoRDGoESH76q87oZ/view.

3. I rescaled the cay data to make it mean 0.

4. For the consumption growth series I use the c series in the same dataset. I then calculate $c_{t+1} - c_t$ and get rid of one observation due to the NA that arises due to 1 lag.

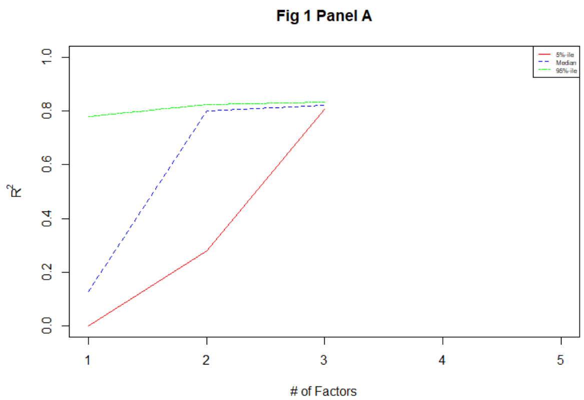
Consumption CAPM

1. I obtain the non-durables and services consumption series from St. Louis FRED website https://fred.stlouisfed.org/

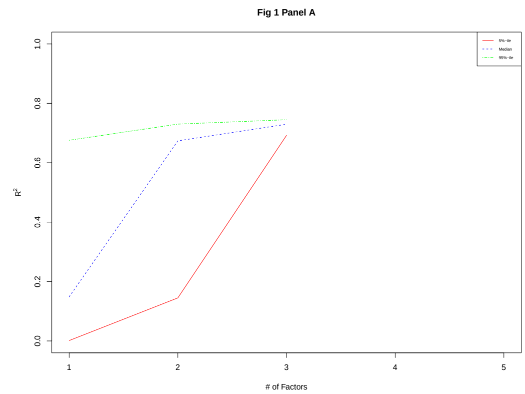
2. This is nominal data, and hence I also get the non-durables deflator and the Services deflator.

3. I delfate each of the nominal series by their corresponding deflators. For this I divide the nominal series element by element by the deflator series for that variable.
4. I then simply add the real non-durables and services consumption series so obtained to obtain the total real consumption C_t
5. I take a log of the variables and obtain $c_t = \log(C_t)$ series.
6. Finally I take the first difference of the c_t series to obtain,

$$\Delta c_{t+1} = c_{t+1} - c_t = \log\left(\frac{C_{t+1}}{C_t}\right)$$



(a) Uncorrelated factors with random weights



(b) Uncorrelated factors with random weights: 1963Q1-2018Q4

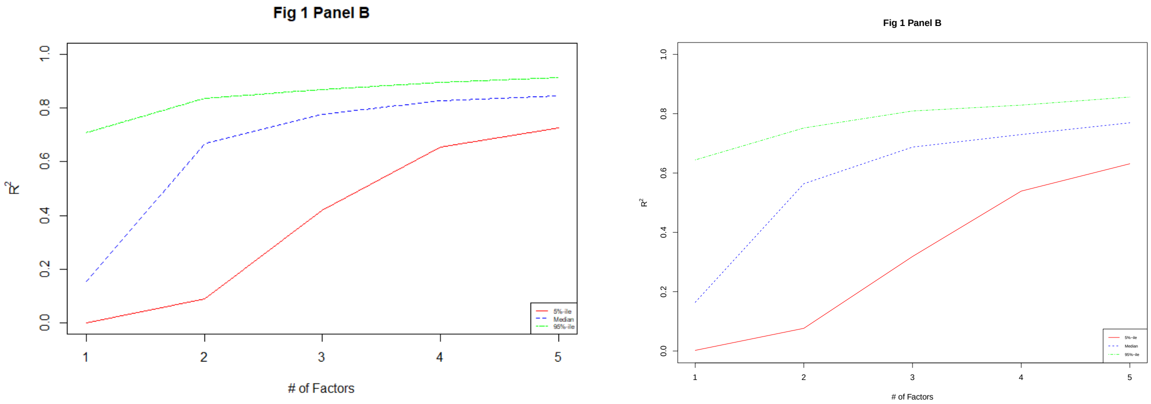
Figure 1: FF25

Method:

$$\mathbf{P}_{T\times K_P} = \mathbf{F}_{T\times K}\mathbf{W}_{K\times K_P}$$

$K_P = \{1, 2, 3\}$ and $K = 3$. The columns of $\mathbf{W}$ are drawn from *iid* standard normal distribution. Now perform the 2 pass regressions with $\mathbf{P}$ as the factors and calculate the R^2. This is conducted $n.sim = 1000$ times to obtain the $\{5\%, 50\%, 95\%\}$-ile values.

The authors perform this exercise 5000 times but I get almost the same estimates by conducting the simulations 1000 times on the data from $1963Q1 - 2004Q4$ (original sample, hereafter). The results do not seem to change drastically for the $1963Q1 - 2018Q4$ (current data, hereafter). However, the R^2 values across all the quantiles seem to be lower than the one in the original sample. This may be due to the factor structure in the data being somewhat distorted in the current data as it includes the financial crisis.



(a) Correlated factors with random weights

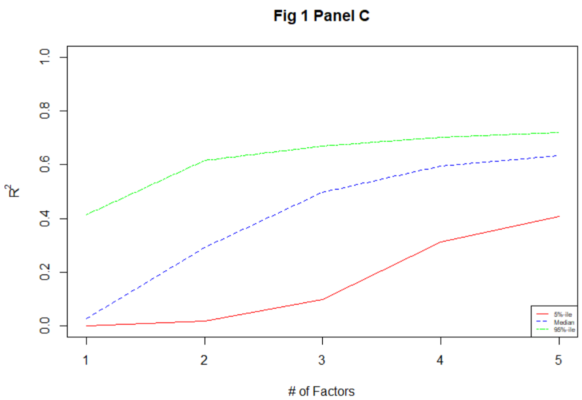
(b) Correlated factors with random weights: 1963Q1-2018Q4

Method:

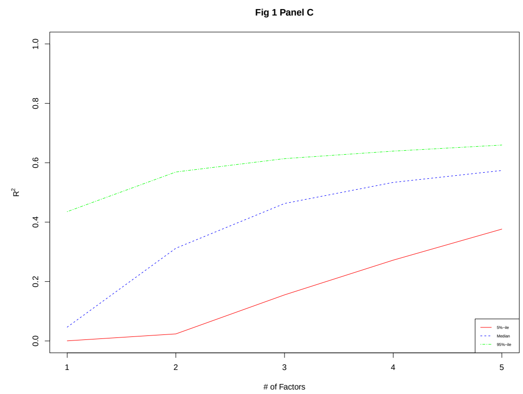
$$\mathbf{P}_{T\times K_P} = \mathbf{R}_{T\times N}\mathbf{W}_{N\times K_P}$$

$K_P = \{1,2,3\}$ and $K = 3$. The columns of $\mathbf{W}$ are drawn from *iid* standard normal distribution. Now perform the 2 pass regressions with $\mathbf{P}$ as the factors and calculate the R^2. This is conducted $n.sim = 1000$ times to obtain the $\{5\%, 50\%, 95\%\}$-ile values.

The goal behind this exercise is to demonstrate that factor mimicking portfolios can be obtained very easily using the test assets. These factor mimicking portfolios also perform very well in terms of the OLS R^2 with the median R^2 with just 5 factors being as high as 0.7 and the 95%-ile with 1 factor equal to nearly 0.6. Again the values are lower for the current data.



(a) Correlated factors with zero mean and random weights



(b) Correlated factors with zero mean and random weights: 1963Q1-2018Q4

Method:

$$\mathbf{P}_{T\times K_P} = \mathbf{R}_{T\times N}\mathbf{W}_{N\times K_P}$$

$K_P = \{1, 2, 3\}$ and $K = 3$. The columns of $\mathbf{W}$ are drawn from *iid* standard normal distribution. Among all these factors only retain the ones which have approximately 0 expected return. In order to expedite my computations I chose this tolerance to be 1 bps (0.01%). Now perform the 2 pass regressions with $\mathbf{P}$ as the factors and calculate the R^2. This is conducted $n.sim = 1000$ times to obtain the $\{5\%, 50\%, 95\%\}$-ile values.

The goal behind this exercise is to demonstrate that factor mimicking portfolios can be obtained very easily using the test assets. Moreover we seem to get high OLS R^2 when in fact the RHS in the second stage should be ≈ 0. These factor mimicking portfolios also perform very well in terms of the OLS R^2 with the median R^2 with just 5 factors being as high as 0.5 and the 95%-ile with 1 factor equal to nearly 0.4. Again the values are lower for the current data.

Figure 2: Sampling Error in R^2

Method: The idea here is to determine what is the confidence interval for the observed in sample R^2. For this we must simulate data of the true $R^2 \in [0,1]$, and for each given true R^2 assuming the model is true sample the factors and the data. Then run the 2 pass procedure to determine the in-sample R^2.

1. For a given true R^2, create the matrix $\mathbf{C} \propto Cov(\mathbf{P}, \mathbf{R})$ such that each column of this matrix accounts for $\frac{1}{K_p}$ fraction of the true R^2 where K_p is the number of factors in $\mathbf{P}$.

2. $$\mathbf{P} = \mathbf{R}W \implies W = V(\mathbf{R})^{-1}Cov(\mathbf{P}, \mathbf{R})$$

   . Retrieve W, hence $\mathbf{P}_{sample} = \mathbf{R}_{sample}W$ by sampling from the returns data.

3. Using this $\mathbf{P}$ run the 2 pass regression, to get the in-sample true R^2. Conducting the second step 10 times and the first step 4000 (In current data, 500 times) times for a total $40,000$ (in current data, 5000) total simulations.

4. Plot the relevant quantiles. Add noise that is 3 times the variance of the factors in the sample. And conduct the same analysis for $1, 3, 5$ factors with noise.

Results

1. In the original sample, the results are replicated almost exactly. Please refer to Figure 2 in the main paper.

2. In the new sample, it can be seen that the R^2 at all the quantiles shift downwards. The condfidence interval for the one factor case without noise the in-sample $R^2 = 0.4 \approx (0.4, 1)$ in the original sample. This confidence interval shifts slightly towards the right, in the current data. This implies somewhat narrower confidence interval in the current data. It is hard to quantify this, because the true R^2 grid has 0.1 as the smallest unit. But it can be seen upon observing the plots that the aforementioned observation indeed holds true.

3. For more number of factors the confidence bands become wider, implying even with the true R^2 as low as 0 in the population we might sometimes end up with the in-sample R^2 as high as 0.6. Thus, we cannot necessarily attach too much value to the in-sample OLS R^2 without knowing the confidence intervals.

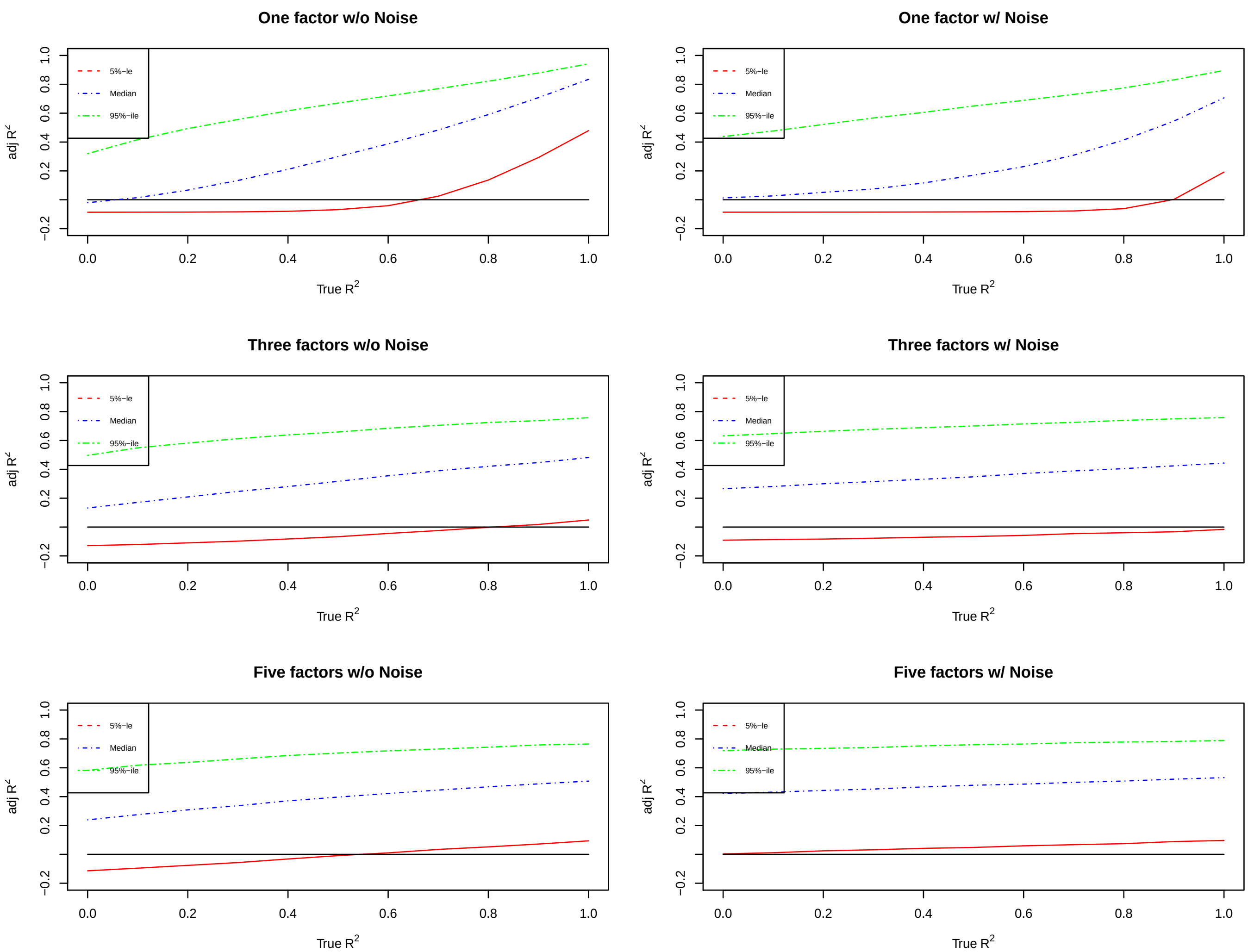


Figure 1: Adj. v/s True R^2: Sampling error in R^2

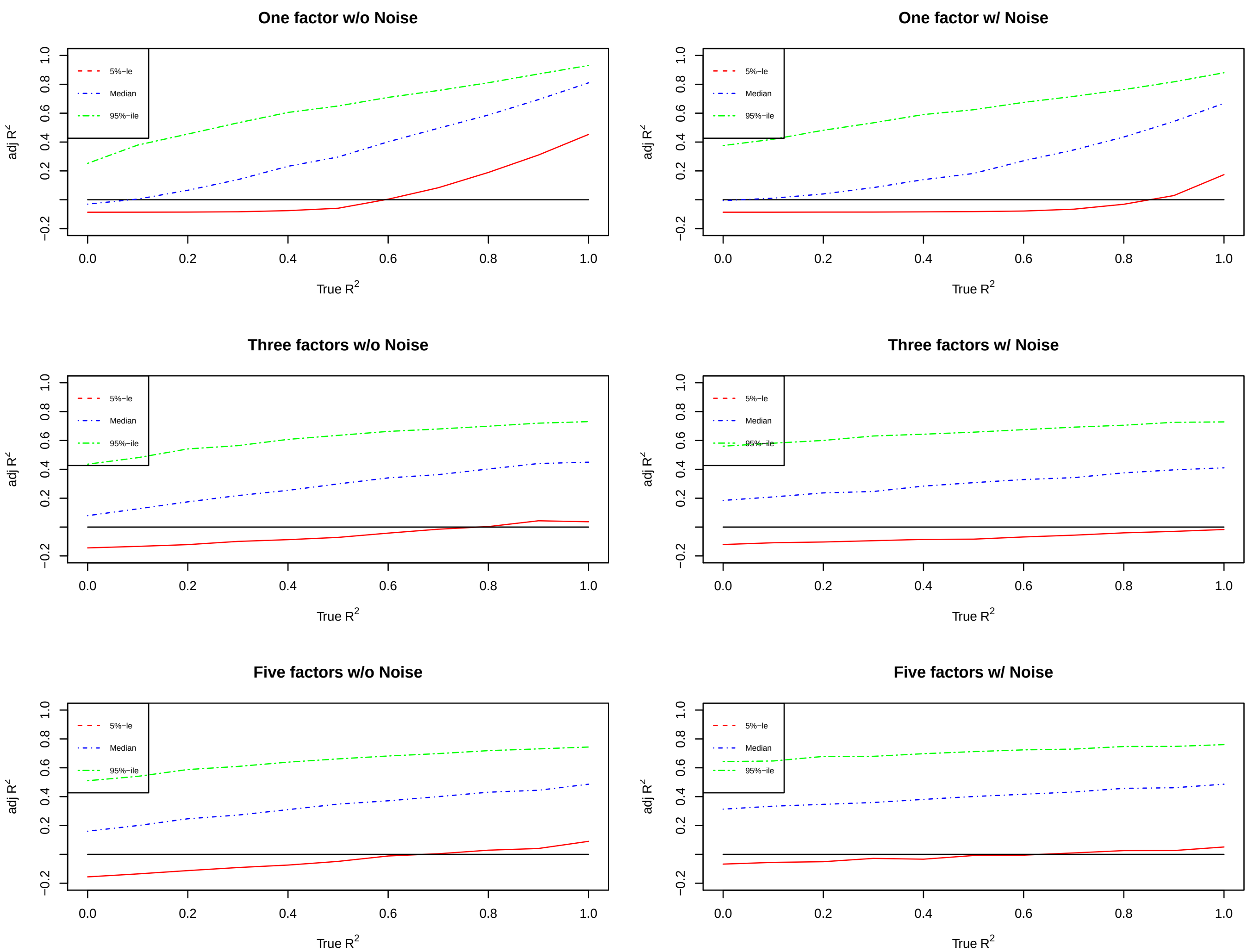


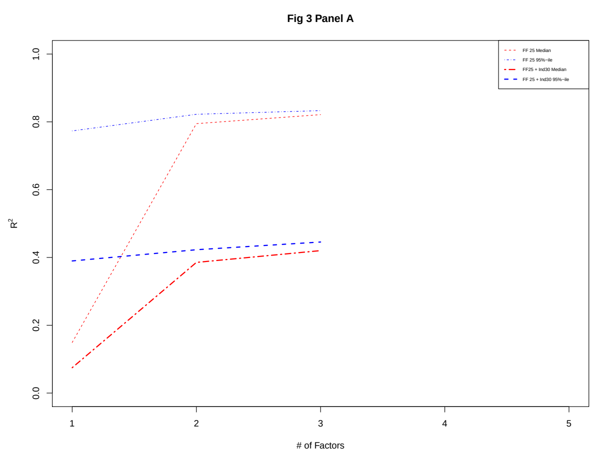
Figure 2: Adj. v/s True R^2: Sampling error in R^2 Sample period 1963Q1-2018Q4

Figure 3: FF25 + Industry 30

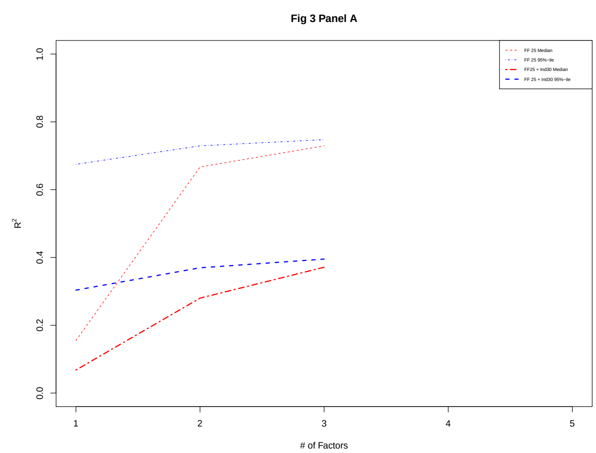
We simply repeat the analysis we conducted in Section(Figure 1: FF25) and expand the set of test assets to include the 30 Industry sorted portfolios. The results for both the cases are shown here for the $\{Median, 95\%\}$-ile, along with the results from Figures(1a,1a,1a).

The results parallel almost exactly the results in the original paper. It is easy to see that in all the panels of this figure, we get lower R^2 with the expanded set of assets. Thus, expanding the set of test assets is likely to mitigate the concerns associated with the strong factor structure in the FF25 test assets.

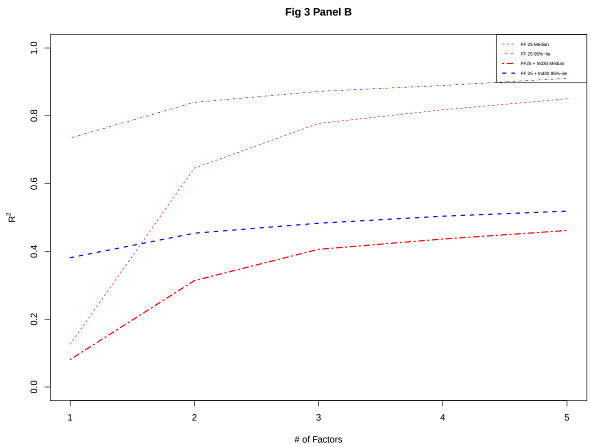
The pattern continues to hold true even in the current data.



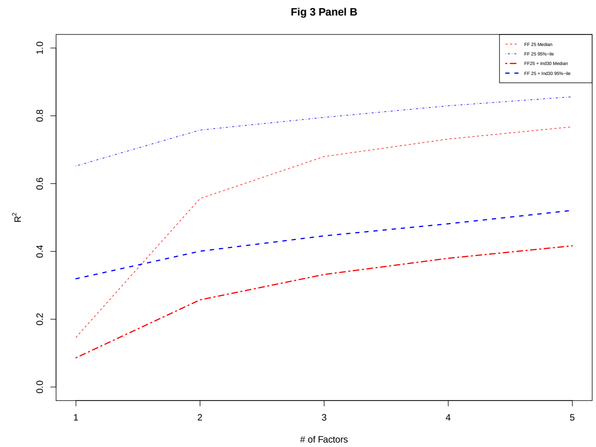
(a) Uncorrelated factors with random weights



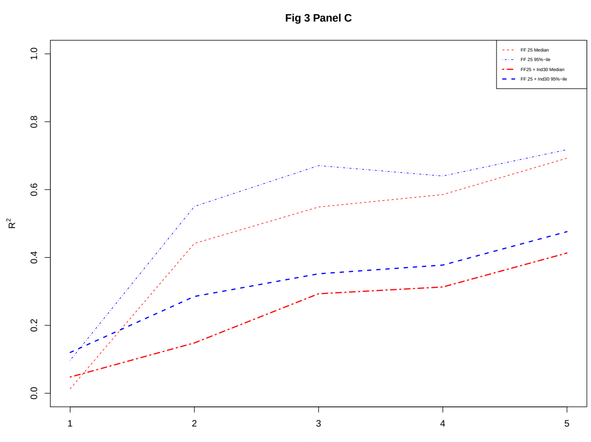
(b) Uncorrelated factors with random weights: 1963Q1-2018Q4



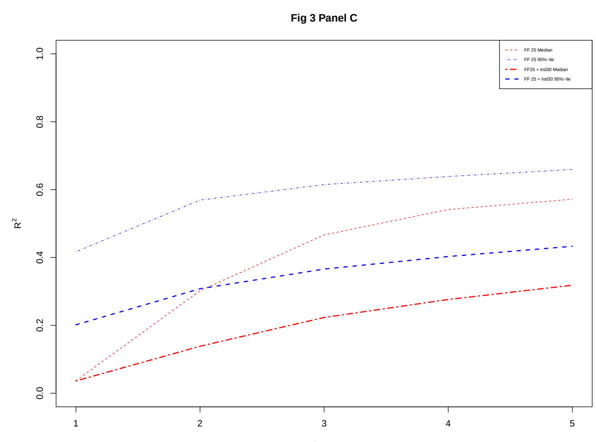
(a) Correlated factors with random weights



(b) Correlated factors with random weights: 1963Q1-2018Q4



(a) Correlated factors with zero mean and random weights



(b) Correlated factors with zero mean and random weights: 1963Q1-2018Q4

Figure 4: OLS vs GLS $\mathbf{R^2}$

Method:

1. First conduct the regular time-series regressions of the first stage to determine the coefficient loadings.

2. Conduct the following regression ,

$$V^{-\frac{1}{2}}\boldsymbol{\mu} = V^{-\frac{1}{2}}\mathbf{C}\boldsymbol{\lambda}_P + \boldsymbol{e}$$

   where $V = Var(\mathbf{R})$. Compute the R^2 of the above regression to get the GLS R^2.

3. I compute $V^{\frac{1}{2}}$ as follows,

$$V^{\frac{1}{2}} = \mathbf{U}\boldsymbol{\Lambda}^{\frac{1}{2}}\mathbf{U}^{-1}$$

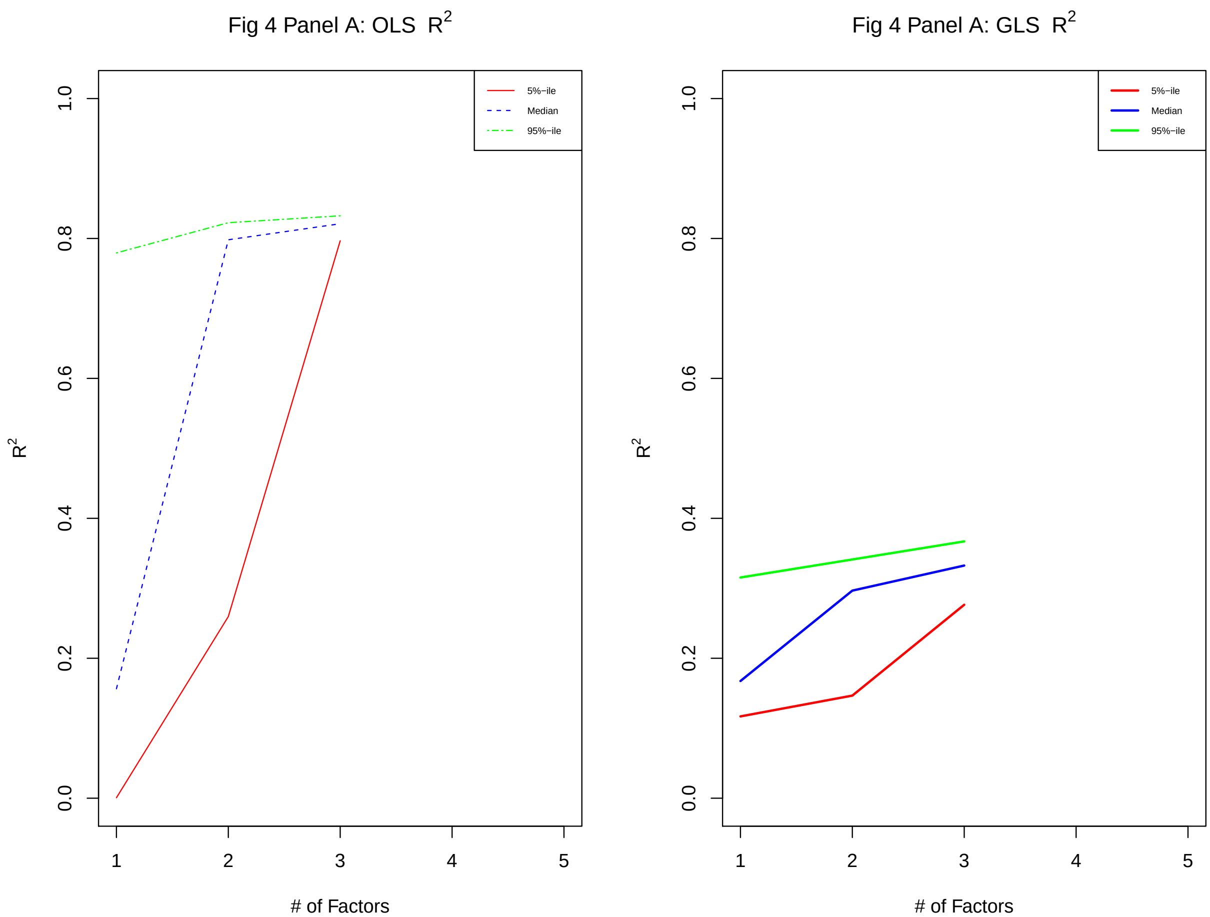
   where $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigen values of $V(\mathbf{R})$. $\mathbf{U}$ has the eigen vectors of $V(\mathbf{R})$ for its columns.
   The above result follows from the fact that $V(\mathbf{R})$ is symmetric and positive semi definite.

1. The GLS R^2 is certainly lower than the OLS R^2 for each specification of Section(Figure 1: FF25).

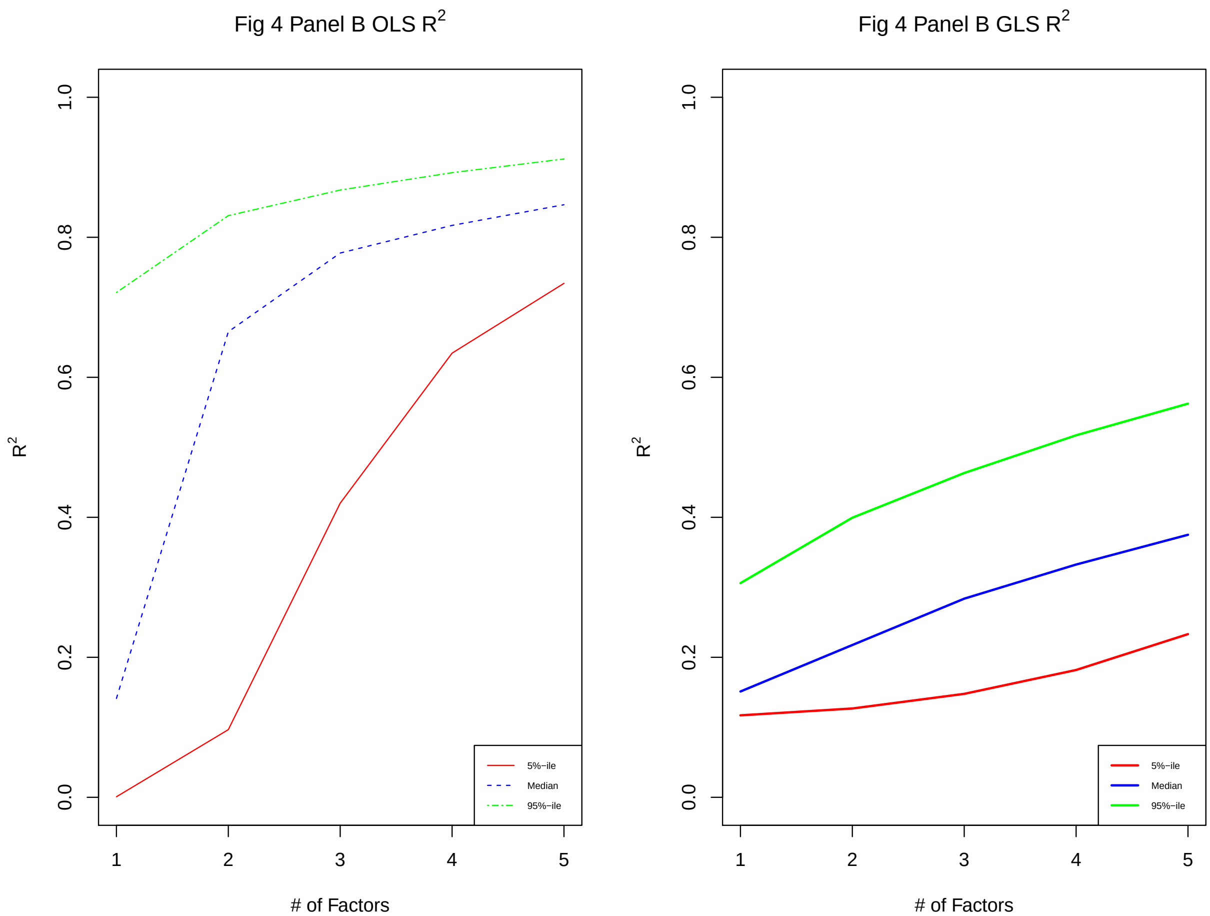
2. As can be seen that the patterns in the GLS R^2 matches that in the paper, however it seems to be shifted upwards by a factor of ≈ 0.1. This could be because I did not explicitly impose the restriction that the factors are mimicking portfolios which have average expected returns 0. This is why even in the case of Figure(3a), I do not get $R^2_{GLS} = 0$ but some value which ≈ 0.1.

3. The authors are not quite clear how they imposed this restriction. Although the pattern in the slope as we increase the number of factors seems to hold true as in the paper.

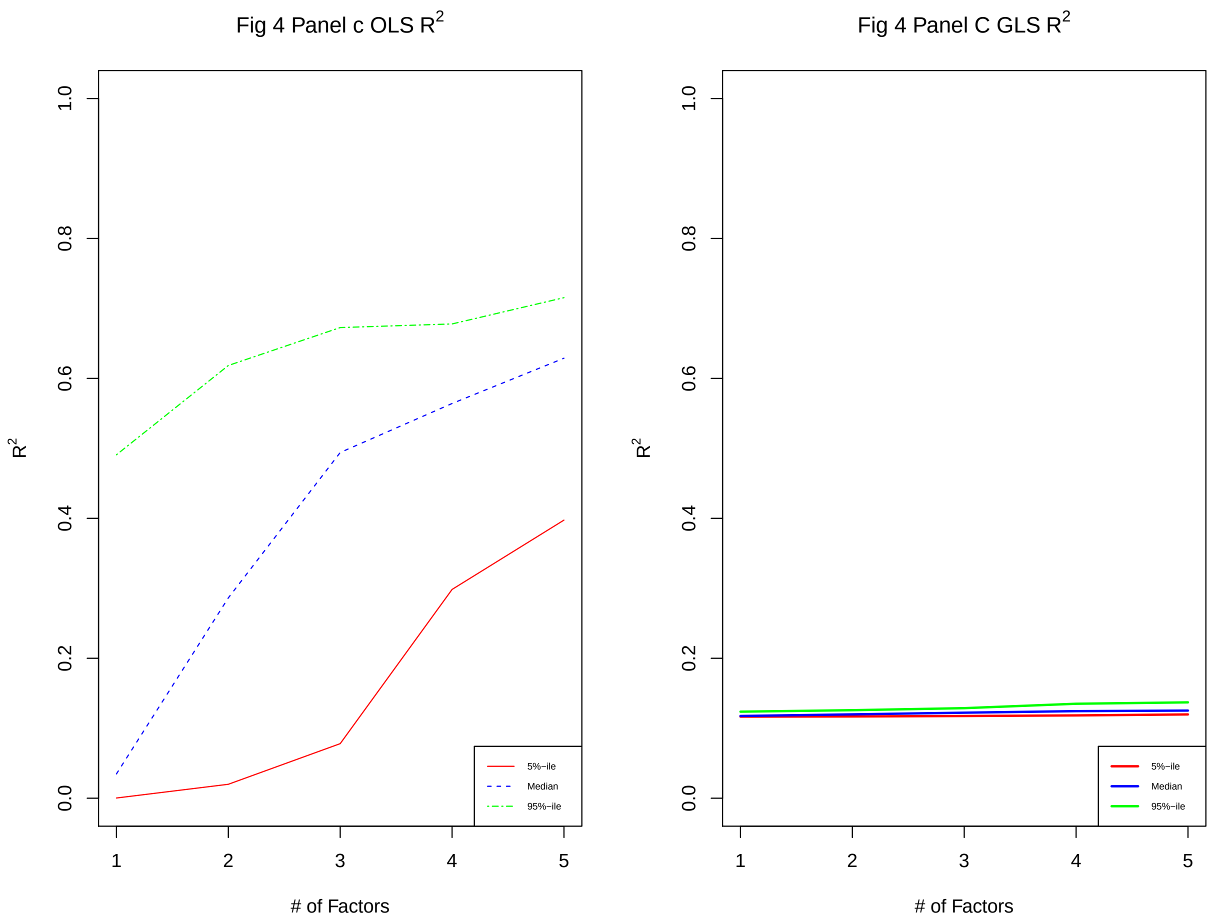
4. The results do not change drastically when we replicate the analysis for the current data.



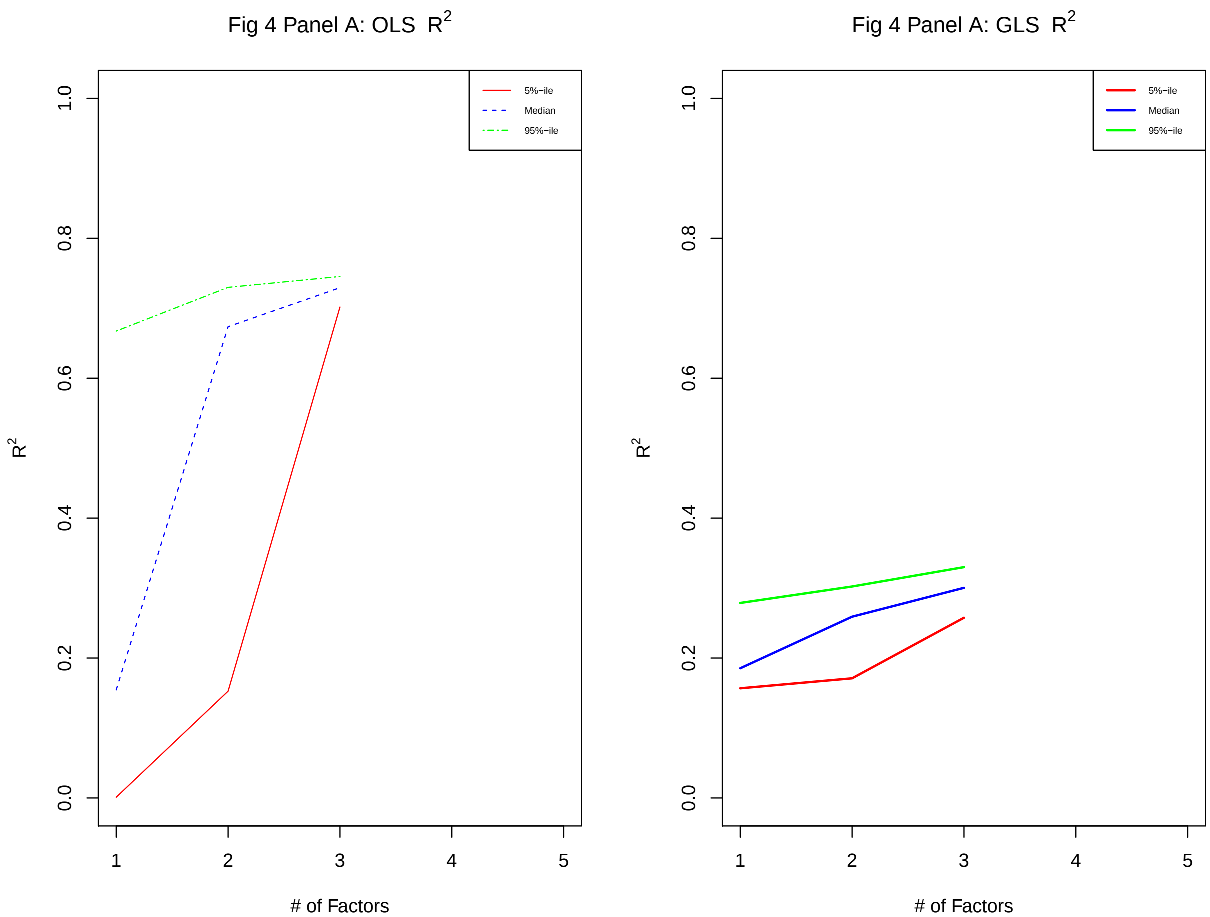
(a) Uncorrelated factors with random weights



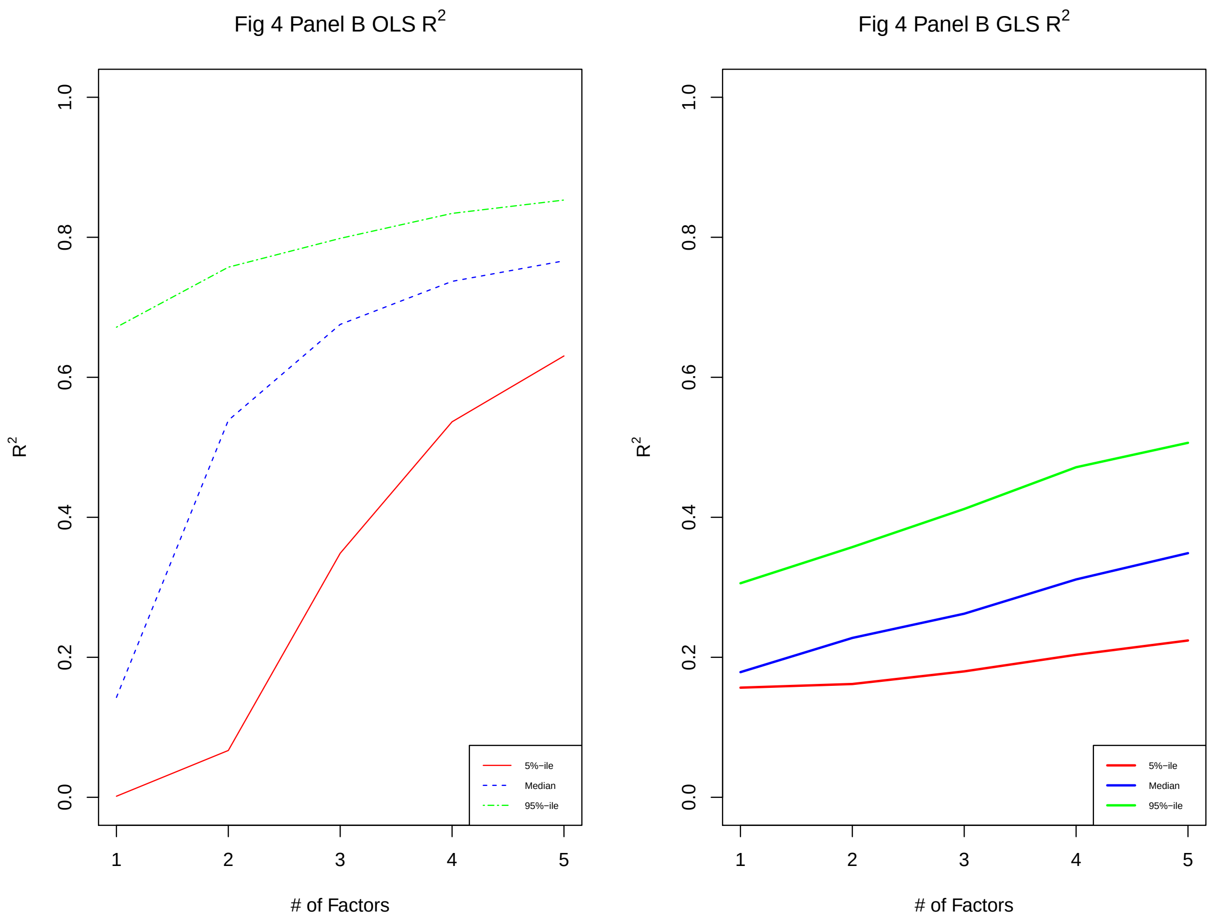
(a) Correlated factors with random weights



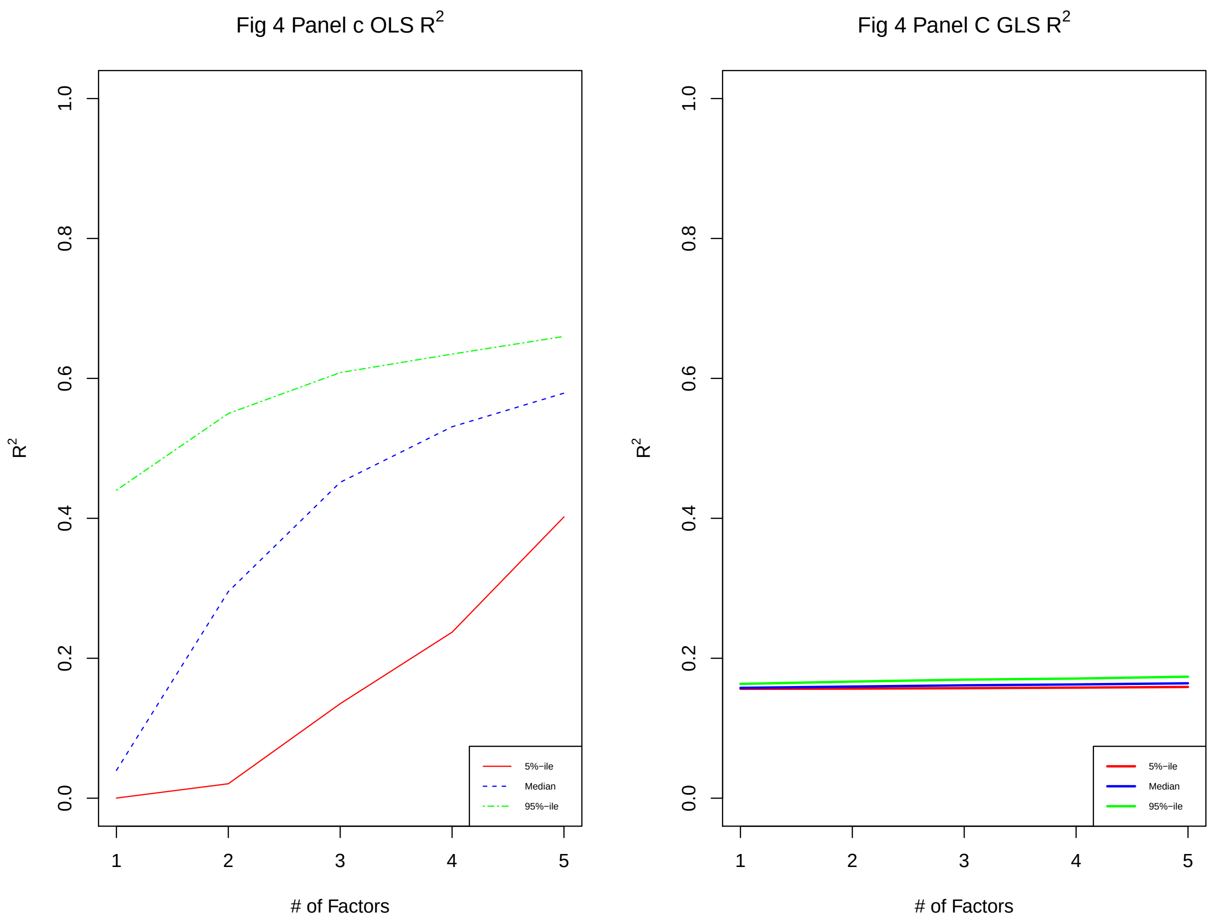
(a) Correlated factors with zero mean and random weights



(a) Uncorrelated factors with random weights: 1963Q1-2018Q4



(a) Correlated factors with random weights: 1963Q1-2018Q4



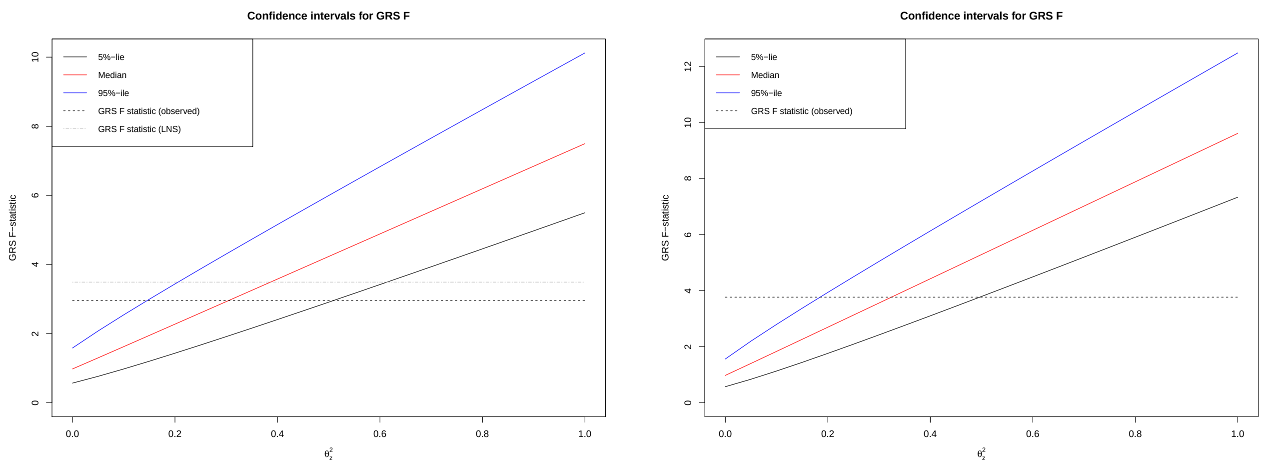
(a) Correlated factors with zero mean and random weights: 1963Q1-2018Q4

Figure 6: GRS F statistic

1. This figure is basically an illustration of how the authors compute different statistics such as the weighted sum of squared pricing errors, e.g. GRS F-statistic, the T^2, and the HJ-distance.

2. They show, how imposing some restrictions on the average factor returns, provides us with economically meaningful interpretation of these statistics.

3. They also illustrate the asymptotic and finite sample distribution of these statistics and how it helps in computing the asymptotic confidence intervals.



(a) 1963Q1-2004Q4

(b) 1963Q1-2018Q4

Figure 3: GRS statistic v/s $\theta_z^2 = \alpha'\Sigma^{-1}\alpha$. I get the value of $GRSF = 2.95$ (3.77 for the latest data) when I use the FF factor model market portfolio. The corresponding confidence interval for θ_z^2 is $(0.15, 0.50)$. Lewellen, Nagel and Shanken (2010) use the value-weighted CRSP market index as the market portfolio. They get this value to be 3.49. The confidence interval for θ_z^2 in that case is $(0.2, 0.6)$

4. The GRS F statistic is given by,

$$c^{-1}\hat{a}'\Sigma_e^{-1}\hat{a}\frac{T-N-K}{N(T-K-1)}$$

   where $c = (1+s_p^2)/T$ and s_p is the maximal Sharpe-ratio. $\Sigma_e = V(\mathbf{e})$. $\hat{a}$ is the sample estimate of the pricing error α. The F-stat follows an asymptotically $F(N, T-N-K)$ distribution with non-centrality parameter $\theta_z^2 = \alpha'\Sigma^{-1}\alpha$.

5. We conduct the simulation to determine the F statistic quantiles for any given θ_z^2.

6. To get the confidence interval of F (or any other statistic such as T^2) we draw a horizontal line and note down the θ_z^2 (q) values of the points where this line intersects the 5%-le and 95%-le lines.

7. My estimate of the in-sample GRS F statistic (CAPM) is 2.95 and in the current data this figure is 3.77 (meaning that we are more likely to reject the null of 0 pricing error in the cross section with the current data). Lewellen, Nagel and Shanken (2010) find this statistic to be 3.49.

8. The confidence interval graphs for T^2 and q statistic are reported in Appendix: T^2 and q Confidence Intervals.

Confidence Intervals for R^2

1. Choose a value of true R_t^2. For this given value of R_t^2 determine

$$h_t = \frac{V_N(\boldsymbol{C\lambda})}{V_N(\boldsymbol{\mu})} \text{ and } \sigma_\alpha^2 = (1 - R_t^2)V_N(\boldsymbol{\mu})$$

   where,

$$\boldsymbol{\mu} = h\mathbf{C}\boldsymbol{\lambda} + \boldsymbol{\alpha} \ , \quad \boldsymbol{\alpha} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I_N}\sigma_\alpha^2)$$

   Choosing these parameters ensures that we have the true R_t^2 in the cross-section of average returns with our factors $\mathbf{P}$.

2. Simulate the $\boldsymbol{\alpha}$ from the normal distribution. Transform the obtained sample sample to make it mean 0, $V(\boldsymbol{\alpha}) = \boldsymbol{I}_N\sigma_\alpha^2$, and $Cov(\boldsymbol{C}, \boldsymbol{\alpha}) = 0$.

3. Transform the factors to be mean 0 ($\tilde{\mathbf{f}}_\mathbf{t}$, $E[\tilde{\mathbf{f}}_\mathbf{t}] = \mathbf{0}$). Sample factors from the the observed empirical sample. This step is different from the one prescribed in HW2Q3. Now simulate

$$\mathbf{R}_t | \tilde{\boldsymbol{f}}_{\boldsymbol{t}} \sim \mathcal{N}(h\mathbf{C}\boldsymbol{\lambda} + \boldsymbol{\alpha} + \boldsymbol{C}\tilde{\boldsymbol{f}}_{\boldsymbol{t}})$$

   The demeaning ensures that in the population our original equation for the cross section $\boldsymbol{\mu} = h\mathbf{C}\boldsymbol{\lambda} + \boldsymbol{\alpha}$ holds true.

4. Run this step 1000 times (instead of 5000 as in the original paper) and compute the R^2 (OLS or GLS) depending on how we generate the $\mathbf{R}_t$ data.

5. Compute the relevant quantiles and plot them against the true R^2.

6. Confidence interval is computed in exactly the same way as we did in the previous section. Draw a horizontal line at the level of observed R^2 and get the R_t^2 coordinates of the points where the horizontal line intersects the lines for the 5%-ile and 95%-ile.

7. The results for the confidence interval for R^2 are in Appendix: R^2 Confidence Intervals.

Empirical Results

The following exercise tests some of the seminal linear factor models and analyzes their performance through the framework developed thus far. Below I replicate the results for

1. CAPM (Mkt factor)
2. Consumption CAPM (Δc factor)
3. FF3 (Mkt, SMB, HML as factors)
4. LL ($\Delta c, cay, \Delta c * cay$ as factors)

I also get the results for these models for the current data. The T^2 statistic p values are computed with a χ^2 distribution with non-centrality parameter q and degree of freedom $N - K - 1$.

Unconditional CAPM Tests:

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	2.9	3.18	-0.03	0.01	65.81	0.39
R_{mkt}	-0.42	-0.38	(0.00,0.52)	(0.00, 0.01)	0.00	(0.14,0.43)

Table 1: FF25 Portfolios as test assets

With the data from $1963Q1 - 2018Q4$

FF25 + 30Ind:

1. All the estimates are close to the values in the original paper.
2. The coefficient estimates in the current data seem to hold up with minor differences.
3. The GLS R^2 confidence bands are narrower.
4. The R^2 confidence bands for OLS are smaller for the FF25 + 30 Industries data.

	Coefficients	t-stat	Rsq OLS	Rsq GLS	Tsq Stat (p-val)	q (confidence interval)
Const.	2.59	2.93	-0.04	-0.02	82.03	0.37
Rm	-0.24	-0.24	(0.00,0.5)	(0.00,0.01)	0.00	(0.17,0.42)

Table 2: FF25 Portfolios as test assets: Current data

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	1.86	2.43	-0.03	-0.1	197.92	1.18
R_m	0.28	0.28	(0.00,0.00)	(0.00,0.04)	(0.00)	(0.63,0.15)

Table 3: FF25 + 30 Industry Portfolios as test assets

	Coefficients	t-stat	Rsq OLS	Rsq GLS	Tsq Stat (p-val)	q (confidence interval)
a	1.83	2.71	-0.03	-0.29	176.63	0.79
Rm	0.26	0.3	(0.00,0.06)	(0.00,0.02)	0	(0.39 , 0.76)

Table 4: FF25 + 30 Industry Portfolios as test assets: Current data

Fama French 3 Factor: Fama and French (1992)

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	2.55	2.15	0.8	0.22	45.36	0.30
R_m	-0.99	-0.73	(0.70,1)	(0.05,0.9)	(0.00)	(0.06,0.31)
SMB	0.80	1.74	-	-	-	-
HML	1.47	3.16	-	-	-	-

Table 5: FF25 Portfolios as test assets

1. All the estimates are close to the values in the original paper.

2. The coefficient estimates in the current data seem to hold up with minor differences.

3. The GLS R^2 confidence bands are narrower.

4. The R^2 confidence bands for OLS are smaller for the FF25 + 30 Industries data.

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	3.41	3.61	0.69	0.17	62.29	0.31
R_m	-1.68	-1.55	(0.54,0.97)	(0.03,0.60)	0.00	(0.06 , 0.31)
SMB	0.57	1.54	-	-	-	-
HML	1.06	2.67	-	-	-	-

Table 6: FF25 Portfolios as test assets: 1963Q1-2018Q4

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	2.06	2.25	0.34	0.11	165.15	1.09
R_m	-0.37	-0.34	(0.01,0.86)	(0.04,0.72)	0.00	(0.49,0.98)
SMB	0.65	1.39	-	-	-	-
HML	0.94	1.99	-	-	-	-

Table 7: FF25 + 30 Industry Portfolios as test assets

	Coefficients	t-stat	R^2 OLS	#Rsq GLS	Tsq Stat (p-val)	q (confidence interval)
Const.	2.87	4.08	0.29	0.26	147.87	0.73
R_m	-1.11	-1.27	(0.04,0.83)	(0.56,0.86)	0.00	(0.31,0.66)
SMB	0.59	1.58	-	-	-	-
HML	0.65	1.59	-	-	-	-

Table 8: FF25 + 30 Industry Portfolios as test assets: 1963Q1-2018Q4

LL 3 factors: Lettau and Ludvigson (2001)

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	4.28	4.25	0.66	0.26	28.84	0.42
Δc	0.02	0.15	(0.47,1.00)	(0.00,0.00)	0.13	(0.00,0.39)
cay	-0.13	-0.31	-	-	-	-
$cay * \Delta c$	0.006	2.25	-	-	-	-

Table 9: FF25 Portfolios as test assets

With the data from $1963Q1 - 2018Q4$

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	1.35	1.21	0.43	0.05	16.89	0.35
Δc	2.64	2.02	(0.90,1.00)	(0.00,0.00)	0.73	(0.00,0.20)
cay	-0.03	-0.08	-	-	-	-
$\Delta c * cay$	0.02	1.51	-	-	-	-

Table 10: FF25 Portfolios as test assets: 1963Q1-2018Q4

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	1.61	2.24	0.03	0.03	437.48	6.44
Δc	-0.17	-1.29	(0.00,0.43)	(0.00,0.03)	0.00	(3.83,5.4)
cay	0.7	2.34	-	-	-	-
$\Delta c * cay$	0.006	3.25	-	-	-	-

Table 11: FF25 + 30 Industries Portfolios as test assets

With the data from $1963Q1 - 2018Q4$:

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	1.78	2.88	0.08	0.24	35.95	0.75
Δc	1.11	1.7	(0.00,1.00)	(0.1,0.6)	0.95	(0,0.49)
cay	-0.18	-0.9	-	-	-	-
$\Delta c * cay$	0.006	1.9	-	-	-	-

1. All the estimates are close to the values in the original paper.

2. The coefficient estimates in the current data are quite different in sign and magnitude. This is because I used my own consumption growth series and coupled that with the cay series obtained from the authors' website. I could not match the data summary figures (See the data section for more details on how I construct the consumption growth series)

3. The null of zero pricing errors is rejected (T^2 stat)in 3 out of 4 cases above. Providing evidence against the model.

4. The GLS R^2 bands are not quite narrow in this case as has been for CAPM and FF3.

5. The R^2 confidence bands for OLS are smaller for the FF25 + 30 Industries data.

Consumption CAPM (all coefficients in decimals)

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	2.7125	3.79	-0.03	0.05	65.62	0.4
Δc	-5e-04	-0.3	(0.00,0.64)	(0.00,0.06)	0.00	(0.14,0.44)

Table 12: FF25 Portfolios as test assets

With the data from $1963Q1 - 2018Q4$:

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	1.773	2.58	-0.01	0.08	78.3	0.37
Δc	0.0011	0.6	(0.00,0.80)	(0.00,0.03)	0.00	(0.16,0.41)

Table 13: FF25 Portfolios as test assets: 1963Q1-2018Q4

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	2.17	3.7	-0.04	0.07	195.56	1.18
Δc	-0.002	-0.02	(0.00,0.00)	(0.00,0.02)	0.00	(0.62,1.13)

Table 14: FF25 + 30 Industries Portfolios as test assets

With the data from $1963Q1 - 2018Q4$:

	Coefficients	t-stat	R^2 OLS	R^2 GLS	T^2 Stat (p-val)	q (confidence interval)
Const.	1.94	3.57	-0.03	0.22	169.32	0.8
Δc	0.0004	0.32	(0.00,0.00)	(0.00,0.00)	0.00	(0.37,0.72)

Table 15: FF25 + 30 Industries Portfolios as test assets: 1963Q1-2018Q4

1. It is hard to match the estimates in the original paper since I use a different consumption series created as part of HW3 (more detail in the data section).

2. The price of consumption growth risk seems to be not significantly different from 0. The model performs very poorly in terms of the cross-sectional fit as well. Which is not surprising given so little empirical support for the unconditional consumption CAPM model.

3. The coefficient estimates in the current data are quite different in sign and magnitude. This is because I used my own consumption growth series. I could not match the data summary figures (See the data section for more details on how I construct the consumption growth series)

4. It is difficult to make inference about the GLS R^2 in this case as I get somewhat strange plots in this particular specification. See Appendix: R^2 Confidence Intervals for more details.

5. The R^2 confidence bands for OLS are smaller for the FF25 + 30 Industries data.

Conclusion

The authors point out that it may not be hard to find factors that may explain the cross-sectional variation in the FF25 Size by B/M portfolios. Figures(1a,1a,1a) demonstrate this point, where they show that even with randomly constructed factors the median R^2 with a few factors tends to be very high. These results are perfectly replicated in the original sample and in the current data. There is no reason for us to believe that the overall patterns in the data from Figures(1-4) are likely to change due to the sample. Only difference is that the R^2 values obtained in the current data is somewhat lower than in the original sample. This may be the case as we now include the finacial crisis in our sample. This episode may have moderately affected the insample factor structure in the data, because of which the R^2 for the model with random factors is also lower.

Similarly, as per the prescribed solutions to the above problems we first expand the set of test assets to include the 30 Industries sorted portfolios. It is clear that the OLS R^2 are lower in case of this larger set of test assets. The result is replicated in the original sample and in the current data (the value of R^2 slightly smaller in the current data). The authors also prescribe to use GLS instead of OLS as it provides a more stringent hurdle to meet and GLS R^2 has some economic significance if we impose certain restrictions on the factor mimicking portfolio returns. The GLS R^2 is certainly lower than the OLS R^2 as is expectd and the patterns of relative increase in the R^2 with increase in the factors is the same here as in the paper. This pattern persists in the original sample as well. However, GLS R^2 I obtain are elevated by ≈ 0.1. This may be because I don't impose the zero average returns restrictions on the factors.
Figure (6) in the original paper reveals the confidence interval for the GRS F-statistic. I am able to replicate the figure but my in sample F-stat value is 2.95 whereas in the paper the authors find this to be 3.49. This value is 3.77 in the current data.

The replication results of CAPM, FF3 match in terms of the coefficient estimates, standard errors, t-stats, the pricing error stats T^2 and also reasonably close in terms of the OLS and GLS R^2. However, the confidence interval of a few GLS R^2 estimates don't seem to match all that well but they still seem reasonably close and do not change the inference.
The CAPM, FF3 results also have similar estimates in case of the current data.
The Lettau and Ludvigson (2001) (LL) original sample results vary very slightly than the ones in the paper as I use a different dataset which was shared for HW1Q3. But the estimates are reasonably close and mirror those in the original LL paper. The estimates of R^2 and T^2 also seem reasonable when compared with the results in the paper. However, the LL model estimates seem to change by quite a bit in the current data. The reason could be the use of a different *cay* and consumption growth series Δc_{t+1} from the authors' website.
The results for the Consumption CAPM are the most off. This again seems to be due to the use

of the consumption series that I use in HW3Q1 using the St.Louis FRED NIPA series. However, the estimates are reasonable in that the consumption price of risk seems not significantly different than 0 and the cross-sectional fit of the model is also quite poor. The results in the original sample and the current data do not change drastically.

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Appendix: Basic Setup

Time-Series Regressions

1. $$R_{i,t} = f_t'\beta_i + \varepsilon_{i,t}$$

   $i \in \{1, 2, \ldots, N\}$. $f_{t,K\times 1}$ (factors), $\beta_{i,K\times 1}$ (time-series loading on factors) and $\varepsilon_{i,t}$ has the standard regression error properties.

2. Stack β columnwise, and R (returns) and f_t factors row-wise.

3. In matrix notation,

   $$\mathbf{R}_{T\times N} = \mathbf{F}_{T\times K}\boldsymbol{\beta}_{K\times N} + \boldsymbol{\varepsilon}$$

Cross-sectional Regressions

The idea is to determine how well our vector of factors f explain the cross-section of average returns.

1. Taking expectation (over time) of the transpose of the above time series equation we have,

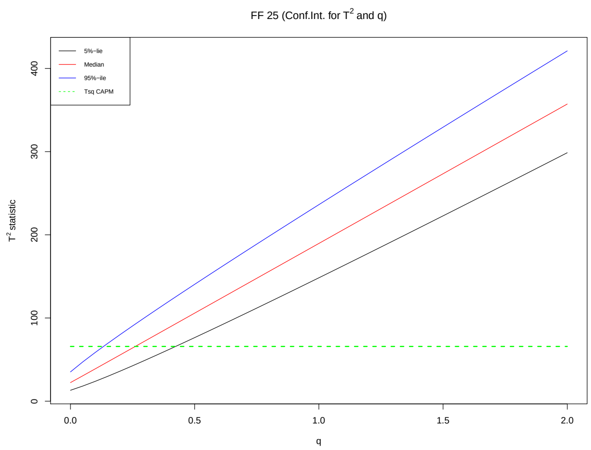
   $$E_t[\boldsymbol{R}] = \boldsymbol{\mu}_{N\times 1} = \boldsymbol{\beta}'_{K\times N} E_t[\mathbf{F}'_{T\times K}] + \mathbf{0}$$

2. Let $\mathbf{B}_{N\times K} \equiv \boldsymbol{\beta}'_{K\times N}$ and $\boldsymbol{\lambda}_{K\times 1} = E_t[\mathbf{F}'_{T\times K}]$. Hence,

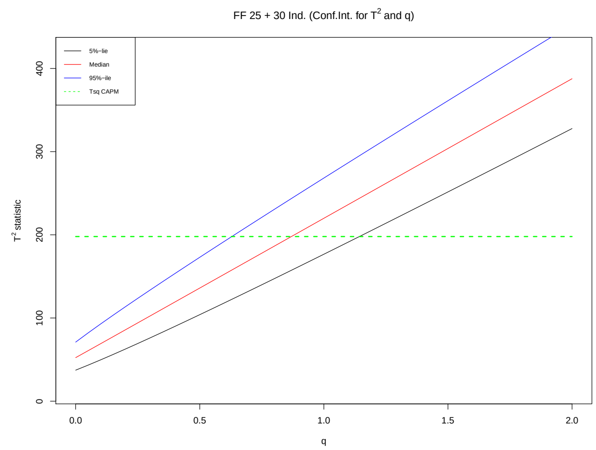
   $$\boldsymbol{\mu}_{N\times 1} = \mathbf{B}_{N\times K}\boldsymbol{\lambda}_{K\times 1}$$

3. Note, here $\mathbf{B}$ is the regressor and λ the coefficients. It is the R^2 (fit) of this last equation that we are interested in.

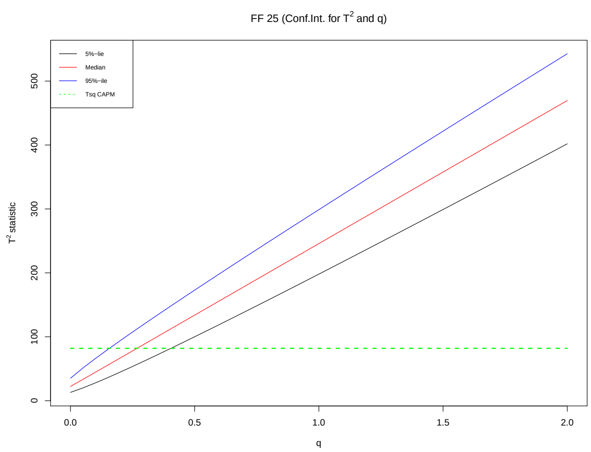
Appendix: T^2 and q Confidence Intervals



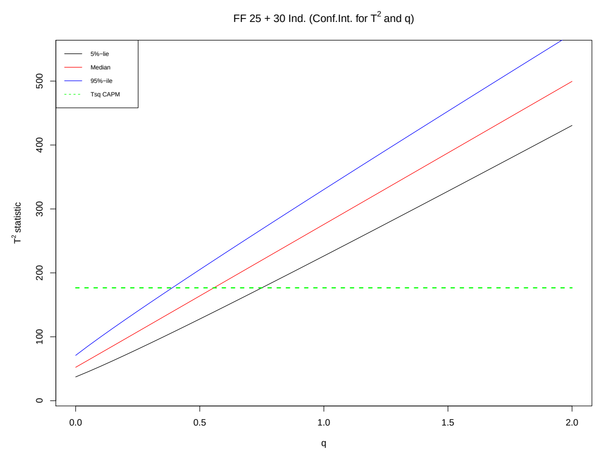
(a) FF25 T^2 vs q unconditional CAPM



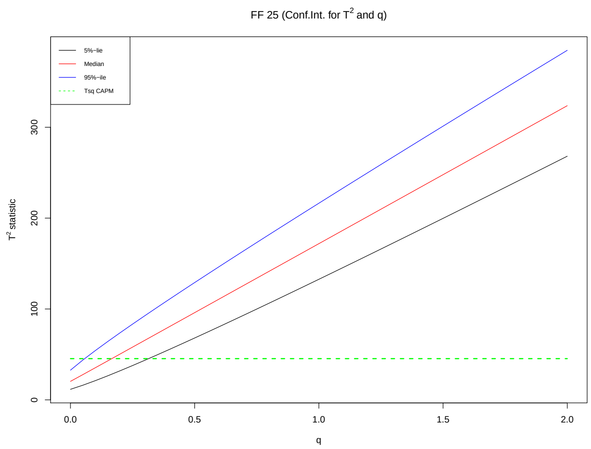
(b) FF25 + 30 Industry T^2 vs q unconditional CAPM



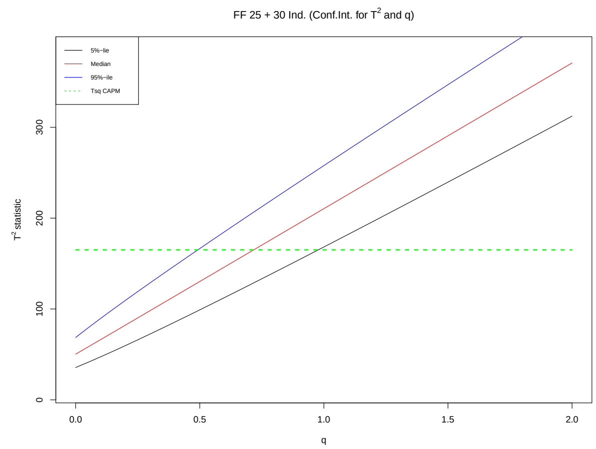
(a) FF25 T^2 vs q unconditional CAPM: 1963Q1-2018Q4



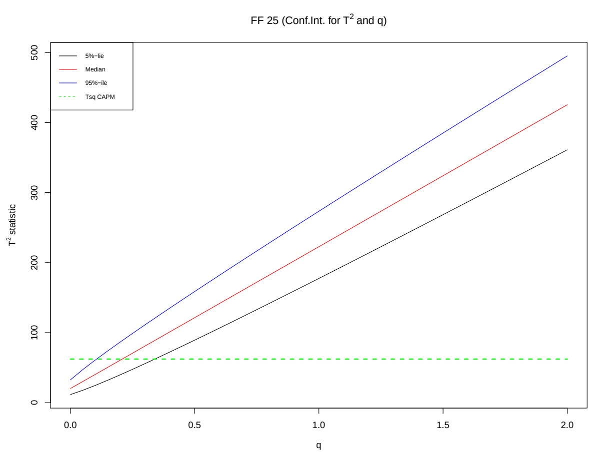
(b) FF25 + 30 Industry T^2 vs q unconditional CAPM: 1963Q1-2018Q4



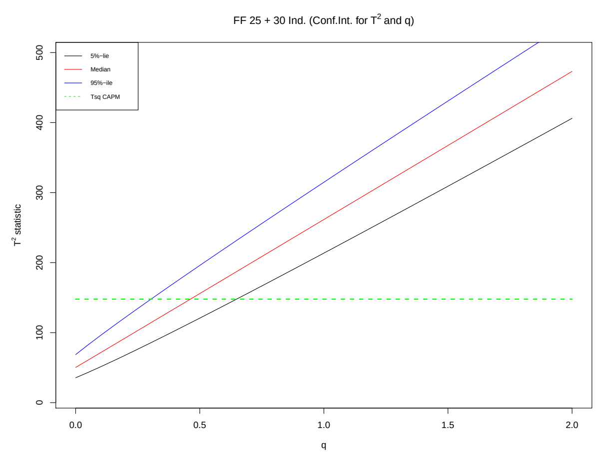
(a) FF25 T^2 vs q FF3 factor



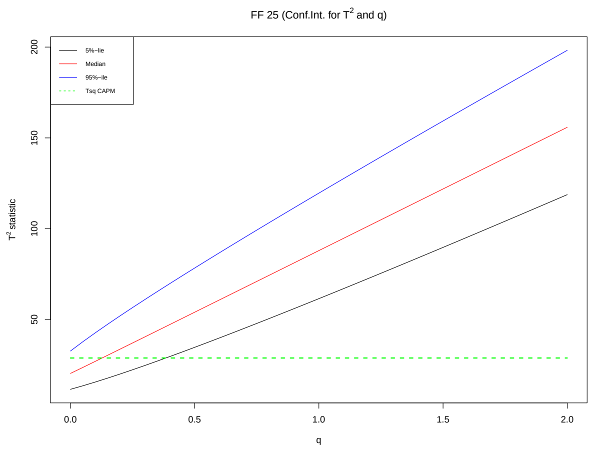
(b) FF25 + 30 Industry T^2 vs q FF3 factor



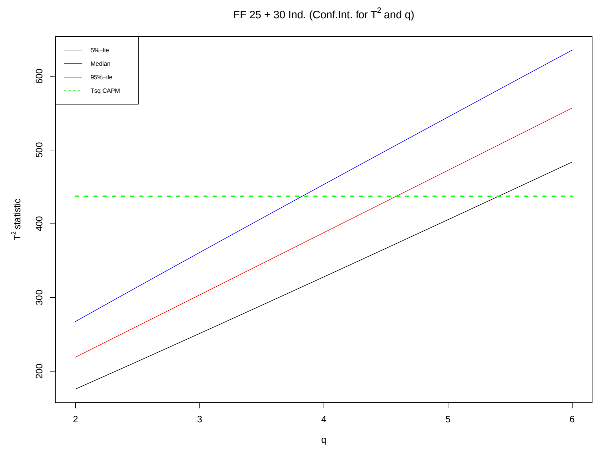
(a) FF25 T^2 vs q FF3 factor: 1963Q1-2018Q4



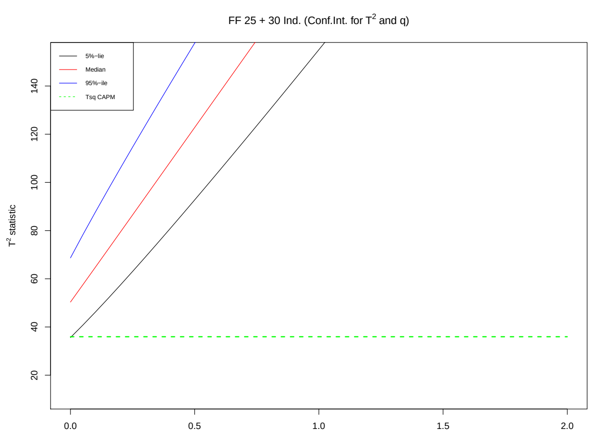
(b) FF25 + 30 Industry T^2 vs q FF3 factor: 1963Q1-2018Q4



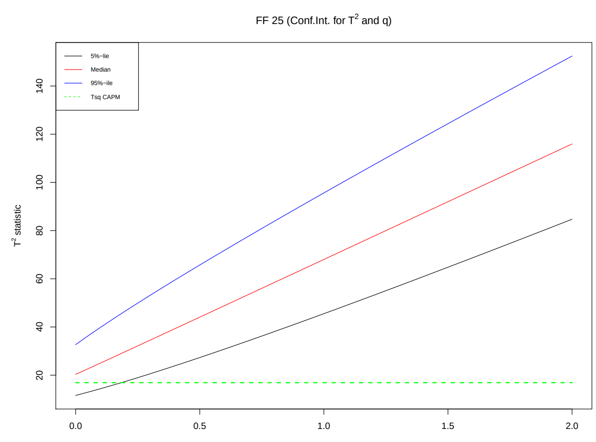
(a) FF25 T^2 vs q LL3 factor



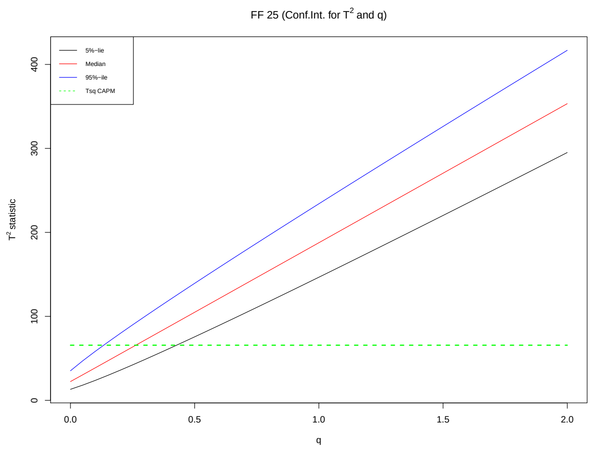
(b) FF25 + 30 Industry T^2 vs q LL3 factor



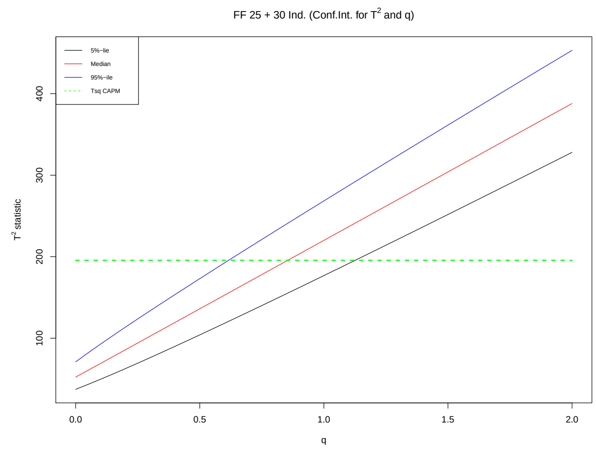
(a) FF25 T^2 vs q LL3 factor: 1963Q1-2018Q4



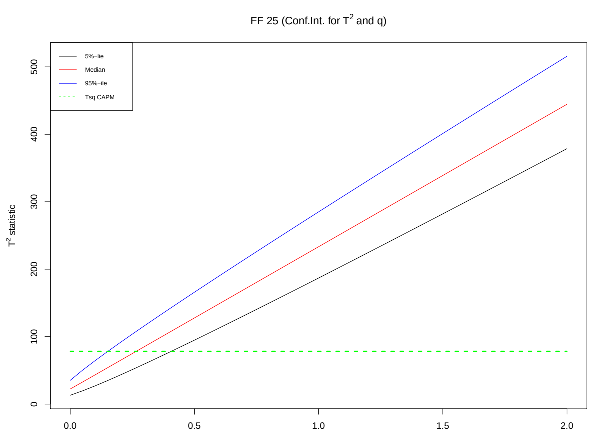
(b) FF25 + 30 Industry T^2 vs q LL3 factor: 1963Q1-2018Q4



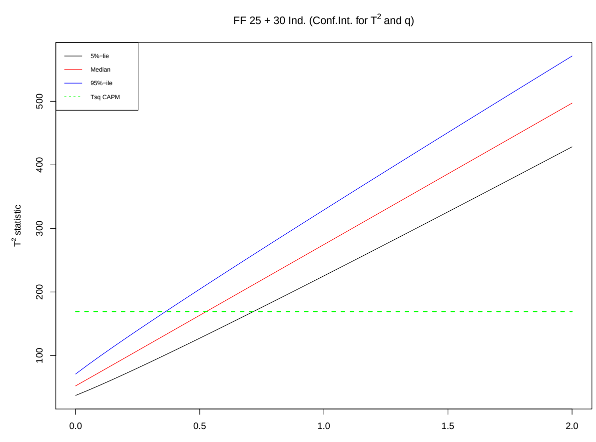
(a) FF25 T^2 vs q CCAPM factor



(b) FF25 + 30 Industry T^2 vs q CCAPM factor



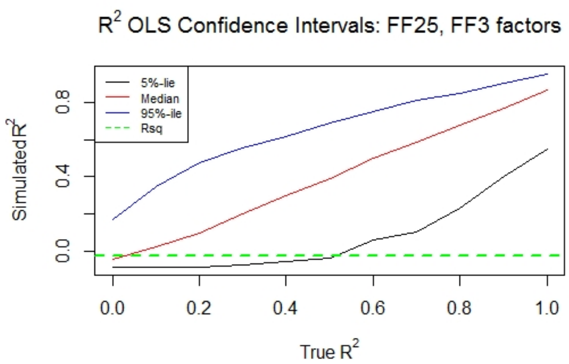
(a) FF25 T^2 vs q CCAPM factor: 1963Q1-2018Q4



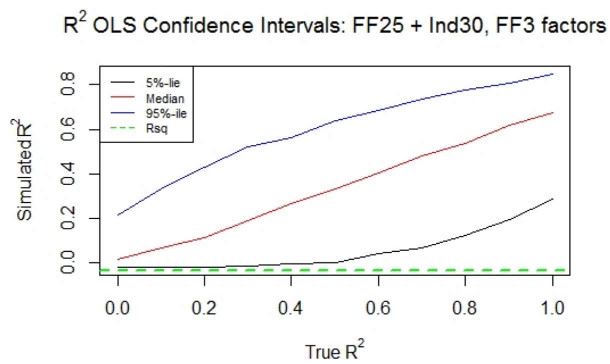
(b) FF25 + 30 Industry T^2 vs q CCAPM factor: 1963Q1-2018Q4

Appendix: R^2 Confidence Intervals

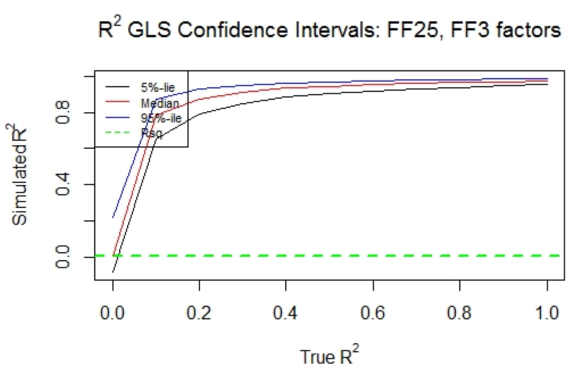
Note: Please ignore the title in the plots. Refer to the captions underneath the figures for an accurate description of the plot



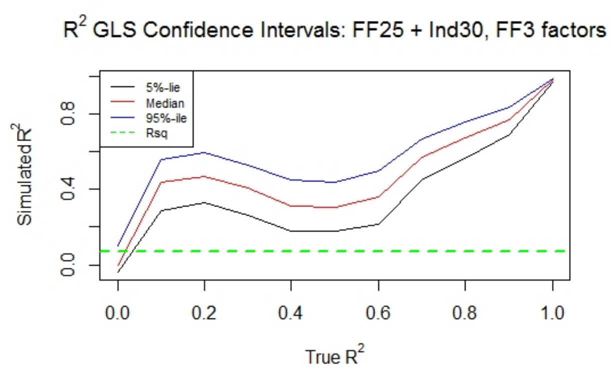
(a) FF25 OLS R^2 vs True R^2 unconditional CAPM



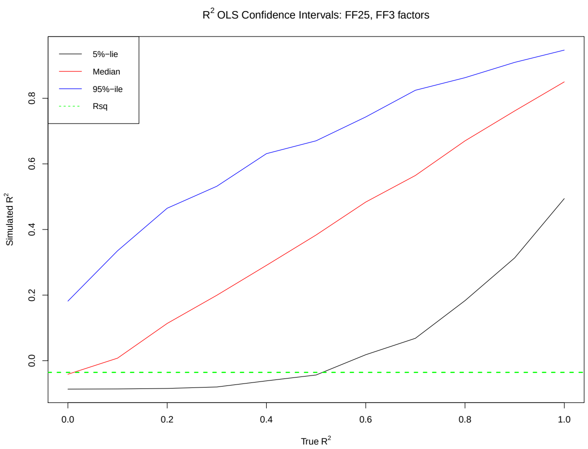
(b) FF25 + 30 Industry OLS R^2 vs True R^2 unconditional CAPM



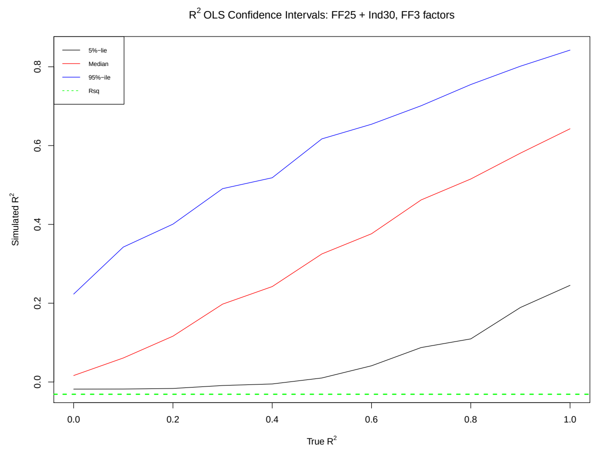
(c) FF25 GLS R^2 vs True R^2 unconditional CAPM



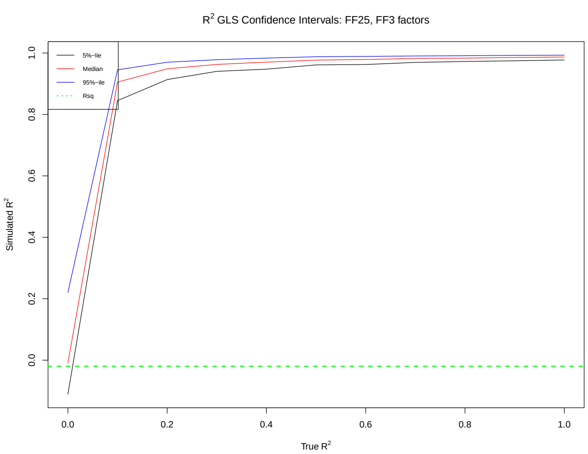
(d) FF25 + 30 Industry GLS R^2 vs True R^2 unconditional CAPM



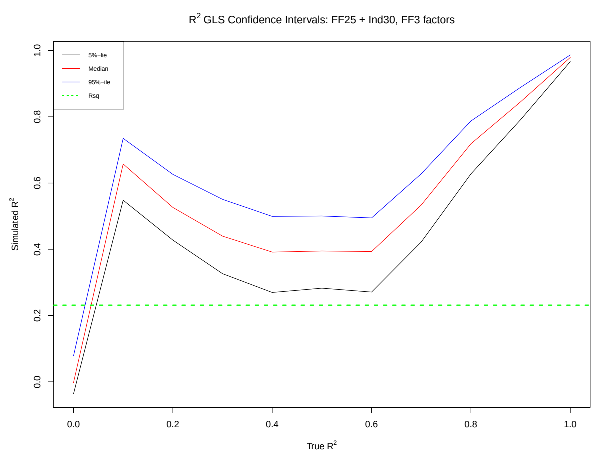
(a) FF25 OLS R^2 vs True R^2 unconditional CAPM: $1963Q1 - 2018Q4$



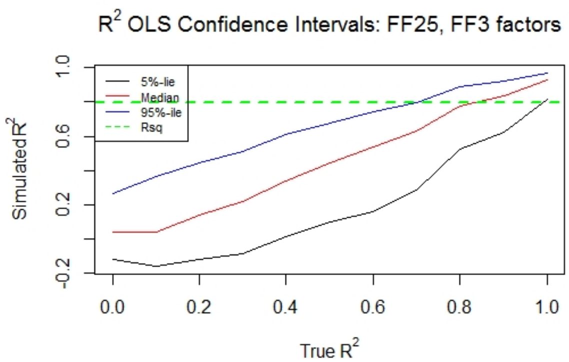
(b) FF25 + 30 Industry OLS R^2 vs True R^2 unconditional CAPM: $1963Q1 - 2018Q4$



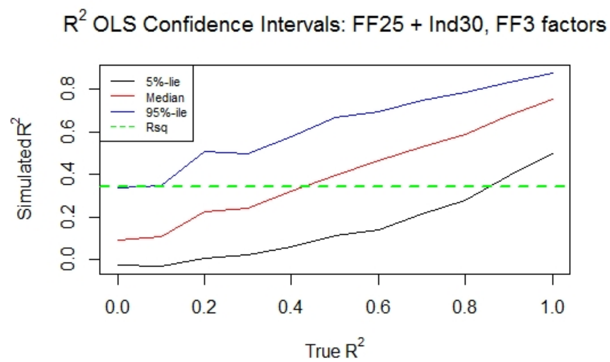
(c) FF25 GLS R^2 vs True R^2 unconditional CAPM: $1963Q1 - 2018Q4$



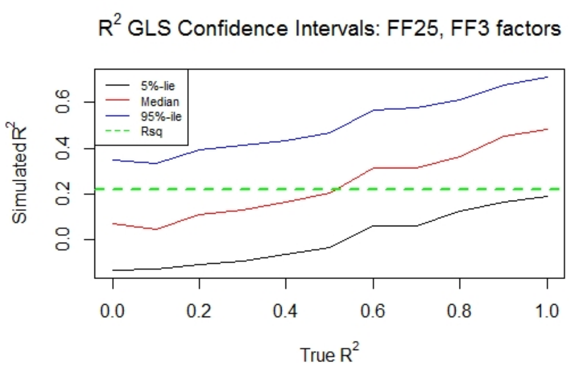
(d) FF25 + 30 Industry GLS R^2 vs True R^2 unconditional CAPM: $1963Q1 - 2018Q4$



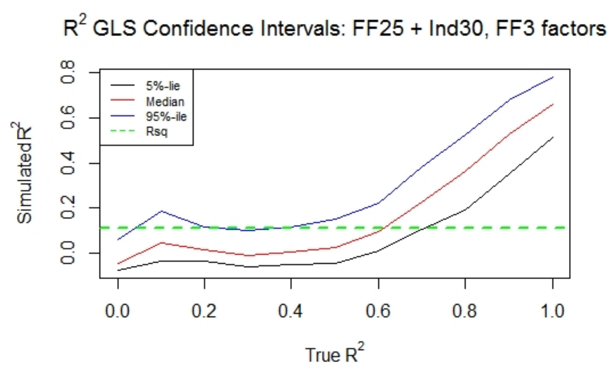
(a) FF25 OLS R^2 vs True R^2 FF3



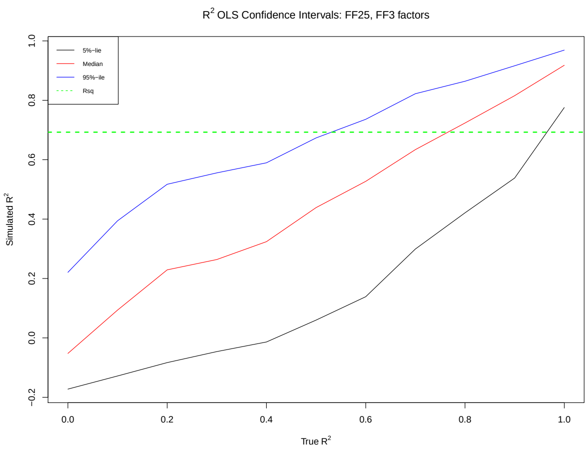
(b) FF25 + 30 Industry OLS R^2 vs True R^2 FF3



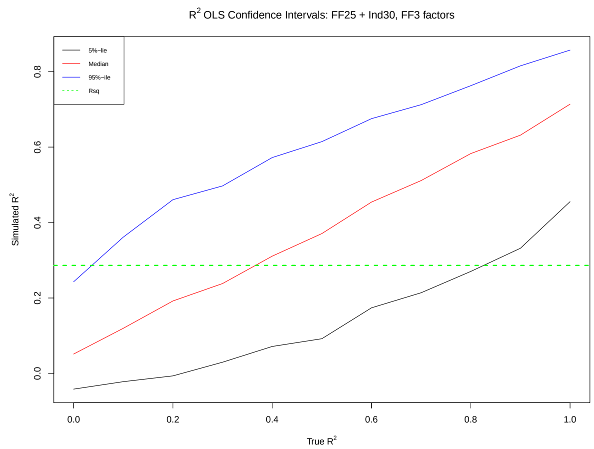
(c) FF25 GLS R^2 vs True R^2 FF3



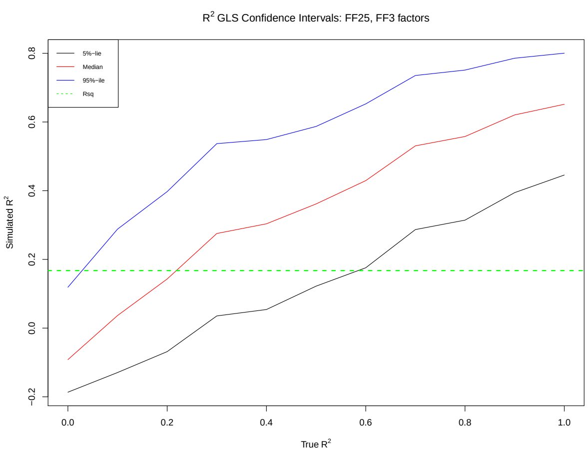
(d) FF25 + 30 Industry GLS R^2 vs True R^2 FF3



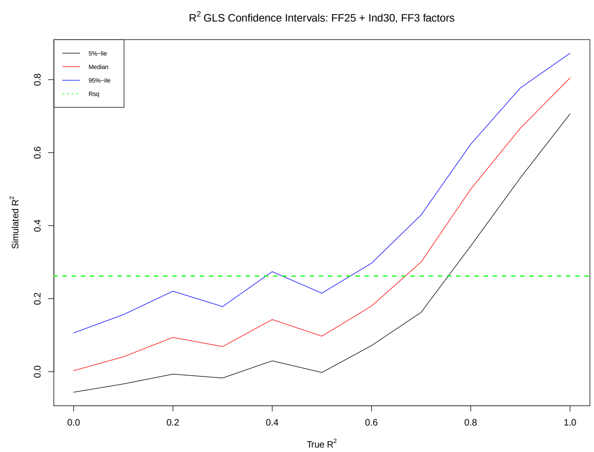
(a) FF25 OLS R^2 vs True R^2 FF3: $1963Q1 - 2018Q4$



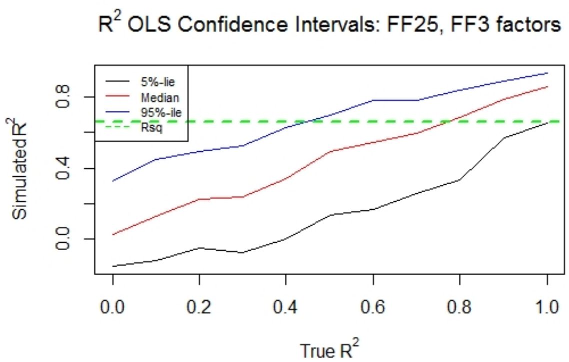
(b) FF25 + 30 Industry OLS R^2 vs True R^2 FF3: $1963Q1 - 2018Q4$



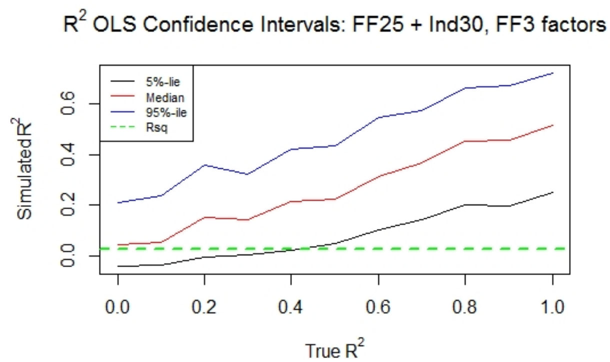
(c) FF25 GLS R^2 vs True R^2 FF3: $1963Q1 - 2018Q4$



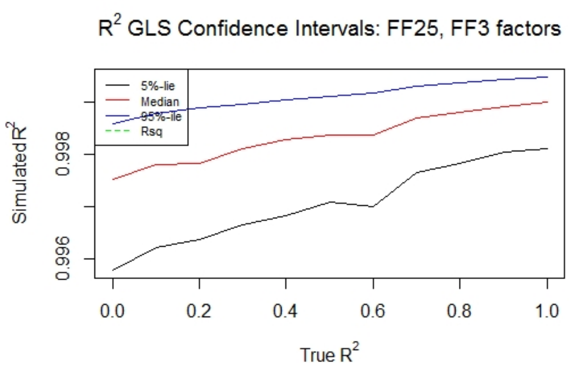
(d) FF25 + 30 Industry GLS R^2 vs True R^2 FF3: $1963Q1 - 2018Q4$



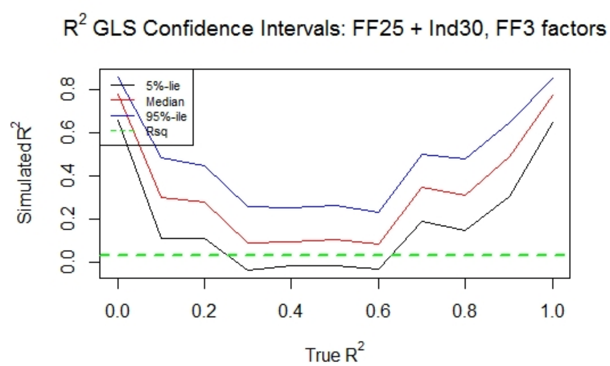
(a) FF25 OLS R^2 vs True R^2 LL



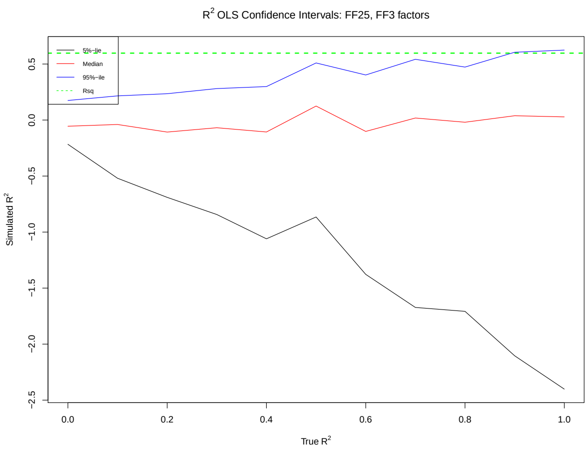
(b) FF25 + 30 Industry OLS R^2 vs True R^2 LL



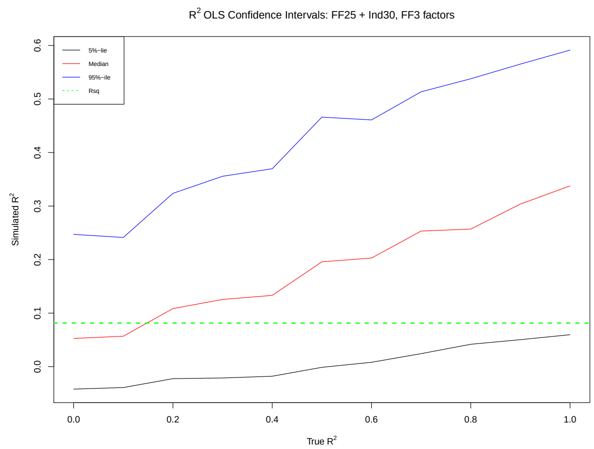
(c) FF25 GLS R^2 vs True R^2 LL



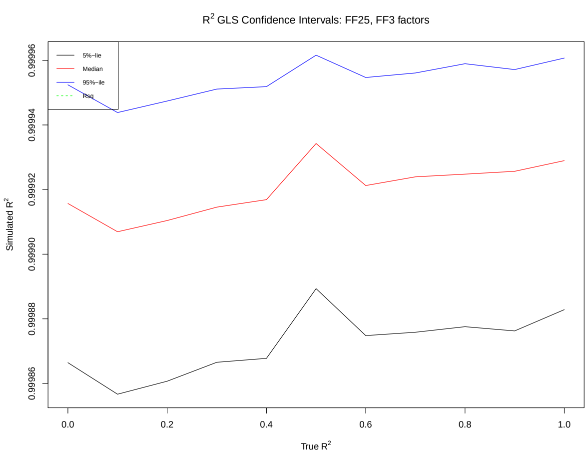
(d) FF25 + 30 Industry GLS R^2 vs True R^2 LL



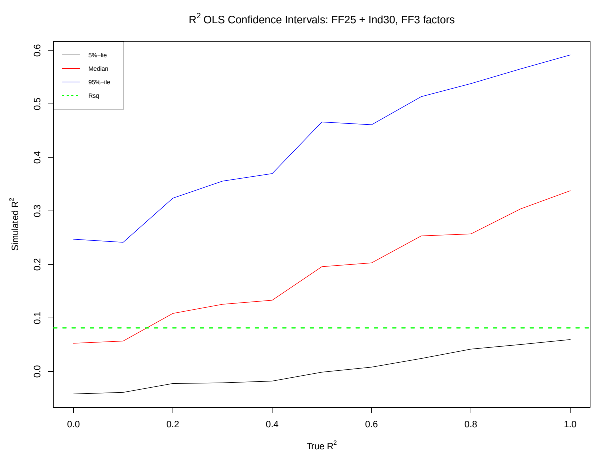
(a) FF25 OLS R^2 vs True R^2 LL: $1963Q1 - 2018Q4$



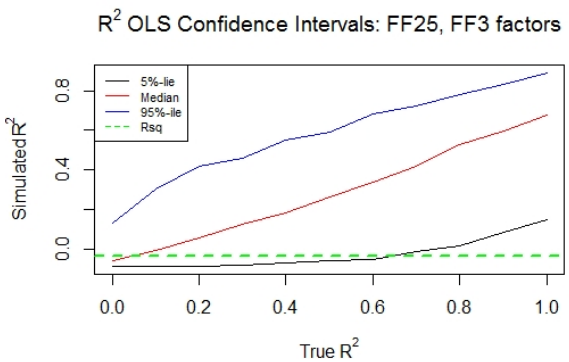
(b) FF25 + 30 Industry OLS R^2 vs True R^2 LL: $1963Q1 - 2018Q4$



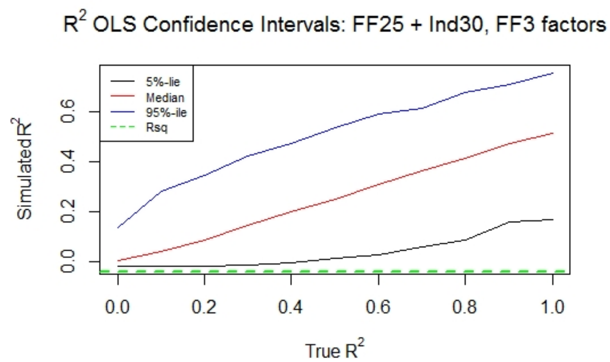
(c) FF25 GLS R^2 vs True R^2 LL: $1963Q1 - 2018Q4$



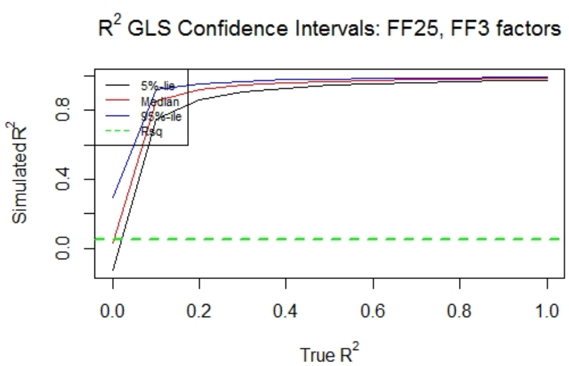
(d) FF25 + 30 Industry GLS R^2 vs True R^2 LL: $1963Q1 - 2018Q4$



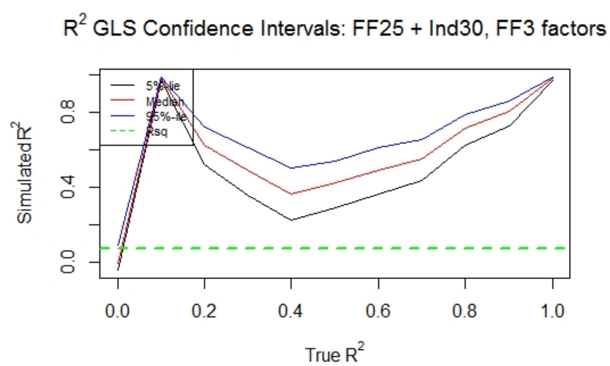
(a) FF25 OLS R^2 vs True R^2 unconditional CAPM



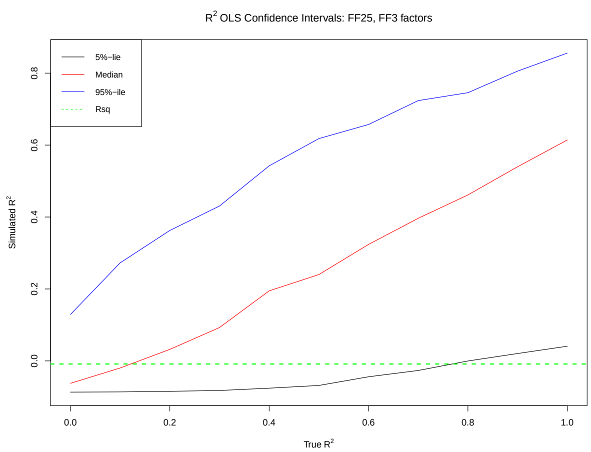
(b) FF25 + 30 Industry OLS R^2 vs True R^2 unconditional CCAPM



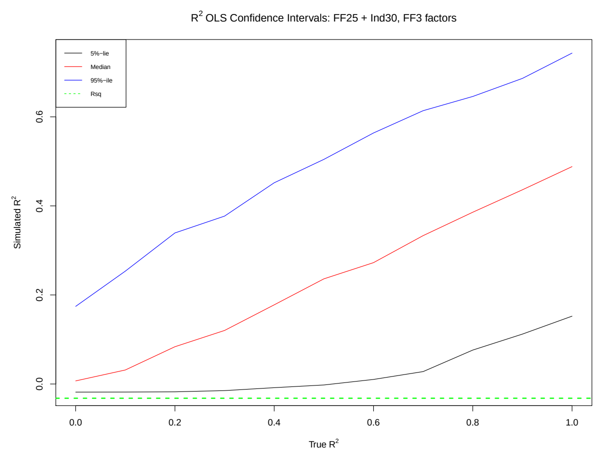
(c) FF25 GLS R^2 vs True R^2 unconditional CAPM



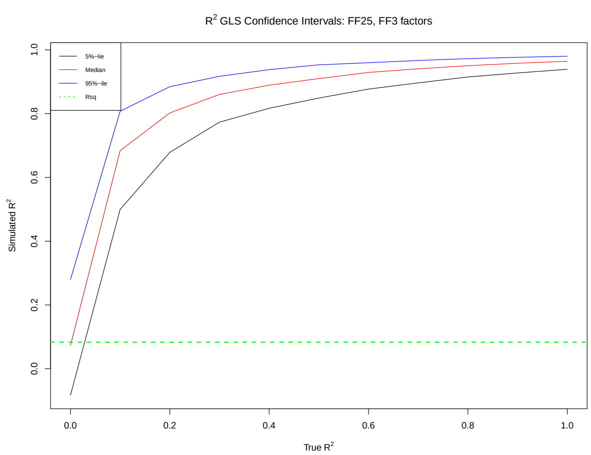
(d) FF25 + 30 Industry GLS R^2 vs True R^2 unconditional CCAPM



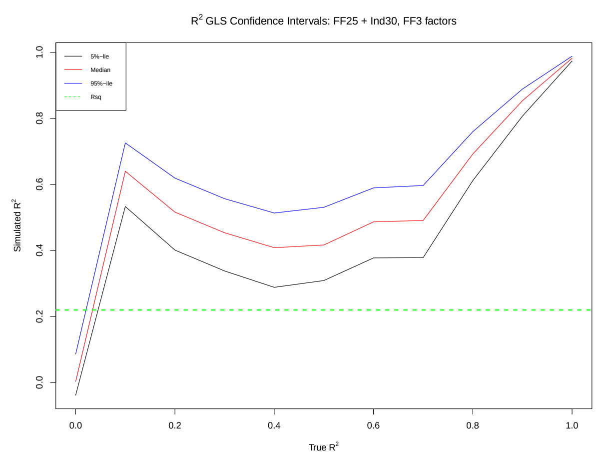
(a) FF25 OLS R^2 vs True R^2 unconditional CCAPM: $1963Q1 - 2018Q4$



(b) FF25 + 30 Industry OLS R^2 vs True R^2 unconditional CCAPM: $1963Q1 - 2018Q4$



(c) FF25 GLS R^2 vs True R^2 unconditional CCAPM: $1963Q1 - 2018Q4$



(d) FF25 + 30 Industry GLS R^2 vs True R^2 unconditional CCAPM: $1963Q1 - 2018Q4$