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UNIVERSITA' DEGLI STUDI DI GENOVA

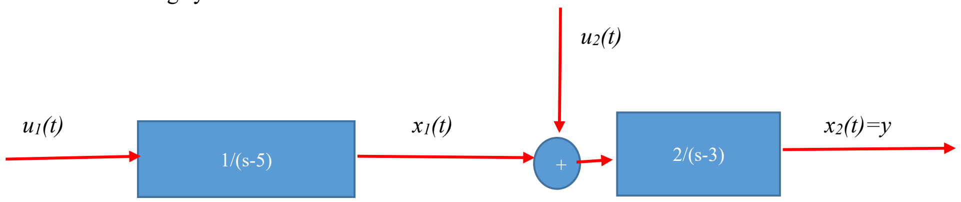
ROBOTICS ENGINEERING

Assignment for the course

"Control of linear multivariable systems"

The computer exercise.

Consider the following system



1) Write the state equations of the system
2) Determine the stationary state-feedback control law having structure

$$\underline{\boldsymbol{u}}(\boldsymbol{t}) = -\boldsymbol{L}\underline{\boldsymbol{x}}(\boldsymbol{t})$$

optimizing the infinite horizon cost function

$$\int_{t_0}^{\infty} \left[\underline{\boldsymbol{x}}^T(\boldsymbol{t})\boldsymbol{V}\underline{\boldsymbol{x}}(\boldsymbol{t}) + \underline{\boldsymbol{u}}^T(\boldsymbol{t})\boldsymbol{P}\underline{\boldsymbol{u}}(\boldsymbol{t})\right] \boldsymbol{dt}$$

where

$$\boldsymbol{Q} = \begin{bmatrix} \mathbf{10} & \mathbf{0} \\ \mathbf{0} & \mathbf{10} \end{bmatrix}$$

and

$$\boldsymbol{P} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{2} \end{bmatrix}$$

3) Check whether this control law ensures that the closed-loop system is asymptotically stable

4) Assume now that the state of the system is not fully accessible (but only the output ***y*** is). Design an asymptotic observer of the state choosing the poles of the observer so that the observer dynamics is considerably “faster” than the dynamics of the original closed-loop system (e.g., ensuring that the time constants of the observer are an order of magnitude smaller than those of that system).

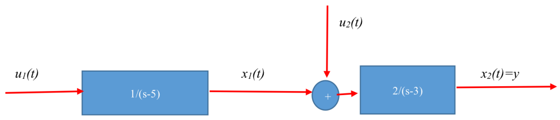
5) Build the overall model (by SIMULINK) of the overall system (the original open-loop system + the feedback controller + the state observer) and analyze the behavior of the system when the input is given by

$$\underline{\boldsymbol{u}}(\boldsymbol{t}) = -\boldsymbol{L}\underline{\boldsymbol{x}}(\boldsymbol{t}) + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \end{bmatrix} \boldsymbol{r}(\boldsymbol{t})$$

being ***r(t)*** an external sinusoidal input.

6) Determine the closed-loop transfer function $\boldsymbol{T}(\boldsymbol{s}) = \boldsymbol{Y}(\boldsymbol{s})/\boldsymbol{R}(\boldsymbol{s})$ (assuming a very fast convergence of the observed system state to the true system state) and analyze the frequency response (for different frequencies of the sinusoidal input).

1) Given the system



$T(s) = \frac{Y(s)}{X(s)} \longrightarrow \quad X_1(s) = \left(\frac{1}{s-5}\right) U_1(s) \qquad X_2(s) = \left(\frac{2}{s-3}\right)\left(X_1(s) + U_2(s)\right)$

$$\left.\begin{array}{l} sX_1(s) - 5X_1(s) = U_1(s) \longrightarrow \dot{x}_1(t) = 5x_1(t) + u_1(t) \\ sX_2(s) - 3X_2(s) = 2X_1(s) + 2U_2(s) \longrightarrow \dot{x}_2(t) = 2x_1(t) + 3x_2(t) + 2u_2(t) \end{array}\right\} \dot{\underline{x}}(t) = \begin{bmatrix} 5 & 0 \\ 2 & 3 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underline{u}(t)$$

$Y(s) = X_2(s) \longrightarrow y(t) = x_2(t) \qquad y(t) = [0 \quad 1]\, \underline{x}(t)$

2) $u(t) = -L\,x(t) = -P^{-1}B^T K\,x(t)$

$$\int_0^{\infty} \left[x^T(t) V x(t) + u^T(t) P u(t)\right] dt$$

Riccati Equation : $KA + A^T K - KBP^{-1}B^T K + V = [0]$

We consider the LTi continuous time system $\dot{\underline{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$. The control objective is the minimization of cost function $J = \frac{1}{2}\int_{t_0}^{\infty}[\underline{x}^T(t)V\underline{x}(t) + \underline{u}^T(t)P\underline{u}(t)]\,dt$. The aim is bringing the system state to zero and keeping the state as low as possible for an arbitrarily long time interval.

Since the system is time invariant and matrices V and P are costant, there is no loss of generality in assuming $t_0 = 0$.

Then, consider the Riccati differential equation $\dot{K}(t) = -K(t)A - A^T K(t) + K(t)BP^{-1}B^T K(t) - V$ under the boundary condition $K(t_f) = [0]$.

Denote $K(t; [0]; t_f)$ the solution of the differential equation with the boundary condition.

We used MATLAB to calculate the state feedback control that minimizes the function.

The lqr function was used in MAtLab and needed AS INPUT: A, B, V, P

AS OUTPUT: K (the solution of the Riccati Equation)

$L = \begin{bmatrix} 11.0707 & 0.7296 \\ 0.7296 & 4.1427 \end{bmatrix} \longleftarrow$ L (optimal gain matrix)

e (poles of the closed loop system $\longrightarrow$ eigenvalues of $P^{-1}B^T K$)

3) In this case, considering the closed loop optimally control with infinite horizon, there is Asymptotic stability if:

- the controllability Matrix $[B \mid AB]$ is FULL RANK.
- Given matrix H such that $V = H^T H$, the pair (A, H) satisfies the condition $rank(Q) = n$ where $Q = \begin{bmatrix} H \\ HA \end{bmatrix}$

Thanks to the Chol MATLAB function we found $H = \begin{bmatrix} 3.1632 & 0 \\ 0 & 3.1632 \end{bmatrix}$

Q has been found of rank 2.

e resulted $e = \begin{bmatrix} -5.6791 + i\,0.490 \\ -5.6781 - i\,0.490 \end{bmatrix}$

$\downarrow$

Since both poles have negative real parts the system is ASIMPTOTICALLY STABLE.

4) The structure of the Identity observer is $\dot{z}(t) = F z(t) + G y(t) + B u(t)$ with $F = A - GC$

In order to arbitrarily assign the eigenvalues of F i refer to the theorem 1.2 in the theory:

Theorem 1.2. By a suitable choice of matrix $\boldsymbol{G}$, one can assign *an arbitrary set of eigenvalues* to matrix $(\boldsymbol{A}-\boldsymbol{GC})$ *if and only if* **the pair** $(\boldsymbol{A},\boldsymbol{C})$ is **completely observable**, that is if and only if the row vectors of the *observability matrix*

$$\boxed{Q = \begin{bmatrix} C \\ CA \\ CA^2 \\ \dots \\ CA^{n-1} \end{bmatrix}} \tag{1.13}$$

span the n-dimensional space (that is, matrix $\boldsymbol{Q}$ has rank n).

I choose the eigenvalues of the observer $(-20,-20)$. Thanks to these eigenvalues we can make the dynamic of the observer faster.

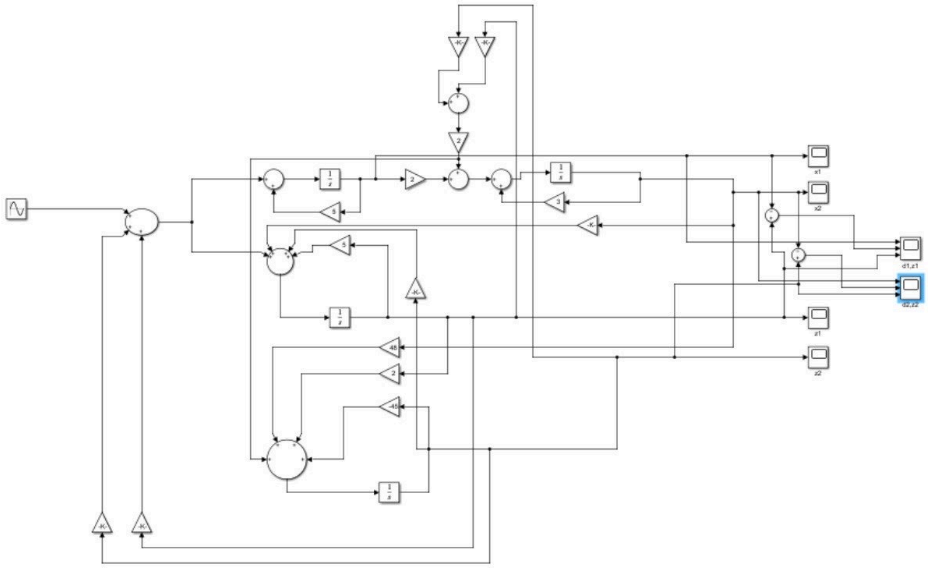
$G = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$ and $F = \begin{bmatrix} 5 & -g_1 \\ 2 & 3-g_2 \end{bmatrix}$ so much so: $\lambda I - F = \begin{bmatrix} \lambda-5 & +g_1 \\ -2 & \lambda-3+g_2 \end{bmatrix} \longrightarrow \det[\lambda I - F] = (\lambda-3+g_2)(\lambda-5) + 2g_1 =$

$= \lambda^2 + (g_2-8)\lambda + 2g_1 - 5g_2 + 15$

To have both poles in $-20 \longrightarrow (\lambda+20)(\lambda+20) = \lambda^2 + 40\lambda + 400 \sim \begin{cases} g_2 - 8 = 40 \longrightarrow g_2 = 48 \\ 2g_1 - 5g_2 + 15 = 400 \quad 2g_1 = 400 + 225 \longrightarrow g_1 = \frac{625}{2} \end{cases}$

$\boxed{G = \begin{bmatrix} 48 \\ 545 \end{bmatrix}}$ so the structure of the observer is: $\dot{z}(t) = \begin{bmatrix} 5 & -\frac{625}{2} \\ 2 & -45 \end{bmatrix} z(t) + \begin{bmatrix} 48 \\ \frac{625}{2} \end{bmatrix} y(t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} u(t)$

5) This is the simulink circuit i made using only integrator blocks.



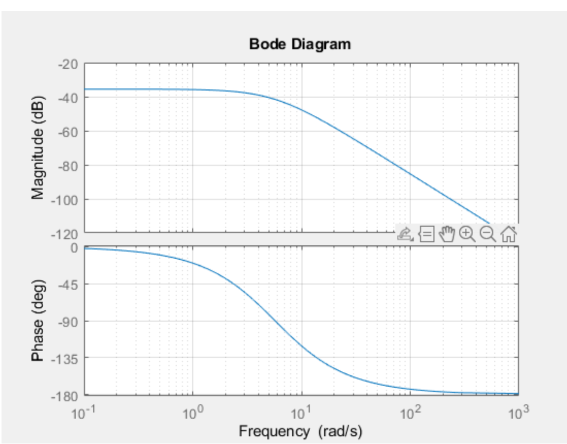
6) As result we have:

$$\begin{cases} \dot{x}(t) = Ax(t) + B\left(-Lx(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t)\right) \\ y = Cx \end{cases} \longrightarrow \begin{cases} \dot{x}(t) = \begin{bmatrix} -6.0707 & -0.7296 \\ 0.5408 & -5.2854 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r(t) \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \end{cases}$$

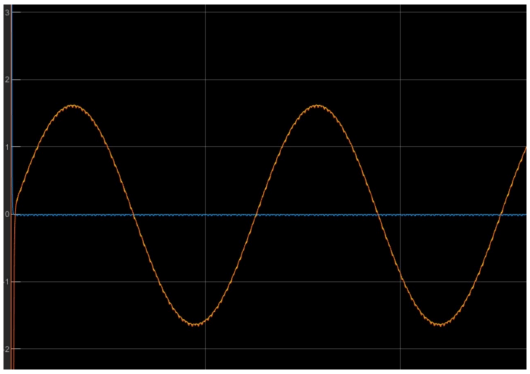
Having said that the transfer function T is:

$$T(s) = \frac{0.5408}{s^2 + 11.36s + 32.48}$$

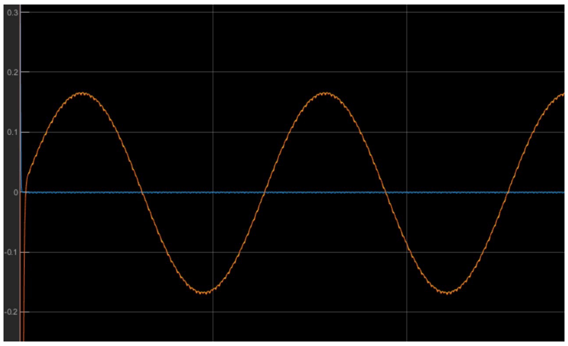
- Bode diagram



- x_1, z_1 and the difference



- x_2, z_2 and the difference



I have tested the system with an external sinusoidal signal :

- $x(t) = \sin(0.1t) \longrightarrow T(j\omega) = 0.0166$
- $x(t) = \sin(t) \longrightarrow T(j\omega) = 0.0162$
- $x(t) = \sin(10t) \longrightarrow T(j\omega) = 0.00402$
- $x(t) = \sin(100t) \longrightarrow T(j\omega) = 5.37 \cdot 10^{-5}$