Assignment 1

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1 Problem 1

Given that

$$\langle x'|X|x\rangle = x\delta\left(x'-x\right) = x'\delta\left(x'-x\right)$$

and

$$[X,P] = \imath\hbar$$

.

$$\langle x'|\left[X,P\right]|x\rangle = \imath\hbar\delta\left(x'-x\right)$$

$$\langle x'|\left[X,P\right]|x\rangle = \langle x'|XP|x\rangle - \langle x'|PX|x\rangle = x'\langle x'|P|x\rangle - x\langle x'|P|x\rangle$$

Therefore,

$$\langle x'|P|x\rangle = \imath\hbar\frac{\delta\left(x'-x\right)}{x'-x} = -\imath\hbar\frac{d}{dx}\left(\delta\left(x-x'\right)\right)$$

2 Problem 2

In the ket representation,

$$\partial_t|\psi(t)\rangle = -\frac{\imath H(t)}{\hbar}|\psi(t)\rangle$$

The wave function $\psi(x,t)$ is defined as $\langle x|\psi(t)\rangle$. Using this and the identity $\int dx'|x'\rangle\langle x'| = 1$,

$$\partial_t\psi(x,t) = -\frac{i}{\hbar}\int dx'\psi(x',t)\langle x|H(t)|x'\rangle$$

Now, let's assume that $H(t)$ is given by $H(t) = \frac{P^2}{2m} + V$.

$$\langle x|H(t)|x'\rangle = \langle x|\frac{P^2}{2m} + V|x'\rangle = \langle x|\frac{P^2}{2m}|x'\rangle + \langle x|V|x'\rangle$$

To calculate the first term on RHS:
Using the identity $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\frac{\imath}{\hbar}px}$ and the operator identity $\int dp'|p'\rangle\langle p'| = 1$, we get

$$\langle x|\frac{P^2}{2m}|x'\rangle = \int dp\frac{p^2}{4m\pi\hbar}e^{\frac{\imath}{\hbar}px}e^{-\frac{\imath}{\hbar}px'}$$

We know that

$$\int dp e^{\frac{\imath}{\hbar}p(x-x')} = 2\pi\hbar\delta(x-x')$$

Using this, therefore

$$\int dx'\psi(x',t)\langle x|\frac{P^2}{2m}|x'\rangle = \frac{\partial^2}{\partial x^2}\int dx'(\frac{\hbar}{\imath})^2\frac{1}{2m}\delta(x-x')\psi(x',t)$$

Therefore, we finally get:

$$-\frac{\hbar}{\imath}\frac{\partial}{\partial t}\psi(x,t) = (-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x))\psi(x,t)$$

(Schrodinger's Equation)

3 Problem 3

For a free non-relativistic particle,

$$H(t) = \frac{P^2}{2m}$$

Using the operator identity in the momentum space, we have

$$\langle x|e^{-\frac{\imath}{\hbar}Ht}|x'\rangle = \int dp\langle x|p\rangle\langle p|e^{-\frac{\imath}{2\hbar m}P^2 t}|x'\rangle$$

Simplifying,

$$\langle x|e^{-\frac{\imath}{\hbar}Ht}|x'\rangle = \int dp\frac{1}{2\pi\hbar}e^{\frac{i}{\hbar}p\left(x-x'\right)-\frac{\imath}{2\hbar m}p^2 t}$$

Resolving the integral, (by completing the square in the exponent):

$$\langle x'|e^{-\frac{\imath}{\hbar}Ht}|x\rangle = \sqrt{-\frac{\imath m}{2t\pi\hbar}}e^{\frac{im(x-x')^2}{2\hbar t}}$$

3.1 Working

$$\frac{ipx}{h} - \frac{ip^2 t}{2hm} = \frac{imx^2}{2ht} - \frac{it\left(p-\frac{mx}{t}\right)^2}{2hm}$$

This implies,

$$e^{\frac{ipx}{h}-\frac{ip^2t}{2hm}} = e^{\frac{imx^2}{2ht}}e^{-\frac{it\left(p-\frac{mx}{t}\right)^2}{2hm}}$$

The following is a standard integral,

$$\int_{-\infty}^{\infty} dx e^{-ix^2} = \sqrt{-\imath\pi}$$

After, performing the appropriate scale transformations, the stated result is obtained.