The next steps:

Overall, about 100 people applied for the PhD position 'Numerical-Relativity Simulations of Black Hole - Neutron Star Systems' at the University of Potsdam/AEI Potsdam. We have gone through all applications and selected a few candidates (including you) that we consider to be excellent. In the final stage of the selection process, we provide a coding exercise and will have an interview.

For the coding exercise, we are providing a problem set in which you are asked to create your own 1+1-dimensional finite differencing code to evolve a single black hole.[1] In case you don't manage to solve everything, please don't be disappointed. For us it is just important to see how you tackle problems and to get an understanding about your background. We have tried to explain things in such a way that you can solve the problem even without any prior knowledge of numerical-relativity, as long as you know general relativity and have a bit of coding experience.

Considering that 'summer time' is also often 'holiday time', we decided not to impose a deadline too soon, but to give you about four weeks for the entire problem set.

After the deadline to hand in the solution (28th of Sept.), you will have a ~ 1 hour online interview in which we will ask you for a 15 minute presentation, in which you should focus on your previous research (8min), your solution of the problem set (5min), and why you want to come to Potsdam/consider yourself as a good candidate (2min)[2]. After your presentation, we will have 45 minutes for discussions. The exact date of the interview will be communicated to you individually via email in the near future.

After this interview, we will finalize our decision as quickly as possible and will let you know. We will also try to ensure that the selected candidate, to whom we will make an offer, will have the opportunity to visit us very soon.

[1] Given that the code you will work with during the PhD (BAM) is written in C, we encourage you to also write the code to this problem set in C. Nevertheless, given your different backgrounds, you can use whatever programming language you want.

[2] Note that the individual times are given for you as a reference, but are not absolutely strict.

A Problem: Evolving the Schwarzschild spacetime

In the following, we will follow the standard 3+1-decomposition of spacetime according to:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(-\alpha^2 + \beta_i\beta^i\right)dt^2 + 2\beta_i dt dx^i + \gamma_{ij}dx^i dx^j, \tag{1}$$

with the lapse function α, the shift vector β^i, and the metric induced into a three-dimensional hypersurface γ_{ij}. Note that here latin indices run only from 1 to 3, while greek indices run from 0 to 3.

For a static and spherically-symmetric system (such as the Schwarzschild spacetime), the spatial part of the metric can be written as:

$$dl^2 = \gamma_{ij}dx^i dx^j = A(r,t)dr^2 + r^2 B(r,t)d\Omega^2, \tag{2}$$

where dl^2 is the line element of the spatial hypersurface, and A and B are functions determined by the field equations of general relativity.

In this case, the extrinsic curvature has only two non-vanishing components $K_A \coloneqq K^r{}_r$ and $K_B \coloneqq K^\theta{}_\theta = K^\phi{}_\phi$.

Given this form of the metric, the ADM (Arnowitt, Deser, Misner) evolution equations in vacuum take the form:

$$\begin{aligned}
\partial_t A &= -2\alpha A K_A, \\
\partial_t B &= -2\alpha B K_B, \\
\partial_t D_A &= -2\alpha\left(K_A D_\alpha + \partial_r K_A\right), \\
\partial_t D_B &= -2\alpha\left(K_B D_\alpha + \partial_r K_B\right), \\
\partial_t K_A &= -\frac{\alpha}{A}\left[\partial_r\left(D_\alpha + D_B\right) + D_\alpha^2 - \frac{D_\alpha D_A}{2} + \frac{D_B^2}{2} - \frac{D_A D_B}{2}\right. \\
&\qquad \left. - A K_A\left(K_A + 2K_B\right) - \frac{1}{r}\left(D_A - 2D_B\right)\right], \\
\partial_t K_B &= -\frac{\alpha}{2A}\left[\partial_r D_B + D_\alpha D_B + D_B^2 - \frac{D_A D_B}{2} - \frac{1}{r}\left(D_A - 2D_\alpha - 4D_B\right)\right. \\
&\qquad \left. - \frac{2(A-B)}{r^2 B}\right] + \alpha K_B\left(K_A + 2K_B\right),
\end{aligned} \tag{3}$$

where we also set $\beta^i = 0$ for simplicity. The auxiliary variables are defined as

$$D_A \coloneqq \partial_r \ln A, \qquad D_B \coloneqq \partial_r \ln B, \qquad D_\alpha \coloneqq \partial_r \ln \alpha, \tag{4}$$

in order to make the system first order in space.

Note that even without prior knowledge on numerical relativity, you can simply consider the set of equations as a system of partial differential equations that you are going to solve.

(A.1) Initial Data

Focusing on the Schwarzschild metric in isotropic coordinates, the spatial part can be written as:

$$dl^2 = \psi^4 \left(dr^2 + r^2 d\Omega^2 \right), \qquad \psi = 1 + \frac{M}{2r}, \tag{5}$$

where M is the mass of the black hole. Note that r here is not the areal radius R in the standard form Schwarzschild metric. These coordinates are called isotropic coordinates.

TASK 1: Show that the relation between r and areal coordinate R is given by:

$$R = r \left(1 + \frac{M}{2r} \right)^2 . \tag{6}$$

(A.2) ADM evolution with geodesic slicing

Use the method of lines to evolve the ADM equations (Eq. 3), i.e., discretize the spatial dimension, e.g., through 2nd (or 4th) order finite-differencing stencils, and consider the equations as ordinary differential equations in the time domain. For the time integration, we suggest that you use a 4th order Runge-Kutta method (but also a simpler forward Euler method would be okay).

To solve the system of equations, take $A = B = \psi$ and $K_A = K_B = 0$ as the initial conditions and employ the geodesic gauge condition, which means:

$$\alpha = 1 = \text{const.} \, . \tag{7}$$

We suggest to use a staggered grid to avoid a point at $r = 0$, e.g.,

$$r_j = \left(j - \frac{1}{2} \right) \Delta r, \quad j = 1, 2, \ldots, N. \tag{8}$$

For the boundary condition, impose the symmetry (asymemtry) condition at $r = 0$:

$$f_{1-i} = \pm f_i, \quad i = 1, 2, \tag{9}$$

with $+$ for A, B, K_A, K_B and $-$ for D_A, D_B.

For the outer boundary, you could use Dirichlet or Sommerfeld boundary conditions, or you could use a 1st order extrapolation.

TASK 2: Write a code as discussed above and perform a few test simulations. Write a short documentation about your results, i.e., visualize your results and try to interpret them. What dou you find. If you have done everything properly, you will see that your code crashes after some time, what is the physical meaning of this?

Suggestion: Since the metric we are evolving is singular in $r = 0$, the variables could have very steep gradients around there and this could lead to a failure of the numerical simulation without any physical reason for it. In order to avoid this issue, it might be necessary/useful to regularize the variables and rewrite the evolution equations as functions of the new variables:

$$\begin{aligned} \tilde{A} &\coloneqq A/\psi^4, & \tilde{B} &\coloneqq B/\psi^4, \\ \tilde{D}_A &\coloneqq D_A - 4\partial_r \ln\psi, & \tilde{D}_B &\coloneqq D_B - 4\partial_r \ln\psi, \end{aligned} \tag{10}$$

in such a way to absorb the well known analytic part in the ψ factor and evolve regular variables. In these new variables the initial conditions are:

$$\begin{aligned} \tilde{A} = \tilde{B} = 1, \\ \tilde{D}_A = \tilde{D}_B = 0, \end{aligned} \tag{11}$$

the variables K_A and K_B are not affected by this transformation.

(A.3) $1 + \log$ slicing

TASK 3: As before, evolve the ADM equations and take $A = B = 1 + \frac{M}{2r}$ and $K_A = K_B = 0$ as the initial conditions. However, this time, employ the $1 + \log$ slicing condition:

$$\partial_t \alpha = -2\alpha K, \tag{12}$$

and use as initial guess $\alpha(0, r) = 1$ for $t = 0$. Write a short documentation about your results.

(A.4) Apparent Horizon

In general, one would compute the event horizon to know the location of the 'black hole's surface', however, this is technically difficult during a numerical-relativity simulation. For this purpose, one uses the concept of an apparent horizon. An apparent horizon can be considered as the outermost trapped null surface. In spherical symmetry the apparent horizon is given by its horizon r_h and can be obtained by solving for

$$\frac{1}{\sqrt{A}}\left(\frac{2}{r} + \frac{\partial_r B}{B}\right) - 2K_B = 0 \tag{13}$$

with $r = r_h$. The corresponding surface of the horizon can be computed through $S_h = 4\pi B r_h^2$.

TASK 4: Evaluate at different times the radius r_h of the apparent horizon in isotropic coordinates and its surface S_h. Are these two quantities static? If not try to interpret your results.