The Zermelo-Fraenkel axioms of set theory

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1. Axiom of Extensionality If X and Y have the same elements, then $X = Y$.
$$\forall u\,(u \in X \equiv i \in Y) \Rightarrow X = Y \tag{1}$$

2. Axiom of the Unordered Pair For any a and b there exists a set (a, b) that contains exactly a and b.
$$\forall a\,\forall b\,\exists c\,\forall x\,(x \in c \equiv (x = a \lor x = b)) \tag{2}$$

3. Axiom of Subsets if φ is a property with parameter p, the for any X and p there exists a set $Y = \{u \in X : \varphi(u, p)\}$ that contains all those $u \in X$ that have the property φ.
$$\forall X\,\forall p\,\exists Y\,\forall u\,(u \in Y \equiv (u \in X \land x = b)) \tag{3}$$

4. Axiom of Sum Set For any X there exists a set $Y = \cup X$, the union of all elements of X.
$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv \exists z\,(z \in X \land \varphi(u, p))) \tag{4}$$

5. Axiom of Power Set For any X there exists a set $Y = P(X)$, the set of all subsets of X.
$$\forall X\,\exists Y\,\forall u\,(u \in Y \equiv u \subseteq X) \tag{5}$$

6. Axiom of Infinity There exists an infinite set.

$$\exists S\,[\emptyset \in S \wedge (\forall x \in S)[x \cup \{x\} \in S]] \tag{6}$$

7. Axiom of Replacement If F is a function, then for any X there exists a set $Y = F[X] = \{F(x) : x \in X\}$.

$$\begin{aligned} &\forall x\,\forall y\,\forall z\,[\varphi(x,y,p) \wedge \varphi(x,z,p) \Rightarrow y = z] \\ \Rightarrow\ &\forall X\,\exists Y\,\forall y\,[y \in Y \equiv \exists x \in X \varphi(x,y,p)] \end{aligned} \tag{7}$$

8. Axiom of Foundations Every nonempty set has an $\in$-minimal element.

$$\exists S\,[S \neq \emptyset \Rightarrow (\exists x \in S)\,S \cap x = \emptyset] \tag{8}$$

9. Axiom of Choice Every family of nonempty sets has a choice function.

$$\forall x \in a\,\exists A(x,y) \Rightarrow \exists y\,\forall x\ \in aA(x,y(x)) \tag{9}$$