Week 0 Solutions

Keane Moraes | Quantum Cryptography | Spring 2021

Julia Lab Exercises

Exercise 1

We're now ready for the first edX Exercise - the result of which you need to enter into the edX grading forms. Consider $|b_0\rangle = \sqrt{\frac{1}{3}}\,|0\rangle + \sqrt{\frac{2}{3}}\,|1\rangle$. What real-valued vectors are orthogonal to this vector? In other words given $|b_0\rangle$ which $|b_1\rangle$ satisfy $\langle b_1|b_0\rangle = 0$?.

Your task is to find parameters $\alpha_1 \geq 0$ and β_1 such that $|b_1\rangle = \begin{pmatrix}\alpha_1\\ \beta_1\end{pmatrix}$ is orthogonal to $|b_0\rangle$.

Take a guess, use Julia to find the vector for you, or compute it by hand! By clicking in the code box just below you can fill in values for α_1 and β_1 and calculate $\langle b_1|b_0\rangle$ to see how you are doing.

To start you off we have set α_1 and β_1 to $\sqrt{\frac{1}{2}}$ and $\sqrt{\frac{1}{2}}$ respectively.

Homework 0

1. (a) In this question we will investigate the simplest of quantum communication tasks: sending a classical bit using a qubit. So let's start by imagining our two favourite protagonists: Alice and Bob. Alice, who we imagine to be a PhD student at CalTech, wants to send some information to Bob, who is a post-doc at TU Delft. However Bob only accepts messages coming through their shared quantum communicator, which can prepare, send, receive and measure qubits. Now imagine Alice wants to send a very simple message, namely a bit (either a 1 or a 0).

   In order to do this she encodes her binary value by preparing a qubit in the standard basis, so

$$0 \rightarrow |0\rangle$$
$$1 \rightarrow |1\rangle$$

   Let's imagine she wants to send a 0, so she prepares the state $|0\rangle$ and sends it to Bob. Now let's imagine that Bob knows Alice sent a qubit encoded in the standard basis (They agreed on this while they were at a conference the month before) and hence when he receives Alice's qubit he measures it in the standard basis. This means he will get either a 1 with probability p_1, or a 0 with probability p_0 as a classical outcome from the machine. What would now be the correct probabilities assigned to these measurement outcomes?

   Solutions

   The probabilities for measurement of the two states are $p_0 = 1$ and $p_1 = 0$.

   (b) Now imagine Alice sends the $|1\rangle$ state to Bob. What would now be the probability distribution of Bob's measurement outcomes if he measures in the standard basis

   Solutions

   The probabilities for this measurement are $p_0 = 0$ and $p_1 = 1$

2. (a) Now lets say Alice was a bit forgetful and forgot that she was supposed to encode the qubit in the standard basis. Instead she chooses to encode her classical bit by preparing states in the Hadamard basis, so

$$|0\rangle \longrightarrow |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$|1\rangle \longrightarrow |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

This means that if Alice wants to send the bit 0 she sends the state $|+\rangle$ to Bob. Bob is unaware of Alice's mistake and measures in the standard basis. He will still get a 1 or a 0 as outcome but what will now be the probability distribution of the measurement outcomes?

Solutions

The probabilities for measurement are $p_0 = 0.5$ and $p_1 = 0.5$

(b) Now imagine Alice sends the bit 1 to Bob. This means she sends him the $|-\rangle$ state. What would now be the probability distribution of Bob's measurement outcomes? (He measures in the standard basis).

Solutions

Once again, the probabilities for measurement are $p_0 = 0.5$ and $p_1 = 0.5$

3. In the last two problems you saw Alice attempting to send a classical bit to Bob by encoding it in a quantum state. However in only one of the problems Bob could actually reliably retrieve Alice's bit from the measurement he makes. Which of these two situations was this?

Solutions

The situation in Problem 1.

4. Now imagine Bob heard through the academic rumour mill that Alice has been wrongfully preparing her states in the Hadamard basis. Luckily Bob, being a smart post-doc, has the ability to apply quantum operations to the qubit he receives before measuring it in the standard basis. What would be the correct unitary operation to apply so that Bob always gets the outcome 0 when Alice sends a $|+\rangle$ state (and that he always gets the outcome 1 when Alice sends a $|-\rangle$ state)?

Solutions

The unitary that does this is the Hadamard basis. This is given by

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

5. Lastly, imagine that Alice's qubit preparation machine is somewhat broken and, when she asks it to prepare a state in $|0\rangle$ zero it actually prepares the state.

$$|\phi\rangle = \frac{\sqrt{2}}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle$$

Now imagine Bob knows this, but his machine is also faulty and he can't correct for the error. What he can do is decide to measure in either the standard basis or in the Hadamard basis. Now imagine Alice sends Bob the above state ϕ. Which one of the bases Bob can measure in (standard or Hadamard) will give him the highest probability of getting the measurement outcome 0 (corresponding to measuring in the $|0\rangle$ state for the standard basis and to the $|+\rangle$ state for the Hadamard basis).

Solutions

Now there is one way of checking this is by finding the probability of measuring $|0\rangle$ or $|+\rangle$ in the Standard or Hadamard basis respectively.