DELFT UNIVERSITY OF TECHNOLOGY

FACULTY OF AEROSPACE ENGINEERING

AE4870B

RE-ENTRY SYSTEMS

Assignment I: Atmospheres

Lecturer: Dr. E. Mooij
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1 Fitting of Model Parameters

The parameters of the exponential atmosphere model can be fitted to the US76 model in order to estimate the atmospheric density profile as a function of the altitude. The exponential atmosphere model is expressed by the following formulation,

$$\rho(h) = \rho_0 e^{\frac{-h}{H_s}} \quad (1) \qquad\qquad \ln \rho(h) = \ln \rho_0 - \frac{h}{H_s} \quad (2)$$

Where h is the altitude, $\rho(h)$ is the density as a function of the altitude, ρ_0 is the reference altitude (surface) density, and $H_s = \frac{RT}{g_0}$ is the scale height (R, the specific gas constant, T the static temperature and g_0 the gravitational constant at the surface of the planet). The parameters ρ_0 and H_s can be fitted to the US76 model based on the *US76.dat* file provided in the assignment based on a least-squares approach, by rewriting Equation (1) into Equation (2),

Equation (2) is of the form $y = a - bx$, meaning that the Linear Least Squares Optimisation approach can be used to fit the data to the model. The general Least-Squares optimisation approach takes the form,

$$X^T X \begin{bmatrix} a \\ b \end{bmatrix} = X^T y \qquad X = \begin{bmatrix} 1 & -h_1 \\ \vdots & \vdots \\ 1 & -h_n \end{bmatrix} \qquad y = \begin{bmatrix} \ln(\rho(h_1)) \\ \vdots \\ \ln(\rho(h_n)) \end{bmatrix}$$

Resolving $(X^T X)^{-1} X^T y$ yields the parameters a and b which can then give ρ_0 and H_s from $\rho_0 = e^a = 1.5045$ kg/m^3 and $H_s = 1/b = 6964.5$ m. Those parameters therefore result in the best fit, which minimises the sum of the squared residuals with the data from the US76 model. However, the approach only takes the absolute difference between the reference data and the model, rather than the relative error. It would be desirable to minimise the relative error, as the same absolute error has a completely different implications for the accuracy of the data when considered at a low versus a high altitude. An alternative approach is to use the *fmincon()* command provided by Matlab and solve the non-linear problem, $d\rho = \frac{\rho_{ref} - \rho_0 e^{\frac{h}{H_s}}}{\rho_{ref}}$ by setting $d\rho = 0$. Based on guess values, the *fmincon()* command uses gradient-based methods to yield a local minimum. This means that the solution is sensitive to the initial condition, which needs to be chosen close enough to the final solution. In this report, the Least Squares solution was used as a guess to feed into *fmincon*. This results in Figure 1 exponential atmosphere model.

The *fmincon* command yields better results than the Least Squares approach, hence that one will be used throughout the rest of this report. This is confirmed by the norm of the $d\rho$ vector which is 534.4305 and 513.1518 for Least Squares and *fmincon* respectively. The model can be seen to be reasonably accurate until an altitude of about 86-87 km: prior to this altitude, the relative error oscillates around the 0% line (which is aimed at), being bounded by relative errors of $\pm 20\%$, which is reasonable considering the simplicity of the model. After this value, the model diverges (and likely would continue to do so if more data points were available for higher altitudes), and the relative error monotonously increases until $\approx -60\%$ at 100 km altitude.

2 Sensitivity Analysis

The parameters determined in the previous section are uncertain when modelling the atmosphere, meaning that the true atmosphere could be better modelled by different values of those parameters (because the values were obtained with respect to a certain model, and the true atmosphere differs from this model). It is therefore important to consider the sensitivity of the model with respect to those parameters. In this purpose, ρ_0 and H_s are both varied by $\pm 15\%$, resulting in four combinations of the two base parameters in minimum (-15%) or maximum ($+15\%$) values. The sensitivity is

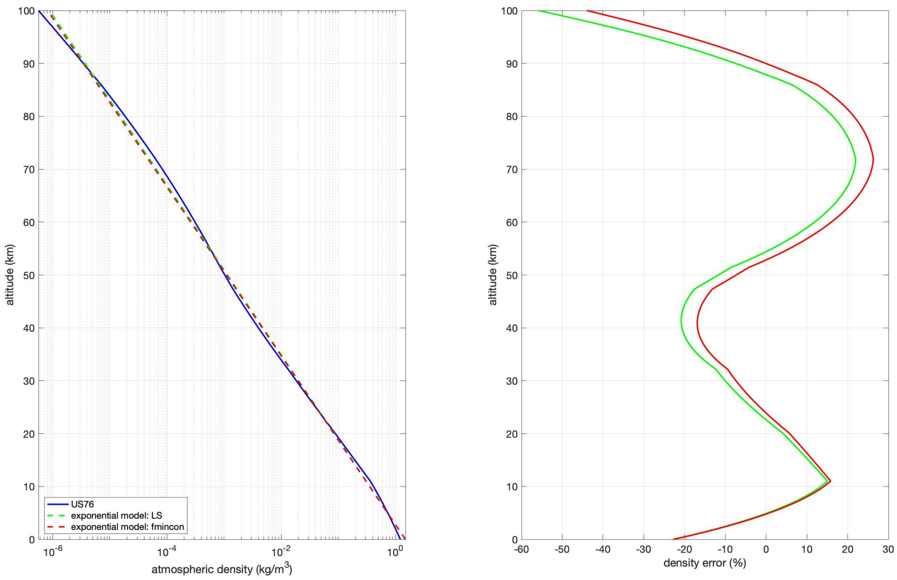
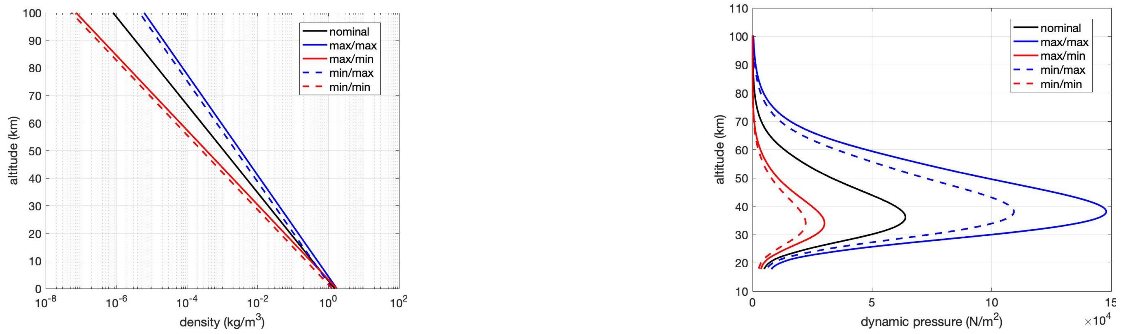


Figure 1: Exponential atmosphere model with $H_s = 6.9265e+03$ m and $\rho_0 = 1.5035$ kg/m^3.



(a) Density variation by $\pm 15\%$ of each parameter.

(b) Dynamic pressure variation by $\pm 15\%$ of each parameter.

Figure 2: Sensitivity analysis from the density and the dynamic pressure. Legend refers to ρ_0/H_s values.

considered for both the density as a function of the altitude and from the dynamic pressure through a specified re-entry trajectory from the assignment. Figure 2 shows the sensitivity of the density and dynamic pressure (of the provided re-entry trajectory) as a function of the altitude. Note that the nominal values used are the ones from the previous section:$H_s = 6.9265e + 03$ m and $\rho_0 = 1.5035$ kg/m^3.

First considering Figure 2a, it is clear that the scale height is the driving parameter for the density (as a function of altitude) of the model, as it defines the slope of the graph in the logarithmic scale. A change of 15% in either direction results in a variation of about one order of magnitude in the density at 100 km of altitude. Furthermore, the model seems to be much less sensitive to the ρ_0 parameter, as can be seen from considering two lines of the same colour in the plot (dashed and continuous), which is relative to a change of 15% in ρ_0. The lines are nearly aligned, resulting in only a small difference in density at a given altitude for a given scale height. Based on this result solely, it can be argued that the scale height needs to be approximated with the largest accuracy, for a given atmosphere.

Following, Figure 2b gives the dynamic pressure as a function of the altitude, for the given re-entry trajectory. This puts into perspective Figure 2a, by considering the main source of loading on the re-entry vehicle. From this plot, it can be seen that the altitude requiring the best accuracy is around 40 km, which is in the $\pm 20\%$ region discussed in section 1 (but on the far side, with a deviation of $\approx 20\%$ in density from the US76 model). It can again be seen that the scale height has the largest influence on the variation, however, this shows that the influence of parameter ρ_0 likely cannot be neglected when considering re-entry applications, which have very high velocities through the atmosphere (although this depends on the application). For the design of a re-entry vehicle, the sensitivity of the model to those parameters would likely be unacceptable, as the entire design would need to be changed if either of the parameters was not approximated appropriately. This would result in an over design of the vehicle for both the thermal and structural aspects, to ensure that it survives the descent even when one of the parameters was not estimated with a high enough degree of accuracy.

It can be argued that a model which is more accurate for the range of altitudes from 20 km to 80 km, by compromising the accuracy at low altitudes (<20 km) and high altitudes (>80 km), would be preferable when considering the design of a re-entry vehicle.

3 Sensitivity on Drag Force

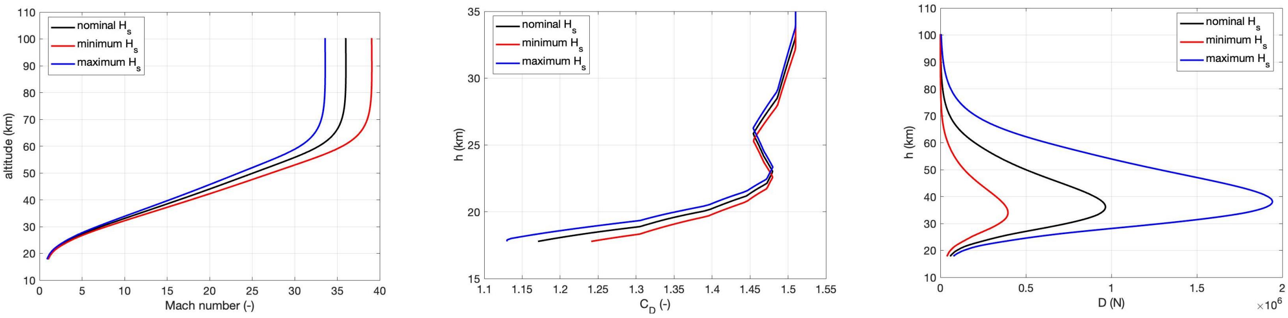
A change in the two base parameters ρ_0 and H_s results in changes in the dynamic pressure, but also in the (isothermal) temperature and therefore the speed of sounding. This change, therefore, propagates to the Mach number (defined as the ratio of the velocity of the vehicle and the speed of sound), which has a direct influence on the drag coefficient. The drag force is given by $D = C_D q$ (q is the dynamic pressure), and therefore changes from the variation of the base parameters.

The scale height is defined as $H_s = \frac{RT}{g_0}$, meaning that a given H_s results in a constant temperature through the atmosphere (isothermal atmosphere assumption). Additionally, the speed of sound for a calorically perfect gases ($T < 800K$, but the relation is assumed to hold for the entire re-entry) is given by $a = \sqrt{\gamma RT}$ where γ is the ratio of specific heat. Considering a variation of the scale height by $\pm 15\%$ gives Table 1. As expected, a larger scale height results in a higher temperature and therefore a larger speed of sound, and inversely for a smaller scale height.

The effects of scale height variation on the Mach number, drag coefficient, and drag force can be considered based on the velocity profile provided in the assignment. This is shown in Figure 3. Note that the drag coefficient was obtained by linearly interpolating the drag coefficient as a function of the Mach number data provided in the assignment. For Mach numbers outside the range provided in the *CD.dat* file, the maximum or minimum Mach number in the data provided is used (depending on whether the Mach number considered is above the maximum or below the minimum of *CD.dat*).

First considering Figure 3a, the effect on the Mach number directly arises from its definition: $M = \frac{V}{\sqrt{\gamma RT}}$, meaning that a larger scale height, results in a larger temperature and a lower Mach number for a given altitude. Logically, the curves have the same shape, and the effect is largest when the velocity of the vehicle is largest as well. That is, at the higher altitudes. As the velocity of the vehicle decreases, the different curves seem to merge together (but stay different, as will be seen below).

Following, the drag coefficient profile can be seen in Figure 3b. A larger scale height results in a higher speed of sound, and therefore a smaller Mach number, which is related to a smaller C_D (for the largest part of the flight, outside



(a) Mach number profile.

(b) Drag coefficient profile.

(c) Drag force profile.

Figure 3: Sensitivity analysis of H_s with $\pm 15\%$. Note: the colours were changed from the original code to match the same maximum (blue) and minimum (red) colours from question 2.

the 2.5-6 Mach range). The effect of the change in the scale height is the largest at low altitudes, where the velocity is the smallest and the drag coefficient is the most dependent on the Mach number. Despite the small difference in Mach number seen from Figure 3a for altitudes close to 20 km, a difference of about 0.1 is found between the maximum and minimum scale height curves. This is explained from considering the $C_D - M$ graph which can be seen from Figure 4, which shows that the largest sensitivity (slope of the graph) of the drag coefficient with respect to the Mach number, is at small Mach numbers. Additionally, the difference in drag coefficient is only significant at altitudes below about 22.5 km, meaning that the drag coefficient is quite accurately known during the phases of high dynamic pressure of the flight discussed earlier (80 to 20 km altitude). This reduces the uncertainty of the drag forces during those flight phases.

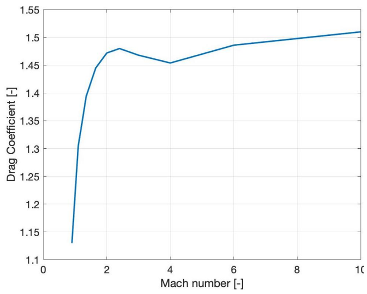


Figure 4: Drag coefficient as a function of Mach number.

Table 1: Temperature and speed of sound for a $\pm 15\%$ variation of the scale height.

	-15%	**Nominal**	**+15%**
Temperature [K]	201.17	236.67	272.18
Speed of sound [m/s]	284.31	308.38	330.70

Finally, the dynamic pressure and the drag coefficient discussed earlier can be combined together to obtain the drag force (assuming a surface of reference of 10 m^2). Here, a larger scale height relates to a smaller C_D (as discussed above) but a much larger dynamic pressure, as seen in question 2. The latter being driving, a larger drag occurs when the scale height is increased. Unsurprisingly, the shape is the same as the dynamic pressure as it is the main driver of the drag. The drag force is directly related to the structural design of the vehicle, and a higher drag force will result in larger maximum temperatures on the re-entry body, which is directly related to the thermal design. As a change of 15% in the main input parameter, the scale height, results in a doubling or halving of the drag force, it can be concluded that this parameter is very sensitive and needs to be very accurately estimated for a given atmosphere in order to design a re-entry vehicle. Furthermore, this graph shows that the point of highest drag force happens at about 40 km above the ground, which would then be used as the starting point of the design of a re-entry vehicle (most critical condition). Note that a mistake was found in the code, as CDmax was defined as "Dmax = CDmax qdynmin Sref", while it should be "Dmax = CDmax qdynmax Sref", as the 'max' subscripted variables are related to the maximum scale height considered. The variables computed with the same scale height (here Hs_{max}) are used together, as they belong to the same situation.

4 Mars and Titan

Similarly to what was done in question 1, the atmosphere of Mars and Titan can be modelled by an exponential atmosphere, by fitting Equation (1) to the data points.

The atmosphere of Mars was considered in two ways: a one layer atmosphere, or a multilayer atmosphere. In the case of a multilayer atmosphere, continuity at the layer's interface is ensured through, $\rho_{0_1} = \rho_{0_2} e^{\frac{\Delta h_{int}}{H_{s_1}}}$. Where Δh_{int} is the height difference between the start and the end of the atmospheric layer '1'. Hence, layer 1 (the new layer that is being computed) only has the scale height as variable, as ρ_0 is directly defined by the interface continuity relation. The equation solved for the new layer is then simple Equation (1) with the ρ_0 definition mentioned above (where ρ_{0_2} is already known). In this case, it was chosen to tackle the layers from up to down (considering the higher altitude layer first). A single layer, and a two layer model are compared in Figure 5.

Clearly the two layer model is more accurate than the one layer one. In the two layers model, the first layer goes from 0 to 36 km, which provides the best fit to the data for the two layer model, with the altitude records provided (all

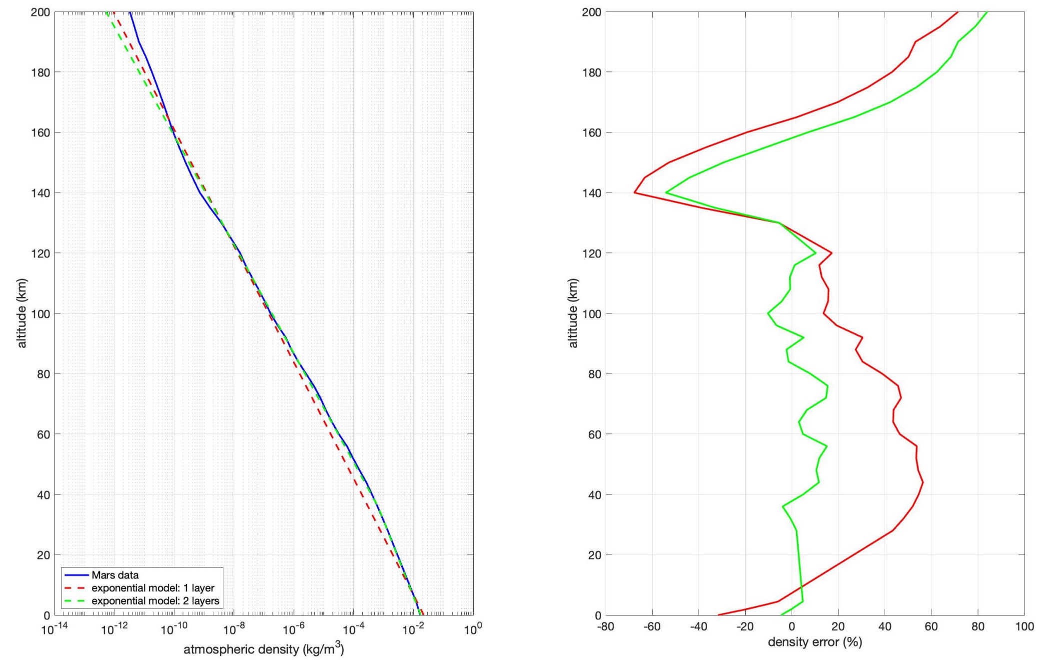


Figure 5: One layer and two layer models for Mars.

possibilities of layer interface altitude were tested, and 36 km yielded the smallest value of the norm of the $d\rho$ vector. Furthermore, it can be seen that the two layer model results in a variation of about $\pm 10\%$ from the measurement data, until 120 km, where the model starts to diverge significantly. The addition of a third layer to the model would likely largely improve the results for the upper layers of the atmosphere (which was not done due to time constraints). The single layer model is described by $\rho_0 = 0.0221$ kg/m^3 and $H_s = 8.37$ km. The two layer model is described by, lower layer: $\rho_0 = 0.0176$ kg/m^3 and $H_s = 10.9184$ km; upper layer: $\rho_0 = 6.5001e-04$ kg/m^3 and $H_s = 7.8352$ km.

A two layer model was implemented to model Titan's highly non-linear atmosphere. The principle being vastly the same as the procedure shown in question 1, in addition to the interface continuity equation. This results in Figure 6. The plots show that Titan's atmosphere is very non-linear, and that the relative error is quite large at all altitudes, compared to the model applied for Mars. The error is contained within $\pm 30\%$ from $\approx$150 to 1000 km for the upper layer, and then diverges. The lower level is clearly more linear and results in more uncertainty of the model, which is shown by the error varying significantly over the altitude. No "trustworthy" range of altitude can be considered. Overall, the accuracy of this model would make it extremely difficult to attempt the design of a Titan re-entry vehicle, without over designing it to ensure a proper survival, similarly to the discussion of the Earth's model (but in worse). The following parameters describe the exponential atmosphere, lower layer: $\rho_0 = 5.1045$ kg/m^3 and $H_s = 19.79$ km; upper layer: $\rho_0 = 0.0043$ kg/m^3 and $H_s = 49.68$ km.

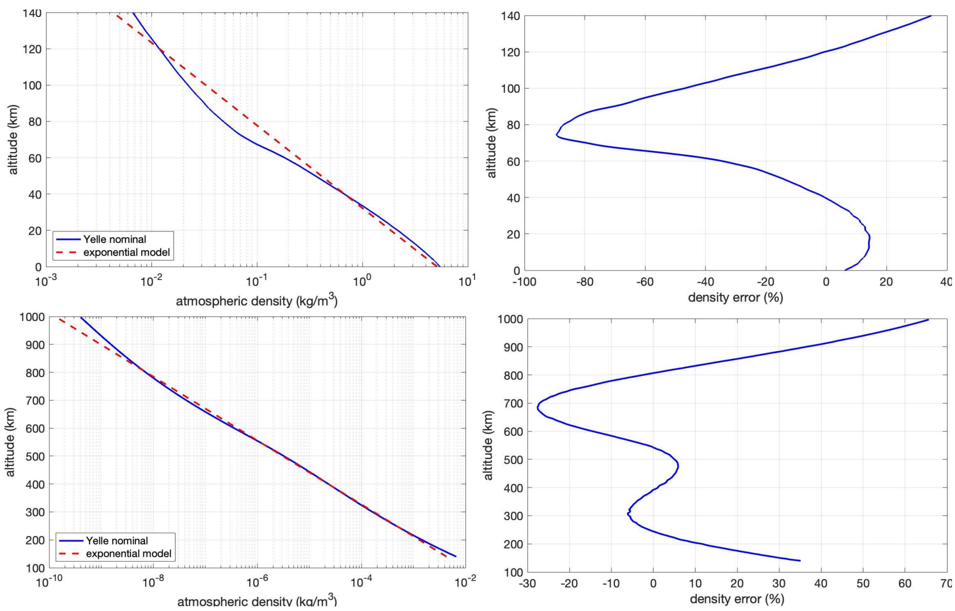


Figure 6: Titan two-layer atmospheric model. L1: 0-140 km; L2: 140-1000 km.