BDMA - Decision Modeling

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This is a summary of the course *Decision Modeling* taught at the Université Paris Saclay - CentraleSupélec by Professor Brice Mayag in the academic year 23/24. Most of the content of this document is adapted from the course notes by Mayag, [1], so I won't be citing it all the time. Other references will be provided when used.

Contents

1 Introduction

1.1 Models

There are many definition of what a model is, depending on the perspective used. For example, a model can be understood as a '*standard or example for imitation or comparison*' or as a '*person employed to wear clothing ot pose with a product for purposes of display and advertising*'. Nonetheless, in the context of decision modeling, which is the one in which we are interested, the definition of model is:

> **Definition 1.1.** A **model** is a *simplified representation of a system or phenomenon, with any hypotheses required to describe the system or explain the phenomenon, often mathematically.*

Models are useful to enhance our understanding of the world to improve our decision making, and they enable us to elaborate a scientific methodology to solve a problem in a duplicable way and with the aim of reducing bias in mind.

A model is said to be **deterministic** if the outcomes are precisely determined through known relationships among states and events. This kind of models always produce the same output when given the same input. On the other hand, a model is **probabilistic** (or stochastic) when all the data that it tries to explain is not known with certainty.

For example, the Newtonian model for gravity is deterministic, while a prediction model for college acceptance is probabilistic.

Deterministic models are used in domains such as Multi-Attribute Decision Making or Linear Programming, among others; while probabilistic models are used in queuing problems, simulations, etc.

We can also classify the data that is used to define a model into **qualitative** and **quantitative**. The former refers to data that is expressed in terms of words, while the latter is data easily expressed using numbers. An example of qualitative data is the hair color of people in class, and for quantitative data is the height of people in class.

We can be more precise in our wording, and call the models that we are talking about **formal models**, which refer to those models that provide a precise statement of the components of the model and their relationships, usually by means of mathematical equations. This make them easy to communicate precisely and the ability to give replicable results. However, being formal does not mean being true. A model can fail to represent the reality that it tries to describe.

1.2 Decision Theory and Decision Analysis

> **Definition 1.2.** A **decision** is a choice that is made about something after thinking about several possibilities.

Decisions appear in many domains, including Mathematics, Economics, Computer Sciences, Psychology,...

> **Definition 1.3. Decision Analysis** consists in trying to provide answers to questions raised by actors involved in a decision process using a model.
>
> B. Roy

In the previous definition, a **Decision Process** refers to a strategy of intervention, such as aid, communication or justification, among others. There are many ways to provide decision aid and no single way to compare methods. This, together with the fact that different models may lkead to different recommendations, makes it hard to assess when a Decision Analysis model is 'good' or, more appropriately, 'suitable'.

Therefore, we cannot compare decision making to solving a well-defined problem, as the former is highly dependent on opinions, interests and, more generally, different human factors involved. In every decision process, there are several possible interventions, among which we can find imagining compromises, communicating, coordinating, controlling, motivating or conducting change.

There are many different models used in Decision Analysis nowadays, with the advantages of:

- Providing a clear language that can be leveraged as a communication tool
- Capturing the essence of a situation, acting a structuration tool
- Answering '*what-if*' questions, serving as a exploration tool

On the other hand, their drawbacks are their high complexity and opaqueness. In addition, in many situations people could argue that such models are not necessary because they already know how to take decisions and they would over-complicate the process; or would ask for higher-level explanations or ideas that are not suitable for formalization; or would rather rely on their intuition.

1.3 Main Steps of Developing a Decision Model

1. **Formulation**: translate the problem scenario into a mathematical model.

   This involves the definition of the problem and the development of a decision model, i.e., the definition of the **variables** or measurable quantities that vary, and the **parameters** or measurable quantities inherent to the problem.

2. **Solution**: solve the mathematical expressions from the formulation.

   This process involves the development of the solutions by correctly manipulating the model to arrive at the best solution, and the testing of the solution, to check that it works as expected and meets the expectations.

3. **Interpretation**: discover the implications of the results.

   This is usually done by conducting **sensitivity analysis**, i.e., testing the different outcomes obtained under a variety of states; and **implementing results**, enacting the solutions and monitoring the performance.

The outlined process is very high level, and there are many possible problems that can arise:

- Defining the problem: we can find conflicting viewpoints that impact differently the stakeholders.
- Model development: it is not always easy to find the formal model that describes the problem at hand, and it is usual to make adaptations.
- Acquering data: can be hard in some scenarios, as well as checking its validity and correctness.
- Developing a solution: we can find many limitations, such as only finding one answer, finding approximate answers, prohibitive computing times,...
- Implementation: it is crucial that the solution is feasible to be implemented, both from a managerial point of view and from the user perspective.

1.4 Decision's Algorithm & Transparency

Decisions made by algorithms can be opaque because of technical and social reasons, in addition to being made purposely opaque to protect intellectual property.

Definition 1.4. An **algorithm** is a sequence of instructions, typically used to solve a class of problems or perform a computation.
It must be:

- Finite: it must eventually solve the problem.
- Well-defined: the series of step must be precise and understandable.
- Effective: it must solve all cases of the problem for which it was defined.

Usually, we find contradictory objectives when developing an algorithm, because simpler algorithms are usually time intensive, while algorithms that are very efficient are very complex are hard to understand.

Definition 1.5. Algorithmic Transparency is the principle that the factors that influence the decisions made by algorithms should be transparent to the people who use, regulate and are affected by systems that employ those algorithms.

This concept is opennes about the purpose, structure and underlying actions of the algorithms used to search for, process and deliver information. Two important properties of transsparency are:

- **Explanability**: systems and institutions that use algorithm decision making are encouraged to produce explanations regarding both the procedures followed by the algorithm and the specific decisions that are made. This is specially relevant in public policy contexts.
- **Accountability**: institutions should be held responsible for decisions made by the algorithms they use, even if it is not feasible to explain in detail how the algorithms produce their results.

2 Preferences as binary relations

2.1 Definitions

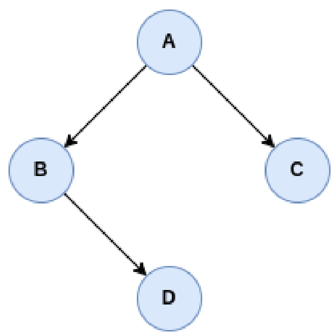
Definition 2.1. Given a set X, a **binary relation**, R, is a subset of ordered pairs of elements in X:

$$R \subseteq X \times X.$$

We can write $(x, y) \in R$ or, equivalently, xRy.

Relations can be expressed as directed graphs. For instance, a relation R of a set X can be represented as the graph $G_R = (N_X, E_R)$, where N_X are the nodes, representing each element in X and E_R are the edges, representing each pair in R. The edges are constructed in such a way that $e = (x, y) \in E_R \iff xRy$.

Example 2.1. Let $X = \{a, b, c, d\}$ and $R = \{(a, b), (a, c), (b, d)\}$, then we can represent this by G_R as:



Another way to represent relations is using matrices. We can construct a matrix M_R by

$$M_R = (m_{xy})_{(x,y) \in X},$$

where

$$m_{xy} = \begin{cases} 1 & if\ xRy \\ 0 & if\ not\,(xRy) \end{cases}.$$

Example 2.2. The previous example can be represented with the following matrix:

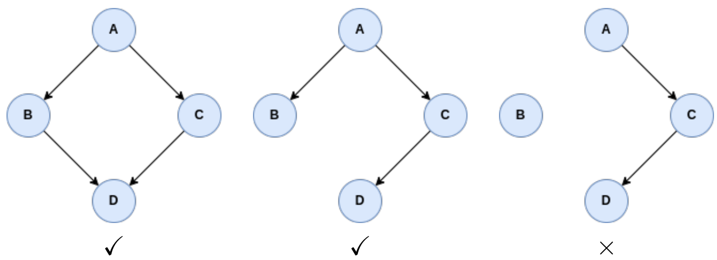
$$M_R = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Depending on how a relation is constructed, it can possess different properties. Some interesting properties are defined as follows:

Definition 2.2. A binary relation R on a set X is said to be:

- **Reflexive** if $xRx, \forall x \in X$.
- **Irreflexive** if $not\,(xRx)\,, \forall x \in X$.
- **Complete** if for every $x, y \in X$, we have xRy or yRx (or both).
- **Weakly complete** if for every $x, y \in X, x \neq y$, we have xRy or yRx (or both).
- **Symmetric** if $[xRy \implies yRx]\,, \forall x, y \in X$.
- **Asymmetric** if $[xRy \implies not\,(yRx)]\,, \forall x, y \in X$.
- **Antisymmetric** if $[xRy \wedge yRx \implies x = y]\,, \forall x, y \in X$.
- **Transitive** if $[xRy \wedge yRz \implies xRz]\,, \forall x, y, z \in X$.
- **Negatively transitive** if $[not\,(xRy) \wedge not\,(yRz) \implies not\,(xRz)]\,, \forall x, y, z \in X$.
- **Semi-transitive** if $[xRy \wedge yRz \implies xRt \vee tRz]\,, \forall x, y, z, t \in X$.

Example 2.3. A semi-transitive relation example.



In addition, we can define paths and cycles on relations, analogously as how it is done in graph theory:

Definition 2.3. A **path** from $x \in X$ to $y \in X$ exists if there are $x_1, ..., x_n \in X$ such that

$$x = x_1Rx_2R \ldots Rx_{n-1}Rx_n = y.$$

A path is called a **cycle** if the it goes from x to x.

For every relation, we can extract two subrelations, as its symmetric part and its asymmetric part:

Definition 2.4. Given a binary relation R on X, we can define its **symmetric part**, I, as

$$xIy \iff [xRy \wedge yRx]\,,$$

and its **asymmetric part**, P, as

$$xPy \iff [xRy \wedge not\,(yRx)]\,.$$

The symmetric part is denoted by I because we can understand this as the all the indiferent pairs of R. In others words, if we understand a relation as a preference over the elements in X, then xRy would mean x *is at least as convenient as* y. Therefore, if we have xRy and yRx, we could think of them as equally convenient, so the decision between them is indiferent. On the other hand, the asymmetric part is denoted by P, from preference, following a similar reasoning.

When we have two different relations, R and R', on the same set, X. We can define their concatenation:

Definition 2.5. Let R and R' be two relations on X. We define their **concatenation** as

$$xR \bullet R'y \iff \exists z \in X : [xRz \wedge zR'y].$$

The following proposition establishes different relationships between the concepts we have seen so far:

Proposition 2.1. *Let R be a binary relation on X. Then:*

- *R transitive $\implies$ $R \bullet R \subset R$.*
- *R asymmetric $\implies$ R irreflexive.*
- *R complete $\iff$ R reflexive and weakly complete.*
- *R asymmetric and negative transitive $\implies$ R transitive.*
- *R complete and transitive $\implies$ R negative transitive.*

Proof. Let's go one by one:

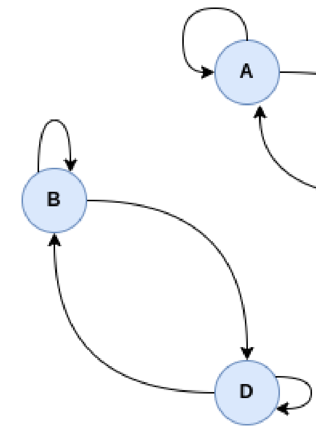
- By definition, we have $xR \bullet Ry \iff \exists z : xRz \wedge zRy$. Since R is transitive, then it must be xRy. Therefore $R \bullet R \subset R$.
- If R is not irreflexive, then there exists $x \in X$ such that xRx, but this is a symmetric relationship, so R cannot be assymetric.
- Trivial.
- By reduction ad absurdum, seeking a contradiction, let's assume that R is not transitive. This means that there exist $x, y, z \in X$ such that $xRy \wedge yRz$ but $not\,(xRz)$. By hypothesis, R is asymmetric, so $xRy \implies not\,(yRx)$. If we combine these two facts, and use the hypothesis that R is negative transitive, we find that
$$not\,(yRx) \wedge not\,(xRz) \overset{neg\ trans}{\implies} not\,(yRz)\,\#$$
This is a contradiction, because we assumed that yRz. Therefore, R must be transitive.
- By reduction ad absurdum, seeking a contradiction, let's assume that R is not negative transitive. This means that there exist $x, y, z \in X$ such that $not\,(xRy) \wedge not\,(yRz)$, but xRz. By hypothesis, R is complete, so $not\,(yRz) \implies zRy$. If we combine the two facts, xRz and zRy, then the transitivity of R gives us $xRy\#$, which is a contradiction with our assumption, $not\,(xRy)$. Therefore, R must be negative transitive.

□

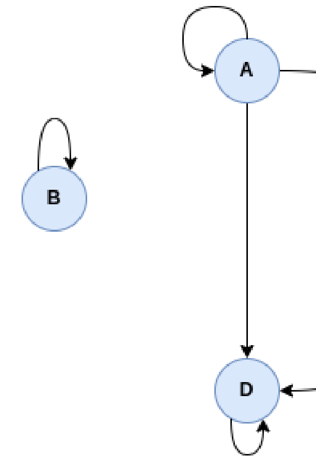
There are some relations that fulfill several of the properties that we have seen, and that hold special characteristics. For instance:

Definition 2.6. Different types of relations
An **equivalence relation** is a relation which is reflexive, symmetric and transitive.
A **preorder** is a relation which is reflexive and transitive.
A **weak order** or a **complete preorder** is a relation which is complete and transitive.
A **total order** or **linear order** is a relation which is complete, antisymmetric and transitive.

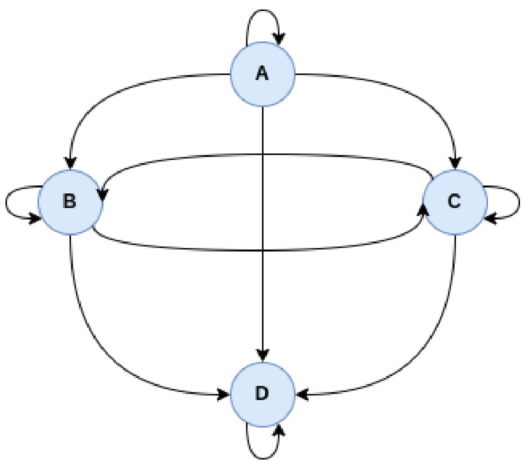
Example 2.4. The relation $R = \{(a, a), (a, c), (c, a), (c, c), (b, b), (b, d), (d, b), (d, d)\}$ is an equivalence relation. Its graph representation is:



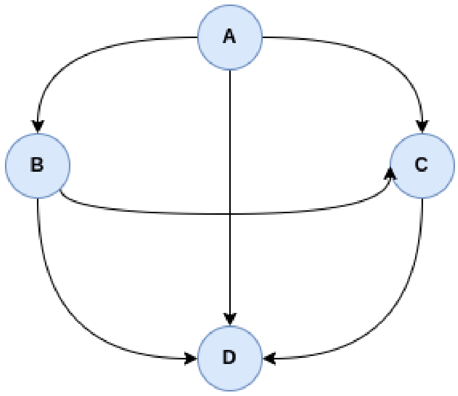
The relation $R = \{(a,a), (a,c), (a,d), (c,c), (c,d), (d,d), (b,b)\}$ is a preorder which is not a complete preorder. Its graph representation is:



The relation $R = \{(a,a), (a,c), (a,d), (c,c), (c,d), (d,d), (b,b), (a,b), (b,d), (b,c), (c,b)\}$ is a complete preorder. Its graph representation is:



The relation $R = \{(a,b), (a,c), (a,d), (b,c), (b,d), (c,d)\}$ is a total order. Its graph representation is:



Exercise 2.1. Let B be a binary relaiton on a set $X = \{a, b, c, d, e, f\}$ defined by

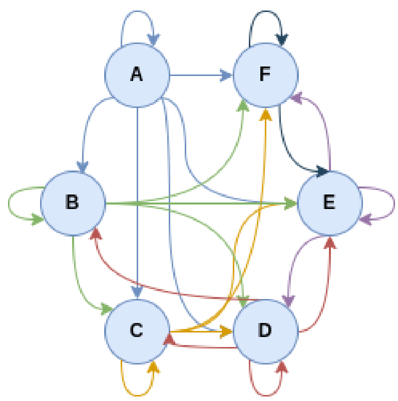
$$\begin{aligned} aBa, aBb, aBc, aBd, aBe, aBf \\ bBb, bBc, bBd, bBe, bBf \\ cBc, cBd, cBe, cBf \\ dBb, dBc, dBd, dBe \\ eBd, eBe, eBf \\ fBe, fBf \end{aligned}$$

Give a matrix and a graph representation of B

The matrix form of B is the following:

$$\begin{array}{c c} & \begin{array}{cccccc} a & b & c & d & e & f \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

And the graph representation:



Is B reflexive? Symmetric? Asymmetric? Transitive? Negative transitive? Semi-transitive?

- Reflexive: Yes, since $xBx, \forall x \in X$.
- Symmetric: No, aBb but *not* (bBa).
- Asymmetric: No, eBf but fBe.

- Transitive: No, dBe and eBf, but not dBf.
- Negative transitive: No, *not* (fBd) and *not* (dBf) but fBf.
- Semi-transitive: Yes. Let's do this little by litte.

  If we take $x = a$, then $aBt, \forall t$, so it will hold. If $y = a$, then $\nexists x$ such that xBa, so it will hold. Same happens if $z = a$. Finally, if $t = a$, then $aBz, \forall z \in X$, so it also holds.

  If we eliminate a (since we have seen all cases in which a is involved), we obtain the relation:

$$\begin{array}{c c} & \begin{array}{ccccc} b & c & d & e & f \end{array} \\ \begin{array}{c} b \\ c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

  Using the same argument, b will work now. When we remove it, we get:

$$\begin{array}{c c} & \begin{array}{cccc} c & d & e & f \end{array} \\ \begin{array}{c} c \\ d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array}$$

  And once again we can use c. When we remove it, we get:

$$\begin{array}{c c} & \begin{array}{ccc} d & e & f \end{array} \\ \begin{array}{c} d \\ e \\ f \end{array} & \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \end{array}$$

  Now we can repeat the same with e, and when we remove it we obtain a two-element relation, which is always semi-transitive.

Exercise 2.2. Let B and B' be two equivalence relations on a set X:

Prove that $B \cap B'$ is an equivalence relation, where

$$x\left(B \cap B'\right) y \iff \left[xBy \wedge xB'y\right], \forall x, y \in X.$$

We need to see that $B \cap B'$ is reflexive, symmetric and transitive:

- Reflexive: take $x \in X$, then $x\left(B \cap B'\right) x \iff xBx \wedge xB'x$, which holds since B and B' are reflexive.
- Symmetric: take $x, y \in X$ such that $x\left(B \cap B'\right) y$. This means that $xBy \wedge xB'y$. Since B and B' are symmetric, we obtain that $yBx \wedge yB'x$, and therefore $y\left(B \cap B'\right) x$.
- Transitive: take $x, y, z \in X$ such that $x\left(B \cap B'\right) y \wedge y\left(B \cap B'\right) z$. This means that $xBy, xB'y, yBz$ and $yB'z$. Now, since B and B' are transitive, we obtain xBz and $xB'z$, so $x\left(B \cap B'\right) z$.

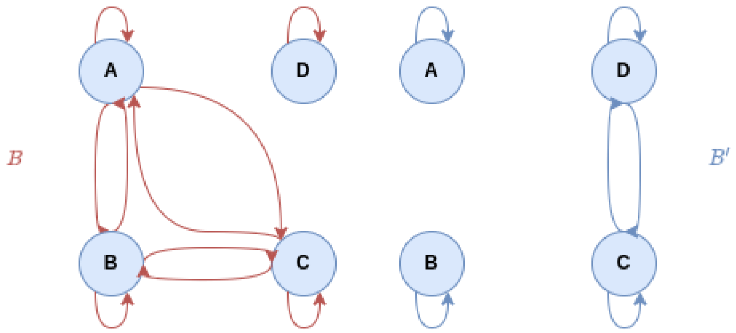
Is $B \cup B'$ an equivalence relation, where

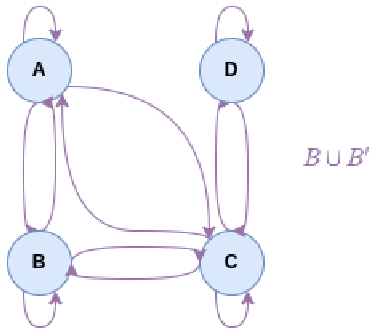
$$x\left(B \cup B'\right) y \iff \left[xBy \vee xB'y\right], \forall x, y \in X?$$

Reflexivity and symmetry are preserved, but what about transitivity? It is not... Take $X = \{a, b, c, d\}$ and the relations

$$B = \{(a,a), (a,b), (a,c), (b,b), (b,a), (b,c), (c,c), (c,b), (c,a), (d,d)\},$$
$$B' = \{(a,a), (b,b), (c,c), (c,d), (d,d), (d,c)\}.$$

Then, they are both equivalence relations, but their union is not. This is shown below:





As we can see, in the union, we find (a, c) and (c, d), but not (a, d), so it is not transitive.

Could we have the same conclusions if B and B' are two complete preorders on a set X?

We are been asked if $B \cap B'$ and $B \cup B'$ are also complete preorders. The answers to both questions is no.

In the case of the intersection, the transitivity is preserved, following the same argument we did for equivalence relations, but completeness is not preserved. To see this, take $X = \{a, b, c, d\}$ and the relations

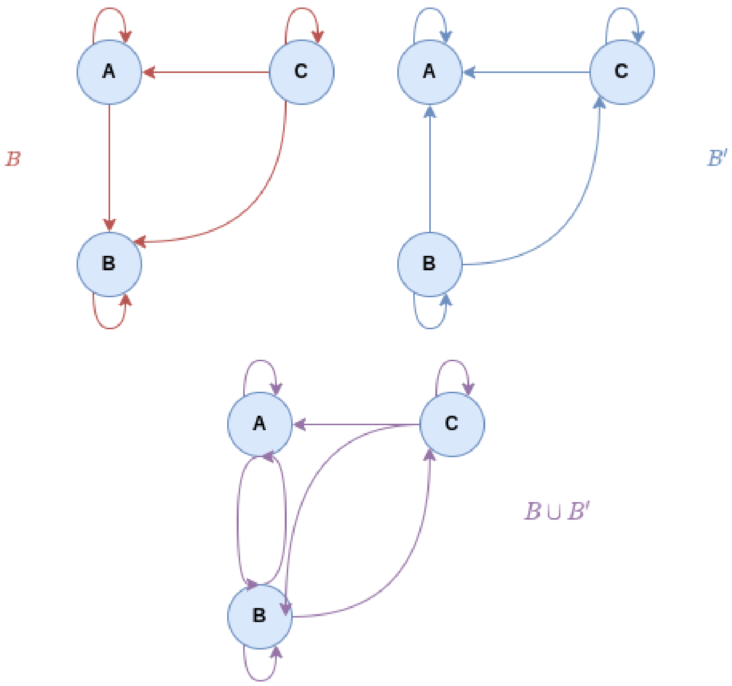
$$B : (a, b) \succ (c, d)$$
$$B' : (c, d) \succ (a, b) .$$

Then, the intersection is

$$B \cap B' : a \equiv b, c \equiv d,$$

which is not complete.

As for the union, the opposite happens. Completeness is preserved, because the relations involved are complete. However, transitivity is not preserved. As a counter example, take the following graph visualization:



As we can see, in the union, we find (a, b) and (b, c) but not (a, c).

Definition 2.7. The **reflexive closure** of a binary relation, R, is

$$r(R) = R \cup Eq,$$

where Eq is the reflexive, relation, i.e., $Eq = \{(x, x) : x \in X\}$.

Therefore, the reflexive closure of a relation is the relation resulting when adding all needed edges for the initial relation to be reflexive. As a example, see Figure 1.

A similar concept is that of symmetric closure:

Definition 2.8. The **symmetric closure** of a relation R is

$$s(R) = R \cup R^c,$$

where R^c is the converse relation, i.e.,

$$xR^c y \iff yRx.$$

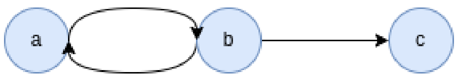
In this case, we are adding all needed edges for the relation to be symmetric. Again, see Figure 1 for an example.

Finally, there is the transitive closure:

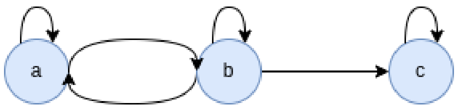
Definition 2.9. The **transitive closure** of a relation R is

$$t(R) = R \cup R^2 \cup R^3 \cdots = \bigcup_{i \geq 1} R^i.$$

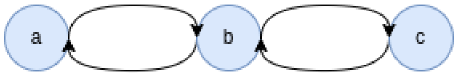
In this case, we add all possible transitive relations by relating all elements with a path between them. Note that if the set X has size N, then we only need to use $N - 1$ step at maximum to reach any reachable node. An example is also shown in Figure 1.



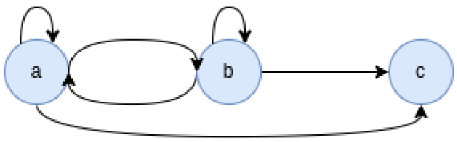
(a) Original relation, R.



(b) Reflexive closure, $r(R)$.



(c) Symmetric closure, $s(R)$.



(d) Transitive closure, $s(R)$.

Figure 1: Closures visualization.

A natural question at this point is if we can get any of our properties by applying some kind of closure, and the answer in the general case is no. However, there are ways to obtain reasonable solutions.

For instance, consider the problem of extending a partial pre-order to a complete pre-order. In such case, we need to define how unrelated elements should be related. One way to solve this is by applying what is called a **topological sorting**, which creates a complete pre-order that respects all the partial pre-order preferences, when there are no cycles. More formally:

Definition 2.10. A **Directed Acyclic Graph (DAG)** is a directed graph with no cycles, i.e., there is no $x \in X$ such that there is a path from x to itself.

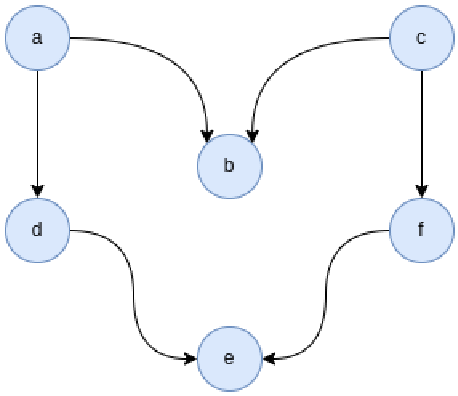
Definition 2.11. A **topological sorting** for a DAG is a linear ordering of vertices such that, for every directed edge, (u, v), vertex u comes before v in the ordering.

There are several ways to perform topological sorting, obtaining different results, and all of them respecting the initial preferences. As a general algorithm, see Algorithm 1. Note that we are first taking those nodes with no outgoing edges, but we could do it also by taking the nodes with no ingoing edges, in which case the list would not need to be reversed at the end.

```
input:
  set X
  relation R (must be DAG)

initialize:
  list L

execute:
  while R is not empty:
    free = {x in X : out_degree(x,R)=0 } # Take all nodes without outgoing edges
    R = R - edges(free) # Remove edges going into nodes in free from R
    list.append(free) # Append each node to the list

  list.append(rest of nodes)

  return list.reverse # The order is reversed
```

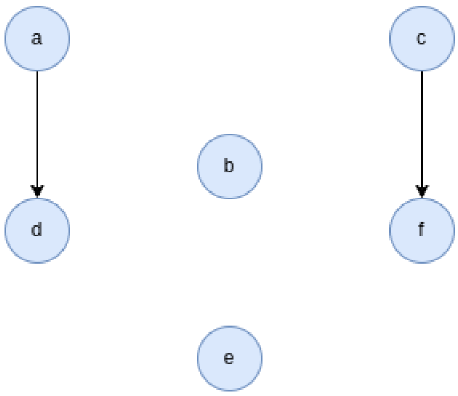
Algorithm 1: Topological sorting.

Example 2.5. Create a topological sorting of the following relation:

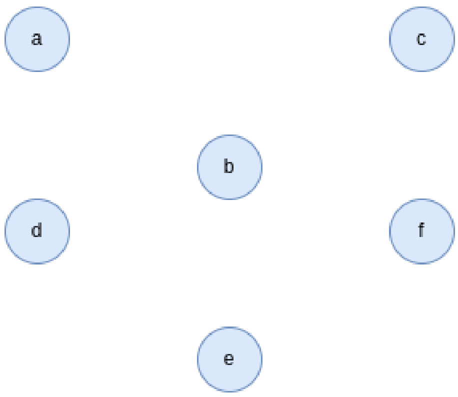


Let's follow our pseudocode:

- Input: $X = \{a, b, c, d, e, f\}$, $R = \{aRb, aRd, cRb, cRf, dRe, fRe\}$
- $L = []$
- $free = \{e, b\}$
- $R = R - edges(\{e, b\}) = \{aRd, cRf\}$



- $L = [e, b]$
- $free = \{d, f\}$
- $R = \{\}$



- $L = [e, b, d, f]$
- Append the rest of nodes: $L = [e, b, d, f, a, c]$.

Therefore, a linear ordering respecting the preferences in the original partial preorder is $c \succsim a \succsim f \succsim d \succsim b \succsim e$.

2.2 Numerical representation

The idea of the numerical representation is to try to construct a binary relation $\succsim$ on a set X such that there exists a numerical function $f : X \to \mathbb{R}$ satisfying the property

$$x \succsim y \iff f(x) \geq f(y).$$

Therefore, we are trying to convey the information given by the relation via a mathematical function. When doing this, we usually assume $\succeq$ to be a preorder.

We also usually use the following notation:

- $x \succsim y$ means *x is at least as good as y.*
- $\succ$ is the asymmetric part of $\succsim$.
- $\sim$ is the symmetric part of $\succsim$.

Cantor proved a theorem that answered this problem:

Theorem 2.1. *Cantor, 1895*
Let X be a countable set and $\succsim$ a binary relation on X. Then:

$$[\exists f : X \to \mathbb{R} | \forall x, y \in X, x \succsim y \iff f(x) \geq f(y)]$$
$$\Updownarrow$$
$$\succsim \text{ is a complete preorder on } X.$$

Proof. $[\implies]$

- Completeness: pick $x, y \in X$, then we compute $f(x)$ and $f(y)$. Since $\mathbb{R}$ is a linearly ordered set, it is either $f(x) \geq f(y)$ or $f(y) \geq f(x)$. This, by hypothesis means that $x \succsim y$ or $y \succsim x$.
- Transitivity: pick $x, y, z \in X$ with $x \succsim y$ and $y \succsim z$. Then, $f(x) \geq f(y)$ and $f(y) \geq f(z)$. The transitivity in the real line gives us $f(x) \geq f(z)$, which by hypothesis means $x \succsim z$.

Therefore, $\succsim$ is a complete preorder on X.

$[\impliedby]$ We assume that $\succsim$ is a complete preorder on X. Since X is countable, we can enumerate its elements, i.e., we can express

$$X = \{x_i : i \in K \subset \mathbb{N}\} = \{x_1, x_2, ...\}.$$

Let $N(y) = \{i \in K : y \succsim x_i\}$, i.e., the set of indices of elements put after y.

Now, we define

$$\begin{array}{rccc} f: & X & \to & \mathbb{R} \\ & y & \mapsto & f(y) = \sum_{i \in N(y)} \frac{1}{2^i} \end{array}.$$

This series is convergent, since $\sum_{n \in \mathbb{N}} \frac{1}{2^n} = 2$ and we are taking a subset of it. Let's see that $x \succsim y \iff f(x) \geq f(y)$:

- If $x \succsim y$, then, the transitivity of the relation ensures that
$$N(x) \supseteq N(y),$$
therefore, it follows that
$$f(x) \geq f(y),$$
simply because we are summing over more elements.
- Conversely, assume that $f(x) \geq f(y)$ but not $x \succsim y$. Since $\succsim$ is complete, then it must be $y \succsim x$, which combined with *not* $(x \succsim y)$ gives us that it must be $N(y) \supsetneq N(x)$. This means that there are elements in $N(y)$ which are not in $N(x)$, so $f(y) > f(x)$ # which is a contradiction. Therefore, it must be $x \succsim y$.

□

Note that this theorem ensures that such a function can only be found if the relation is a complete preorder, and therefore all partial preorders that are not complete cannot be assigned such a function. Note also that the defined function is not unique (for instance, add 1 to this function, and it still works).

In addition, the following proposition proves that even if there are infinite possible such functions, they are all related:

Proposition 2.2. *Let $\succsim$ be a complete preorder on X, representable by a function $f : X \to \mathbb{R}$. Then, the following two properties are equivalent:*

1. *$v : X \to \mathbb{R}$ is a function representing $\succsim$.*
2. *There exists a strictly increasing function $\varphi : f(X) \to \mathbb{R}$ such that $v = \varphi \circ f$.*

Proof. [2 $\implies$ 1] $x \succsim y \overset{f \text{ represents } \succsim}{\iff} f(x) \geq f(y) \overset{\varphi \text{ strictly increasing}}{\iff} \varphi(f(x)) \geq \varphi(f(y)) \overset{\text{definition of } g}{\iff} g(x) \geq g(y)$.

[1 $\implies$ 2] Define $\varphi : f(X) \to \mathbb{R}$ as
$$\varphi(u) = v(x) : f(x) = u.$$

We have to see that φ is well-defined, strictly increasing and that, indeed, $v = \varphi \circ f$.

- Well-defined: For each $u \in f(X)$, there exist $x \in X$ such that $f(x) = u$, so $\{v(x) : f(x) = u\}$ is non-empty and therefore $\varphi(u)$ is defined.
- Strictly increasing: Take $u_1, u_2 \in f(X)$ with $u_1 > u_2$. Then, there exist $x_1, x_2 \in X$ with $f(x_1) = u_1$ and $f(x_2) = u_2$. This means that $x_1 \succsim x_2$ and not $x_2 \succsim x_1$, since $f(x_1) > f(x_2)$. But then, it is $v(x_1) > v(x_2)$, and so $\varphi(u_1) > \varphi(u_2)$.
- The equality: Take $x \in X$, then
$$\varphi(f(x)) = v(x) \,|f(x) = f(x) = v(x).$$

□

Remark 2.1. This representation functions are **ordinal scales**, since the importance does not rely on the absolute values they provide, but in comparating the values for different elements.

References

[1] Brice Mayag. Decision modeling. Lecture Notes.