Simulation et Applications Interactives

Physically Based Simulation

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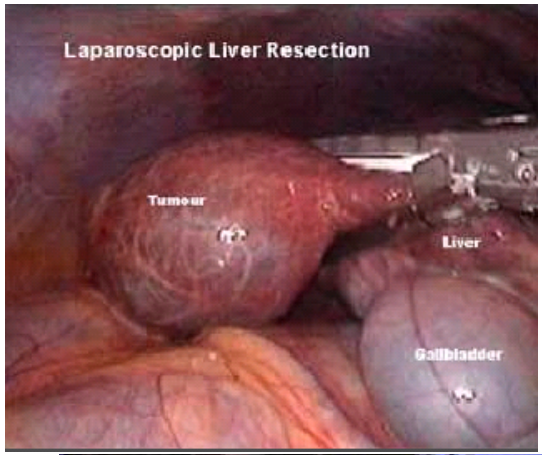
Geometry vs Physics

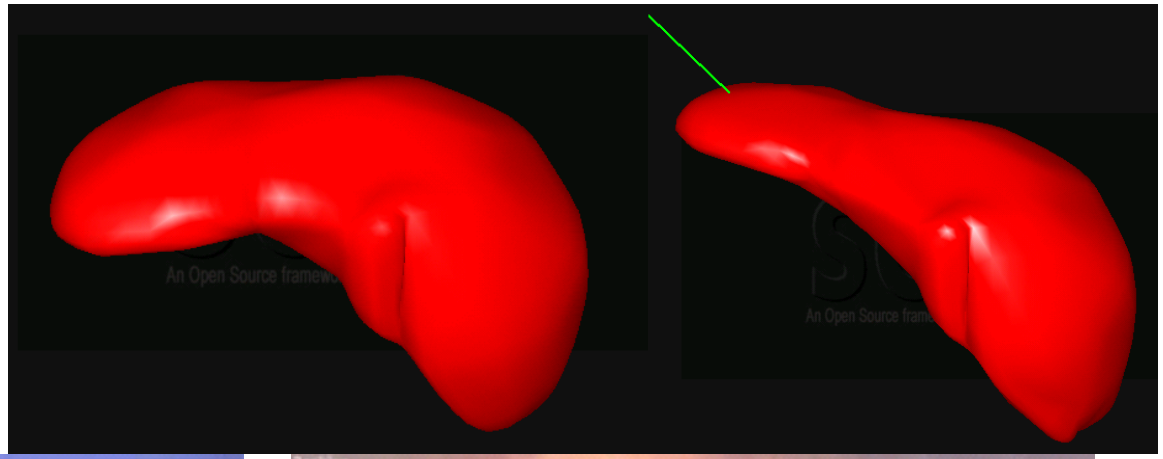
- Geometrically based animation
  - Describes the deformation by some mathematical function
- Physically based animation
  - Describes the causes of the deformation (rather than the deformation itself) by using the laws of physics
  - Also referred to as simulation

Pros & cons of Physics

- High level control for the animator
  - Easy to control a lot of objects
  - Hard to precisely obtain a specific effect
- Simulation is time-consuming
- Simulation might be unstable
- More realistic deformations
- Possibility to obtain dynamic deformations
- Possibility of interaction

Examples of Applications





Overview

- Continuum Mechanics
- Mass-Spring Systems
- Finite Elements
- Continuous Forces
- Particle Systems
- Introduction to Collision Detection and Response

Continuum Mechanics

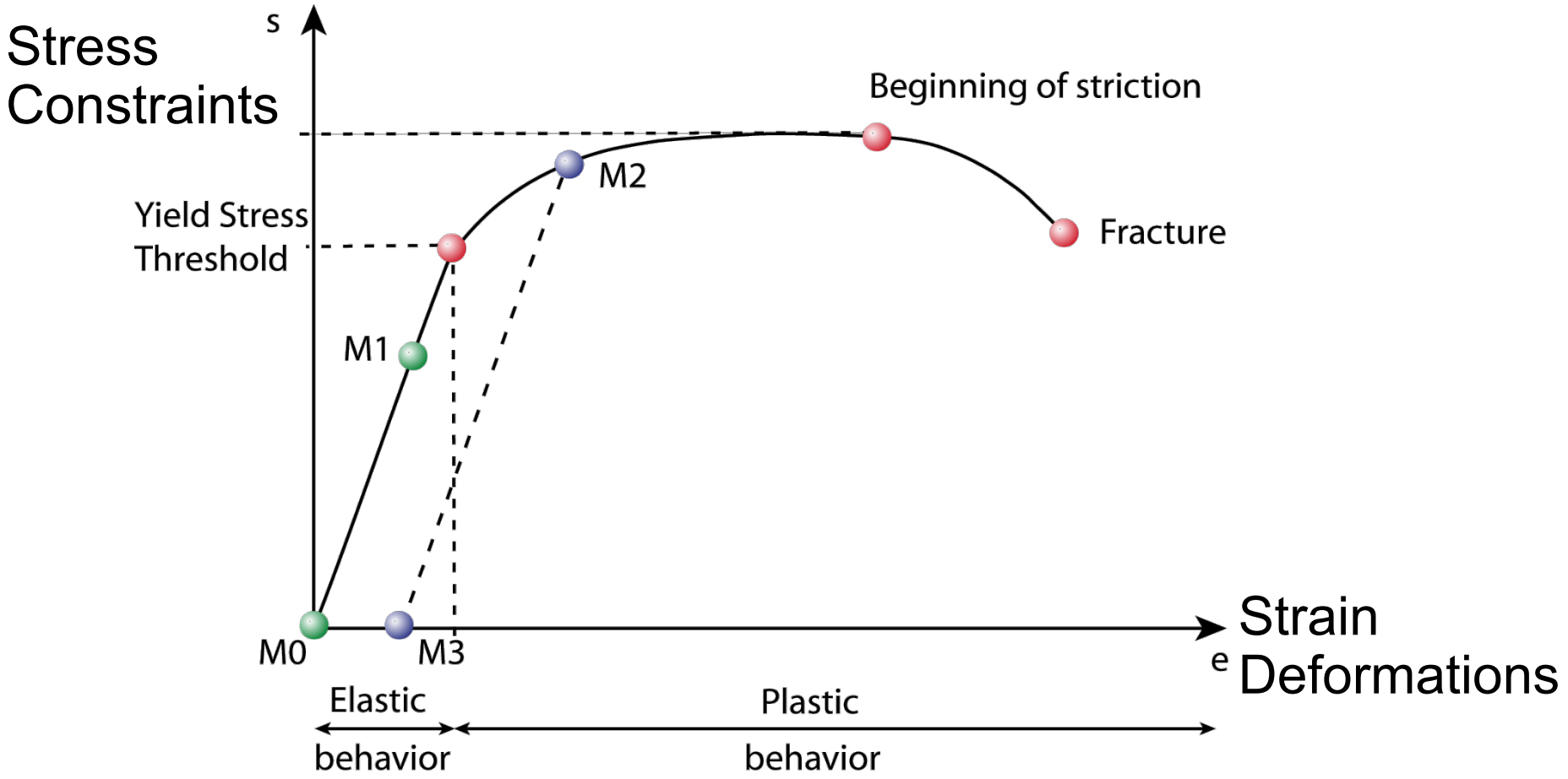
Continuum Mechanics

- Object first described in its reference position (rest state)
- Motion described as a continuous series of deformations between the rest state and the deformed state over time
- Two formalisms:
  - Lagrangian (solids)
  - Eulerian (fluids)

Lagrangian Description

- Object is composed of “particles” or “material points” i
- At each time t, the position of i is described a $P_i = P0_i + U_i$
- $P0_i$ = rest position of i
- U_i = displacement vector of i
- Function valid for any point in the solid
  - Best adapted if structure does NOT change over time (solid not fluid)
- Continuous description

Solid Deformation: Strain-Stress Curve



Deformation

Types

- Elasticity
- Viscosity
- Plasticity
- Fracture

Models

Continuous / Discrete

- Continuous model is not adapted to Computer Graphics (analytical solution difficult to compute -- if possible)
- Discretization in space
  - Mass-Springs, Finite elements
- Discretization in time
  - Time step and numerical integration

Particle

- Single point with a mass associated, usually represented as a sphere

- Position x in m (meters)
- Velocity v in $m.s^{-1}$
- Acceleration a in $m.s^{-2}$
- Mass m in Kg

Notations

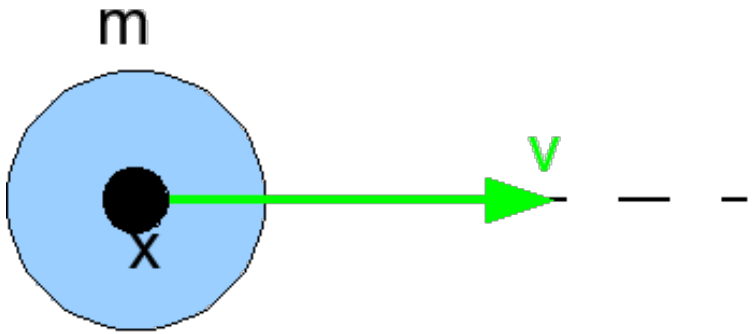
- Let $x(t)$ be the trajectory of a particle over time
- $\dot{x}(t) = \frac{dx}{dt}$ its velocity (and tangent to the trajectory)
- $\ddot{x}(t) = \frac{d^2x}{dt^2}$ its acceleration
- For simplification, we will also use:
- p(t), the position of a particle at time t
- v(t) its velocity at time t
- a(t) its acceleration at time t

Newton's Laws

First Law of Motion

Law of inertia

An isolated system (no forces applied) has a constant velocity

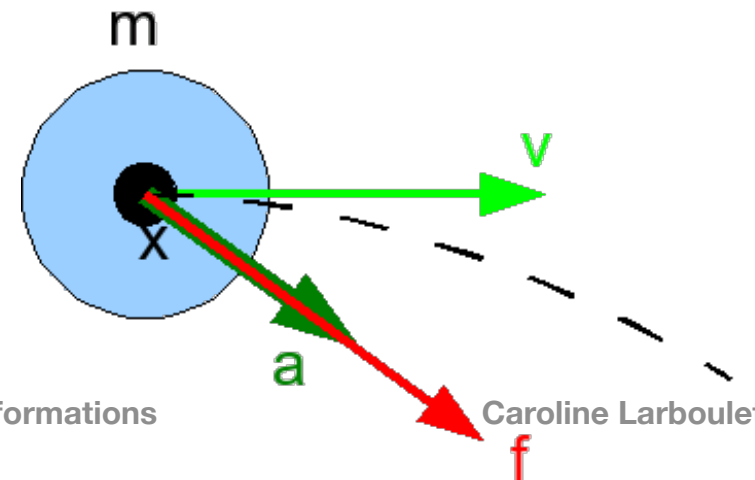


Newton's Laws

Second Law of Motion

$$\vec{F} = m \cdot \vec{a}$$

The acceleration of a particle only depends on the Forces (F is net force in Kg.m/s^2) applied to it and its mass m



Newton's Laws

Third Law of Motion

Interaction Principle

In every interaction, there is a pair of forces, with an equal and opposite direction, on both objects (i.e. the net force applied to an isolated system is null)

$$f_{1\to 2} = -f_{2\to 1}$$

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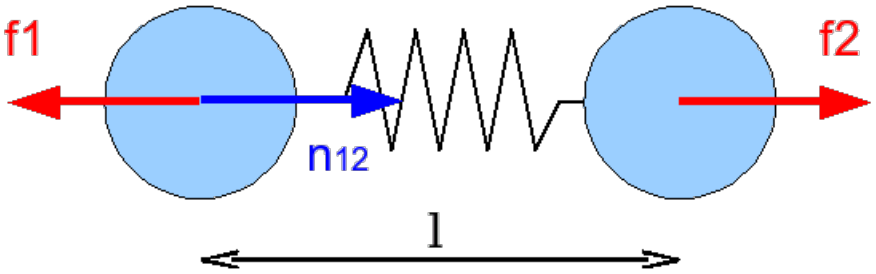
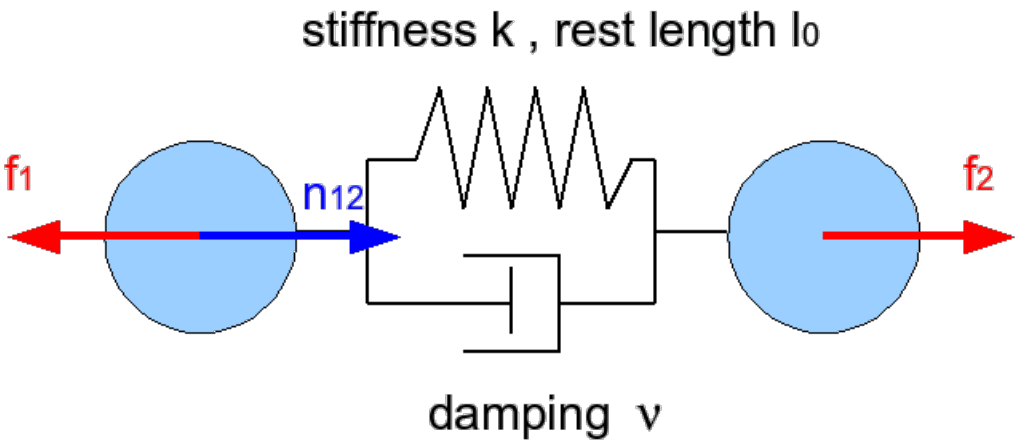
Mass-Spring Systems

Linear Spring (1D)

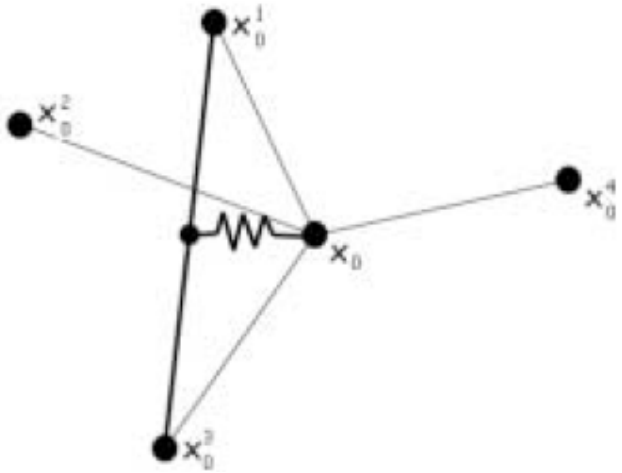
- F = - k u
- k = stiffness
- u = elongation



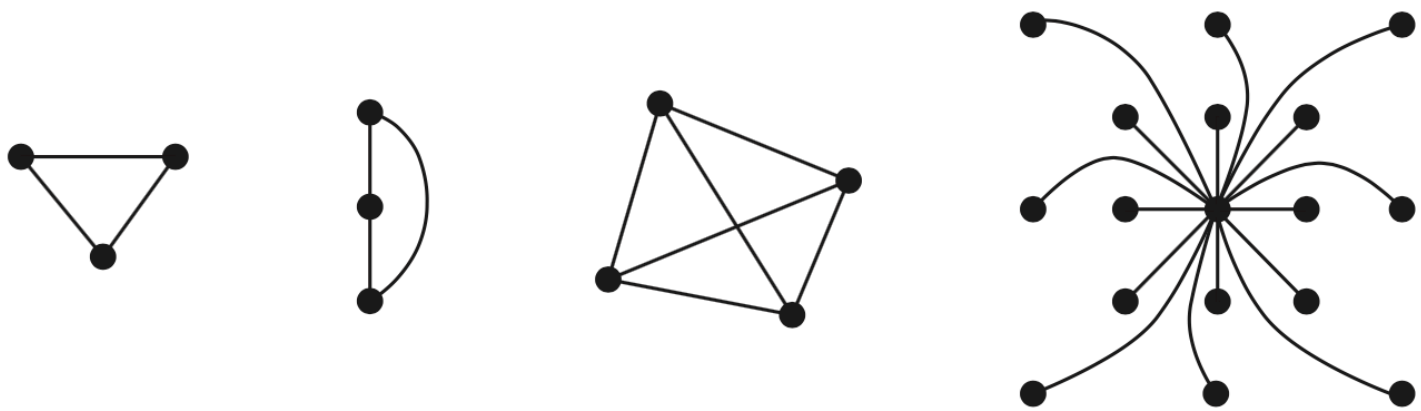
- If damping: acts on the velocity and in opposite direction
- F = - k u + d v
- !! To the vector dir in 3D !!

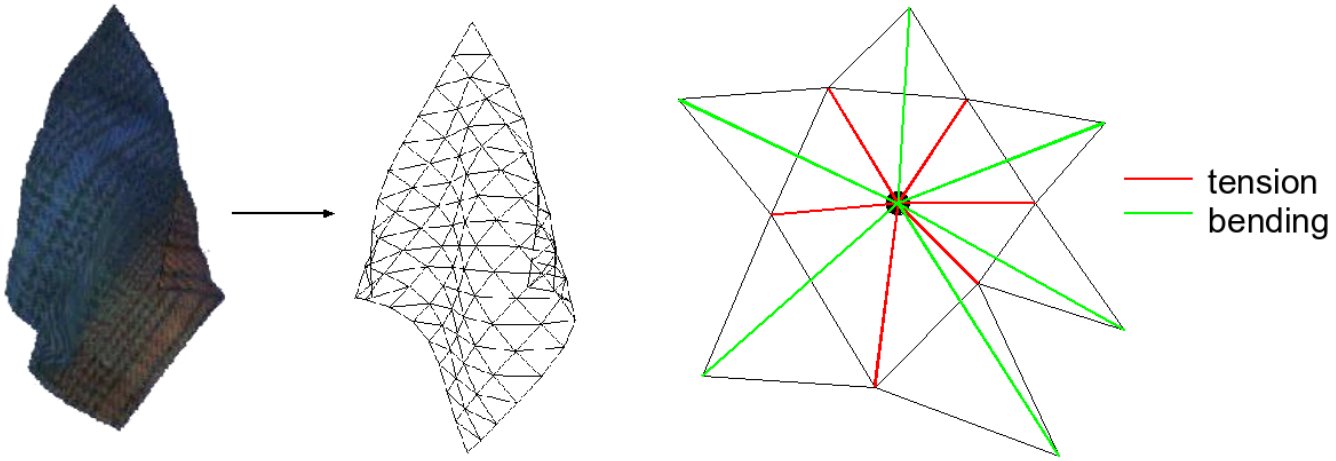


Angular Spring



Mass-Spring Systems





Continuous object Discrete model Particle neighborhood

Mass-Spring - Overview

- Animate() //runs at least at dt

{

  for each particle

  {

    - //Compute the forces (gravity, springs, other)
    - //Compute the acceleration (F=ma)
    - //Integrate velocities and positions (Euler, RK2 ...): a -> v -> p

  }

}

Forces

- Gravity
  - P = m G with m in Kg and G = 9.81 N.Kg-1
- Elastic

$$Fs = -(k(||P_2 - P_1|| - l_0)).n_{12}$$

$$Fd = (d(v_2 - v_1).n_{12}).n_{12}$$

$$F = (-k(||P_2 - P_1|| - l_0) + (d(v_2 - v_1).n_{12})).n_{12}$$

$$n_{12} = \frac{P_2 - P_1}{||P_2 - P_1||}$$

Numerical Integration

- Euler explicit

$$v_{t+dt} = v_t + dt \cdot a_t$$

$$p_{t+dt} = p_t + dt \cdot v_t$$

- Very unstable if more than one spring
  - Use the new Vt+1 for more stability

$$p_{t+dt} = p_t + dt \cdot v_{t+dt}$$

- Simple and fast but unstable
  - Use an implicit or hybrid scheme instead

Numerical Integration

- Euler Explicit
- Runge Kutta (RK2)
- Runge Kutta (RK4)

Example:

$$x(t) = 100\ e^{(0.1t)}$$
$$v(t) = dx/dt\ x(t) = 10\ e^{(0.1t)}$$
$$a(t) = dx/dt\ v(t) = e^{(0.1t)}$$

Euler Explicit = RK1

```
Runge Kutta 1
y'=f(y,t) and y(t0) = y0
h is the time-step
yn+1 = yn + h f(t, yn)

This corresponds to Euler:
p(t+dt) = p(t) + dt*v(t)
v(t+dt) = v(t) + dt*a(t)
```

- Example on board

Runge Kutta 2 (RK2)
Mid-point

- Example on board …

```
Runge Kutta 2

k1 = f(tn, yn)
k2 = f(tn+h/2, yn + h/2*k1)

yn+1 = yn + h*k2

At k1:
p_k1(t) = p(t)
v_k1(t) = v(t)  //slope k1
a_k1(t) = a(t)  //Computed with current p and v

At k2:
p_k2(t+dt/2) = p(t) + v_k1*dt/2
v_k2(t+dt/2) = v(t) + a_k1*dt/2    //slope k2
a_k2(t+td/2) //Computed with current p_k2 and v_k2
```


Runge Kutta 4 (RK4)

- **Example continued …**

```
k1 = f(tn, yn)
k2 = f(t + h/2, yn + h*k1/2)
k3 = f(t + h/2, yn + h*k2/2)
k4 = f(t + h, yn + h*k3)
yn+1 = yn + (k1 + 2 k2 + 2 k3 + k4)*h/6
p(t+dt) = p(t) + dt/6 ( k1 + 2 k2 + 2 k3 + k4 )
```

k1 is the slope at the beginning of the interval;
k2 is the slope at the midpoint of the interval, using slope k1 to determine the value of y at the point tn + h/2 using Euler's method;
k3 is again the slope at the midpoint, but now using the slope k2 to determine the y-value;
k4 is the slope at the end of the interval, with its y-value determined using k3. The easiest way to go it to store various positions, velocities and derivatives for each runge kutta step (1 to 4).

Implicit Schemes

- Computed using “future” values of variables
- Needs to solve a system of equations
- Euler implicit

$$\begin{aligned} \dot{x}(t+h) &= \dot{x}(t) + h\ddot{x}(t+h) = \dot{x}(t) + hf(t+h)/m \\ x(t+h) &= x(t) + h\dot{x}(t+h) \end{aligned}$$

Use of mass-springs

- Mushroom example
- Burning Paper
- Chadwick: FFD coupled with mass-springs

Mass-Spring under Maya

Overview

- Continuum Mechanics
- Mass-Spring Systems
- **Finite Elements**
- Continuous Forces
- Particle Systems
- Introduction to Collision Detection and Response

Finite Elements

Finite Elements

- More continuous model than mass-springs
- Discretization in elements, but continuous functions inside the elements to compute
  - Its mass M
  - Its stiffness K
  - Its damping D (if applicable)

FEM

- The basis functions have to be chosen carefully
  - Tradeoff between correct approximation of the material behavior
  - And the ease of computation (the displacements of the nodes need to be expressed as a linear combination of the basis functions)

Basis functions

- To have simple equations, it is common to use functions that are linear by piece
  - Value 1 at node i
  - Value 0 at all other nodes
- There are other possibilities (Hermite …)

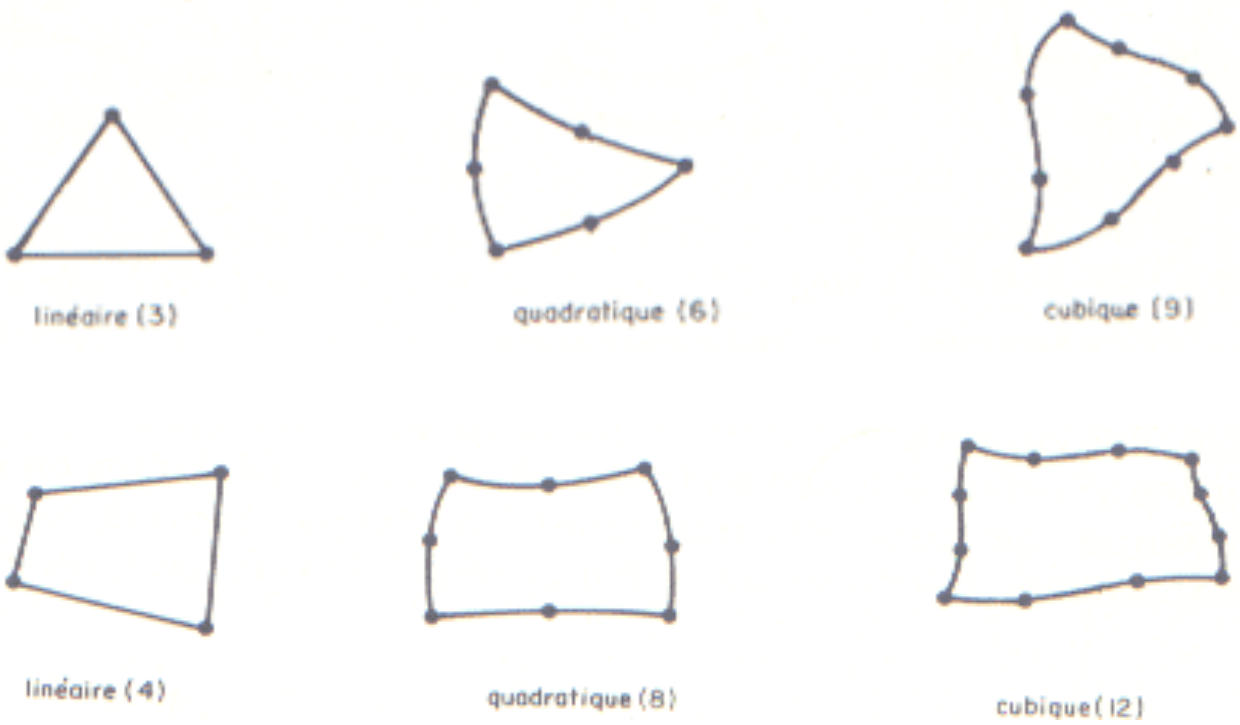
Basis Functions ...

- Over time, we need to compute the displacement u_i of the n nodes a_i
- Projected on the basis of functions, we can compute the position P of any point inside the element:

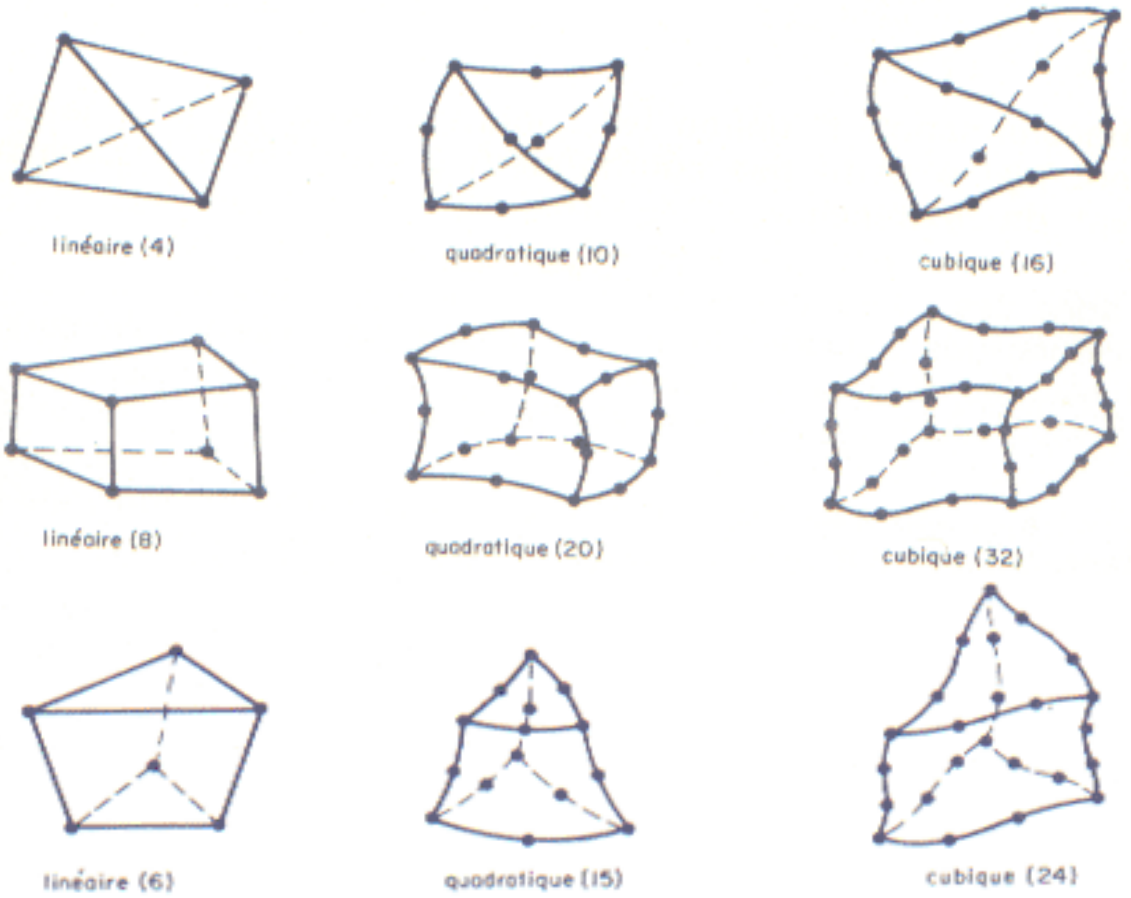
$$P = \sum_i a_i \Phi_i$$

Φ_i are the basis functions

Finite Elements 2D



Finite Elements 3D



Equilibrium Equation

- By trying to minimize the error wrt the behavior of the material (strain-stress curve), we obtain

$$F = K\,u$$

K is the stiffness matrix [n*n] that depends on the material expressed (Young, Poisson) in the basis

F is the vector of forces [n*1] expressed in the basis

u is the displacement vector [n*1] of all u_i's

Example of K matrix

Elongation $= \Delta \overrightarrow{u} = \overrightarrow{u_2} - \overrightarrow{u_1}$

$\overrightarrow{F_1} = k\overrightarrow{u_1} - k\overrightarrow{u_2}$

$\overrightarrow{F_2} = -k\overrightarrow{u_1} + k\overrightarrow{u_2}$

F_1, F_2, u_1, u_2 are respectively the forces and the displacements for each node.

Axial forces applied to the spring

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = K \,.\, u$$

To solve the equation

- K is the stiffness matrix

$$F = K \,.\, u$$

$$u = K^{-1} \,.\, F$$

- Computing K^{-1} is the costly step. It is common to pre-invert this matrix.

Same with Damping

- If damping is used, it modifies the velocities. It affects the strain-stress curve (acceleration != 0)
- The equilibrium equation becomes

$$F = Ma + Dv + Ku$$

M: mass matrix

D: damping matrix

a and v: acceleration and velocity

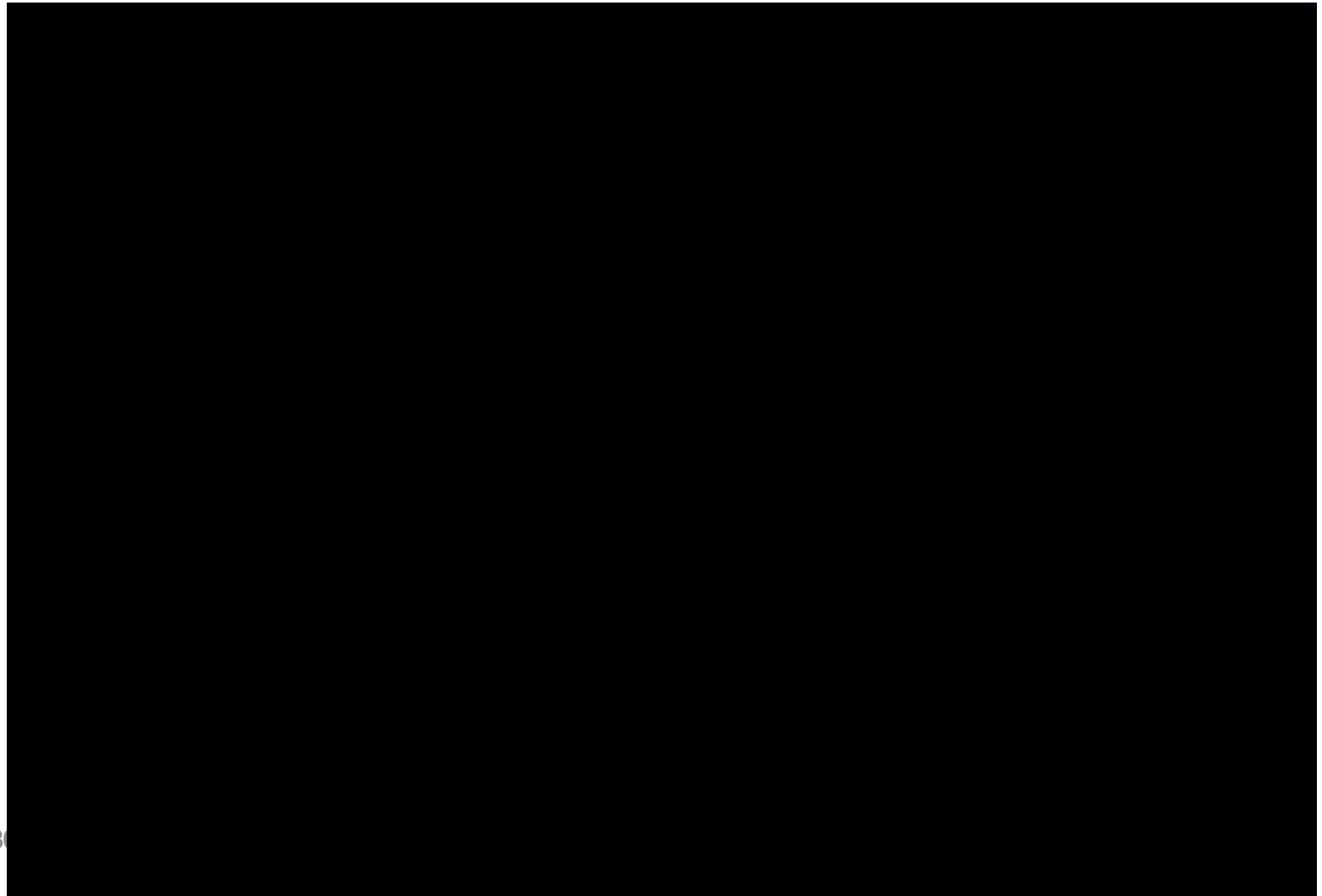
=> Necessary for simulating dynamics

FEM

Capell

FEM + Modal Analysis

DyRT



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Continuous Forces

Singularities

- Continuous forces to create force fields
  - Source
  - Sink
  - Vortex
  - Directional

Singularities

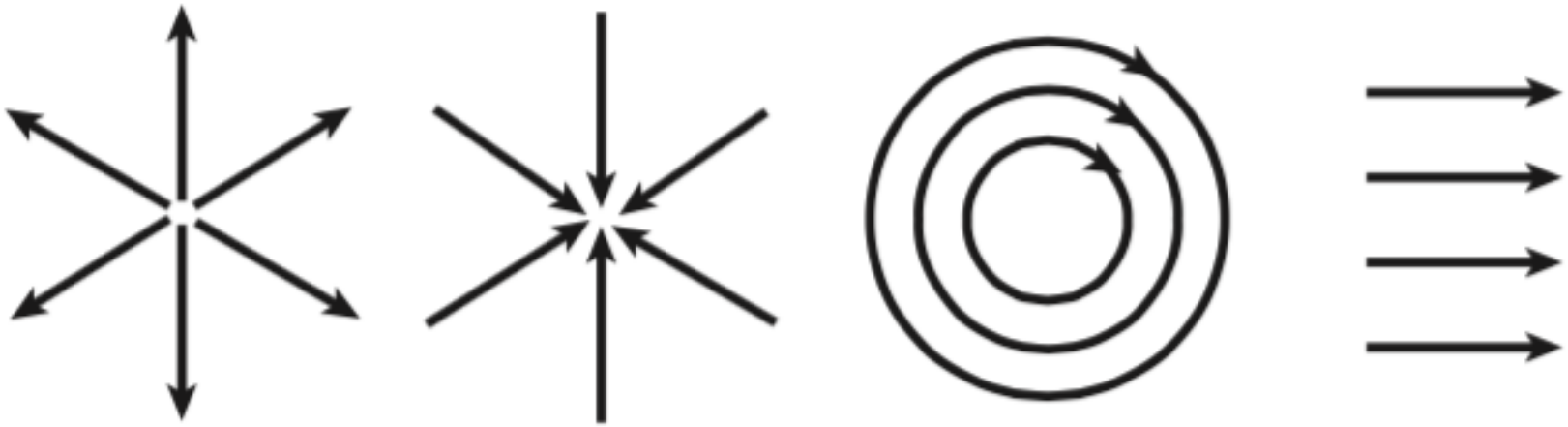
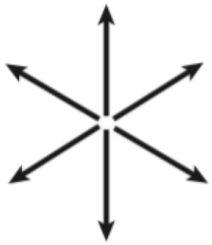


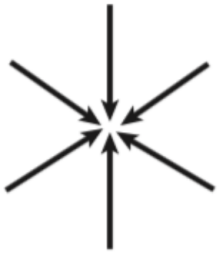
Figure 3: The four types of singularities used to model the fluid's velocity field. From left to right: Source, Hole, Whirlwind and Directional Field.

Source



- $$\mathbf{S}(\mathbf{p}) = \frac{\Phi}{\frac{\Phi}{\Phi_{max}} + d(\mathbf{p},\mathbf{C})^2} \frac{\vec{\mathbf{C}}\mathbf{p}}{||\vec{\mathbf{C}}\mathbf{p}||}$$

Sink

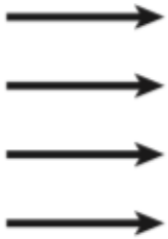


$$\mathbf{H}(\mathbf{p}) = -\mathbf{S}(\mathbf{p}) = -\frac{\Phi}{\frac{\Phi}{\Phi_{max}} + d(\mathbf{p},\mathbf{C})^2} \frac{\vec{\mathbf{Cp}}}{||\vec{\mathbf{Cp}}||}$$

Vortex

$$\mathbf{W}(\mathbf{p}) = \pm \frac{\Phi}{\frac{\Phi}{\Phi_{max}} + d(\mathbf{p},\mathbf{C})^2} \frac{\vec{\mathbf{C}}\mathbf{p} \times \vec{R}}{||\vec{\mathbf{C}}\mathbf{p} \times \vec{R}||}$$

Directional



- Force move things in a given direction
- Can be a sin or cos for wave effects

$$\mathbf{D}(\mathbf{p}) = \Phi(\mathbf{p}, t).\vec{v};$$

Example

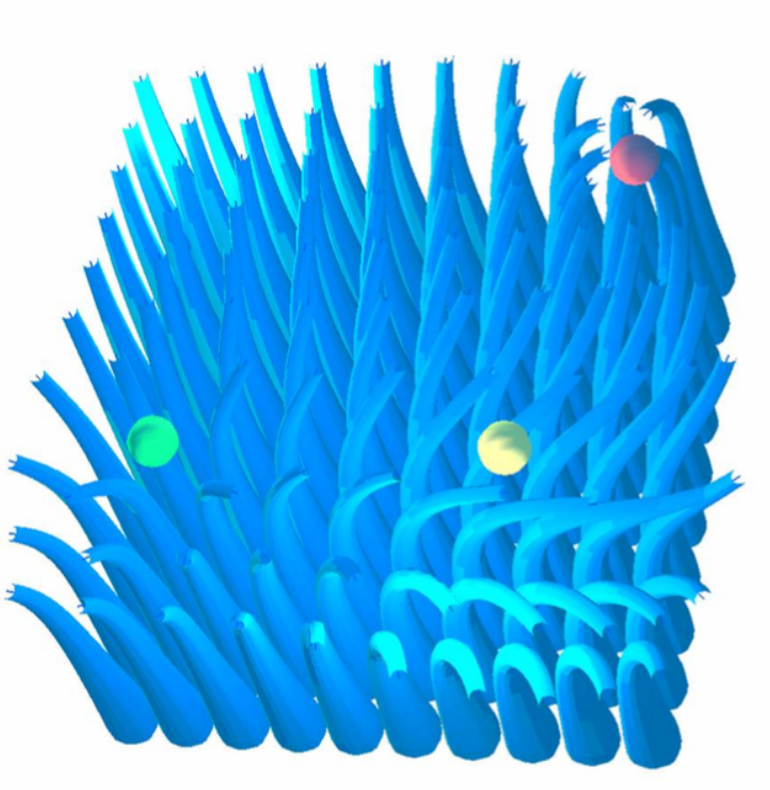


Figure 12: Influence of a Source (green), a Hole (red) and a Whirlwind (yellow) on a group of identical fibers.

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Particle System

- Emitter emits particles
  - Spawning rate (how many particles are generated per time unit)

$$n = (t - t_{last}) / \text{ rate}$$

  - Spawning position (emitter pos)
  - Particles initial velocity
    - Magnitude and direction
  - Particles lifetime

Particle System

- Particles live their lives
- Before being animated, check if alive or dead
- If alive, update velocity / position
  - Can be very simple: vel -= 0.2
  - Or sophisticated: forces, a, v, p
- Check for collisions with other objects

Particle System

- Rendering of particles
  - As points (color / size according to age)
  - As billboards (quad always facing the user)
  - Metaballs (offline rendering) for liquids
  - 3D mesh (slower)

Particle System

- In practice: a lot of particles 10^5
- Use a table to store the particles
- Max number of particles
  - Fixed size table
  - Not need to create/destroy particles, just replace them

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- **Introduction to Collision Detection and Response**

Introduction to Collision Detection and Response

Collision Detection

Motivation

Teschner, Kimmerle, Heidelberger, Zachmann, Raghupathi, et al., Collision Detection for Deformable Objects. Computer Graphics Forum, 2005.

- Why detect collisions and respond ?
  - To change object trajectories
  - To change object shapes
  - To prevent interpenetration of mesh (can lead to unstable simulation)
- Two types of collisions
  - Object vs object
  - Self-collision

Detection

- Collisions can be detected on any type of mesh, for any type of animation
  - What is tested is whether there are interpenetrations at a given time t
- However, depending on the type of surface/volume, it can be
  - Easy (fast) or hard (slow) to compute
  - Approximated or exact
  - Different types of responses

Polygonal Models

- Not feasible for each polygon versus each polygon (too many ray intersections to compute !)
  - Possibility of using Octrees, KD-trees
  - Approximation by bounding boxes or implicit surfaces (spheres) : BVH
  - Make use of a stochastic approach

Bounding Volume Hierarchies (BVH)

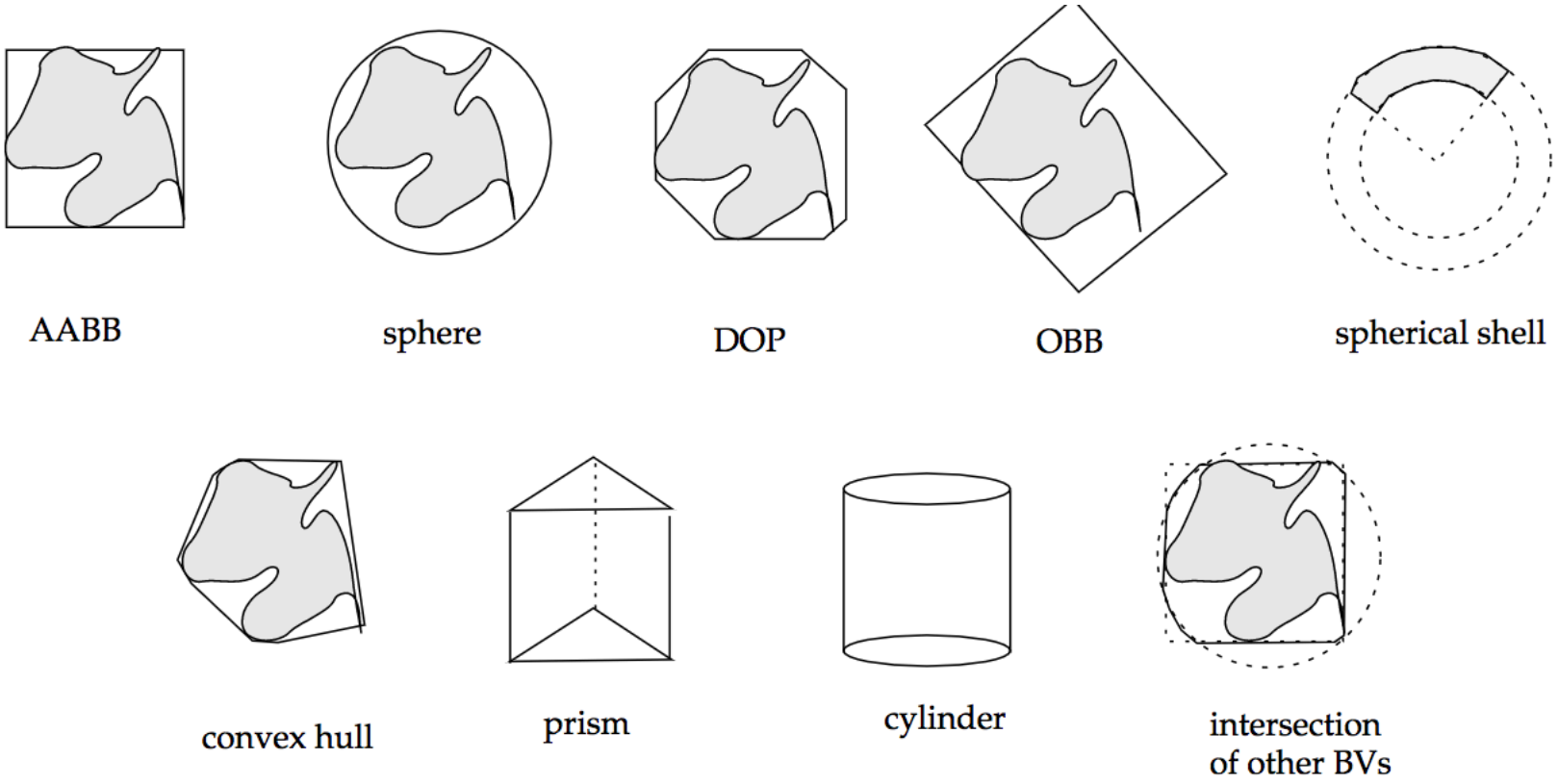
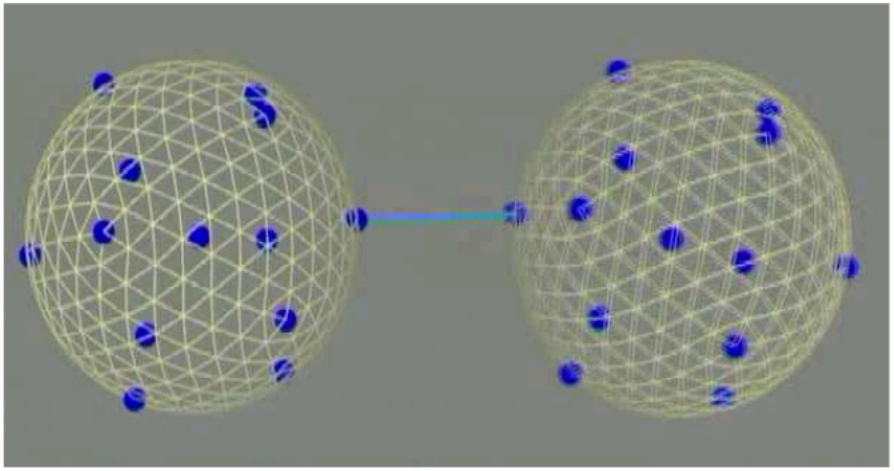
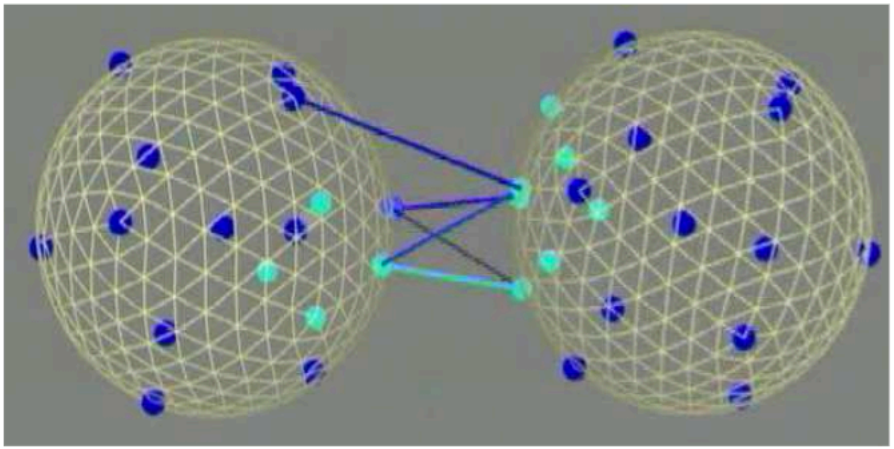


Figure 4: *A variety of bounding volumes has been proposed for hierarchy-based collision detection.*

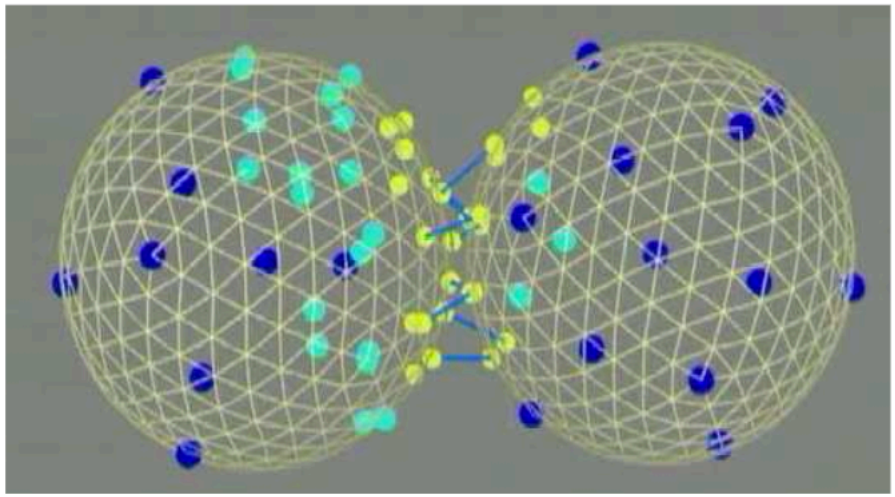
Stochastic Approach



(a)



(b)



(c)

Figure 10: *Illustration of stochastic collision detection between two volumetric bodies: Pairs of features (depicted by lines) across different resolutions (different colored points) being tracked as the objects approach each other.*

Subdivision Surfaces

- By nature, the collision detection and response can be done on the coarse polygonal mesh
  - Good approximation
  - Use same techniques as for polygons

Parametric Surfaces

- Curves are polynomial functions of degree 3:
  - Intersection not easy to compute

  Forget it !
- However, for normal, regular and positive splines (ex. B-spline), you can use the convex hull: probably the best approximation you can get !

Implicit Surfaces

- The type of surface that is maybe most appropriate to collision detection
  - Penetration can be computed exactly
  - Zone of contact can also be computed exactly
- However, not practical to render
  - Use implicit surfaces to approximate your object

Bounding Sphere

- Orientation independent
- Center = mesh center
- Radius = furthest vertex
- Effective for convex, oval bodies
- Bounding ellipsoid for more elongated bodies

Collision Response

Collision Response

- Apply repulsion forces
  - Penalty Method for physically based simulations: put a spring between the two colliding solids with a rest length of 0 (or epsilon) and high stiffness

Using Penalty Method

Selle et al., A mass spring model for hair simulation, SIGGRAPH 2008

Using Penalty Method

Selle et al., A mass spring model for hair simulation, SIGGRAPH 2008

Using Penalty Method

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Using Penalty Method

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Collision Response

- Modify speed for solids or particles
  - Elastic Collision
  - Soft Collision
  - Collision of solid/particle with a plane

Elastic Collision

- 2 particles with m_1 and m_2 collide
- v_1 and v_2 before collision
- v_1' and v_2' after collision
- Quantity of Motion is preserved:

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$

- Kinetic Energy is preserved:

$$m_1v_1^2 + m_2v_2^2 = m_1v_1'^2 + m_2v_2'^2$$

Elastic Collision

$$m_1(v_1-v_1') = m_2(v_2'-v_2)$$

$$m_1(v_1^2-v_1'^2) = m_2(v_2'^2-v_2^2)$$

$$v_1+v_1' = v_2+v_2'$$

$$u' = -u$$

$u = v_2 - v_1$ (relative speed before collision)

$u' = v_2' - v_1'$ (relative speed after collision)

Elastic Collision

$$v'_1 = \frac{2m_2v_2 + v_1(m_1 - m_2)}{m_1 + m_2} \qquad v'_2 = \frac{2m_1v_1 + v_2(m_2 - m_1)}{m_1 + m_2}$$

- If $m_1 = m_2$, the solids exchange their velocities
- $v_1' = v_2$ and $v_2' = v_1$

Soft Collision

- If colliding solids interpenetrate, they glue together
- $v_2' = v_1'$
- There is a loss of kinetic energy
- Coefficient of restitution e (empirical)

$$e = \frac{v_1' + v_2'}{v_1 + v_2}$$

Soft Collision
with the Floor

- Let a ball fall from height h with a restitution coefficient e
- Relative speed u = speed v of the ball (the ground does not move)
- Let h' be the new height

$$\frac{1}{2}mu^2 = mgh$$

$$h' = e^2 h$$

$$\frac{1}{2}mu'^2 = mgh'$$

Collision with a Plane

- Collision detection
- P: point on the plane
- Q: particle
- n: normal of the plane

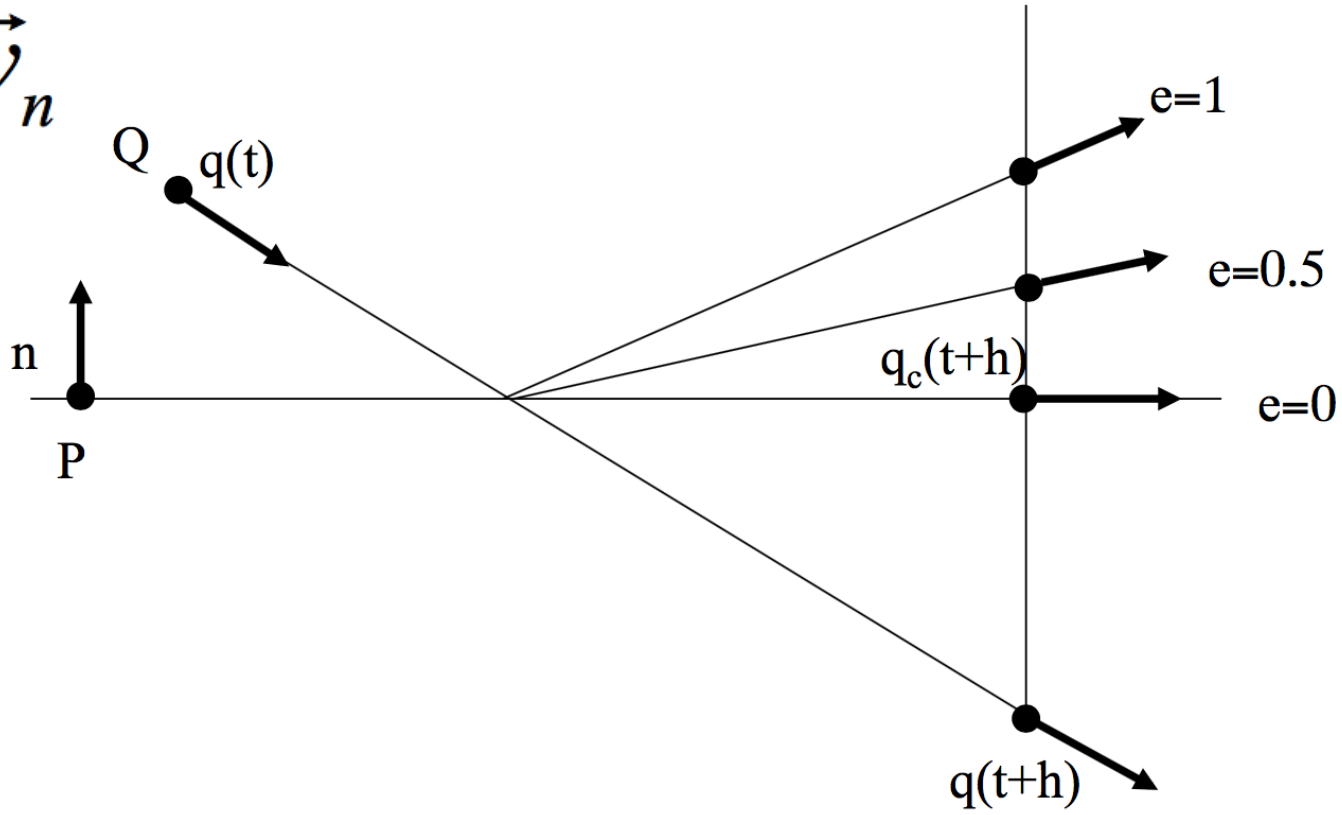
$$\vec{PQ}.\vec{n} < 0$$

- Tangential speed unchanged
- Normal speed inverted

Collision with a Plane

$$\vec{v} = \vec{v}_t + \vec{v}_n$$

$$\vec{v} \leftarrow \vec{v}_t - e\vec{v}_n$$



Collision with a Plane

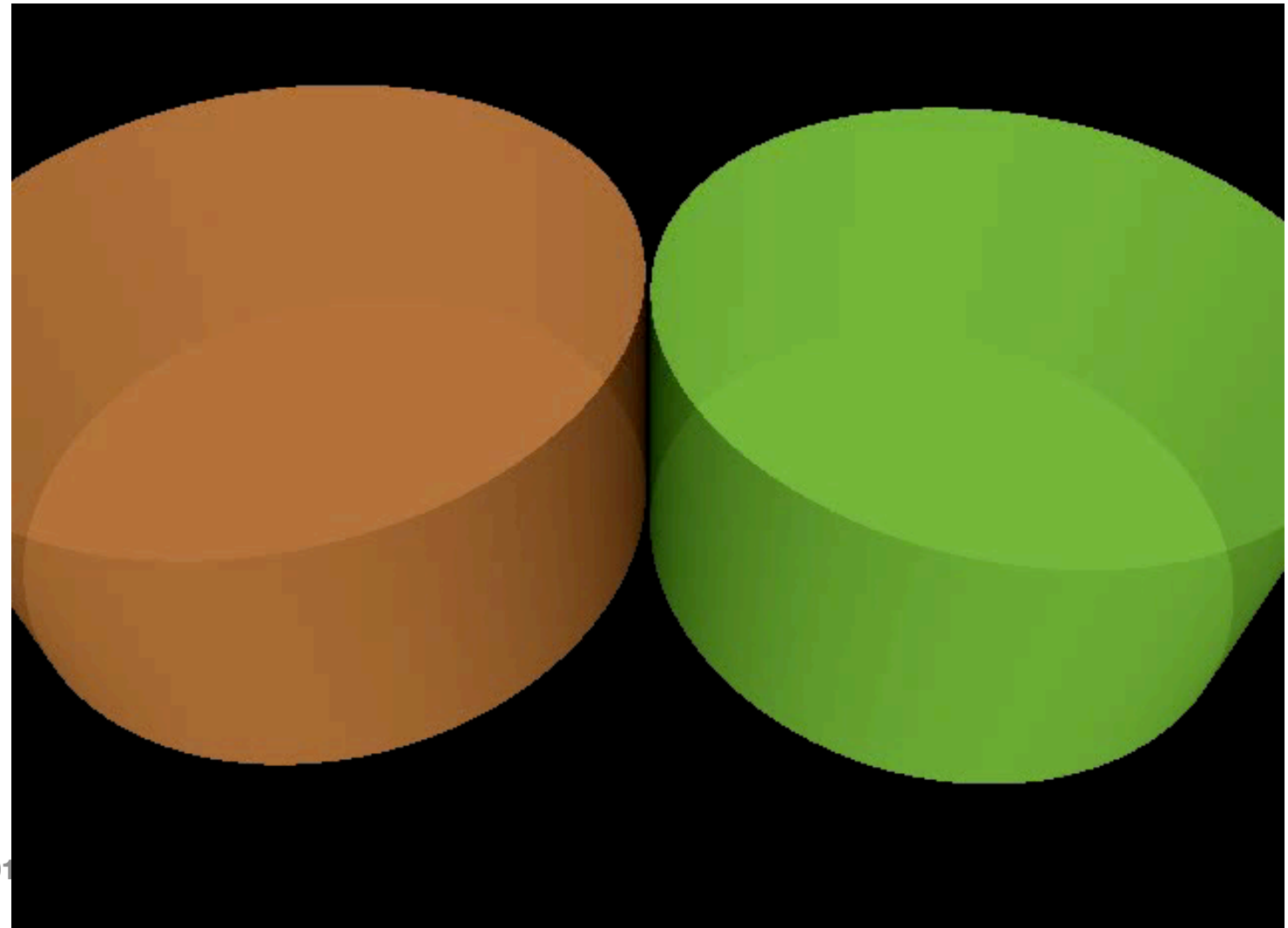
- e = 0: collision absorbed
- e = 1: perfectly elastic collision

Collision Response

- Apply some displacement along the colliding normal
  - For geometrically based techniques
  - Example with distance fields

2D Distance Field

Hoff III et al. Fast and Simple 2D Geometric Proximity Queries Using Graphics Hardware, 2001



2D Distance Field

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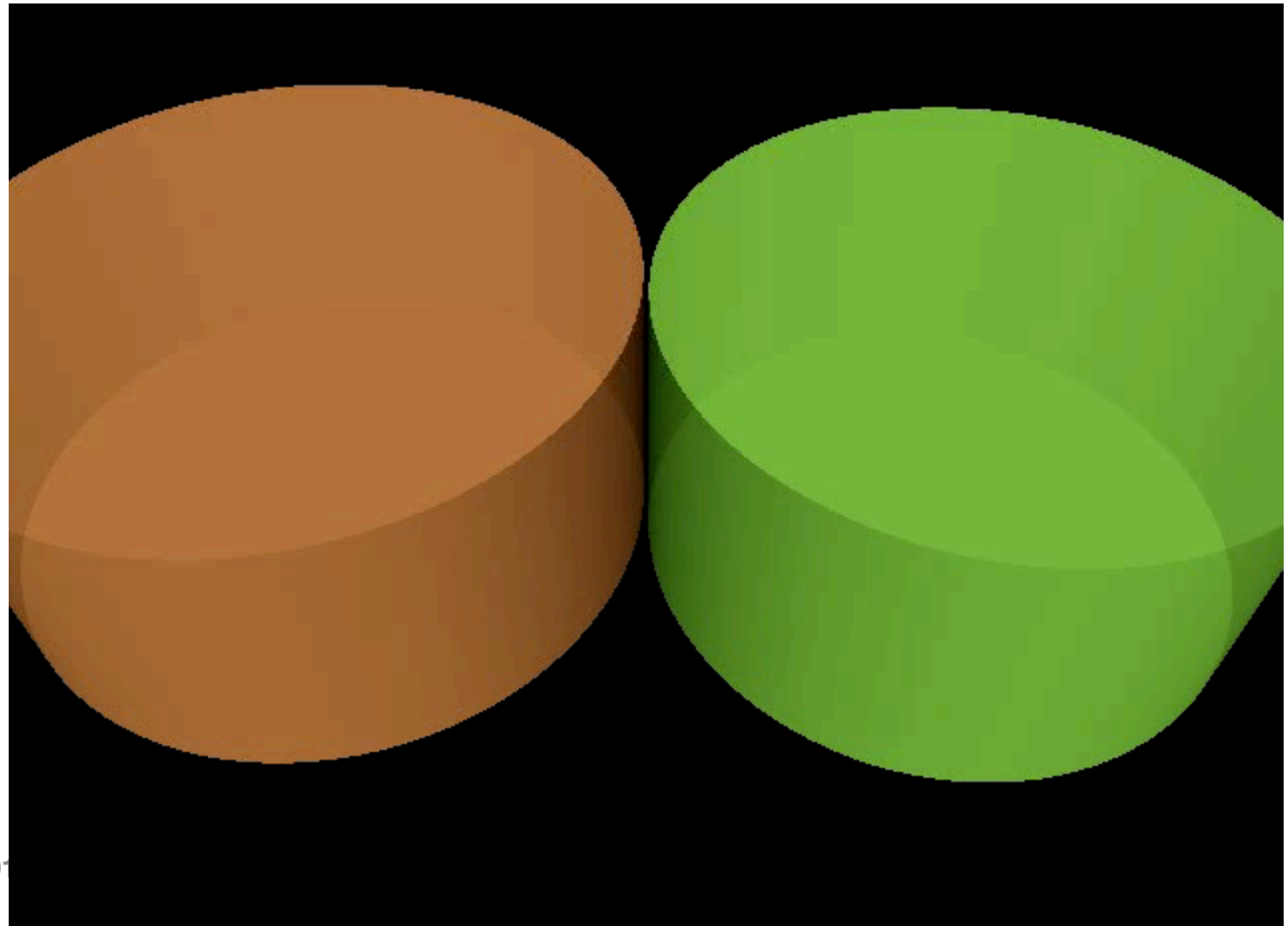

2D Distance Field

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