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The changing face of adaptive control: The use of multiple models ☆

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ABSTRACT

Adaptive systems that continuously monitor their own performance and adjust their control strategies to improve it, have been studied for over 50 years. The theory of such systems is now commonly referred to as classical adaptive control. Such control is now well established and is found to be satisfactory when the uncertainty in the system to be controlled (i.e. the plant) is small.

During the past 15 years several attempts were made to extend this general methodology to systems with large uncertainties, by using multiple models to identify the plant. Among these, two general methods based on "switching" and "switching and tuning" have emerged as the leading contenders. Recently, a radically different approach was proposed by the authors (Han & Narendra, 2010b), in which the multiple models are used to play a significantly larger role in the decision making process, resulting in substantial improvement in performance.

In this paper, which is tutorial in nature, the three methods based on multiple models are critically examined. At the same time, alternative methods using fixed and adaptive models are also proposed. In all cases, detailed simulation studies of adaptation in different environments are presented. Theoretical explanations are given, where available, for the wide spectrum of performances observed in the simulation studies.

1. Introduction

In this paper an attempt is made to examine critically the use of multiple models for the adaptive control of linear dynamical systems with large uncertainties. The latter represent a class of systems for which classical adaptive control (adaptive control using a single identification model) is generally found to be unsatisfactory. Our objectives are to determine, on the practical side, the improvement in performance that results from the use of multiple models, and on the theoretical side to investigate the effect of using multiple models on both stability and robustness. Since the number of models needed generally has a bearing on the feasibility of the methods proposed in practical applications, we shall also attempt to discuss the tradeoffs involved. While numerous approaches based on multiple models have been proposed in the past, two methods (Morse, 1996; Narendra & Balakrishnan, 1997) have emerged over the years as the principal contenders. Our aim in this paper is to formulate new methods which use different information structures obtained from the different models, and discuss the relative performance of such methods as compared to the two mentioned earlier. What follows is consequently a fairly exhaustive comparative study of both the practical and theoretical aspects of different schemes using multiple models, for the identification and control of linear systems.

1.1. Historical background

The concept of using multiple models is not new in control theory. Multiple Kalman filters were studied in the 1970s to improve the accuracy of the state estimate in control problems (Athans et al., 1977; Lainiotis, 1976; Magill, 1965). This was followed by several applications (Lane & Maybeck, 1994; Li & Bar-Shalom, 1993; Moose, Landingham, & McCabe, 1979; Yu, Roy, Kaufman, & Bequette, 1992). In all cases, no switching was involved and no stability results were reported. In the late 1980s, switching was introduced in the context of adaptive control by Martensson (1986). Around this time, two types of switching schemes were proposed. In the first, termed direct switching (Fu & Barmish, 1986; Martensson, 1986; Miller, 1994; Miller & Davison, 1989; Poolla & Cusumano, 1988), the emphasis was on when to switch to the next controller in a pre-determined sequence, based directly on the output. However, it was soon realized that such schemes have little practical utility. Indirect switching, which constitutes the second class, uses multiple models to determine both when and to which controller to switch. These are far more attractive for practical applications. This was first proposed by Middleton, Goodwin, Hill,

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and Mayne (1988) and later modified in by Morse, Mayne, and Goodwin (1992) and Weller and Goodwin (1994). The use of switching and tuning using multiple models was first introduced by Narendra and Balakrishnan (1992) and later studied extensively by Narendra and Balakrishnan (1994, 1997). At the same time Morse (1993) studied the use of multiple fixed models for switching.

The ideas proposed, and the proofs of stability provided in Narendra and Balakrishnan (1992, 1994, 1997) and Morse (1996, 1997), have been responsible for the development of general methodologies for adaptive control based only on switching (Morse, 1996) and on both switching and tuning (Narendra & Balakrishnan, 1997). Both methodologies have found wide application in different fields, including flight control problems (Bošković & Mehra, 1999).

It is also worth pointing out that the success of the simulation studies reported in Narendra and Balakrishnan (1992) were primarily responsible for the detailed study of switching and tuning methods included in Narendra and Balakrishnan (1994, 1997). As a continuation of the effort started in 1994, extensive investigations of the use of multiple models for adaptive control have been carried out by the authors in the past 3 years, and these, in turn, led to the radically new developments reported in Narendra and Han (2010) and Han and Narendra (2010a, 2010b).

1.2. Motivation and need for multiple models

It is generally accepted that the classical adaptive control methods that have been developed are both stable and robust for time-invariant plant when the parametric uncertainty is small (Goodwin & Sin, 1984; Ioannou & Sun, 1996; Krstić, Kanellakopoulos, & Kokotović, 1995; Middleton et al., 1988; Narendra & Annaswamy, 1989; Tao, 2003). The need for analytic tools for reacting effectively to large uncertainties has been arising in a variety of fields including biology and medicine, economics and finance, and various engineering problems such as energy management, aircraft and automotive control, and security. These problems can be cast as adaptive decision making (or control) problems in which an unknown plant parameter vector varies rapidly with large amplitude. The objective in such cases is to estimate the variations accurately in the presence of disturbances, and take appropriate control action. In many of these applications, it has been found that when the parameter errors are large, classical adaptive control methods result in large and oscillatory responses. The limitations of classical adaptive control also became evident when the plant parameters varied rapidly with time. With increasing interdependence of different types of systems, such problems began to occur with even greater frequency. Classical adaptive control was found to be too slow and consequently incapable of coping with the new family of problems.

It was to address the above class of problems that the use of multiple models was originally proposed. A vast number of contributions were made in which different assumptions were made regarding the plant to be controlled. Among these, two methods emerged as general approaches to the problem of adaptively controlling a general linear time invariant plant with large uncertainties. The first involved switching between fixed models (referred to as switching), and the second involved switching between fixed and adaptive models (referred to as switching and tuning). A block diagram illustrating this kind of multiple models schemes is shown in Fig. 1. N identification models with a series-parallel structure are set up to identify the unknown plant, and N controllers are constructed each corresponding to one of the identification models. Based on the identification errors, one of the controller is chosen using a predefined selection criterion and subsequently used to generate the control input. As mentioned earlier, both fixed models, adaptive models, and their combinations were proposed for use in the identification process. Each of these methods and a few variations are discussed separately in Sections 3–6. To provide a benchmark for comparing the performances of these different schemes we present a relatively simple adaptive control problem and describe all the steps involved in controlling it with a single identification model. Following this, simulation studies on a simple second order system are presented. The same example is also used throughout the paper to compare the performance of the different schemes.

2. Classical adaptive control problem

An LTI plant Σ_p is described by the state equations

$$\Sigma_p : \dot{x}_p(t) = A_p x_p(t) + bu(t) \tag{1}$$

where $x_p(\cdot) : \mathbb{R}^+ \to \mathbb{R}^n$, $u(\cdot) : \mathbb{R}^+ \to \mathbb{R}$. $A_p \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ are in companion form and defined as

$$A_p = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pn} \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{2}$$

The parameters a_{pi} $(i = 1, 2, \ldots, n)$ are unknown and have to be estimated from observations on $x_p(t)$ (which is assumed to be accessible) and $u(t)$. Let $\theta_p = [a_{p1}, a_{p2}, \ldots, a_{pn}]^T$ be the unknown parameter vector. It is assumed that bounds on θ_p is known and that $\theta_p \in \mathcal{S}_\theta = \{\theta_p : \underline{\theta}_p \leqslant \theta_p \leqslant \bar{\theta}_p\}$.

A reference model Σ_m is described by the differential equation

$$\Sigma_m : \dot{x}_m(t) = A_m x_m(t) + br(t) \tag{3}$$

where $r(\cdot) :\ \mathbb{R}^+ \to \mathbb{R}$ is a known bounded piecewise continuous reference signal. The matrix A_m is also in companion form, is stable, and has the last row $[a_{m1}, a_{m2}, \ldots, a_{mn}] = \theta_m^T$. Assuming that $\theta_p \in \mathcal{S}_\theta$, where $\mathcal{S}_\theta$ is a compact set in parameter space, the objective is to determine the input $u(\cdot)$ to the plant such that all signals in the system are bounded and $\lim_{t\to\infty}[x_p(t) - x_m(t)] = 0$.

An identification model Σ can be set up as

$$\Sigma :\ \dot{x}(t) = A_m x(t) + [A(t) - A_m]x_p(t) + bu(t) \tag{4}$$

where $A(t)$ is a matrix also in companion form, with the last row consisting of adjustable parameters $[a_1(t), a_2(t), \ldots, a_n(t)] = \theta^T(t)$ (the estimates of the plant parameters). Defining $\theta(t) - \theta_p = \tilde{\theta}(t)$ to be the parameter error and $x(t) - x_p(t) = e(t)$ to be the identification error, the error equation can be written as

$$\dot{e}(t) = A_m e(t) + b\tilde{\theta}^T(t)x_p(t). \tag{5}$$

If a Lyapunov function candidate is set up as

$$V(e, \tilde{\theta}) = e^T Pe + \tilde{\theta}^T\tilde{\theta} \tag{6}$$

where P is the positive definite matrix solution of the Lyapunov equation $A_m^T P + PA_m = -Q,\ Q = Q^T > 0$, it follows that

$$\dot{V}(e, \tilde{\theta}) = -e^T Qe + 2e^T Pb\tilde{\theta}^T x_p + 2\tilde{\theta}^T\dot{\tilde{\theta}}, \tag{7}$$

so the adaptive law

$$\dot{\theta}(t) = -x_p(t)e^T(t)Pb \tag{8}$$

results in $\dot{V}(e, \tilde{\theta}) = -e^T Qe \leqslant 0$. This assures the stability of the identifier and consequently the boundedness of both the identification error $e(t)$ and the parameter error $\tilde{\theta}(t)$ (and hence $\theta(t)$).

In order to assure the stability of the plant, and consequently the boundedness of $x_p(t)$, using well known results from adaptive control, the feedback control law is chosen as

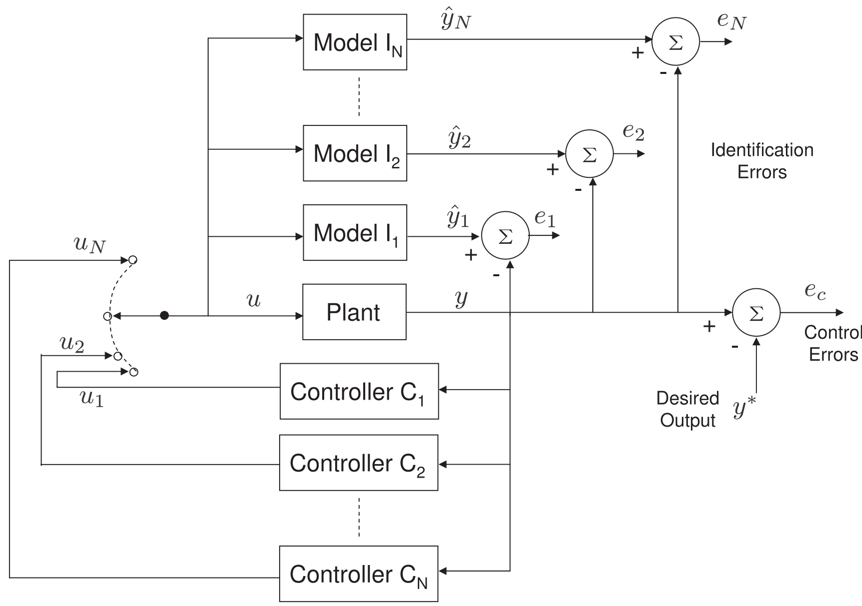


Fig. 1. Block diagram of adaptive control using multiple models.

$$u(t) = k^T(t)x_p(t) + r(t) \tag{9}$$

where

$$k(t) = \theta_m - \theta(t). \tag{10}$$

This yields the control error equation

$$\dot{e}_c(t) = A_m e_c(t) - b\tilde{\theta}^T(t)x_p(t) \tag{11}$$

where $e_c(t) = x_p(t) - x_m(t)$.

Alternatively, if the direct control approach is used

$$\dot{e}_c(t) = A_m e_c(t) + b\tilde{k}^T(t)x_p(t) \tag{12}$$

where $\tilde{k}(t) = k(t) - k^*$ and $k^* = \theta_m - \theta_p$. The adaptive law for adjusting $k(t)$ can be derived as

$$\dot{k}(t) = -x_p(t)e_c^T(t)Pb. \tag{13}$$

Following standard arguments in adaptive control, it follows that $e_c \in \mathcal{L}^2 \cap \mathcal{L}^\infty$ which assures the boundedness of $x_p(t)$ and $\dot{e}_c(t)$. By using Barbalat's lemma it follows that $\lim_{t\to\infty} e_c(t) = 0$. Or the state $x_p(t)$ of the plant follows the state $x_m(t)$ of the reference model asymptotically.

Further, it is well known that $\tilde{\theta}_i(t)$ tends to zero if $u(t)$ is persistently exciting, so that $\lim_{t\to\infty}\theta_i(t) = \theta_p$.

Example 1. We illustrate the classical adaptive control approach discussed thus far using the following example. An unstable dynamical system in $\mathbb{R}^2$ is described by the differential equation (1) with unknown parameter vector $\theta_p = [4, 0]^T$. The reference model is stable with $\theta_m = [-6, -5]^T$ and has poles at -3 and -2. θ_p is known to belong to the set $\mathcal{S}_\theta$ where $\mathcal{S}_\theta = [-15, 15] \times [-15, 15] \in \mathbb{R}^2$. Three experiments were carried out where a single identification model was initialized at $\theta_i(0) = [-5, 5]^T$, $\theta_i(0) = [-15, 15]^T$, and $\theta_i(0) = [7, 3]^T$, respectively. The parameter space trajectories in each experiment and the corresponding system responses are illustrated in Figs. 2–4, respectively. An adaptive gain $\gamma = 10$ was used in each case.

It is seen from Fig. 3 that when the identification model is initialized far away from the plant in parameter space, the trajectory is highly nonlinear and the corresponding system response is unsatisfactory with a large transient and slow convergence. However, as shown in Fig. 4, if it is started in the vicinity of the plant, the identification model adapts in a linear fashion and the resulting performance is satisfactory.

3. Multiple adaptive models with switching

Instead of using a single model for identification and control as described in Section 2, N identification models $\Sigma_1, \Sigma_2, \ldots, \Sigma_N$ can be set up to provide N estimates of the unknown plant parameter vector as

$$\Sigma_i: \ \dot{x}_i(t) = A_m x_i(t) + [A_i(t) - A_m]x_p(t) + bu(t) \tag{14}$$

with $x_i(t_0) = x_p(t_0)$ $(i \in \Omega = \{1, 2, \ldots, N\})$.

Comment: The N identification models are described by identical differential equations with the same initial state as the plant but with different initial values of the parameter vectors. The former condition is realizable since it is assumed that the plant states are accessible.

From the above assumptions it follows that the identification errors $e_i(t) = x_i(t) - x_p(t)$ $(i \in \Omega)$ satisfy the error differential equations

$$\dot{e}_i(t) = A_m e_i(t) + b\tilde{\theta}_i^T(t)x_p(t) \tag{15}$$

with $\theta_i(t_0) = \theta_{i0}$ and $e_i(t_0) = 0$ $(i \in \Omega)$.

The corresponding stable adaptive laws have the form

$$\dot{\theta}_i(t) = \dot{\tilde{\theta}}_i(t) = -x_p(t)e_i^T(t)Pb. \tag{16}$$

Since N models are operating in parallel, the question arises as to how the information obtained is to be used to control the system at every instant. This becomes particularly relevant when the plant is unstable. Using classical adaptive control theory, any one of the estimates can be used to stabilize the system. It was suggested in Narendra and Balakrishnan (1992, 1994, 1997) that performance indices of the form

$$\begin{aligned} &J_i(t) = \textstyle\int_{t_0}^t \|e_i(\tau)\|^2 d\tau \\ &\text{or } J_i(t) = \alpha\|e_i(t)\|^2 + \beta\textstyle\int_{t_0}^t \|e_i(\tau)\|^2 d\tau \qquad (i \in \Omega) \\ &\text{or } J_i(t) = \alpha\|e_i(t)\|^2 + \beta\textstyle\int_{t_0}^t e^{-\lambda(t-\tau)}\|e_i(\tau)\|^2 d\tau \end{aligned} \tag{17}$$

could be used to compare the different estimates and provide a basis for the choice of the control parameter vector. If one of the

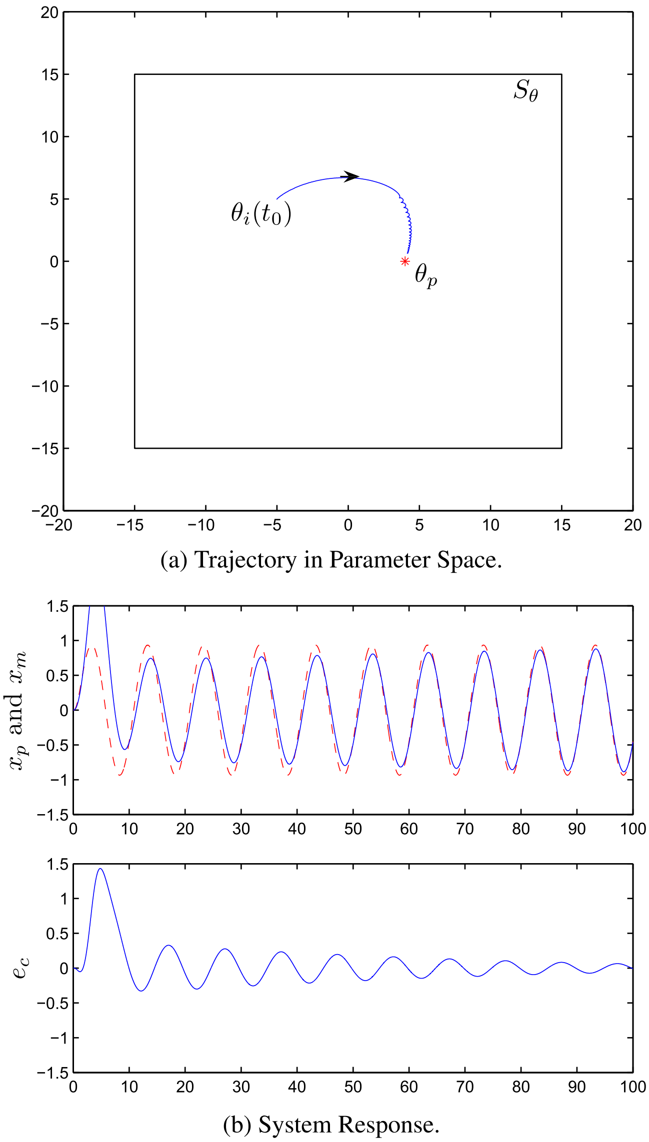


Fig. 2. Classical adaptive control with $\theta_i(t_0) = [-5, 5]^T$.

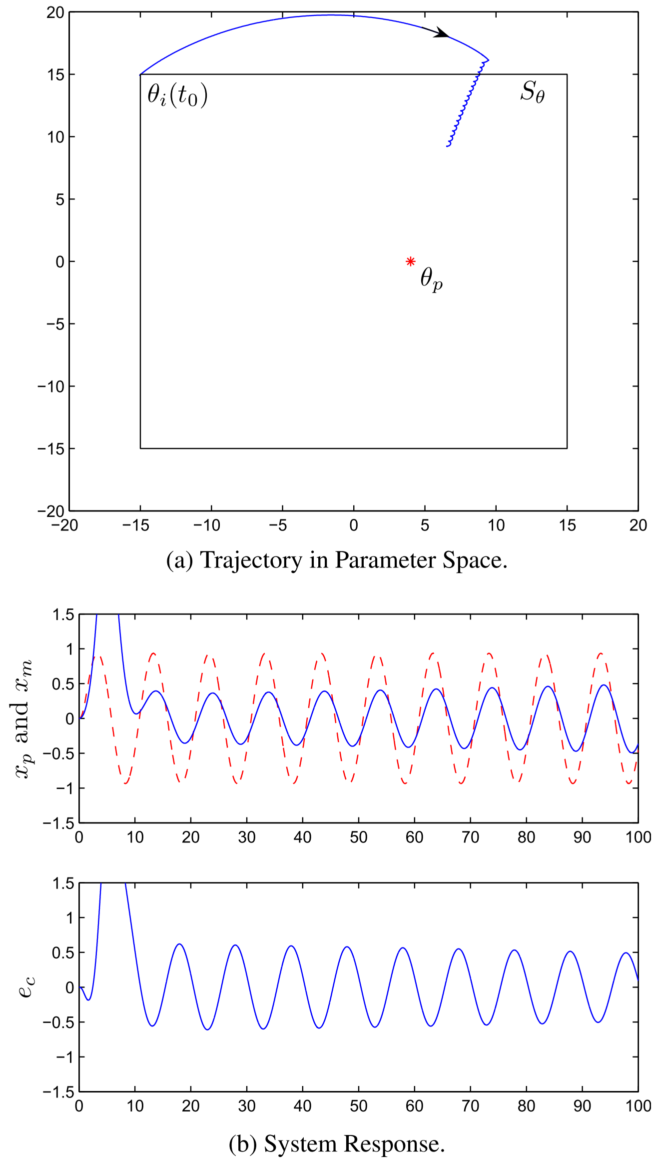


Fig. 3. Classical adaptive control with $\theta_i(t_0) = [-15, 15]^T$.

models corresponding to $\min_i\{J_i(t)\}$ is chosen as the "best" at any instant according to one of the criteria in (17), it can, in turn, be used to select the controller parameter. It was shown in Narendra and Balakrishnan (1997) that if an arbitrarily small but finite dwell time is used, switching between different parameters results in the stability of the overall system and the asymptotic convergence of the control error to zero. As in classical adaptive control, the parameter estimates need not converge to the plant parameter vector θ_p but do so if the reference input is persistently exciting.

Comment: The efficacy of the control depends upon how rapidly (and how accurately) the plant parameter can be estimated. When the number of models N is small and the region of uncertainty $\mathcal{S}_\theta$ is large, the improvement in the transient behavior of the system, over that realized using a single model, may not be significant.

4. Multiple fixed models with switching

In this case N fixed models are used instead of N adaptive models as described in (14) with $A_i(t) \equiv A_{i0}$.

While switching is discontinuous, fast, but coarse, it is well known that for switching to result in stability, a necessary condition is that there is at least one fixed model near the plant which when stabilized using a controller, will also stabilize the plant. Therefore, the method calls for a large number of models (Vinnicombe, 1993). The above number also increases exponentially with the dimension of the unknown parameter vector.

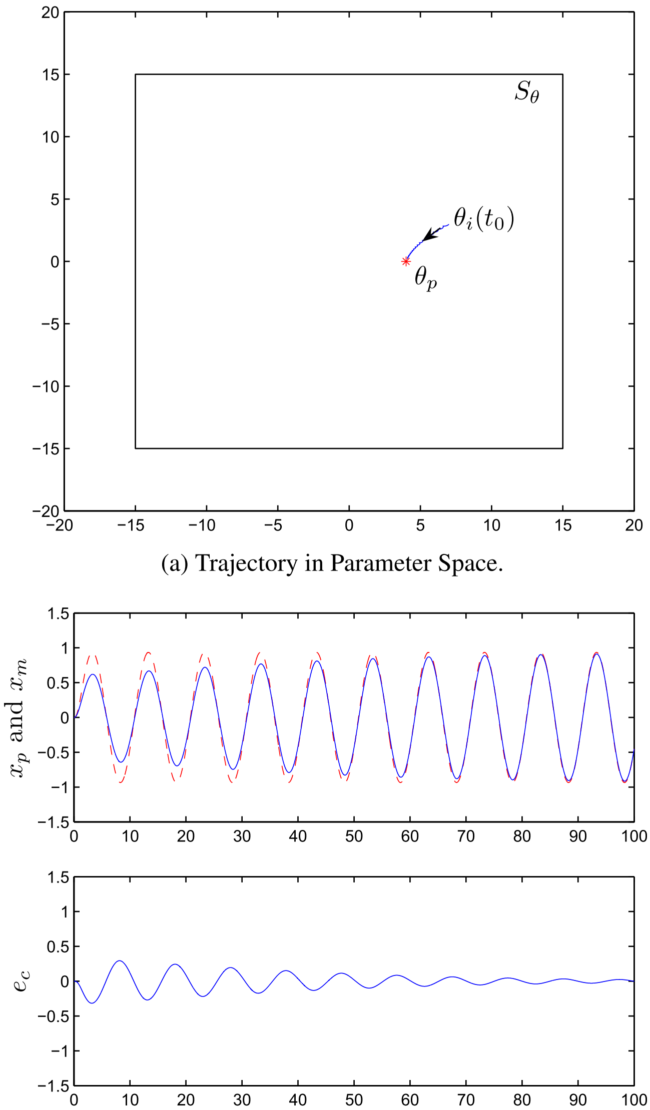
We illustrate this method using the same example as in Example 1 (see Fig. 5).

Example 2. In Fig. 5, 16 fixed models were uniformly distributed in the uncertainty region $\mathcal{S}_\theta$. It was observed that the switching never stopped; instead, asymptotically, it continued to switch between parameters θ_7, and θ_{11}. When only four fixed models were used, switching continued indefinitely between parameters θ_1, and θ_4, and the system became unstable.

Analytically, from Eqs. (1), (9), and (10) it follows that

$$\dot{x}_p(t) = [A_p + b(\theta_m - \theta_i)^T]x_p + br. \tag{18}$$

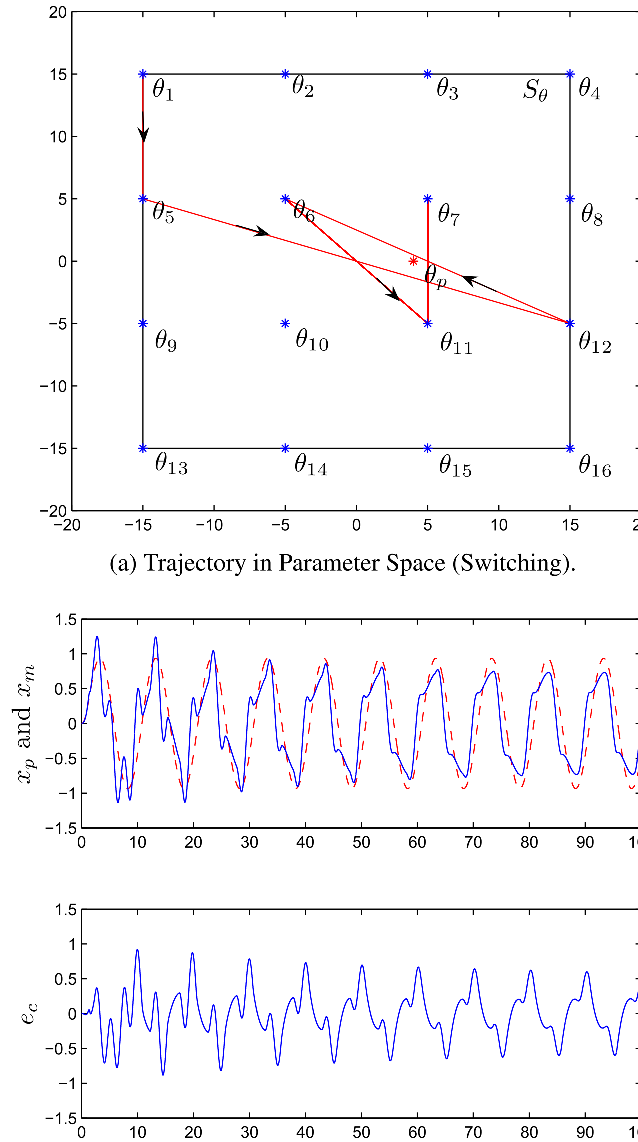
For the example under consideration, when 16 fixed models were used, since



(a) Trajectory in Parameter Space.

(b) System Response.

Fig. 4. Classical adaptive control with $\theta_i(t_0) = [7, 3]^T$.



(a) Trajectory in Parameter Space (Switching).

(b) System Response.

Fig. 5. Multiple (16) fixed models.

$$[A_p + b(\theta_m - \theta_7)^T] = \begin{bmatrix} 0 & 1 \\ -7 & -10 \end{bmatrix} \tag{19}$$

and

$$[A_p + b(\theta_m - \theta_{11})^T] = \begin{bmatrix} 0 & 1 \\ -7 & 0 \end{bmatrix} \tag{20}$$

the controller that stabilized models θ_7 and θ_{11} also stabilized the unknown plant.

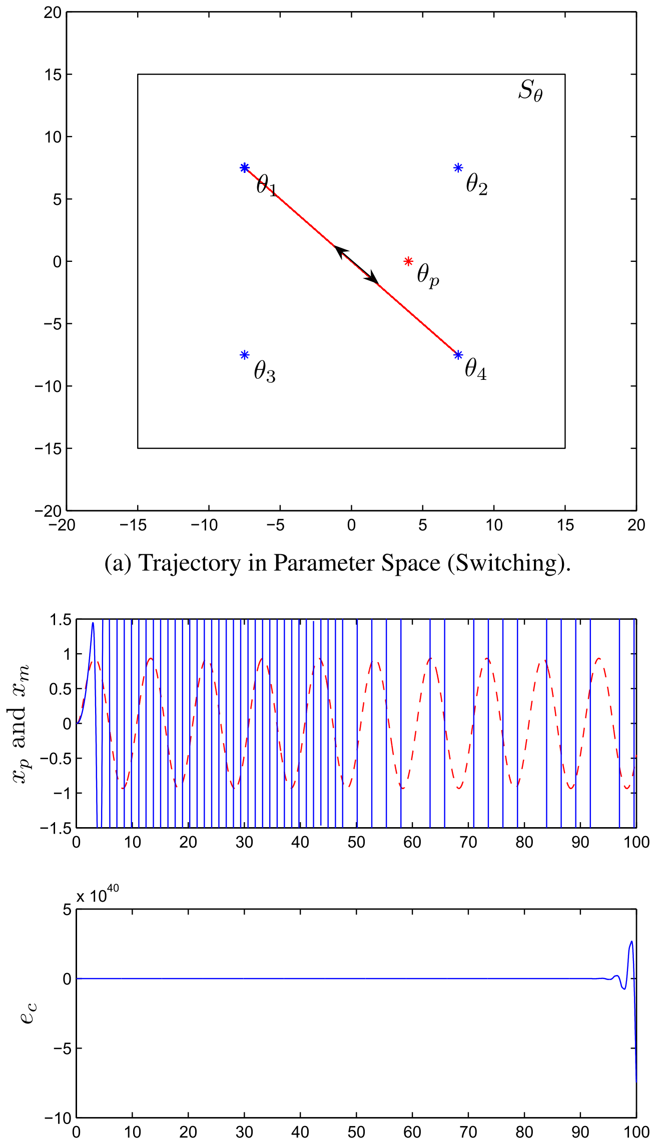
When four fixed models were used, there was only a single model among them, i.e. θ_2 which when stabilized would also assure the stability of the overall system. However, θ_2 was never selected by the controller (Fig. 6).

5. Multiple models with switching and tuning

"Switching and tuning" utilizing both fixed and adaptive models was proposed in Narendra and Balakrishnan (1992, 1994, 1997). A large number of fixed models are distributed in the region of uncertainty, and based on the responses of the plant and the models, one of the models is chosen at every instant as the "best" fit for the unknown plant using one of the J_i indices (17). The integral square error or a modification of that has been used by many authors as the index of performance. In "switching and tuning", an adaptive model is initialized from the location of the fixed model chosen, and the parameters of the latter determine the control to be used.

Example 3. The example showed in Examples 1 and 2 was simulated using multiple models with switching and tuning. In Fig. 7, nine fixed models were uniformly distributed in the uncertainty region $\mathcal{S}_\theta$, one free-running adaptive model and one re-initialized adaptive model were both started at one of the corners of $\mathcal{S}_\theta$. The re-initialized adaptive model switched to the fixed model which was closest to the plant and started adaptation from there. An adaptive gain $\gamma = 10$ was used for both adaptive models which was the same as in Example 1. The resulting performance was seen to be satisfactory.

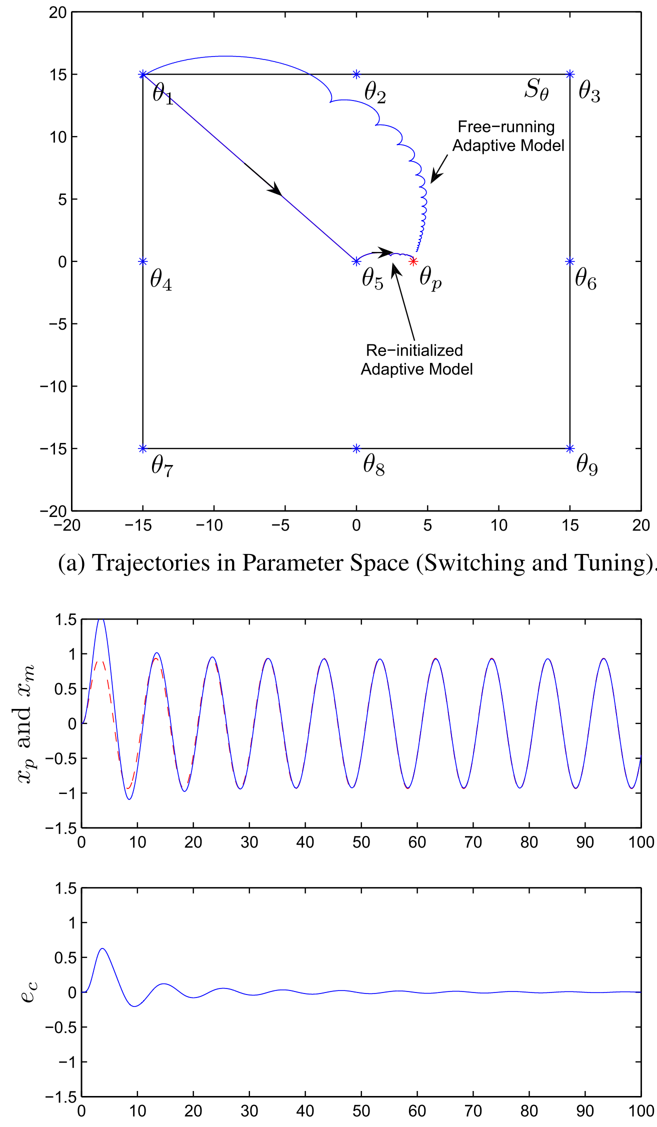
As seen from this section, when the number of models that can be used is sufficiently large, the "switching and tuning" scheme performs satisfactorily, and no specific consideration is needed



(a) Trajectory in Parameter Space (Switching).

(b) System Response.

Fig. 6. Multiple (4) fixed models.



(a) Trajectories in Parameter Space (Switching and Tuning).

(b) System Response.

Fig. 7. Nine fixed models and two adaptive models.

concerning the location of the fixed models. However, when the number of models available is relatively small compared with the size of the uncertainty region, the question naturally arises as to whether the fixed models can be redistributed so that they are used more effectively to result in improved performance. This idea is discussed in the following section.

6. Redistribution of fixed models

If fixed models with parameter vectors θ_i are used to estimate the plant parameter vector θ_p, it was shown (Eq. (15)) that the output identification errors $e_i(t)$ and parameter error vectors $\tilde{\theta}_i$ are related by the differential equation

$$\dot{e}_i = A_m e_i + b\tilde{\theta}_i^T x_p.$$

A vast number of procedures for estimating $\tilde{\theta}_i$ have been reported in the past. For the following discussion, we merely express $e_i(t)$ as $e_i(t) = H(t)\tilde{\theta}_i$, (where $H(t) = \int_0^t e^{A_m(t-\tau)} b x_p^T(\tau) d\tau$) so that $e_i(t)$ and $\tilde{\theta}_i$ are linearly related. Since $A_m,\ b$ are known, and $x_p(t)$ is accessible, $H(t)$ can be computed at every instant of time. This implies that the integral square error $J_i(t)$, of the ith model has the form

$$J_i(t) = \int_0^t e_i^T(\tau)e_i(\tau)d\tau = \tilde{\theta}_i^T\left[\int_0^t H^T(\tau)H(\tau)d\tau\right]\tilde{\theta}_i = \tilde{\theta}_i^T W(t)\tilde{\theta}_i \qquad (21)$$

which is a quadratic (time-varying) function of the unknown parameter vector $\tilde{\theta}_i$. Assuming that the symmetric matrix $W(t)$ is positive definite for $t \geqslant T$, it follows that the performance indices of all the models are merely points on a time-varying quadratic surface, whose minimum corresponds to the plant. This can consequently be used as a basis for the redistribution of the models.

6.1. Redistribution Method I

It was earlier assumed that $W(t)$ is positive definite for $t \geqslant T$. It consequently follows that $J_i(T) = \tilde{\theta}_i^T W(T)\tilde{\theta}_i$ is a quadratic function of $\tilde{\theta}_i$. At time $t = T$, the values $J_i(T)$ represent points on a surface $\tilde{\theta}^T W(T)\tilde{\theta}$, and the objective is to determine the location of the minimum of the function using the observed data (which corresponds to the unknown plant parameter vector θ_p). Several heuristic approaches were considered, but most of them required explicit knowledge of the symmetric matrix $W(T)$ (which can be theoretically computed). The following method, which merely assumes the quadratic nature of the surface (is simple, and scales readily to higher dimensions) was chosen as the most satisfactory one.

An example of performance index function is shown in Fig. 8.

Assume that $(N-2)$ fixed models and two adaptive models are set up as in the "switching and tuning" scheme. If the ith fixed model θ_i (corresponding to model M_i) has the minimum index of performance (i.e. $J_i = \min_j\{J_j\}$), all the other $(N-3)$ fixed models M_j $(j \neq i)$ are redistributed in parameter space along the straight lines connecting θ_j to θ_i with the new location determined by

$$\bar{\theta}_j = \frac{\sqrt{J_j}}{\sqrt{J_i}+\sqrt{J_j}}\theta_i + \frac{\sqrt{J_i}}{\sqrt{J_i}+\sqrt{J_j}}\theta_j. \tag{22}$$

In Fig. 9 M_1 represents the model with the minimum index of performance, P represents the plant, and M_2 to M_4 represent three additional fixed models which are redistributed. The performance indices corresponding to the new model positions are smaller, and the model closest to the plant has a smaller index than M_1. The free-running adaptive models are unaffected by the redistribution. Adaptation in the new setting is initiated at one of the fixed models. Stability of the overall system is assured by using two adaptive models, as in Narendra and Balakrishnan (1997).

Example 4. In this example the same problem simulated in the previous examples are now shown with redistribution using Method I (Fig. 10). The fixed models are redistributed every $T = 5$ units of time. It is seen that all nine models converged to the plant after only two iterations which leads to satisfactory performance. However, it is worth noting that this is not always the case. In many situations, depending upon the location of models and the unknown plant, it may require significantly larger number of iterations of the procedure for one of the models to be sufficiently close to the plant.

6.2. Redistribution Method II

If $\mathcal{S}_\theta$ is the region of uncertainty, and if, after observing the response of a set of fixed models over an interval of length T it can be concluded that the plant lies in a subregion $\mathcal{S}_\theta^1 \subset \mathcal{S}_\theta$, then all the fixed models can be redistributed in $\mathcal{S}_\theta^1$. While several heuristic approaches were considered, very few combined mathematical

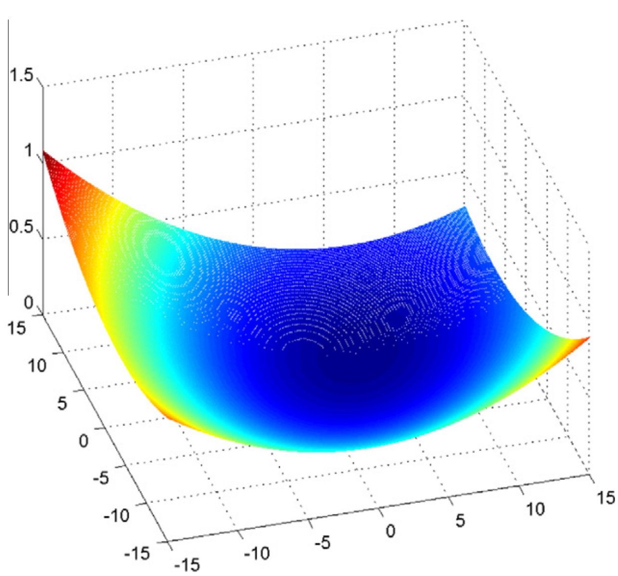


Fig. 8. Performance index function.

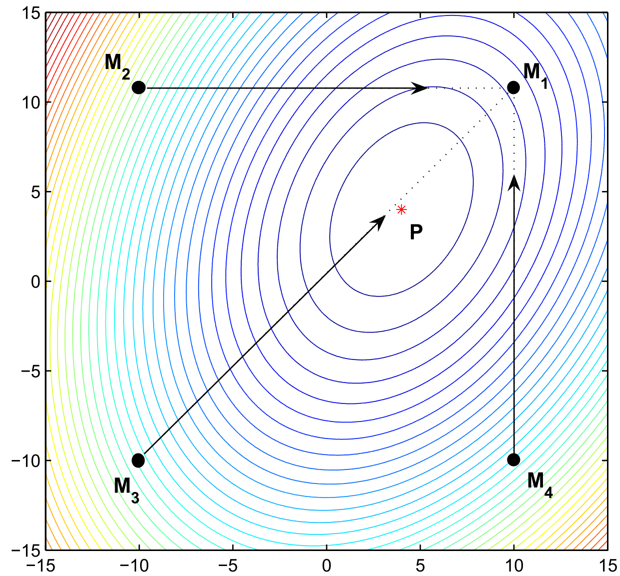


Fig. 9. Performance index function contours.

tractability, simplicity, and scalability (for extension to higher dimensions). One of the latter is briefly described below.

$\mathcal{S}_\theta$ is partitioned into a uniform lattice with the N models provided. Using fixed models the performance index J_i $(i = 1, 2, \ldots, N)$ is evaluated at the vertices of the lattice. Using an appropriate criterion the location of the plant in a subregion is concluded. One such scheme computed $\bar{J}_k$ corresponding to the kth subregion, where $\bar{J}_k$ is the sum of the J_i at its vertices. However, it is hard to provide a theoretical justification for the procedure. A two-dimensional lattice in $\mathcal{S}_\theta$ with four subregions, and the subregion $\mathcal{S}_\theta^1$ in $\mathcal{S}_\theta$ is shown in Fig. 11. $\bar{J}$ corresponding to $\mathcal{S}_\theta^1$ is $J_5 + J_6 + J_8 + J_9$.

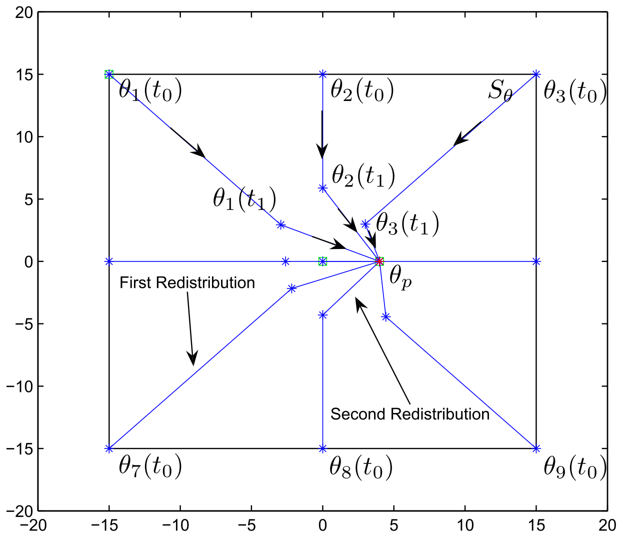
Comment: In the schemes discussed thus far, the responses of fixed models over an interval T are used (i) to redistribute the models or (ii) to determine the subregion $\mathcal{S}^1$ containing the plant. T is consequently a critical adaptive parameter in both cases.

As in Redistribution Method I, stability of the overall system is guaranteed if two adaptive models are used.

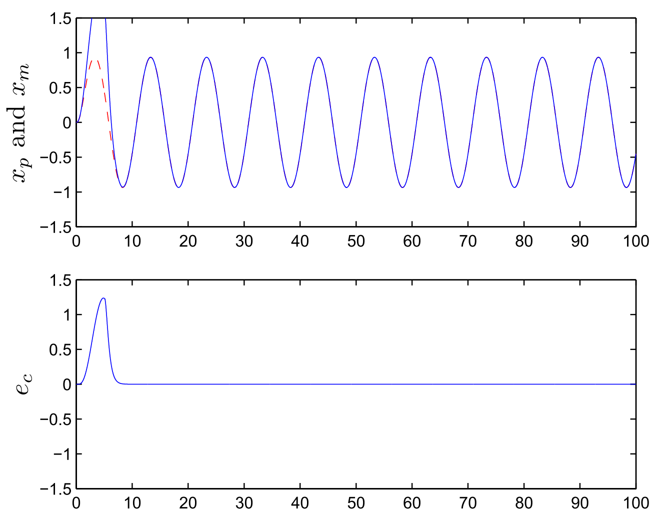
Example 5. In this example the same problem simulated in the previous examples are now shown with redistribution using Method II (Fig. 12). The fixed models are redistributed every $T = 7.5$ units of time. All nine models were found to converge to the plant. However, a large number of iterations are required for any one of the models to be sufficiently close to the plant.

The different methods using redistribution of multiple models represent useful extensions of the "switching and tuning" scheme. As shown in Han and Narendra (2009) through extensive simulation studies, the redistribution based methods generally improve performance compared with the "switching and tuning" method which justifies the additional effort and complexity involved. However, since the methods are still only extensions and variations of the "switching and tuning" method, they suffer from the following disadvantages:

(i) the number of models required still increases exponentially with the dimension of the unknown plant parameter vector θ_p,
(ii) the algorithm is more complex than the "switching and tuning" method,
(iii) re-initialization is required in a time-varying environment, and
(iv) only partial theoretical justification is currently available.



(a) Trajectories in Parameter Space.



(b) System Response.

Fig. 10. Multiple fixed models with Redistribution Method I.

7. Multiple models with second level adaptation: A new approach

Despite the tremendous success of the multiple-model based methods discussed earlier, they all suffer from several shortcomings. The new approach proposed by the authors (Han & Narendra, 2010b), using multiple models with second level adaptation, represents a radical departure from those methods and leads to significantly improved performance. Furthermore, the new approach also possesses great potential for further developments as a general methodology.

7.1. Motivation for the new approach

From the preceding discussion it is seen that for "switching" to result in stability, a necessary condition is that there is at least one fixed model sufficiently close to the plant which, when stabilized using a controller, will also stabilize the plant. Similarly, "switching and tuning" is based on the assumption that adaptation from the model closest to the plant will result in improved performance. In adaptive control, it is generally assumed that the region $\mathcal{S}$ of uncertainty in parameter space is bounded and is known. When

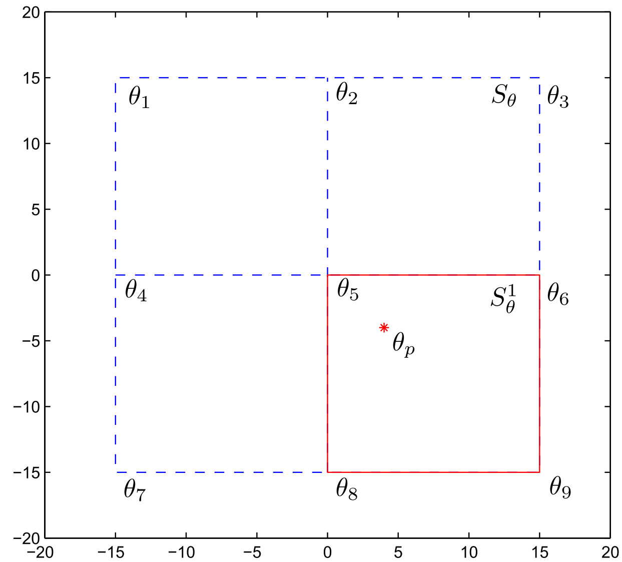


Fig. 11. Two-dimensional lattice.

the number of models used is large, and uniformly distributed in $\mathcal{S}$, one or more models will lie in the neighborhood (suitably defined) of the plant in parameter space. Since adaptive control is known to perform well when parametric errors are small, it is not surprising that both the procedures described above result in substantial improvement in performance.

In more realistic problems, where the dimension of the plant and the dimension of the unknown parameter vector are large, the number of models needed to satisfy the conditions described above becomes prohibitively large, since the above number increases exponentially with the dimension of the unknown parameter vector. Assuming that N is the number of models that the designer would consider "reasonable" and also that this number is not adequate to achieve satisfactory response when the parameters are unknown, the question naturally arises as to how the resources (i.e. models) can be used more effectively.

Another major drawback of the above two methods is that very little information provided by all the models is actually used in the decision process. A large number of models are created but the control input at any instant is based entirely on one model which is closest to the plant according to some metric. The procedures for redistributing the fixed models attempt to incorporate information provided by more than one model, and result in improved performance. Nevertheless the question still remains as to whether the information provided by all the models can be utilized more efficiently to identify and control the plant.

It is also worth noting that in the procedures described so far, switching is an indispensable component. However, as has been observed in many practical applications, the discontinuities in the control signal caused by extensive switching can be undesirable (Kuipers & Ioannou, 2010).

Finally, a common characteristic of all the earlier methods is that the identification and control are strongly coupled. The identification model chosen dictates the choice of the controller. This coupling leaves little freedom for new designs to emerge.

In summary, earlier methods using multiple models suffer from several disadvantages which are listed below:

(i) the number of models required is large and grows exponentially with the dimension of θ_p,
(ii) the information provided by every model is not used efficiently,

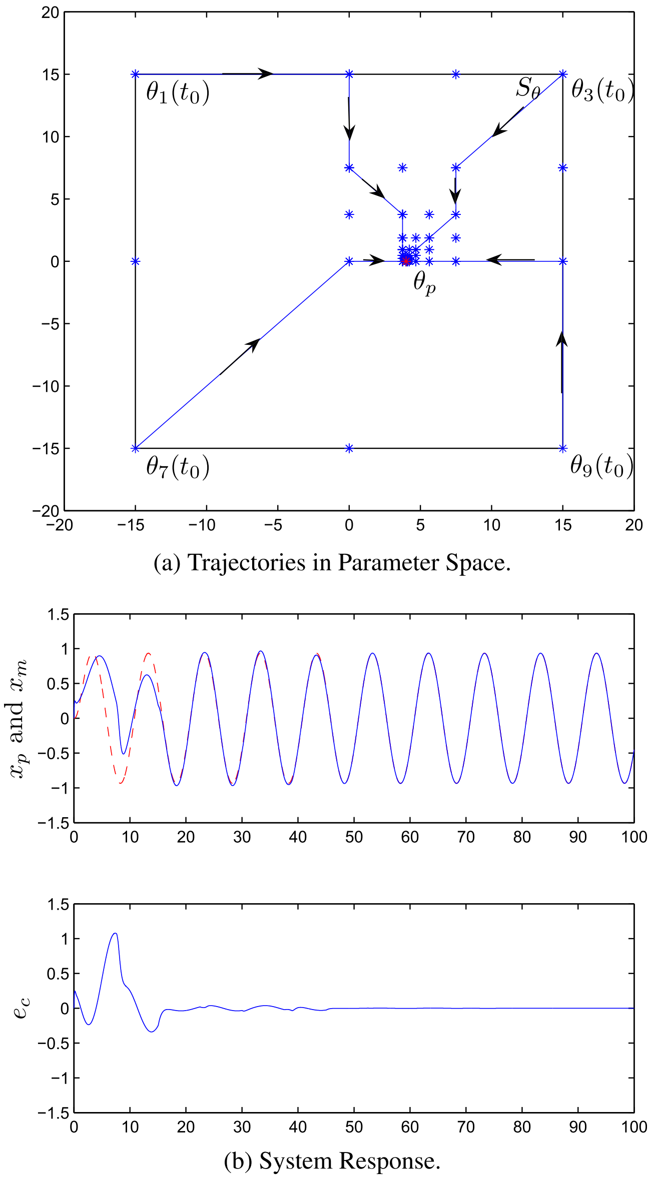


Fig. 12. Multiple fixed models with Redistribution Method II.

(iii) switching results in discontinuous control signals, and
(iv) identification and control are coupled.

The new approach described in this section addresses all the above concerns as stated below:

(i) the number of identification models needed to implement the method is comparable to the dimension of the unknown plant parameter vector,
(ii) information provided by all the models is used efficiently,
(iii) the control signal is continuous and smooth.
(iv) identification and control are decoupled, which provides greater freedom in the determination of the control input, and consequently in new designs.

7.2. Convex hull property

In Section 3, N estimates $\theta_1, \theta_2, \ldots, \theta_N$ of the plant parameter vector θ_p were generated. Since the plant Σ_p is linear, it follows that any convex combination of the estimates is also an estimate of θ_p so that

$$\theta_0(t) = \sum_{i=1}^{N} \beta_i \theta_i(t) \tag{23}$$

can be considered as a virtual model, where β_i are nonnegative constant coefficients satisfying $\sum_{i=1}^{N} \beta_i = 1$.

As shown in Han and Narendra (2010b), any arbitrary convex combination of the N models can be considered as a virtual model with the same properties as the real models.

The above discussion concerning virtual models defined as the convex combination of the N real models leads to the following theorem which was first introduced in Narendra and Han (2010).

Theorem 1. *If N adaptive identification models described in (14) are adjusted using adaptive laws (8) with initial conditions $\theta_i(t_0)$ and initial states $x_i(t_0) = x_p(t_0)$, and if the plant parameter vector θ_p lies in the convex hull $\mathcal{K}(t_0)$ of $\theta_i(t_0)$ $(i \in \Omega)$, then θ_p lies in the convex hull $\mathcal{K}(t)$ of $\theta_i(t)$ $(i \in \Omega)$ for all $t \geqslant t_0$.*

7.3. Second level adaptation

In the previous section it was shown that the plant parameter vector θ_p can be expressed as

$$\theta_p = \sum_{i=1}^{N} \alpha_i \theta_i(t_0) = \sum_{i=1}^{N} \alpha_i \theta_i(t) t \geqslant t_0 \tag{24}$$

for $\sum_{i=1}^{N} \alpha_i = 1$ and $\alpha_i \geqslant 0$.

The question naturally arises as to whether we can generate better estimate $\bar{\theta}(t)$ of the form

$$\bar{\theta}(t) = \sum_{i=1}^{N} \hat{\alpha}_i(t) \theta_i(t) \tag{25}$$

where the estimates $\hat{\alpha}_i(t)$ are generated in the second level, such that control using $\bar{\theta}(t)$ will lead to improved performance. This idea was proposed and investigated in Han and Narendra (2010b). Different schemes for generating the estimates $\alpha_i(t)$ were proposed and one of them is give below.

If we define $E(t) = [e_1(t), e_2(t), \ldots, e_N(t)]$ and $\alpha = [\alpha_1, \alpha_2, \ldots, \alpha_N]^T$ and $\ell = [1, 1, \ldots, 1]^T \in \mathbb{R}^N$, it follows from the convexity condition that $\ell^T \alpha = 1$. We then have

$$M(t)\alpha = h \tag{26}$$

where $M(t)$ can be expressed in the form

$$M(t) = \begin{bmatrix} E(t) \\ \hline \ell^T \end{bmatrix} \tag{27}$$

and $h = [0, 0, \ldots, 0, 1]^T$.

An estimation model is set up as

$$M(t)\hat{\alpha}(t) = \hat{h}(t) \tag{28}$$

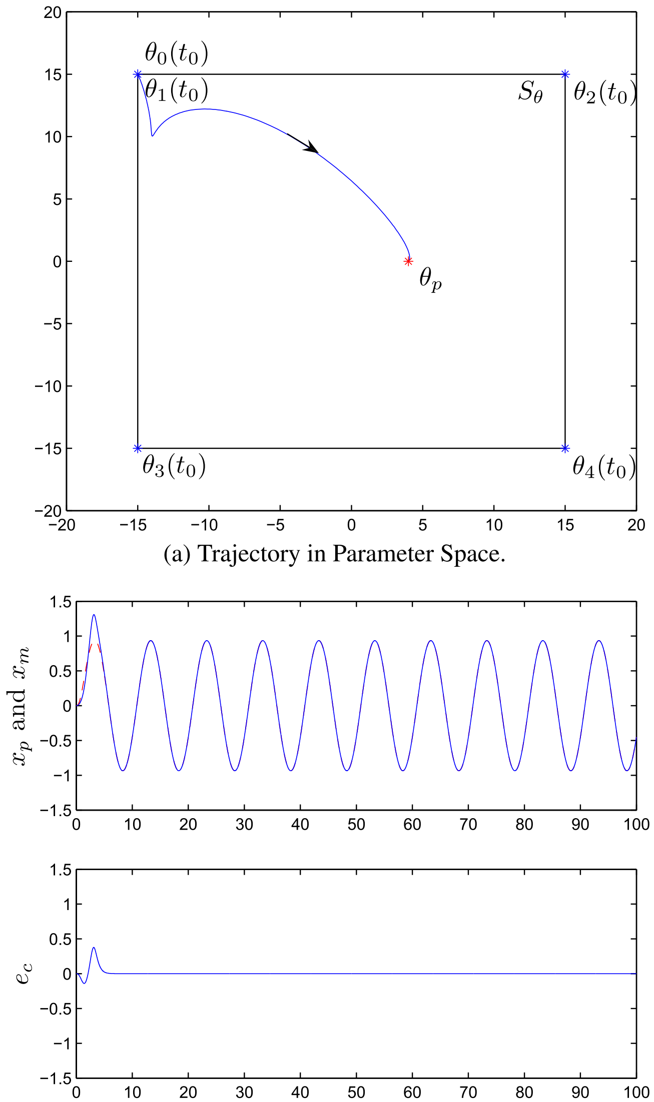
where $\hat{\alpha}(t)$ is the estimate of α and is adjusted using the adaptively law

$$\dot{\hat{\alpha}}(t) = -M^T(t)M(t)\hat{\alpha}(t) + M^T(t)h. \tag{29}$$

7.4. Stability

The following stability theorem which was stated and proved in Han and Narendra (2010a) assures the overall stability of the system when second level adaptation is used.

Theorem 2. *If the assumptions in Theorem 1 are satisfied, and the control signal is generated algebraically based on the plant parameter estimate*



(a) Trajectory in Parameter Space.

(b) System Response.

Fig. 13. Multiple models with second level adaptation.

$$\bar{\theta}(t) = Proj_{\bar{\theta}(t)\in S_\theta}\left\{\sum_{i=1}^{N}\hat{\alpha}_i(t)\theta_i(t)\right\} \tag{30}$$

for nonnegative piecewise differentiable $\hat{\alpha}_i(t)$ *which satisfy* the condition

$$\sum_{i=1}^{N}\hat{\alpha}_i(t) = 1 \tag{31}$$

then the overall system is asymptotically stable.

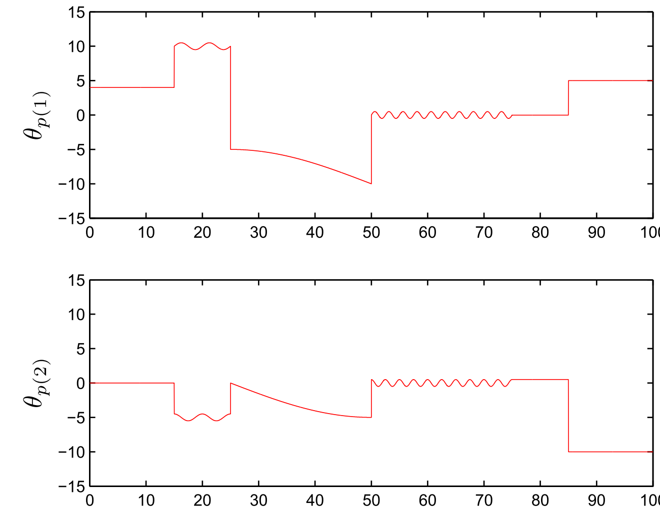


Fig. 14. Time-varying plant parameters.

From Theorem 2, any convex combination of the N estimates results in a control parameter vector which assures stability. This decouples the stability and performance issues, and $\alpha_i(t)$ can be chosen primarily to improve performance. Advantages gained by using second level adaptation over conventional adaptation is discussed in the following section.

7.5. Comparison of first and second level adaptation

As shown in Han and Narendra (2010b), if the performance index $J_i(t)$ is chosen as

$$J_i(t) = \int_{t_0}^{t}\|e_i(\tau)\|^2 d\tau \tag{32}$$

then the second level adaptation will always generate a performance index no larger than the one generated by a model with only first level adaptation, i.e. $J_{II}(t) \leqslant J_I(t)$.

Example 6. In this example the same problem simulated in earlier examples is considered (Fig. 13). Second level adaptation is used to identify and control the system. The first level is fixed and the second level adaptation gain is $\gamma_\alpha = 2$.

8. Comparison of different methods

Speed, accuracy, stability, and robustness are the features sought after in any efficient adaptive system. If the indirect approach is used, these depend upon the speed and accuracy with which the unknown plant parameter vector θ_p can be estimated. A summary of performance of the different schemes is given in Table 1.

Table 1
Comparison of different methods.

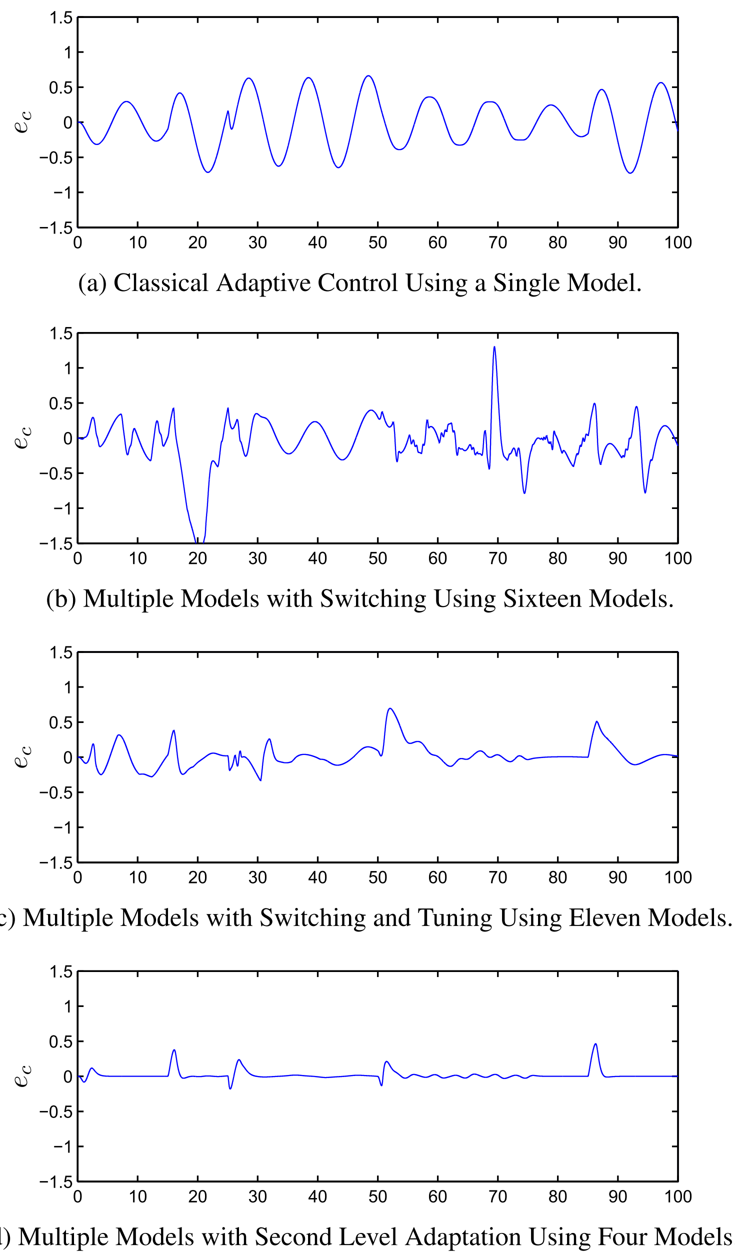
	Classical adaptive control	Switching	Switching and tuning	Redistribution	Second level adaptation
Stability	Stable	Stable	Stable	Stable	Stable
Speed	Very slow	Fast	Fast	Faster	Very fast
Accuracy	Accurate	Inaccurate	Accurate	Accurate	Accurate
Robustness	Not robust	Not robust	Robust	Not robust	Very robust
Continuity	Continuous	Discontinuous	Discontinuous	Discontinuous	Continuous
Complexity	$\mathcal{O}(1)$	$\mathcal{O}(c^n)$	$\mathcal{O}(c^n)$	$\mathcal{O}(c^n)$	$\mathcal{O}(n)$
Overall performance	Poor	Fair	Good	Good	Very good

It is seen from the second row that all the schemes are stable. It is well known that "classical adaptive control" is stable (1980). While "switching" using adaptive models is stable (Narendra & Balakrishnan, 1994), "switching" using fixed models is stable only if sufficiently large number of models are used (Morse, 1996, 1997; Vinnicombe, 1993). The "switching and tuning" scheme was proved to be stable in Narendra and Balakrishnan (1997), and the stability property of the "redistribution" scheme, which is a natural extension of the latter was investigated in Han and Narendra (2009). Finally, stability of "second level adaptation" was proved by the authors in Han and Narendra (2010a, 2010b).

The speed of convergence of the various schemes is included in the third row. It is well known that "classical adaptive control" is very slow, especially when the parametric error is large. "Switching" and "switching and tuning" result in faster response. As an extension of the "switching and tuning" scheme, "redistribution" further improves the speed of convergence as shown in Han and Narendra (2009). Simulation studies have shown that the new approach, i.e. "second level adaptation", results in significantly faster identification (Han & Narendra, 2010b).

A method is deemed accurate if the resulting control error is asymptotically zero. Therefore, all the methods listed except "switching" using fixed models are accurate. The "switching" method will be accurate only if an infinite number of models can be used. In Morse (1996), however, only set point control was attempted, and in such cases the output error tends to zero.

Robustness of each of the methods is compared qualitatively based on their performance in a rapidly time-varying environment. Simulation studies have been carried out in the past to evaluate their response to rapid time-variations of the unknown plant parameters. Not surprisingly, "classical adaptive control" and "switching" fail to cope with such situations. "Switching and tuning" is robust when the time-variation of the unknown parameter is not too rapid. "Redistribution" methods are robust only when the redistribution intervals are carefully chosen. This requires prior knowledge of the time-variations. This eventually renders the



(a) Classical Adaptive Control Using a Single Model.

(b) Multiple Models with Switching Using Sixteen Models.

(c) Multiple Models with Switching and Tuning Using Eleven Models.

(d) Multiple Models with Second Level Adaptation Using Four Models.

Fig. 15. Comparison of different schemes.

"redistribution" method a non-robust one. The "second level adaptation" method is shown to be very robust under rapid time-variations, including large discontinuous parameter changes. This point is illustrated below in Example 7.

By continuity we mean the continuity of the control signal generated by each of the methods considered. Therefore, any method involving switching will result in a discontinuous control signal. In the methods listed in Table 1, only "classical adaptive control" and "second level adaptation" methods will generate continuous and smooth control signals.

Finally, a comparison of the number of models required to implement each of the schemes is presented. "Classical adaptive control" requires only a single model. The number of models required for "switching", "switching and tuning", and "redistribution" methods all grow exponentially with the dimension 'n' of the plant parameter vector θ_p. Therefore, their complexity is denoted as $\mathcal{O}(c^n)$, where c is some positive integer. In contrast to this, "second level adaptation" requires only '$n + 1$' models which is significantly smaller than the previous three if 'n' is large.

The last row of Table 1 summarizes the overall performance of each method considering all the six aspects discussed earlier. The new approach, discussed in Section 7, appears to be significantly better than all the other methods in all aspects considered.

Example 7. In this final example, the performance of all the methods discussed in the paper are compared while controlling a rapidly time-varying plant. The time-varying parameter $\theta_p(t)$ is shown in Fig. 14. (It is worth stressing that the stability analysis of the different methods has not been investigated thus far, when the plant parameters are time-varying. Hence, the experiment was carried out merely to obtain empirical evidence of their performance in such contexts.)

It is seen that the time-variations include (i) small but rapid variations, (ii) large but slow variations, and (iii) large discontinuities over subintervals. The two redistribution based methods are not included here since their performance depends on prior information about the time-variations. Further, to better facilitate the "switching" and "switching and tuning" methods for dealing with rapid time-variations, the switching was based on the third criterion in (17). The multiple models with second level adaptation scheme was used as described in the paper.

As shown in Fig. 15, the multiple models with second level adaptation scheme gives far superior performance as compared to all the other methods.

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