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Neutron-star mergers in scalar-tensor theories of gravity

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Contents

Conventions and notation 3

1 Scalar-tensor theories of gravitation 5

1.1 General overview on scalar-tensor theories . . . 5
1.1.1 Introduction . . . 5
1.1.2 Action in Jordan frame . . . 6
1.1.3 Field equations in Jordan frame . . . 7
1.1.4 Boundary terms . . . 13
1.2 Jordan frame and Einstein frame . . . 15
1.2.1 Conformal transformation . . . 15
1.2.2 Definition of Einstein frame . . . 17
1.2.3 Field equations in Einstein frame . . . 18
1.2.4 Generalization to multi-scalar-tensor theories . . . 19
1.3 Stress-energy tensor conservation . . . 20
1.3.1 Conformal transformation law . . . 20
1.3.2 Einstein frame . . . 21
1.3.3 Jordan frame . . . 22
1.4 The Brans-Dicke model . . . 23
1.5 Scalar-tensor PN parameters . . . 24
1.5.1 DEF gravity . . . 27
1.6 TOV scalar metric . . . 28
1.6.1 Equation of state . . . 32
1.7 Nonperturbative scalarization . . . 33
1.8 Constraints from binary pulsars . . . 36

2 ST correction at the 3PN order 37

2.1 Effective-one-body approach . . . 37
2.2 Dynamics . . . 38
2.2.1 Effective metric . . . 39
2.2.2 Effective Hamiltonian . . . 41
2.2.3 Spin interactions . . . 42
2.2.4 Tidal interactions . . . 43
2.2.5 Hamilton's equations . . . 45
2.2.6 Circular approximation . . . 46
2.3 Effective A potential at the 3PN order . . . 47
2.3.1 Tidal term . . . 52
2.3.2 Instantaneous term . . . 52
2.3.3 Tail term . . . 56
2.3.4 Resummation of the A and D functions . . . 57

2.4 Effective Dynamics 64
2.4.1 Last Stable Orbit 65
2.4.2 Light Ring 68

3 Scalar-tensor waveform **70**
3.1 `TEOBResumS` waveform 70
3.1.1 Multipolar expansion 70
3.1.2 GR factorization 72
3.1.3 ST 2PN correction 74
3.2 2PN correction to the orbital phase 78
3.3 Results: scalar-tensor waveform 80
3.3.1 GWtest:H4 81
3.3.2 GW170817:ENG 90
3.3.3 GW170817:SLy 99

4 Conclusions and future prospects **109**

Bibliography **110**

Abstract

The observation of the inspiral and merger of compact binaries by the LIGO-Virgo collaboration ushered a new era in the study of strong-field gravity. These observations provide for the first time accurate measures of the parameters of the source and can therefore be used to constraint alternative gravitation theories. In particular, the detection a gravitational wave (GW) signal from a binary neutron star inspiral and merger (GW170817) has enabled tests of strong-field dynamics of compact binaries in presence of matter. Due to their relative simplicity, scalar-tensor (ST) theories are a good framework to study the strong-field dynamics of modified gravity. Binary neutron stars (BNS) in scalar-tensor (ST) theories present marked differences with respect to general relativity (GR) in both the late-inspiral and merger phases. In particular, non-perturbative phenomena related to the dynamical or spontaneous scalarization of isolated neutron stars take place in the late or early stages of the evolution of binary systems, with important effects in the ensuing dynamics. This thesis will be focused on the analysis of BNS in context of the ST theories of gravity.

We will work in the context of the effective-one-body (EOB) framework in ST theories to construct a waveform model for inspiralling neutron stars that could be used to put constraints on the parameters of the theory using GW170817-like data.

In the first part of this thesis we will recall some ST theories basics, focusing our attention on a restricted class of them: GR with a single massless additional scalar field. We will elaborate on the scalar-tensor action in the two main frames: Jordan frame and Einstein frame. It is possible to switch from one frame to the other through a conformal transformation of the metric. In Jordan frame the action presents a scalar-tensor non-minimally coupled term and the metric is directly coupled with the matter fields. In Einstein frame, instead, the ST non-minimally coupled term is converted into a minimally coupled one, but the metric in this frame will no longer be directly coupled with the matter fields. We will show the non-trivial field equations in the both frames, the metric conformal transformation calculations and the stress-energy tensor conservation. In this chapter we will recall also the Brans-Dicke theory, i.e. the simplest class of scalar-tensor theories introduced by C. Brans and R. Dicke in the '60, and some basics of non-perturbative phenomena which take place in isolated neutron stars such as spontaneous scalarization during the early stages of binary neutron stars evolution (or dynamical scalarization during the late stages). These modified theories of gravity are very interesting in order to test general relativity. Important differences with respect to Einstein's theory can be revealed by pulsar timing and laser-interferometer gravitational waves detectors due to the scalar charges carrying by neutron stars in these modified theories of gravity.

In second chapter we will make calculations in post-Newtonian (PN) approximations. As well as in GR the EOB approach, introduced by T. Damour and A. Buonanno in 1998, maps the two-body problem into a test particle moving within an external effective metric. In Einstein's theory of gravity the unique spherical solution in vacuum

is the Schwarzschild metric, where $g_{tt} = -g^{rr}$. In EOB framework the binary systems dynamic presents non-trivial components of effective metric which are no longer equals and they differ with respect to the Schwarzschild solution through PN terms. g_{tt} allows us to compute energy on circular orbits as function of angular momentum and orbital frequency. In this ST work we take a recent 3PN result given by L. Bernard for the conserved energy on circularized orbits and non-spinning bodies in order to find the 3PN correction to g_{tt} by comparing the two results. We will start approximating Hamilton's equations to obtain the angular momentum as function of EOB radial coordinate. The EOB circular Hamiltonian gives to us the orbital frequency and therefore we will calculate the gauge invariant energy as function of frequency in ST theories up to 3PN order.

In order to construct a waveform model for inspiralling neutron stars in scalar-tensor theories we will collect the recent 2PN waveform information given by N. Sennett. The general relativistic parts will be factorized out expanding the corrections up to 2PN order, splitting them into amplitude and phase. Therefore, we will make explicit the ST correction in waveform and in energy flux emitted by the BNS system.

Our aim is then to construct a scalar-tensor EOB model that combines the most recent state-of-the-art `TEOBResumS` model, i.e. full general relativistic EOB approach matched with numerical relativity, with these ST corrections. Furthermore, the current LIGO-Virgo collaboration, and next-generation interferometers such as the Laser Interferometer Space Antenna (LISA), require an highly precision waveform templates. Present and future experiments could then give new and more rigid constraints to the scalar-tensor parameters in order to confirm these theories as possible extensions of General Relativity or to rule them out.

Conventions and notation

Units, indices and metric signature. We will always work in natural units in which $G_\star = c = \hbar = 1$, where $G_\star$ is the Newton constant, c is the speed on light in vacuum and $\hbar$ denotes the reduced Planck constant. If we had to write them explicitly we would indicate it.

Spacetime indices are denoted by Greek letters, $\alpha, \beta, \gamma, \cdots, \mu, \nu, \rho, \cdots = 0, \cdots, 3$. Spatial indices are instead denoted by Latin letters, $i, j, k, \cdots = 1, \cdots, 3$.
We use the Minkowski signature [1, 2]

$$\eta_{\mu\nu} = diag(-, +, +, +).$$

This is the usual choice in the GWs physics, while the opposite signature is a common choice in QFT.

4-vector and derivatives. We use also the usual notations:

$$x^\mu = (x^0, \mathbf{x}) = (t, \mathbf{x}),$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_t, \partial_i),$$

$$\dot{f}(x) = \partial_t f(x),$$

$$f^{\prime \cdots \prime}(u) = f^{(n)}(u) = \frac{\partial^n}{\partial u^n} f.$$

Metric tensor. Our Lorentzian manifold is denoted as $(\mathcal{M}, g_{\mu\nu})$ [1, 3], where $g_{\mu\nu}$ is the *metric tensor* in a certain coordinate system. The *inverse metric* $g^{\mu\nu}$ is defined by $g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu$.

We denote the *determinant of the metric* by $g\,(< 0)$. The invariant integration measure, therefore, reads $\sqrt{-g} d^4x$.
Moreover, the contraction rules follow the conventions:

$$g_{\mu\nu} x^\mu y^\nu = x_\mu y^\mu = xy, \qquad\qquad g_{\mu\nu} x^\mu x^\nu = x^2.$$

We always adopt the Einstein summation convention on the repeated indices.

Christoffel symbols and curvature tensors. Let us now briefly recall the definition of the main ingredients to build the *Einstein-Hilbert action.*
The *Christoffel symbols* in Levi-Civita connection ($\nabla_\rho g_{\mu\nu} = 0$, $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$) read

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu} \right).$$

The *covariant derivative* ∇_μ is given in terms of the Christoffel symbols in the Levi-Civita connection written above.

The *Riemann Tensor* is defined by

$$R^{\rho}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\sigma\nu} - \partial_\nu \Gamma^{\rho}_{\sigma\mu} + \Gamma^{\rho}_{\lambda\mu}\Gamma^{\lambda}_{\sigma\nu} - \Gamma^{\rho}_{\lambda\nu}\Gamma^{\lambda}_{\sigma\mu},$$

$$R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^{\lambda}_{\sigma\mu\nu}.$$

The *Ricci tensor* and the *Ricci scalar* are

$$R_{\mu\nu} = g^{\rho\lambda} R_{\rho\mu\lambda\nu}, \qquad R = g^{\mu\nu} R_{\mu\nu}.$$

Stress-energy tensor, Einstein-Hilbert action and field equations. The *stress-energy tensor* can be obtained from the variation of the matter action

$$S_m[\Psi_m, g_{\mu\nu}] = \int d^4x \sqrt{-g}\mathcal{L}_m(\Psi_m, g_{\mu\nu})$$

with respect to $g^{\mu\nu}$:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m[\Psi_m, g_{\rho\sigma}]}{\delta g^{\mu\nu}}.$$

From the Einstein-Hilbert action

$$S_{EH}[g_{\mu\nu}] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} R, \qquad \kappa = 8\pi$$

and, from the definition of the *Einstein tensor*

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu},$$

we can read the *Einstein field equations* (EFE)

$$G_{\mu\nu} = 8\pi T_{\mu\nu}.$$

The covariant conservation of the stress-energy tensor $\nabla^\mu T_{\mu\nu} = 0$, in general relativity, follows by the *Bianchi identity* $\nabla^\mu G_{\mu\nu}$, which is a purely geometric identity.

Classical scalar field theory. We also recall the classical massive, real, single-scalar field theory in flat spacetime [4]:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2 \right).$$

The variation with respect to ϕ gives the *scalar field equation*

$$\left(\Box + m^2\right)\phi = 0,$$

where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ is the flat (or Minkowskian) laplacian operator.

The general relativistic extension is given by

$$S[\phi] = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2 \right).$$

The curved scalar field equation is the same as the flat one, with the only prescription: $\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$.
Notice that $\nabla_\mu\phi = \partial_\mu\phi$ due to the fact that ϕ is a scalar field. In the classical scalar field equation arise the curved laplacian because we have to variate also the determinant of the metric, present in the invariant integration measure.

Chapter 1

Scalar-tensor theories of gravitation

1.1 General overview on scalar-tensor theories

1.1.1 Introduction

In General Relativity (GR), gravity is mediated by a single rank-2 tensor field, or the spacetime metric tensor $g_{\mu\nu}$ [5–7], namely a massless spin-2 field in a Quantum Field Theory (QFT) point of view. GR was one of the most successful theories. It was continuously confirmed since 1919 with the first experiment on light deviation due to the solar curvature [8], to the most recent Gravitational Waves (GWs) experiments. However, there is no reason to think that gravitational action could not contain other fields. One is then free to add such additional fields: scalar, vector, tensor, or even higher rank tensor. The simplest scenario that one could study is the addition of a single scalar field (or, more generally, a set of scalar fields) and we refer to these case as *scalar-tensor theories of gravity* [9–11].

Clearly the effect of such additional scalar field needs to be suppressed and the constraints to these scalar degrees of freedom will be given by the experiments. Most scalar-tensor theories are called *metric theories of gravity*, namely theories that respect the Einstein weak-equivalent principle. However, these theories generally violate the strong-equivalence principle. Important violations of the strong-equivalence principle will arise in the class of single massless scalar-tensor theories studied in Ref. [9] through *non-perturbative strong-field effects* in Neutron Stars (NSs), such as the *spontaneous scalarization phenomenon* [12]. Constraints to non-perturbative strong-field effects in scalar-tensor theories are strongly given by Pulsar timing and by laser-interferometer GWs detector [13].

The study of these modified theories of gravity is also theoretical justified by the fact that they are equivalent to the $f(R)$ theories, and also arise from the size of compactified dimensions in $(4+n)$-dimensional gravity. Then, scalar-tensor theories of gravitation, are often considered as the prototype of the modified theories of gravity [11].

The experimental constraints are important to excluding or verifying the presence of these scalar degrees of freedom. Gravitational Waves (GWs) emitted by inspiralling and merging of Binary Neutron Stars (BNSs) is the best tool to comprehending the strong-field effects and additional terms in the radiation, i.e. dipolar radiation which is not present in GR, due to the scalar extension to general relativity [13].

The detection of gravitational wave events require highly accurate templates for the gravitational waveform. Current templates are made by merging different approaches: both analytical with Post-Newtonian (PN) approximation [14] assisted by resummation approach such Effective-One-Body (EOB) approach [15], and numerical relativity.

The waveform templates are currently used by the Laser Interferometer Gravitational-Wave Observatory (LIGO)-Virgo Interferometer collaboration for matching to GW170817-like data (BNSs) [16] to put constraints on the parameters of the modified theories of gravity, such ST which we will use in our calculations.

The purpose of this work is to focus on a restrict class of scalar-tensor theories, i.e. single, massless, scalar field (we will cite sometimes the general case but we will not use it in the main calculations). In such theories we will find the third PN order correction to the effective metric in the EOB framework, in non-spinning and quasi-circular case, by comparing with the already known 3PN energy [17]. Therefore we will use the new best available PN conservative dynamics with the 2PN order multipole expansion of the waveform in scalar-tensor theories [18] to build a model ready to use.

We will factorize out the GR part in order to expand the scalar-tensor correction up to 2PN order, by also splitting it into amplitude and phase. In the last part of this work we will implement all these corrections into the `TEOBResumS` model [19–35], i.e. EOB model matched with numerical relativity, with "tensor" tidal effects in General Relativity. The "scalar" ones will be found at the Leading Order (LO) in terms of the scalar Love numbers, which do not exist yet. Thus we will not implement them in this work and we leave this to future works.

We will also discuss the recent constraints on one of the most interesting scalar-tensor theory, which we will call *Damour-Esposito Farèse* (DEF) model [9, 12, 36]. This theory allows the spontaneous scalarization of neutron stars for some values of the parameter as we will see, i.e. allows the generation of scalar charge which does not exist in general relativity.

1.1.2 Action in Jordan frame

Let us focus on a (massless) mono-scalar-tensor theory, without any cosmological contribution, as we introduced above. This theory is given by the general action [10]

$$\begin{aligned} S[g_{\mu\nu}, \phi, \Psi_m] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \Big(f(\phi) R - g(\phi) \left(\nabla\phi\right)^2 \Big) \\ + S_m \left[\Psi_m, h(\phi) g_{\mu\nu}\right], \end{aligned} \tag{1.1.1}$$

where f, g and h are arbitrary functions of the scalar field of the theory ϕ and S_m is the action of the non-minimally coupled matter fields Ψ_m.

The "square" in kinetic term of the scalar field ϕ is given by

$$(\nabla\phi)^2 \equiv g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi. \tag{1.1.2}$$

The function $h(\phi)$ can be absorbed into the conformal metric

$$\tilde{g}_{\mu\nu} = h(\phi) g_{\mu\nu}. \tag{1.1.3}$$

The conformal frame generated by this transformation is usually called *Jordan frame* (JF) in which the matter fields are minimally coupled to the metric and, as well as we will show below, it is the physical frame.

Here, in JF, the metric is identified with $\tilde{g}_{\mu\nu}$ in order to differentiate it with respect to the Einstein frame (EF) one, which we will define in the following section. We will also omit the "$\sim$" to simplify the notation. In this conformal frame test-particle follow geodetics given by the conformal metric to which they are coupled.

By redefinition of the scalar field ϕ, Without Loss Of Generality (WLOG), the function $f(\phi)$ can be set to ϕ. Rewriting also the function $g(\phi)$, the scalar-tensor action (1.1.1) in the conformal Jordan frame read

$$S[g_{\mu\nu}, \phi, \Psi_m] = \frac{1}{2\kappa} \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega(\phi)}{\phi} (\nabla \phi)^2 \right) + S_m [\Psi_m, g_{\mu\nu}] \,, \tag{1.1.4}$$

where we omitted the " $\sim$ " in the redefined scalar field ϕ and in the conformal metric $g_{\mu\nu}$. Moreover the "square" $(\nabla\phi)^2$ is given by the conformal metric $g_{\mu\nu}$.

$\omega(\phi)$ fixes uniquely the scalar-tensor theory and is often called "*coupling parameter*". As we will see later, the constraints on theory will be given on the logarithm derivatives of this parameter.

Eq. (1.1.4) is the Jordan [37], Firez [38], Brans-Dicke [39, 40] (or simply Brans-Dicke) generalization, in which ω is a constant (we will discuss it in follow sections), currently used as a prototype of single, massless, scalar-tensor theories. As well as in classical and quantum field theory it is known the freedom to absorb the dimension of coupling parameter into the kinetic term. The ϕ^{-1} factor is needed in order to make a dimensionally coherent scalar kinetic term with a dimensionless coupling parameter $\omega(\phi)$.

In this frame, matter field Ψ_m are directly coupled with the metric $g_{\mu\nu}$. Examples of matter action S_m can be

$$S_m = - \int d^4x \sqrt{-g} \left(g^{\mu\nu} \nabla_\mu \bar{\Psi} \nabla_\nu \Psi + m^2 \bar{\Psi} \Psi \right) \tag{1.1.5}$$

for a complex scalar field,

$$S_m = - \int d^4x \sqrt{-g} \left(i \bar{\Psi} g^{\mu\nu} \gamma_\mu \partial_\nu \Psi + m^2 \bar{\Psi} \Psi \right) \tag{1.1.6}$$

for a spin 1/2 spinor field and

$$S_m = - \sum_A \int ds_A(g_{\mu\nu}) m_A(\phi) \tag{1.1.7}$$

for a set of massive "skeletonized" extended bodies. Here $ds_A = \sqrt{-g_{\mu\nu} dx_A^\mu dx_A^\nu}$ where $dx_A^\mu(s_A)$ denotes the position of body A and, in scalar-tensor theories, particle mass (or energy-mass) $m_A(\phi)$ is a function of the scalar field and its logarithmic derivatives enter in binary dynamics, as we will see in following sections.

1.1.3 Field equations in Jordan frame

The field equations are given by varying the action $S[g_{\mu\nu}, \phi, \Psi_m]$, from Eq. (1.1.4), with respect to $g^{\mu\nu}$ and to ϕ.

Tensor field equations. By omitting the argument in function $\omega(\phi)$, the tensor field equations read, by varying with respect to $g^{\mu\nu}$:

$$\begin{aligned}
0 &= \delta_g S \\
&= \frac{1}{2\kappa}\int d^4x \left[\delta_g\sqrt{-g}\left(\phi R - \frac{\omega}{\phi}(\nabla\phi)^2\right) + \sqrt{-g}\left(\phi\delta_g R - \frac{\omega}{\phi}\delta_g(\nabla\phi)^2\right)\right] \\
&+ \delta_g S_m[\Psi_m, g_{\rho\sigma}] \\
&= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[\phi G_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}\frac{\omega}{\phi}g_{\mu\nu}(\nabla\phi)^2\delta g^{\mu\nu} - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi\delta g^{\mu\nu}\right. \\
&+ \left.\frac{2\kappa}{\sqrt{-g}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m(\Psi_m, g_{\rho\sigma})\right)}{\delta g^{\mu\nu}}\delta g^{\mu\nu}\right] + \underbrace{\frac{1}{2\kappa}\int d^4x\sqrt{-g}\phi g^{\mu\nu}\delta R_{\mu\nu}}_{\equiv\delta_g\tilde{S}},
\end{aligned} \tag{1.1.8}$$

where we defined $\delta_g \equiv \int \delta g^{\mu\nu}\delta/\delta g^{\mu\nu}$ and we used the standard general relativistic results [3]

$$\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu} \tag{1.1.9}$$

and

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}. \tag{1.1.10}$$

We will omit the subscript "g" when the object that is going to be varied does not presents ϕ in it.

The last variation term in Eq. (1.1.8), $\delta_g\tilde{S}$, is non trivial, similarly to the general relativistic Gibbons-Hawking boundary term (or Gibbons-Hawking-York) [41,42]. Let us now focus on this term.

Well known GR calculations give to us the variation of Riemann tensor components in Levi-Civita connection after a variation $\delta g^{\mu\nu}$. In fact, by using the identity

$$\nabla_\mu\left(\delta\Gamma^\rho_{\nu\sigma}\right) = \partial_\mu\left(\delta\Gamma^\rho_{\nu\sigma}\right) + \Gamma^\rho_{\mu\lambda}\delta\Gamma^\lambda_{\nu\sigma} - \Gamma^\lambda_{\mu\nu}\delta\Gamma^\rho_{\lambda\sigma} - \Gamma^\rho_{\nu\lambda}\delta\Gamma^\lambda_{\mu\sigma} \tag{1.1.11}$$

we get

$$\delta R^\rho_{\sigma\mu\nu} = \nabla_\mu\left(\delta\Gamma^\rho_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\rho_{\mu\sigma}\right). \tag{1.1.12}$$

Therefore, we can write the variation of Ricci tensor that we are looking for, i.e. *Palatini identity* [43]:

$$\delta R_{\sigma\nu} = \nabla_\mu\left(\delta\Gamma^\mu_{\nu\sigma}\right) - \nabla_\nu\left(\delta\Gamma^\mu_{\mu\sigma}\right). \tag{1.1.13}$$

To simplify the notation we can define [11]

$$\Gamma_\sigma \equiv \Gamma^\mu_{\mu\sigma} = \frac{1}{2}g^{\alpha\beta}\partial_\sigma g_{\alpha\beta} = \frac{\partial_\sigma\sqrt{-g}}{\sqrt{-g}}, \tag{1.1.14}$$

presents in the last term of Palatini identity (1.1.13). The variation term $\delta_g\tilde{S}$ reads

$$\delta_g\tilde{S} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\phi g^{\mu\nu}\left[\nabla_\rho\left(\delta\Gamma^\rho_{\mu\nu}\right) - \nabla_\mu\left(\delta\Gamma_\nu\right)\right]. \tag{1.1.15}$$

By integrating by parts we obtain

$$\nabla_\rho\left(\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) = \nabla_\rho\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} + \phi g^{\mu\nu}\nabla_\rho\delta\Gamma^\rho_{\mu\nu}, \tag{1.1.16}$$

$$\nabla_\mu\left(\phi g^{\mu\nu}\delta\Gamma_\nu\right) = \nabla_\mu\phi g^{\mu\nu}\delta\Gamma_\nu + \phi g^{\mu\nu}\nabla_\mu\delta\Gamma_\nu. \tag{1.1.17}$$

In order to use the Stokes' theorem, we first write the identity

$$\int d^4x\sqrt{-g}\nabla_\rho\left(\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) = \int d^4x\partial_\rho\left(\sqrt{-g}\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right), \tag{1.1.18}$$

$$\int d^4x\sqrt{-g}\nabla_\mu\left(\phi g^{\mu\nu}\delta\Gamma_\nu\right) = \int d^4x\partial_\mu\left(\sqrt{-g}\phi g^{\mu\nu}\delta\Gamma_\nu\right). \tag{1.1.19}$$

In fact, by using the definition of Γ_ρ from Eq. (1.1.14), we get

$$\begin{aligned}\nabla_\rho A^\rho &= \partial_\rho A^\rho + \Gamma_\rho A^\rho \\ &= \partial_\rho A^\rho + \frac{\partial_\rho\sqrt{-g}}{\sqrt{-g}}A^\rho,\end{aligned} \tag{1.1.20}$$

For each vector field A^ρ.

Therefore, Eq. (1.1.20) implies:

$$\sqrt{-g}\nabla_\rho A^\rho = \partial_\rho\left(\sqrt{-g}A^\rho\right). \tag{1.1.21}$$

The Stokes' theorem then gives

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\rho\left(\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) = \int_{\mathcal{M}} d^4x\partial_\rho\left(\sqrt{-g}\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}\right) = \int_{\partial\mathcal{M}} d\sigma_\rho\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu}, \tag{1.1.22}$$

$$\int_{\mathcal{M}} d^4x\sqrt{-g}\nabla_\mu\left(\phi g^{\mu\nu}\delta\Gamma_\nu\right) = \int_{\mathcal{M}} d^4x\partial_\mu\left(\sqrt{-g}\phi g^{\mu\nu}\delta\Gamma_\nu\right) = \int_{\partial\mathcal{M}} d\sigma_\mu\phi g^{\mu\nu}\delta\Gamma_\nu, \tag{1.1.23}$$

where $d\sigma_\mu = \epsilon n_\mu\sqrt{|h|}d^3y$. h is the determinant of (4-1)-dimensional induced metric in the boundary $\partial\mathcal{M}$ (see Ref. [44]), n_μ is the unit normal to $\partial\mathcal{M}$ and $\epsilon = n^\mu n_\mu = g^{\mu\nu}n_\mu n_\nu = \pm 1$.

These boundary terms are similar to Gibbons-Hawking term [41, 42] and we will comment them in following section.

Then, once settle the boundary terms, Eq. (1.1.15) becomes

$$\begin{aligned}\delta_g\tilde{S} &= -\frac{1}{2\kappa}\int d^4x\sqrt{-g}g^{\mu\nu}\left[\nabla_\rho\phi\delta\Gamma^\rho_{\mu\nu} - \nabla_\mu\phi\delta\Gamma_\nu\right] \\ &= -\frac{1}{2\kappa}\int d^4x\sqrt{-g}\nabla_\mu\phi\left[g^{\rho\sigma}\delta\Gamma^\mu_{\rho\sigma} - g^{\mu\nu}\delta\Gamma_\nu\right].\end{aligned} \tag{1.1.24}$$

We want now to express the variation of Christoffel symbols in terms of the initial variation of metric $\delta g^{\mu\nu}$. This in order to get Eq. (1.1.24) in the form (see Refs. [45–47])

$$\frac{1}{2\kappa}\int d^4x\sqrt{-g}\left(\cdots_{\mu\nu}\right)\delta g^{\mu\nu}. \tag{1.1.25}$$

Let us now enunciate a Lemma that will be used to write the action variation as in Eq. (1.1.25):

Lemma:

$$\delta\Gamma^\mu_{\rho\sigma} = -\frac{1}{2}\Big(g_{\sigma\nu}\nabla_\rho\left(\delta g^{\mu\nu}\right) + g_{\rho\nu}\nabla_\sigma\left(\delta g^{\mu\nu}\right) - g_{\rho\alpha}g_{\sigma\beta}g^{\mu\lambda}\nabla_\lambda\left(\delta g^{\alpha\beta}\right)\Big). \tag{1.1.26}$$

$$\delta\Gamma_\nu = -\frac{1}{2}g_{\rho\sigma}\nabla_\nu(\delta g^{\rho\sigma}). \tag{1.1.27}$$

Proof:
Let us first define

$$A_{(\mu\nu)} \equiv \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right) \tag{1.1.28}$$

in order to simplify calculations.

Variation of Christoffel symbols then read

$$\begin{aligned} \delta\Gamma^{\mu}_{\rho\sigma} &= \frac{1}{2}\delta g^{\mu\lambda}\big(\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma}\big) + \frac{1}{2}g^{\mu\lambda}\delta\big(\partial_\rho g_{\sigma\lambda} + \partial_\sigma g_{\rho\lambda} - \partial_\lambda g_{\rho\sigma}\big) \\ &= \delta g^{\mu\lambda}\partial_{(\rho}g_{\sigma)\lambda} - \frac{1}{2}\delta g^{\mu\lambda}\partial_\lambda g_{\rho\sigma} + g^{\mu\lambda}\partial_{(\rho}(\delta g_{\sigma)\lambda}) - \frac{1}{2}g^{\mu\lambda}\partial_\lambda(\delta g_{\rho\sigma}) \end{aligned} \tag{1.1.29}$$

We can write the partial derivatives of metric variations in terms of their covariant derivatives:

$$\begin{aligned} \nabla_\lambda(\delta g_{\rho\sigma}) &= \partial_\lambda(\delta g_{\rho\sigma}) - \Gamma^{\nu}_{\lambda\rho}\delta g_{\nu\sigma} - \Gamma^{\nu}_{\lambda\sigma}\delta g_{\rho\nu} \\ &= \partial_\lambda(\delta g_{\rho\sigma}) - 2\Gamma^{\nu}_{\lambda(\rho}\delta g_{\sigma)\nu}. \end{aligned} \tag{1.1.30}$$

Hence, by remembering the torsion-free condition on our manifold $\Gamma^{\nu}_{\lambda\sigma} = \Gamma^{\nu}_{\sigma\lambda}$, Christoffel symbols variation (1.1.29) becomes

$$\begin{aligned} \delta\Gamma^{\mu}_{\rho\sigma} &= \delta g^{\mu\lambda}\partial_{(\rho}g_{\sigma)\lambda} - \frac{1}{2}\delta g^{\mu\lambda}\partial_\lambda g_{\rho\sigma} + g^{\mu\lambda}\big(\nabla_{(\rho}(\delta g_{\sigma)\lambda}) + \Gamma^{\nu}_{(\rho\sigma)}\delta g_{\nu\lambda}\cancel{+\Gamma^{\nu}_{\lambda(\rho}\delta g_{\sigma)\nu}}\big) \\ &- \frac{1}{2}g^{\mu\lambda}\big(\nabla_\lambda(\delta g_{\rho\sigma})\cancel{+2\Gamma^{\nu}_{\lambda(\rho}\delta g_{\sigma)\nu}}\big) \\ &= \delta g^{\mu\lambda}\partial_{(\rho}g_{\sigma)\lambda} - \frac{1}{2}\delta g^{\mu\lambda}\partial_\lambda g_{\rho\sigma} + g^{\mu\lambda}\Gamma^{\nu}_{\rho\sigma}\delta g_{\nu\lambda} \\ &+ g^{\mu\lambda}\big(\nabla_{(\rho}(\delta g_{\sigma)\lambda}) - \frac{1}{2}\nabla_\lambda(\delta g_{\rho\sigma})\big), \end{aligned} \tag{1.1.31}$$

where we used the covariant derivative's expression (1.1.30), with "$(\rho\sigma)$" notation (1.1.28), and $\Gamma^{\nu}_{(\rho\sigma)} = \Gamma^{\nu}_{\rho\sigma}$, because they are symmetric with respect to $\rho \leftrightarrow \sigma$.

Remembering the identities

$$0 = \delta\left(\delta^{\mu}_{\sigma}\right) = \delta\left(g_{\sigma\alpha}g^{\mu\alpha}\right) = g^{\mu\alpha}\delta g_{\sigma\alpha} + g_{\sigma\alpha}\delta g^{\mu\alpha}, \tag{1.1.32}$$

$$0 = \delta g_{\sigma\alpha} + g_{\mu\alpha}g_{\sigma\beta}\delta g^{\mu\beta}, \tag{1.1.33}$$

we obtain

$$\begin{aligned} \delta\Gamma^{\mu}_{\rho\sigma} &= \delta g^{\mu\alpha}g_{\alpha\lambda}\Gamma^{\alpha}_{\rho\sigma} + g^{\mu\lambda}\Gamma^{\nu}_{\rho\sigma}\delta g_{\nu\lambda} + g^{\mu\lambda}\big(\nabla_{(\rho}(\delta g_{\sigma)\lambda}) - \frac{1}{2}\nabla_\lambda(\delta g_{\rho\sigma})\big) \\ &= \delta g^{\mu\alpha}g_{\alpha\lambda}\Gamma^{\alpha}_{\rho\sigma} - g^{\mu\lambda}\Gamma^{\nu}_{\rho\sigma}g_{\alpha\lambda}g_{\nu\beta}\delta g^{\alpha\beta} + g^{\mu\lambda}\big(\nabla_{(\rho}(\delta g_{\sigma)\lambda}) - \frac{1}{2}\nabla_\lambda(\delta g_{\rho\sigma})\big) \\ &= \cancel{\delta g^{\mu\alpha}g_{\alpha\lambda}\Gamma^{\alpha}_{\rho\sigma}}\cancel{-\Gamma^{\nu}_{\rho\sigma}g_{\nu\beta}\delta g^{\mu\beta}} + g^{\mu\lambda}\big(\nabla_{(\rho}(\delta g_{\sigma)\lambda}) - \frac{1}{2}\nabla_\lambda(\delta g_{\rho\sigma})\big) \\ &= -g^{\mu\lambda}\big(\nabla_{(\rho}(g_{\sigma)\beta}g_{\alpha\lambda}\delta g^{\alpha\beta}) - \frac{1}{2}\nabla_\lambda(g_{\rho\alpha}g_{\sigma\beta}\delta g^{\alpha\beta})\big) \\ &= -\big(g_{\alpha(\sigma}\nabla_{\rho)}\delta g^{\mu\alpha} - \frac{1}{2}g_{\rho\alpha}g_{\sigma\beta}g^{\mu\lambda}\nabla_\lambda\delta g^{\alpha\beta}\big). \end{aligned} \tag{1.1.34}$$

By making explicit the $(\rho\sigma)$ notation we get

$$\delta\Gamma^{\mu}_{\rho\sigma} = -\frac{1}{2}\big(g_{\alpha\sigma}\nabla_\rho\delta g^{\mu\alpha} + g_{\alpha\rho}\nabla_\sigma\delta g^{\mu\alpha} - g_{\rho\alpha}g_{\sigma\beta}g^{\mu\lambda}\nabla_\lambda\delta g^{\alpha\beta}\big), \tag{1.1.35}$$

which proves Eq. (1.1.26).

In order to obtain Eq. (1.1.27) we set $\mu = \rho$ and $\sigma \to \nu$ in Eq. (1.1.35):

$$\begin{aligned}\delta\Gamma_\nu &= -\frac{1}{2}\big(g_{\alpha\nu}\nabla_\mu\delta g^{\mu\alpha} + g_{\alpha\mu}\nabla_\nu\delta g^{\mu\alpha} - g_{\mu\alpha}g_{\nu\beta}g^{\mu\lambda}\nabla_\lambda\delta g^{\alpha\beta}\big)\\ &= -\frac{1}{2}\big(\cancel{g_{\alpha\nu}\nabla_\mu\delta g^{\mu\alpha}} + g_{\alpha\mu}\nabla_\nu\delta g^{\mu\alpha}\cancel{-g_{\nu\beta}\nabla_\alpha\delta g^{\alpha\beta}}\big)\\ &= -\frac{1}{2}g_{\alpha\mu}\nabla_\nu(\delta g^{\alpha\mu}),\end{aligned} \tag{1.1.36}$$

which ends the proof.

□

Then, combining Eqs. (1.1.26) and (1.1.27) in Eq. (1.1.24), we get

$$\begin{aligned}-(2\kappa)\delta_g\tilde{S} &= -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\mu\phi\Big[g^{\rho\sigma}\Big(g_{\sigma\nu}\nabla_\rho\left(\delta g^{\mu\nu}\right) + g_{\rho\nu}\nabla_\sigma\left(\delta g^{\mu\nu}\right)\\ &- g_{\rho\alpha}g_{\sigma\beta}g^{\mu\lambda}\nabla_\lambda\left(\delta g^{\alpha\beta}\right)\Big) - g^{\mu\nu}g_{\alpha\beta}\nabla_\nu(\delta g^{\alpha\beta})\Big]\\ &= -\frac{1}{2}\int d^4x\sqrt{-g}\nabla_\mu\phi\Big[\nabla_\nu\left(\delta g^{\mu\nu}\right) + \nabla_\nu\left(\delta g^{\mu\nu}\right)\\ &- g_{\alpha\beta}g^{\mu\lambda}\nabla_\lambda\left(\delta g^{\alpha\beta}\right) - g^{\mu\nu}g_{\alpha\beta}\nabla_\nu(\delta g^{\alpha\beta})\Big]\\ &= -\int d^4x\sqrt{-g}\nabla_\mu\phi\Big[\nabla_\nu\left(\delta g^{\mu\nu}\right) - g^{\mu\nu}g_{\alpha\beta}\nabla_\nu(\delta g^{\alpha\beta})\Big]\\ &= -\int d^4x\sqrt{-g}\Big[\nabla_\mu\phi\nabla_\nu\left(\delta g^{\mu\nu}\right) - \nabla_\rho\phi g^{\rho\sigma}g_{\mu\nu}\nabla_\sigma(\delta g^{\mu\nu})\Big].\end{aligned} \tag{1.1.37}$$

By integrating by parts the covariant derivative acting on $\delta g^{\mu\nu}$, one can write

$$\nabla_\nu(\nabla_\mu\phi\delta g^{\mu\nu}) = \nabla_\nu\nabla_\mu\phi\delta g^{\mu\nu} + \nabla_\mu\phi\nabla_\nu\left(\delta g^{\mu\nu}\right), \tag{1.1.38}$$

$$\begin{aligned}\nabla_\sigma(g^{\rho\sigma}\nabla_\rho\phi g_{\mu\nu}\delta g^{\mu\nu}) &= g^{\rho\sigma}\nabla_\sigma\nabla_\rho\phi g_{\mu\nu}\delta g^{\mu\nu} + \nabla_\rho\phi g^{\rho\sigma}g_{\mu\nu}\nabla_\sigma(\delta g^{\mu\nu})\\ &= g_{\mu\nu}\Box\phi\delta g^{\mu\nu} + g_{\mu\nu}\nabla_\rho\phi\nabla^\rho(\delta g^{\mu\nu}).\end{aligned} \tag{1.1.39}$$

Boundary terms are, respectively, given by:

$$\int_{\partial\mathcal{M}} d\sigma_\nu\nabla_\mu\phi\delta g^{\mu\nu} \tag{1.1.40}$$

and

$$\int_{\partial\mathcal{M}} d\sigma_\sigma g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\delta g^{\mu\nu}. \tag{1.1.41}$$

These integrals give zero because the metric variation enter only as $\delta g^{\mu\nu}$ which is zero on the boundary $\partial\mathcal{M}$.

Hence, because $\nabla_\nu\nabla_\mu\phi = \nabla_\mu\nabla_\nu\phi$ (see Eq. (1.3.9)):

$$\delta_g\tilde{S} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big[-\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\Box\phi\Big]\delta g^{\mu\nu}. \tag{1.1.42}$$

Therefore, the stationary action principle applied to extra term $\tilde{S}$ gives the following additional term in left hand side of tensorial field equations:

$$g_{\mu\nu}\Box\phi - \nabla_\mu\nabla_\nu\phi. \tag{1.1.43}$$

Finally, from Eqs. (1.1.8) and (1.1.43), the tensorial field equations read

$$\phi G_{\mu\nu} = 8\pi T_{\mu\nu} - \left(\Box\phi + \frac{1}{2}\frac{\omega}{\phi}\left(\nabla\phi\right)^2\right) g_{\mu\nu} + \nabla_\mu\nabla_\nu\phi + \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi, \tag{1.1.44}$$

or

$$\phi G_{\mu\nu} = 8\pi T_{\mu\nu} - \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\phi - T^\phi_{\mu\nu}, \tag{1.1.45}$$

where we defined an useful effective "scalar" stress-energy tensor [10]:

$$T^\phi_{\mu\nu} = \frac{\omega}{\phi}\left(\frac{1}{2}g_{\mu\nu}\left(\nabla\phi\right)^2 - \nabla_\mu\phi\nabla_\nu\phi\right). \tag{1.1.46}$$

Scalar field equation. Now, by defining $\delta_\phi \equiv \int \delta\phi\delta/\delta\phi$ we can varying the action (1.1.4) with respect to ϕ:

$$\begin{aligned} 0 &= \delta_\phi S \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R\delta\phi - \delta_\phi\left(\frac{\omega}{\phi}\right)\left(\nabla\phi\right)^2 - \frac{\omega}{\phi}\delta_\phi\left(\nabla\phi\right)^2\right] + \delta_\phi S_m[\Psi_m, g_{\rho\sigma}] \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\left[R\delta\phi - \left(\frac{\omega}{\phi}\right)'\delta\phi\left(\nabla\phi\right)^2 - 2\frac{\omega}{\phi}g^{\mu\nu}\nabla_\mu\delta\phi\nabla_\nu\phi\right] \\ &= \frac{1}{2\kappa}\int d^4x\sqrt{-g}\delta\phi\left[R - \left(\frac{\omega}{\phi}\right)'\left(\nabla\phi\right)^2 + 2g^{\mu\nu}\left(\left(\frac{\omega}{\phi}\right)'\nabla_\mu\phi\nabla_\nu\phi + \frac{\omega}{\phi}\nabla_\mu\nabla_\nu\phi\right)\right], \end{aligned} \tag{1.1.47}$$

where we used the identity $\nabla_\mu\left(\frac{\omega}{\phi}\right) = \left(\frac{\omega}{\phi}\right)'\nabla_\mu\phi$ and we neglected the boundary term.

Therefore Eq. (1.1.47) gives

$$R + \left(\frac{\omega'}{\phi} - \frac{\omega}{\phi^2}\right)\left(\nabla\phi\right)^2 + 2\frac{\omega}{\phi}\Box\phi = 0. \tag{1.1.48}$$

The field equations must be considered with the on-shell condition on the matter fields:

$$\frac{\delta S_m[\Psi_m, g_{\mu\nu}]}{\delta\Psi_m} = 0. \tag{1.1.49}$$

By taking the trace of Eq. (1.1.44), the Ricci scalar reads

$$R = \frac{3}{\phi}\Box\phi + \frac{\omega}{\phi^2}\left(\nabla\phi\right)^2 - \frac{8\pi T}{\phi}, \tag{1.1.50}$$

where, as usual, the trace of the stress-energy is $T = g^{\mu\nu}T_{\mu\nu}$. Therefore, eliminating R in Eqs. (1.1.44) and (1.1.48) with the expression in Eq. (1.1.50) we have respectively

$$\phi R_{\mu\nu} = 8\pi\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right) + \frac{1}{2}g_{\mu\nu}\Box\phi + \nabla_\mu\nabla_\nu\phi + \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi, \tag{1.1.51a}$$

$$\Box\phi = \frac{1}{2\omega+3}\left(8\pi T - \omega'\left(\nabla\phi\right)^2\right). \tag{1.1.51b}$$

1.1.4 Boundary terms

In this section we briefly comment the boundary terms appeared in Eqs. (1.1.15),(1.1.37). The explicit variation of $\tilde{S}$, defined in Eq. (1.1.8), reads

$$\begin{aligned}(2\kappa)\delta_g\tilde{S} =& \int_{\mathcal{M}} d^4x\sqrt{-g}\Big[-\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\Box\phi\Big]\delta g^{\mu\nu} + \int_{\partial\mathcal{M}} d\sigma_\rho\phi g^{\mu\nu}\delta\Gamma^\rho_{\mu\nu} \\ &- \int_{\partial\mathcal{M}} d\sigma_\mu\phi g^{\mu\nu}\delta\Gamma_\nu + \int_{\partial\mathcal{M}} d\sigma_\nu\nabla_\mu\phi\delta g^{\mu\nu} - \int_{\partial\mathcal{M}} d\sigma_\sigma g_{\mu\nu}g^{\rho\sigma}\nabla_\rho\phi\delta g^{\mu\nu} \\ =& \int_{\mathcal{M}} d^4x\sqrt{-g}\Big[-\nabla_\mu\nabla_\nu\phi + g_{\mu\nu}\Box\phi\Big]\delta g^{\mu\nu} \\ &+ \int_{\partial\mathcal{M}} d\sigma_\mu\Big[\phi\Big(g^{\rho\sigma}\delta\Gamma^\mu_{\rho\sigma} - g^{\mu\nu}\delta\Gamma_\nu\Big) + \nabla_\nu\phi\Big(\delta g^{\mu\nu} - g^{\mu\nu}g_{\rho\sigma}\delta g^{\rho\sigma}\Big)\Big].\end{aligned} \tag{1.1.52}$$

Let us define as δS_B the boundary integral over $\partial\mathcal{M}$ in Eq. (1.1.52). By definition, metric variation used in stationary action principle has the property:

$$\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0 = \delta g^{\mu\nu}|_{\partial\mathcal{M}}, \tag{1.1.53}$$

then only the first two terms of δS_B have non-zero value because Eq. (1.1.53) does *not* implies that $\partial_\rho\delta g_{\mu\nu}|_{\partial\mathcal{M}} = 0 = \partial_\rho\delta g^{\mu\nu}|_{\partial\mathcal{M}}$. Now we take:

$$\delta A^\mu = g^{\rho\sigma}\delta\Gamma^\mu_{\rho\sigma} - g^{\mu\nu}\delta\Gamma_\nu, \tag{1.1.54}$$

hence

$$\begin{aligned}\delta_g S_B &= \int_{\partial\mathcal{M}} d\sigma_\mu\phi\delta A^\mu \\ &= \int_{\partial\mathcal{M}} d^3y\sqrt{|h|}\epsilon n_\mu\phi\delta A^\mu.\end{aligned} \tag{1.1.55}$$

We can use the condition $\delta g^{\mu\nu} = 0$, Eq. (1.1.53), to compute $\delta\Gamma^\mu_{\rho\sigma}$ on $\partial\mathcal{M}$, in order to get δA^μ:

$$\delta\Gamma^\mu_{\rho\sigma}|_{\partial\mathcal{M}} = g^{\mu\lambda}\partial_{(\rho}\delta g_{\sigma)\lambda} - \frac{1}{2}g^{\mu\lambda}\partial_\lambda\delta g_{\rho\sigma}. \tag{1.1.56}$$

Keeping in mind the symmetry property of $(\rho\sigma)$ notation, and renaming some indices, we obtain

$$\begin{aligned}\delta A^\mu|_{\partial\mathcal{M}} &= g^{\rho\sigma}g^{\mu\lambda}\partial_\rho\delta g_{\sigma\lambda} - \frac{1}{2}g^{\rho\sigma}g^{\mu\lambda}\partial_\lambda\delta g_{\rho\sigma} - g^{\mu\nu}g^{\rho\lambda}\partial_{(\rho}\delta g_{\nu)\lambda} + \frac{1}{2}g^{\mu\nu}g^{\rho\lambda}\partial_\lambda\delta g_{\rho\nu} \\ &= \frac{1}{2}g^{\rho\sigma}g^{\mu\lambda}\partial_\rho\delta g_{\sigma\lambda} - g^{\rho\sigma}g^{\mu\lambda}\partial_\lambda\delta g_{\rho\sigma} + \frac{1}{2}g^{\mu\nu}g^{\rho\lambda}\partial_\lambda\delta g_{\rho\nu} \\ &= g^{\rho\sigma}g^{\mu\lambda}\left(\partial_\rho\delta g_{\sigma\lambda} - \partial_\lambda\delta g_{\rho\sigma}\right).\end{aligned} \tag{1.1.57}$$

Remembering the ADM decomposition (see Ref. [44]) we can use the identity $g^{\rho\sigma} = h^{\rho\sigma} + \epsilon n^\rho n^\sigma$ in Eq. (1.1.57). Now, $\partial_\rho\delta g_{\sigma\lambda}$ is a derivative tangent to $\partial\mathcal{M}$ while n^ρ is normal by definition, then the $n^\rho n^\sigma$ parts disappear. Furthermore, $\partial_\rho\delta g_{\sigma\lambda}|_{\partial\mathcal{M}}$ is non-zero but its tangential projection on $\partial\mathcal{M}$ has zero value as well as $\delta g^{\mu\nu}|_{\partial\mathcal{M}}$. In ADM formulation this projection is given by

$$e^\rho_a\partial_\rho\delta g_{\sigma\lambda} = 0, \tag{1.1.58}$$

where $a, b, \dots = 1, 2, 3$ and $e^\rho_a \equiv \partial x^\rho/\partial y^a$ [44].

Hence

$$h^{\rho\sigma}\partial_\rho\delta g_{\sigma\lambda} = h^{ab}e^\rho_a e^\sigma_b \partial_\rho \delta g_{\sigma\lambda} = 0. \tag{1.1.59}$$

Therefore, Eq. (1.1.57) becomes

$$n_\mu \delta A^\mu|_{\partial\mathcal{M}} = -n^\lambda h^{\rho\sigma}\partial_\lambda g_{\rho\sigma}. \tag{1.1.60}$$

It can be shown that Eq. (1.1.60) is equal to the variation of the extrinsic curvature $K = h^{\mu\nu}K_{\mu\nu}$ evaluated on $\partial\mathcal{M}$, defined in ADM decomposition by $2K_{\mu\nu} = \mathcal{L}_n g_{\mu\nu}$, where $\mathcal{L}_n$ is the Lie derivative with respect to n.

In fact, we obtain

$$\begin{aligned} K|_{\partial\mathcal{M}} &= h^{\mu\nu}\nabla_{(\mu}n_{\nu)} \\ &= h^{\mu\nu}\left(\partial_\mu n_\nu - \Gamma^\lambda_{\mu\nu}n_\lambda\right). \end{aligned} \tag{1.1.61}$$

Hence

$$\begin{aligned} \delta K|_{\partial\mathcal{M}} &= -h^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}n_\lambda \\ &= -h^{\mu\nu}n_\lambda\Big(g^{\lambda\rho}\partial_{(\mu}\delta g_{\nu)\rho} - \frac{1}{2}g^{\lambda\rho}\partial_\rho\delta g_{\mu\nu}\Big) \\ &= \frac{1}{2}h^{\mu\nu}n_\lambda g^{\lambda\rho}\partial_\rho \delta g_{\mu\nu}, \end{aligned} \tag{1.1.62}$$

where in last step we used Eq. (1.1.59) to remove the projection of metric variation's partial derivative.

Then, by comparing Eqs. (1.1.60) and (1.1.62), we can write

$$n_\mu\delta A^\mu|_{\partial\mathcal{M}} = -2\delta K|_{\partial\mathcal{M}}. \tag{1.1.63}$$

Finally, the boundary action is the Gibbons-Hawking term with a scalar coupling with the extrinsic curvature, as well as with the Ricci scalar in (1.1.4).

Therefore, the real total scalar-tensor action is given by

$$\begin{aligned} S_{tot} &= S + S^\phi_{GH} \\ &= \frac{1}{2\kappa}\int_\mathcal{M} d^4x\sqrt{-g}\Big[\phi R - \frac{\omega}{\phi}\left(\nabla\phi\right)^2\Big] + S_m[\Psi_m, g_{\mu\nu}] \\ &- \frac{1}{\kappa}\int_{\partial\mathcal{M}} d^3y\sqrt{|h|}\epsilon\phi K. \end{aligned} \tag{1.1.64}$$

Comments. Variation calculations made in the previous sections are valid even in more general theories, such $f(R)$, which are equivalents to scalar-tensor. This calculation formalism is also called *metric formalism* [45, 47].

Remembering the first scalar-tensor action, that we wrote in Eq.(1.1.1), the non-minimally coupling is given by $f(\phi)R$. All variation calculations in this theory are the same to what we done above. In fact, the additional terms in field equations read in Eq.(1.1.43) becomes

$$\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f(\phi). \tag{1.1.65}$$

Similarly, in $f(R)$ theories, this extra term read:

$$\left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)f'(R), \tag{1.1.66}$$

where $f'(R) = df(R)/dR$. Here, the terms that in GR give the Einstein tensor $G_{\mu\nu}$ give instead $f'(R)R_{\mu\nu} - 1/2g_{\mu\nu}f(R)$, which reduces to $G_{\mu\nu}$ in GR limit, i.e. $f(R) = R$. We also see that when $f(R) = R$ the extra term (1.1.66) vanishes, as we expect.

1.2 Jordan frame and Einstein frame

In the previous section we analyzed the massless, mono scalar-tensor action, and its field equations, within the Jordan frame. As we will see below, this is also called as the *physical frame*, as we introduced within the introduction.

Here we will show the other very used frame, called *Einstein frame* (EF). This one is obtained by a conformal transformation of the metric and the scalar field in order to rewrite the ST action. The new Einstein frame metric will be minimally coupled with the new scalar field but will no longer be directly coupled with the matter fields.

1.2.1 Conformal transformation

Physical metric, $g_{\mu\nu}$, can be written in terms of the Einstein frame metric $g^{\star}_{\mu\nu}$ [9]:

$$g_{\mu\nu} = A^2(\phi) g^{\star}_{\mu\nu}, \tag{1.2.1a}$$
$$g^{\mu\nu} = A^{-2}(\phi) g^{\star\mu\nu}. \tag{1.2.1b}$$

The function $A(\phi)$ will be the only object which defines the scalar-tensor theory in Einstein frame and we will call it *coupling parameter* or *coupling function*.

In 4D spacetime[1], Eq. (1.2.1) implies this transformation law for the determinant of the metric

$$\sqrt{-g} = A^4 \sqrt{-g^{\star}}. \tag{1.2.2}$$

It is interesting to write the scalar-tensor action in EF, i.e. in terms of $g^{\star}_{\mu\nu}$ instead of $g_{\mu\nu}$ in (1.1.4).

In this section we will follow the notation introduced by Y. Fujii and K. Maeda in [11] but with an opposite definition of the coupling parameter A. This will lead to opposite signs in some terms as we will discuss later.

To obtain the transformation laws we have to transform under (1.2.1) the curvature tensors until the Ricci scalar. To make this transformation we have to start from the Christoffel symbols.

Remembering that the scalar field in JF is a function of the spacetime points x^{μ}, from Eq. (1.2.1), we have

$$\begin{aligned} \partial_\mu g_{\lambda\nu} &= \partial_\mu \left(A^2 g^{\star}_{\mu\nu}\right) \\ &= 2A\partial_\mu A g^{\star}_{\lambda\nu} + A^2 \partial_\mu g^{\star}_{\lambda\nu} \\ &= A^2 \left(2 g^{\star}_{\lambda\nu}\partial_\mu f + \partial_\mu g^{\star}_{\lambda\nu}\right) \end{aligned} \tag{1.2.3}$$

and

$$\partial_\mu g^{\lambda\nu} = A^{-2}\left(-2g^{\star\lambda\nu}\partial_\mu f + \partial_\mu g^{\star\lambda\nu}\right), \tag{1.2.4}$$

where we defined

$$f = log\, A. \tag{1.2.5}$$

Then, from (1.2.3), the Christoffel symbols reads

$$\begin{aligned} \Gamma^{\rho}_{\mu\nu} &= \Gamma^{\star\rho}_{\mu\nu} + g^{\star\rho\lambda}\left(g^{\star}_{\lambda\nu}\partial_\mu f + g^{\star}_{\lambda\mu}\partial_\nu f - g^{\star}_{\mu\nu}\partial_\lambda f\right) \\ &= \Gamma^{\star\rho}_{\mu\nu} + \left(\delta^{\rho}_{\nu}\partial_\mu f + \delta^{\rho}_{\mu}\partial_\nu f - g^{\star}_{\mu\nu}\partial^{\rho} f\right). \end{aligned} \tag{1.2.6}$$

[1] For the aims of this thesis we can limit our attention to the 4D spacetime. Generalizations to n spacetime dimensions modify some transformation exponents and, within this section, would trivially generalize the results.

We found an overall factor A^2 in (1.2.3), which is cancelled out by A^{-2} in (1.2.6) when we include the transformed inverse "starry" metric $g^{\mu\nu} = A^{-2} g^{\star\mu\nu}$.

Notice, however, that we used a different notation in the conformal transformation, i.e. we used the Eq. (1.2.1) instead of $g_{\mu\nu} = A^{-2} g^{\star}_{\mu\nu}$ used by [11]. This implies an opposite sign in some terms of calculations.

In 4D spacetime[2] $g_{\mu\nu} g^{\mu\nu} = g^{\star}_{\mu\nu} g^{\star\mu\nu} = 4$. Therefore, from Eqs. (1.2.3), (1.2.4), we have the following transformation law

$$\begin{aligned}\Gamma_\mu &= \Gamma^\star_\mu + g^{\star\rho\sigma} g^\star_{\rho\sigma} \partial_\mu f \\ &= \Gamma^\star_\mu + 4\partial_\mu f,\end{aligned} \tag{1.2.7}$$

where Γ_μ was defined in Eqs. (1.1.14) and we also write

$$\begin{aligned}C^\mu \equiv g^{\rho\sigma}\Gamma^\mu_{\rho\sigma} &= g^{\mu\lambda} g^{\rho\sigma} \partial_\rho g_{\sigma\lambda} - g^{\mu\lambda}\Gamma_\lambda \\ &= -\partial_\rho g^{\mu\rho} - g^{\mu\lambda}\Gamma_\lambda,\end{aligned} \tag{1.2.8}$$

$$C^\mu = A^{-2}\left(C^{\star\mu} - 2g^{\star\mu\nu}\partial_\nu f\right). \tag{1.2.9}$$

Now, from Eq. (1.2.6), it follows that the Riemann tensor transforms as

$$\begin{aligned}R^\rho_{\sigma\mu\nu} = R^{\star\rho}_{\sigma\mu\nu} &+ \partial_\mu\left(\delta^\rho_\nu\partial_\sigma f + \delta^\rho_\sigma\partial_\nu f - g^\star_{\sigma\nu}\partial^\rho f\right) - \partial_\nu\left(\delta^\rho_\mu\partial_\sigma f + \delta^\rho_\sigma\partial_\mu f - g^\star_{\sigma\mu}\partial^\rho f\right) \\ &+ \Gamma^{\star\rho}_{\lambda\mu}\left(\delta^\lambda_\nu\partial_\sigma f + \delta^\lambda_\sigma\partial_\nu f - g^\star_{\sigma\nu}\partial^\lambda f\right) + \left(\delta^\rho_\mu\partial_\lambda f + \delta^\rho_\lambda\partial_\mu f - g^\star_{\lambda\mu}\partial^\rho f\right)\Gamma^{\star\lambda}_{\sigma\nu} \\ &+ \left(\delta^\lambda_\nu\partial_\sigma f + \delta^\lambda_\sigma\partial_\nu f - g^\star_{\sigma\nu}\partial^\lambda f\right)\left(\delta^\rho_\mu\partial_\lambda f + \delta^\rho_\lambda\partial_\mu f - g^\star_{\lambda\mu}\partial^\rho f\right) \\ &- \Gamma^{\star\rho}_{\lambda\nu}\left(\delta^\lambda_\mu\partial_\sigma f + \delta^\lambda_\sigma\partial_\mu f - g^\star_{\sigma\mu}\partial^\lambda f\right) - \left(\delta^\rho_\nu\partial_\lambda f + \delta^\rho_\lambda\partial_\nu f - g^\star_{\lambda\nu}\partial^\rho f\right)\Gamma^{\star\lambda}_{\sigma\mu} \\ &- \left(\delta^\lambda_\mu\partial_\sigma f + \delta^\lambda_\sigma\partial_\mu f - g^\star_{\sigma\mu}\partial^\lambda f\right)\left(\delta^\rho_\nu\partial_\lambda f + \delta^\rho_\lambda\partial_\nu f - g^\star_{\lambda\nu}\partial^\rho f\right).\end{aligned} \tag{1.2.10}$$

Therefore, the Ricci tensor and the Ricci scalar read

$$\begin{aligned}R_{\sigma\nu} = R^\mu_{\sigma\mu\nu} = R^\star_{\sigma\nu} &+ \left(2\partial_\nu\partial_\sigma f - \partial_\mu g^\star_{\sigma\nu}\partial^\mu f - g^\star_{\sigma\nu}\partial_\mu g^{\star\mu\lambda}\partial_\lambda f - g^\star_{\sigma\nu}\partial_\mu\partial^\mu f\right) \\ &- \left(4\partial_\nu\partial_\sigma f + \partial_\nu\partial_\sigma f - \partial_\nu\partial_\sigma f\right) + \left(\Gamma^\star_\nu\partial_\sigma f + \Gamma^\star_\sigma\partial_\nu f - g^\star_{\sigma\nu}\Gamma^\star_\lambda\partial^\lambda f\right) \\ &+ 4\Gamma^{\star\lambda}_{\sigma\nu}\partial_\lambda f + \left(8\partial_\nu f\partial_\sigma f - 4g^\star_{\sigma\nu}\partial^\lambda f\partial_\lambda f\right) \\ &- \left(\Gamma^\star_\nu\partial_\sigma f + \Gamma^{\star\mu}_{\sigma\nu}\partial_\mu f - g^\star_{\sigma\mu}\Gamma^{\star\mu}_{\lambda\nu}\partial^\lambda f\right) \\ &- \left(\Gamma^{\star\lambda}_{\sigma\nu}\partial_\lambda f + \Gamma^\star_\sigma\partial_\nu f - g^\star_{\lambda\nu}\Gamma^{\star\lambda}_{\sigma\mu}\partial^\mu f\right) \\ &- \left(6\partial_\nu f\partial_\sigma f - 2g^\star_{\sigma\nu}\partial^\lambda f\partial_\lambda f\right),\end{aligned} \tag{1.2.11}$$

$$\begin{aligned}A^2 R = g^{\star\sigma\nu}R_{\sigma\nu} &= R^\star + \left(-2\partial_\mu\partial^\mu f - 2\Gamma^\star_\mu\partial^\mu f - 4\partial_\mu g^{\star\mu\lambda}\partial_\lambda f\right) - 4\partial_\nu\partial^\nu f \\ &- 2\Gamma^\star_\nu\partial^\nu f + 2C^{\star\lambda}\partial_\lambda f - 6\partial_\nu f\partial^\nu f \\ &= R^\star - 4\Gamma^\star_\mu\partial^\mu f - 6\partial_\mu\partial^\mu f - 4\partial_\mu g^{\star\mu\lambda}\partial_\lambda f \\ &+ 2C^{\star\lambda}\partial_\lambda f - 6\partial_\nu f\partial^\nu f \\ &= R^\star - 6\partial_\mu\partial^\mu f - 6\partial_\mu f\partial^\mu f - 6\partial_\mu g^{\star\mu\lambda}\partial_\lambda f \\ &- 3g^{\star\rho\sigma}\partial_\mu g^\star_{\rho\sigma}\partial^\mu f.\end{aligned} \tag{1.2.12}$$

[2]The generalization to the nD spacetime trivially extends some coefficients in calculations. This is not of our interest within this work.

Or, equivalently,

$$R_{\sigma\nu} = R^{\star}_{\sigma\nu} - 2\nabla^{\star}_{\mu}\nabla^{\star}_{\nu} f + 2\nabla^{\star}_{\mu} f \nabla^{\star}_{\nu} f - \left(2\nabla^{\star}_{\rho} f \nabla^{\star\rho} f + \Box_{\star} f\right) g^{\star}_{\sigma\nu}, \tag{1.2.13}$$

$$R = A^{-2}\left(R^{\star} - 6\nabla^{\star}_{\mu} f \nabla^{\star\mu} f - 6\Box_{\star} f\right), \tag{1.2.14}$$

where $\nabla^{\star}_{\mu}$ is the covariant derivative with respect to the EF metric $g^{\star}_{\mu\nu}$ and $\Box_{\star} = g^{\star\mu\nu}\nabla^{\star}_{\mu}\nabla^{\star}_{\nu}$.

Therefore, the scalar-tensor non-minimally coupled action

$$S_{nm}[g_{\mu\nu}, \phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\phi R \tag{1.2.15}$$

becomes

$$S_{nm}[g^{\star}_{\mu\nu}, \phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-g^{\star}}A^2\phi\left(R^{\star} - 6\nabla^{\star}_{\mu} f \nabla^{\star\mu} f - 6\Box_{\star} f\right). \tag{1.2.16}$$

1.2.2 Definition of Einstein frame

The non-minimally coupling to the Ricci scalar in Eq. (1.2.16) can be transformed into a minimally coupled term, to the conformal metric, by making a specific choice of the coupling function A, [10, 11]:

$$A^2 = \frac{1}{\phi}. \tag{1.2.17}$$

This choice defines the transformation between the Jordan frame and the Einstein frame in a general massless mono-scalar-tensor theory of gravity.

It must be note that, by defining the action in Jordan frame (1.1.4), we adopted the usual choice for the coupling scalar-tensor term: ϕR. In a general case we can write it as $F(\phi)R$ and therefore the transformation in Eq. (1.2.17) becomes $A^2 = F^{-1}$.

Observing that the term with $\Box_{\star} f$ in (1.2.16) disappears by integrating by parts and, applying the transformation (1.2.17) to the total ST action (1.1.4), we obtain

$$\begin{aligned} S[g^{\star}_{\mu\nu}, \phi, \Psi_m] = &\frac{1}{2\kappa}\int d^4x\sqrt{-g^{\star}}\left(R^{\star} - 2(3+2\omega(\phi))g^{\star\mu\nu}\nabla^{\star}_{\mu} f\nabla^{\star}_{\nu} f\right) \\ &+ S_m[\Psi_m, g^{\star}_{\mu\nu}/\phi]. \end{aligned} \tag{1.2.18}$$

We now can transform the kinetic term of the scalar field into the classical one, i.e. without the scalar coupling $\omega(\phi)$, by making the definition of a new scalar field φ, as function of ϕ [9]:

$$\alpha(\varphi) \equiv \frac{\partial f(\varphi)}{\partial\varphi} = \frac{1}{\sqrt{3+2\omega(\phi)}}, \tag{1.2.19}$$

together with the condition (1.2.17), where A is functional of φ.

In Eq. (1.2.19) we defined the useful functional derivative of the logarithm of the coupling parameter A and we used a different definition of φ to obtain a different factor in the scalar kinetic term as we will show below.

By comparing Eqs. (1.2.18) and (1.2.19) with Eq. (1.1.51b) it can be also seen the same factor $3 + 2\omega(\phi)$.

Therefore, in terms of the definition of φ, given in Eqs. (1.2.19), (1.2.17), the action (1.2.18) becomes

$$S[g^{\star}_{\mu\nu}, \varphi, \Psi_m] = \frac{1}{2\kappa}\int d^4x\sqrt{-g^{\star}}\left(R^{\star} - 2g^{\star\mu\nu}\nabla^{\star}_{\mu}\varphi\nabla^{\star}_{\nu}\varphi\right) + S_m[\Psi_m, A^2(\varphi)g^{\star}_{\mu\nu}]. \quad (1.2.20)$$

This is the single, massless scalar-tensor theory in the Einstein Frame, i.e. the action (1.1.4) after applying the Conformal transformation of the metric (1.2.1).

Sometimes the definition of the new scalar field φ, defined in (1.2.19), is done so that the scalar kinetic term has the classical 1/2 factor. Clearly a multiplicative numerical factor in the lagrangian density is irrelevant but in QFT the 1/2 factor is conventionally used to obtain the usual normalization of the propagator. In this work we follow the usual convention used in the classical modified theories of gravity [9].

As we shall see below, in Einstein frame the field equations take a simpler form.

Moreover it can be seen that, in the absence of any matter fields Ψ_m, the scalar-tensor action in Einstein frame (1.2.20) can be seen to be conformally related to GR in the presence of a massless scalar field because we have removed the non-minimally coupling with the Ricci scalar through the conformal transformation (1.2.1).

Similarly, for instance, to the study of the Reissner-Nordström metric, in which we add the non-abelian Yang-Mills kinetic term in curved spacetime to the Einstein-Hilbert action, i.e. $\mathcal{L} = -\frac{1}{4}\sqrt{-g^{\star}}g^{\star\mu\rho}g^{\star\nu\sigma}F^{\star}_{\mu\nu}F^{\star}_{\rho\sigma}$, $F^{\star}_{\mu\nu} = \nabla^{\star}_{\mu}A_{\nu} - \nabla^{\star}_{\nu}A_{\mu}$ [48].

The GR limit gives the same scenario, i.e. Einstein-Hilbert in the presence of extra fields, Eq. (1.2.20) in vacuum is an example of this. In Einstein frame the GR limit is realized by fixing the only scalar-tensor parameter, i.e. $A(\varphi)$:

$$A(\varphi) = const. \quad (1.2.21)$$

In Jordan frame the GR limit can be read in the equation which defines φ, Eq. (1.2.19), using the Einstein frame one (1.2.21):

$$\omega \to +\infty. \quad (1.2.22)$$

1.2.3 Field equations in Einstein frame

Now, from the extremization of the action in Einstein frame (1.2.20) respectively with respect to $g^{\star\mu\nu}$ and φ, we get the conformal transformed field equations:

$$G^{\star}_{\mu\nu} = 8\pi T^{\star}_{\mu\nu} - g^{\star}_{\mu\nu}\left(\nabla^{\star}\varphi\right)^2 + 2\nabla^{\star}_{\mu}\varphi\nabla^{\star}_{\nu}\varphi, \quad (1.2.23a)$$

$$\Box_{\star}\varphi = -4\pi\alpha(\varphi)T^{\star}, \quad (1.2.23b)$$

with $T^{\star}_{\mu\nu} = -2(-g^{\star})^{-1/2}\delta S_m/\delta g^{\star\mu\nu}$ and its EF trace is given by $T^{\star} = g^{\star\mu\nu}T^{\star}_{\mu\nu}$.

Eq. (1.2.23a) can be immediately found keeping in mind the comments on boundary terms done previously. In order to find Eq. (1.2.23b) we write

$$\begin{aligned}
0 &= \delta_{\phi}S \\
&= \frac{1}{2\kappa}\int d^4x\sqrt{-g^{\star}}\left(-4g^{\star\mu\nu}\nabla^{\star}_{\mu}\delta\varphi\nabla^{\star}_{\nu}\varphi\right) + \delta_{\phi}S_m \\
&= \frac{1}{2\kappa}\int d^4x\sqrt{-g^{\star}}\Big[4g^{\star\mu\nu}\delta\varphi\nabla^{\star}_{\mu}\nabla^{\star}_{\nu}\varphi \\
&+ \frac{2\kappa}{\sqrt{-g^{\star}}}\frac{\delta\left(\sqrt{-g}\mathcal{L}_m(\Psi_m, A^2(\varphi)g^{\star}_{\rho\sigma})\right)}{\delta\varphi}\delta\varphi\Big],
\end{aligned} \quad (1.2.24)$$

where we integrated by parts the kinetic scalar term removing the boundary integral because $\delta\varphi|_{\partial\mathcal{M}} = 0$.

Then, remembering the metric conformal transformation: $g_{\mu\nu} = A^2(\varphi)g^\star_{\mu\nu}$, $\sqrt{-g} = A^4(\varphi)\sqrt{-g^\star}$ (see Eq.(1.2.1) and Eq.(1.2.2)), we have

$$\begin{aligned}\frac{-2}{\sqrt{-g^\star}}\frac{\delta S_m[\Psi_m, A^2(\varphi)g^\star_{\rho\sigma}]}{\delta\varphi} &= A^4(\varphi)\frac{-2}{\sqrt{-g}}\frac{\delta S_m[\Psi_m, g_{\rho\sigma}])}{\delta g^{\mu\nu}}\frac{\delta g^{\mu\nu}}{\delta\varphi} \\ &= A^4(\varphi)T_{\mu\nu}\left(-2A^{-3}(\varphi)\frac{\partial A(\varphi)}{\partial\varphi}g^{\star\mu\nu}\right) \\ &= -2A^2(\varphi)T_{\mu\nu}A^{-1}(\varphi)\frac{\partial A(\varphi)}{\partial\varphi}g^{\star\mu\nu},\end{aligned} \tag{1.2.25}$$

where, from Eq.(1.2.19), $\alpha \equiv A^{-1}\frac{\partial A}{\partial\varphi}$. Therefore

$$\begin{aligned}\frac{-2}{\sqrt{-g}}\frac{\delta S_m[\Psi_m, A^2(\varphi)g^\star_{\rho\sigma}]}{\delta\varphi} &= -2A^2(\varphi)T_{\mu\nu}\alpha(\varphi)g^{\star\mu\nu} \\ &= -2\alpha(\varphi)T^\star.\end{aligned} \tag{1.2.26}$$

In last step we used the transformation law of stress-energy tensor in Eq.(1.3.2) and the usual notation $T^\star = T^\star_{\mu\nu}g^{\star\mu\nu}$.

Then, Eqs.(1.2.24),(1.2.25) give the scalar field equation (1.2.23b).

In the non-homogeneous term of (1.2.23b) is present the derivative of the logarithm of A, the coupling parameter which gives the Conformal transformation (1.2.17). α can be read in terms of the Jordan frame coupling in (1.2.19).

Similarly to what was done in (1.1.50), we can found the Ricci scalar by tanking trace of (1.2.23a) with respect to $g^{\star\mu\nu}$:

$$R^\star = 2\left(\nabla^\star\varphi\right)^2 - 8\pi T^\star. \tag{1.2.27}$$

Therefore, the tensor field equation (1.2.23a) can be written by eliminating $R^\star$, found in (1.2.27):

$$R^\star_{\mu\nu} = 2\nabla^\star_\mu\varphi\nabla^\star_\nu\varphi + 8\pi\left(T^\star_{\mu\nu} - \frac{1}{2}T^\star g^\star_{\mu\nu}\right). \tag{1.2.28}$$

It is clear that the field equations, for the metric tensor $g^\star_{\mu\nu}$ (1.2.28) and for the scalar field φ (1.2.23b), in Einstein frame are simpler that those in Jordan frame (1.1.51a), (1.1.51b).

1.2.4 Generalization to multi-scalar-tensor theories

In this work we use the single, massless, scalar-tensor theories to make our calculation. Here, following the work done by T. Damour [9], we briefly report the general modified theory of gravity with a set of scalar fields and with a potential (or cosmological term).

In Einstein frame the action reads

$$\begin{aligned}S[g^\star_{\mu\nu}, \varphi^a, \Psi_m] = &\frac{1}{2k}\int d^4x\sqrt{-g^\star}\left(R^\star - 2g^{\star\mu\nu}\gamma_{ab}(\varphi^c)\nabla^\star_\mu\varphi^a\nabla^\star_\nu\varphi^b - 2\Lambda(\varphi^a)\right) \\ &+ S_m[\Psi_m, A^2(\varphi^a)g^\star_{\mu\nu}],\end{aligned} \tag{1.2.29}$$

where γ_{ab} is the metric of this "σ-model":

$$d\sigma^2 = \gamma_{ab}(\varphi^c)d\varphi^a d\varphi^b, \tag{1.2.30}$$

with $a, b, c, \cdots = 1, \cdots, n$ and n is the dimension of the scalar fields space.

In Eq. (1.2.29) we used the usual "-2" factor in the cosmological term, in contrast to what was done in [9].

The field equations are $4+n$ instead of $4+1$ of the single scalar field case. Together with the on-shell condition on the matter fields Ψ_m (1.1.49), Eq. (1.2.29) gives

$$G^\star_{\mu\nu} + \Lambda(\varphi^a) g^\star_{\mu\nu} + \gamma_{ab}(\varphi^c) \left(g^\star_{\mu\nu} g^{\star\rho\sigma} \nabla^\star_\rho \varphi^a \nabla^\star_\sigma \varphi^b - 2\nabla^\star_\mu \varphi^a \nabla^\star_\nu \varphi^b \right) = 8\pi T^\star_{\mu\nu}, \tag{1.2.31}$$

$$\Box_\star \varphi^a - \frac{1}{2}\Lambda^a(\varphi^b) + g^{\star\mu\nu} \gamma^a_{bc}(\varphi^d) \nabla^\star_\mu \varphi^b \nabla^\star_\nu \varphi^c = -4\pi\alpha^a(\varphi^b) T^\star, \tag{1.2.32}$$

where

$$\Lambda^a = \gamma^{ab} \frac{\partial \Lambda}{\partial \varphi^b}, \tag{1.2.33}$$

$$\alpha^a = \gamma^{ab} \frac{\partial\, log\, A}{\partial \varphi^b}. \tag{1.2.34}$$

Moreover

$$\gamma^a_{bc} = \frac{1}{2}\gamma^{ad} \left(\partial_b \gamma_{cd} + \partial_c \gamma_{bd} - \partial_d \gamma_{bc} \right) \tag{1.2.35}$$

are the Christoffel symbols of the "σ-model" metric γ_{ab} (1.2.30).

We will do not elaborate on more this generalization because it is not the core of this work, and we will always use the simpler case that we discussed above.

1.3 Stress-energy tensor conservation

Here we will briefly comment the conservation of the stress-energy tensor both in Jordan and Einstein frame. This will justify what we introduced above, i.e. that the JF is the physical one. In general relativity, the Bianchi identity implies the covariant conservation of the stress-energy tensor. We will see that this is also true in JF but it does not in EF.

1.3.1 Conformal transformation law

The conformal transformation between the Jordan and Einstein frame was defined by Eq. (1.2.17): $g_{\mu\nu} = A^2 g^\star_{\mu\nu}$. This condition gives $\sqrt{-g} = A^4\sqrt{-g^\star}$, as written in Eq. (1.2.2).

In order to find the transformation law of the stress-energy tensor under the Eq. (1.2.17) we compute

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta S_m}{\delta g^{\mu\nu}} = \frac{-2}{A^4\sqrt{-g^\star}} \underbrace{\frac{\delta g^{\star\mu\nu}}{\delta g^{\mu\nu}}}_{=A^2} \frac{\delta S_m}{\delta g^{\star\mu\nu}}. \tag{1.3.1}$$

Therefore we get

$$T^\star_{\mu\nu} = A^2 T_{\mu\nu}, \tag{1.3.2}$$

where $T_{\mu\nu}$ is the stress-energy tensor in Jordan frame, while $T^\star_{\mu\nu}$ in the Einstein frame. Moreover Eqs.(1.2.17), (1.2.2) and (1.3.2) give the following transformation laws:

$$\begin{aligned} T^{\star\mu\nu} &= g^{\star\mu\rho} g^{\star\nu\sigma} T^\star_{\rho\sigma} = A^2 g^{\mu\rho} A^2 g^{\nu\sigma} A^2 T_{\rho\sigma} \\ &= A^6 T^{\mu\nu}, \end{aligned} \tag{1.3.3}$$

$$\begin{aligned} T_{\mu}^{\star\nu} &= g^{\star\nu\rho}T_{\mu\rho}^{\star} = A^2 g^{\nu\rho}A^2 T_{\mu\rho} \\ &= A^4 T_{\mu}^{\nu}, \end{aligned} \tag{1.3.4}$$

$$\begin{aligned} \sqrt{-g^{\star}}T_{\mu}^{\star\nu} &= \sqrt{-g^{\star}}g^{\star\nu\rho}T_{\mu\rho}^{\star} = A^{-4}\sqrt{-g}A^2 g^{\nu\rho}A^2 T_{\mu\rho} \\ &= \sqrt{-g}T_{\mu}^{\nu} \end{aligned} \tag{1.3.5}$$

and

$$\begin{aligned} T^{\star} &= g^{\star\mu\nu}T_{\mu\nu}^{\star} = A^2 g^{\mu\nu}A^2 T_{\mu\nu} \\ &= A^4 T. \end{aligned} \tag{1.3.6}$$

We now want to write the conservation of the stress-energy tensor in Jordan and Einstein frame.

1.3.2 Einstein frame

Let us start from the Einstein frame. Remembering the Bianchi identity $\nabla^{\star\mu}G_{\mu\nu}^{\star} = 0$ [1,3], we can find the covariant derivative of $T_{\mu\nu}^{\star}$ by applying $\nabla^{\star\mu}$ to Eq. (1.2.23a). By using the Levi-Civia connection condition for the metric, i.e. $\nabla_{\rho}^{\star}g_{\mu\nu}^{\star} = 0$, we obtain

$$\begin{aligned} 8\pi\nabla^{\star\mu}T_{\mu\nu}^{\star} &= \cancelto{0}{\nabla^{\star\mu}G_{\mu\nu}^{\star}} + \nabla_{\nu}^{\star}\left(\nabla^{\star}\varphi\right)^2 - 2\nabla^{\star\mu}\left(\nabla_{\mu}^{\star}\varphi\nabla_{\nu}^{\star}\varphi\right) \\ &= 2\nabla_{\nu}^{\star}\nabla^{\star\mu}\varphi\nabla_{\mu}^{\star}\varphi - 2\left(\Box_{\star}\varphi\nabla_{\nu}^{\star}\varphi + \nabla_{\mu}^{\star}\varphi\nabla^{\star\mu}\nabla_{\nu}^{\star}\varphi\right) \\ &= -2\Box_{\star}\varphi\nabla_{\nu}^{\star}\varphi + 2\nabla_{\mu}^{\star}\varphi g^{\star\mu\sigma}\left[\nabla_{\nu}^{\star},\nabla_{\sigma}^{\star}\right]\varphi, \end{aligned} \tag{1.3.7}$$

where we defined the usual commutator of operators acting on a scalar field f:

$$\left[A_{\mu}, B_{\nu}\right] f = A_{\mu}B_{\nu}f - B_{\nu}A_{\mu}f. \tag{1.3.8}$$

The commutator in Eq. (1.3.7) gives a zero value because φ is a scalar field and $\nabla^{\star}$ is the Einstein frame covariant derivative in the Levi-Civita connection, i.e. $\Gamma_{\sigma\nu}^{\star\lambda} = \Gamma_{\nu\sigma}^{\star\lambda}$. In fact we have

$$\left[\nabla_{\nu}^{\star},\nabla_{\sigma}^{\star}\right]\varphi = \nabla_{\nu}^{\star}\partial_{\sigma}\varphi - \nabla_{\sigma}^{\star}\partial_{\nu}\varphi = \partial_{\nu}\partial_{\sigma}\varphi - \Gamma_{\nu\sigma}^{\star\lambda}\partial_{\lambda}\varphi - \partial_{\sigma}\partial_{\nu}\varphi + \Gamma_{\sigma\nu}^{\star\lambda}\partial_{\lambda}\varphi = 0. \tag{1.3.9}$$

Therefore Eq. (1.3.7) becomes

$$8\pi\nabla^{\star\mu}T_{\mu\nu}^{\star} = -2\Box_{\star}\varphi\nabla_{\nu}^{\star}\varphi. \tag{1.3.10}$$

Using now the scalar field equation (1.2.23b) we can eliminate $\Box_{\star}\varphi$ in Eq. (1.3.10):

$$\nabla^{\star\mu}T_{\mu\nu}^{\star} = \alpha T^{\star}\nabla_{\nu}^{\star}\varphi. \tag{1.3.11}$$

Therefore in Einstein frame the stress-energy tensor is not in general covariantly conserved except in the GR limit (1.2.21). In fact, if $A = const$ we have $\alpha = A^{-1}\delta A/\delta\varphi = 0$.

1.3.3 Jordan frame

Let us now focus on the same calculation in Jordan frame, i.e. let us take the Jordan frame covariant derivative ∇^μ of the Eq. (1.1.44):

$$\begin{aligned} 8\pi\nabla^\mu T_{\mu\nu} &= G_{\mu\nu}\nabla^\mu\phi + \left(\nabla_\nu\Box\phi - \Box\nabla_\nu\phi\right) \\ &+ \nabla^\mu\left(\frac{1}{2}g_{\mu\nu}\frac{\omega}{\phi}\left(\nabla\phi\right)^2 - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi\right). \end{aligned} \tag{1.3.12}$$

In (1.3.12) we used the Bianchi identity for the Einstein tensor $\nabla^\mu G_{\mu\nu} = 0$.

Therefore, remembering the definition of $T^\phi_{\mu\nu}$ given in (1.1.46), the last term on the right hand of (1.3.12) can be written as

$$\begin{aligned} \nabla^\mu T^\phi_{\mu\nu} &= \frac{1}{2}\nabla_\nu\left(\frac{\omega}{\phi}\right)\left(\nabla\phi\right)^2 + \frac{\omega}{\phi}\nabla_\nu\nabla^\mu\phi\nabla_\mu\phi - \nabla^\mu\left(\frac{\omega}{\phi}\right)\nabla_\mu\phi\nabla_\nu\phi \\ &- \frac{\omega}{\phi}\left(\Box\phi\nabla_\nu\phi + \nabla_\mu\phi\nabla^\mu\nabla_\nu\phi\right) \\ &= -\frac{1}{2}\left(\frac{\omega}{\phi}\right)'\left(\nabla\phi\right)^2\nabla_\nu\phi + \cancel{\frac{\omega}{\phi}\nabla_\nu\nabla^\mu\phi\nabla_\mu\phi} \\ &- \frac{\omega}{\phi}\left(\Box\phi\nabla_\nu\phi + \cancel{\nabla_\mu\phi\nabla^\mu\nabla_\nu\phi}\right) \\ &= \frac{1}{2}\left(-\left(\frac{\omega}{\phi}\right)'\left(\nabla\phi\right)^2 - 2\frac{\omega}{\phi}\Box\phi\right)\nabla_\nu\phi, \end{aligned} \tag{1.3.13}$$

where we used $\nabla_\nu\left(\frac{\omega}{\phi}\right) = \left(\frac{\omega}{\phi}\right)'\nabla_\nu\phi$ in the first and last terms of the first line of (1.3.13) to find the first term of the third line. We also used Eq.(1.3.9) to remove the terms shown in third line.

Remembering the Ricci scalar in Jordan frame found above in Eq. (1.1.50), we can rewrite the covariant derivative of $T^\phi_{\mu\nu}$ as

$$\nabla^\mu T^\phi_{\mu\nu} = \frac{1}{2}R\nabla_\nu\phi = \frac{1}{2}Rg_{\mu\nu}\nabla^\mu\phi. \tag{1.3.14}$$

Notice that, in literature, the definition of $T^\phi_{\mu\nu}$ could be done with a different 1/2 factor or an opposite sign.

Let us focus now on the second couple of terms in Eq.(1.3.12). Those terms can be written as the commutator $[\nabla_\nu, \Box]$ acting on the scalar field φ.

In fact, we can compute

$$[\nabla_\nu, \Box]\,\phi = g^{\rho\sigma}\left[\nabla_\nu, \nabla_\rho\nabla_\sigma\right]\phi = g^{\rho\sigma}\left(\nabla_\rho\left[\nabla_\nu, \nabla_\sigma\right]\phi + \left[\nabla_\nu, \nabla_\rho\right]\nabla_\sigma\phi\right). \tag{1.3.15}$$

In the last step of Eq. (1.3.15) we used the identity $[A, BC] = B\,[A, C] + [A, B]\,C$, valid for all operators A, B and C.

The first commutator in Eq. (1.3.15) gives zero because ϕ is a scalar field and we are in the Levi-Civita connection, i.e. $\Gamma^\lambda_{\sigma\nu} = \Gamma^\lambda_{\nu\sigma}$. Similarly to what was done in Eq. (1.3.9).

To conclude the calculation of (1.3.15) we have to use

$$\left[\nabla_\nu, \nabla_\rho\right]V_\sigma = -R^\lambda_{\sigma\nu\rho}V_\lambda, \tag{1.3.16}$$

for every 1-form field V_σ [3]. This gives

$$[\nabla_\nu, \Box]\,\phi = -g^{\rho\sigma} R^{\lambda}_{\sigma\nu\rho} \nabla_\lambda \phi = -R_{\nu\lambda} \nabla^\lambda \phi, \tag{1.3.17}$$

where we used the antisymmetry of Riemann tensor with respect to the exchange of the first two and the last two indices: $R_{\lambda\sigma\nu\rho} = -R_{\sigma\lambda\nu\rho}$ and $R_{\lambda\sigma\nu\rho} = -R_{\lambda\sigma\rho\nu}$ [3].

By merging all these terms into Eq. (1.3.12) we gain

$$\begin{aligned} 8\pi \nabla^\mu T_{\mu\nu} &= G_{\mu\nu} \nabla^\mu \phi + [\nabla_\nu, \Box]\,\phi + \nabla^\mu T^{\phi}_{\mu\nu} \\ &= G_{\mu\nu} \nabla^\mu \phi - R_{\mu\nu} \nabla^\mu \phi + \frac{1}{2} R g_{\mu\nu} \nabla^\mu \phi. \end{aligned} \tag{1.3.18}$$

This gives the result for the covariant derivative of stress-energy tensor

$$\nabla^\mu T_{\mu\nu} = 0. \tag{1.3.19}$$

Therefore, the Jordan frame stress-energy tensor is always covariantly conserved. Indeed, this frame is called as "physical frame" as we introduced above.

1.4 The Brans-Dicke model

One special class of scalar-tensor theories was introduced by C.H. Brans and R.H. Dicke [39, 40] of which, the theories described in Eq. (1.1.4), were future generalizations.

These sub-class of theories are characterized by the presence of only one coupling parameter. On the contrast, their extensions that we are considering in this work, defined by Eq. (1.1.4) (in Jordan frame) or Eq. (1.2.20) (in Einstein frame), contain a coupling function of the scalar field. In JF these restrictions simply read as:

$$\omega(\phi) = \omega_{BD} = const, \tag{1.4.1}$$

where $\omega(\phi)$ is the coupling function defined in Eq. (1.1.4). The Einstein frame version can be read from Eq. (1.2.19), in which the EF coupling is given in terms of the JF one. This, therefore, implies:

$$\alpha(\varphi) = \alpha_{BD} = const. \tag{1.4.2}$$

This simplified case is then called: *Brans-Dicke model*.

In the following sections we will show that the ST perturbation corrections depend on the logarithmic derivatives of the coupling function. Within the Brans-Dicke model, there is only the α_{BD} ST parameter and this implies that the EF function, from which it is derived, reads:

$$A_{BD}(\varphi) = e^{\alpha_{BD}\varphi}. \tag{1.4.3}$$

The Field equations in Jordan frame, from Eqs. (1.1.44), (1.1.51b), read:

$$\phi G_{\mu\nu} = 8\pi T_{\mu\nu} - \left(\Box\phi + \frac{1}{2} \frac{\omega_{BD}}{\phi} (\nabla\phi)^2 \right) g_{\mu\nu} + \nabla_\mu \nabla_\nu \phi + \frac{\omega_{BD}}{\phi} \nabla_\mu \phi \nabla_\nu \phi, \tag{1.4.4a}$$

$$\Box\phi = \frac{1}{2\omega_{BD} + 3} 8\pi T, \tag{1.4.4b}$$

where ω_{BD} is the only ST constant parameter within these theories.

1.5 Scalar-tensor PN parameters

The PN perturbation theory is a low-velocity expansion [49, 50], namely $(v/c) \ll 1$ (here we reintroduced the light speed c to make explicit the dimensionless expansion parameter). Using the classical virial theorem we can translate this dimensionless velocity parameter into a distance one:

$$\left(\frac{v}{c}\right)^2 = \frac{\tilde{G}_{AB} M}{R} \equiv u, \tag{1.5.1}$$

where R is the EOB radial coordinate (see Eq. (2.2.4) below within the EOB section) and M denotes the total mass of the binary system that we consider. The ST factor (bodies-dependent) $\tilde{G}_{AB}$ is a deviation with respect to GR and it can be read in Eq. (1.5.10) in terms of the parameters that we are going to examine.

The weak-field expansion $u \ll 1$ is called post-Minkowskian (PM) theory [51, 52]. Blanchet-Damour approach [15] is based on a mixing of these perturbation theories applied in different zones, i.e. PN in the near-region and PM in the far-region, and then matching them in the intermediate region.

Now, in the Einstein frame, from Eq. (1.2.23b), it is clear that $\alpha(\varphi)$ measures the coupling strength between the scalar field and the matter. In Refs. [9, 12, 53] it has been shown that all weak-field deviation from GR, i.e. any PN order, can be written in terms of the logarithmic derivative of the coupling function, $\alpha(\varphi)$, and its derivatives valued at the asymptotic value of scalar field. This new parameter, called φ_0, denotes the asymptotic value of φ at the spatial infinity. Using a term dear to the spontaneous symmetry breaking theory, φ_0 is the Vacuum Expectation Value (VEV) of φ.

We will define the coupling function derivatives as [9]:

$$\alpha_0 \equiv \alpha(\varphi_0), \qquad \beta_0 \equiv \frac{\partial \alpha}{\partial \varphi}|_{\varphi_0}, \qquad \beta_0' \equiv \frac{\partial \beta}{\partial \varphi}|_{\varphi_0}, \qquad \beta_0'' \equiv \frac{\partial \beta'}{\partial \varphi}|_{\varphi_0}, \tag{1.5.2a}$$

$$\beta_0^{(n)} \equiv \frac{\partial \beta^{(n-1)}}{\partial \varphi}|_{\varphi_0} \equiv \frac{\partial^{n+1} \alpha}{\partial \varphi^{n+1}}|_{\varphi_0}. \tag{1.5.2b}$$

The subscript "0" always denotes the parameter evaluated in the Einstein frame VEV φ_0.

We can remember that the Jordan frame transformation is defined by (see Eqs.(1.2.1), (1.2.19)):

$$g_{\mu\nu} = A^2(\varphi) g^{\star}_{\mu\nu}, \qquad \alpha(\varphi) = \frac{d \log A(\varphi)}{d\varphi} = \frac{1}{\sqrt{3 + 2\omega(\phi)}} \tag{1.5.3}$$

and the scalar transformation $\varphi(\phi)$ is defined by $\phi = A^{-2}(\varphi)$ (see Eq.(1.2.17)). Therefore, the scalar VEV in Einstein frame is given by $\varphi_0 = \varphi(\phi_0)$.

Parameters	Jordan Frame [17, 18, 54]	Einstein Frame [9, 55]
Weak-field		
G	$\frac{4+2\omega_0}{\phi_0(3+2\omega_0)}$	$(1+\alpha_0^2)A_0^2$
ζ	$\frac{1}{4+2\omega_0}$	$\frac{\alpha_0^2}{1+\alpha_0^2}$
λ_1	$\frac{\zeta^2}{1-\zeta}\phi_0\frac{d\omega}{d\phi}\vert_0$	$\frac{\beta_0}{2(1+\alpha_0^2)}$
λ_2	$\frac{\zeta^3}{1-\zeta}\phi_0^2\frac{d^2\omega}{d\phi^2}\vert_0$	$\frac{1}{4(1+\alpha_0^2)^2}\left(-2\alpha_0^2\beta_0+4\beta_0^2-\alpha_0\beta_0'\right)$
λ_3	$\frac{\zeta^4}{1-\zeta}\phi_0^3\frac{d^3\omega}{d\phi^3}\vert_0$	$\frac{1}{8(1+\alpha_0^2)^3}\left(8\alpha_0^4\beta_0+24\beta_0^3+6\alpha_0^3\beta_0'\right.$ $\left.-13\alpha_0\beta_0\beta_0'+\alpha_0^2(-24\beta_0^2+\beta_0'')\right)$
Strong-field		
s_A	$\frac{d\,log\,m_A(\phi)}{d\log\phi}\vert_0$	$\frac{1}{2}-\frac{\alpha_A}{2\alpha_0}$
s'_A	$\frac{d^2\,log\,m_A(\phi)}{d\log\phi^2}\vert_0$	$\frac{-\alpha_A\beta_0}{4\alpha_0^3}+\frac{\beta_A}{4\alpha_0^2}$
s''_A	$\frac{d^3\,log\,m_A(\phi)}{d\log\phi^3}\vert_0$	$-\frac{3\alpha_A\beta_0^2}{8\alpha_0^5}+\frac{3\beta_A\beta_0}{8\alpha_0^4}+\frac{\alpha_A\beta_0'}{8\alpha_0^4}-\frac{\beta_A'}{8\alpha_0^3}$
s'''_A	$\frac{d^4\,log\,m_A(\phi)}{d\log\phi^4}\vert_0$	$-\frac{15\alpha_A\beta_0^3}{16\alpha_0^7}+\frac{15\beta_0^2\beta_A}{16\alpha_0^6}+\frac{5\alpha_A\beta_0\beta_0'}{8\alpha_0^6}$ $-\frac{\beta_A\beta_0'}{4\alpha_0^5}-\frac{3\beta_0\beta_A'}{8\alpha_0^5}-\frac{\alpha_A\beta_0''}{16\alpha_0^5}+\frac{\beta_A''}{16\alpha_0^4}$
	$\hat{s}_{A,B}\equiv 1-2s_{A,B}$	
Binaries		
N		
α	$1-\zeta+\zeta\hat{s}_A\hat{s}_B$	$\frac{1+\alpha_A\alpha_B}{1+\alpha_0^2}$
1PN		
γ	$-2\alpha^{-1}\zeta\hat{s}_A\hat{s}_B$	$-\frac{2\alpha_A\alpha_B}{1+\alpha_A\alpha_B}$
β_A	$\alpha^{-2}\zeta\hat{s}_B^2(\lambda_1\hat{s}_A+2\zeta s'_A)$	$\frac{\alpha_B^2\beta_A}{2(1+\alpha_A\alpha_B)^2}$
β_B	$\alpha^{-2}\zeta\hat{s}_A^2(\lambda_1\hat{s}_B+2\zeta s'_B)$	$\frac{\alpha_A^2\beta_B}{2(1+\alpha_A\alpha_B)^2}$
2PN		
δ_A	$\alpha^{-2}\zeta(1-\zeta)\hat{s}_A^2$	$\frac{\alpha_A^2}{(1+\alpha_A\alpha_B)^2}$
δ_B	$\alpha^{-2}\zeta(1-\zeta)\hat{s}_B^2$	$\frac{\alpha_B^2}{(1+\alpha_A\alpha_B)^2}$
$\chi_A\equiv-\frac{\epsilon_A}{4}$	$\alpha^{-3}\zeta\hat{s}_B^3(\tilde{\chi}_0\hat{s}_A-6\zeta\lambda_1 s'_A+2\zeta^2 s''_A)$	$-\frac{\alpha_B^3\beta_A'}{4(1+\alpha_A\alpha_B)^3}$
$\chi_B\equiv-\frac{\epsilon_B}{4}$	$\alpha^{-3}\zeta\hat{s}_A^3(\tilde{\chi}_0\hat{s}_B-6\zeta\lambda_1 s'_B+2\zeta^2 s''_A)$	$-\frac{\alpha_A^3\beta_B'}{4(1+\alpha_A\alpha_B)^3}$
	$\tilde{\chi}_0\equiv\lambda_2-4\lambda_1^2+\zeta\lambda_1$	
3PN		
κ_A	$\alpha^{-4}\zeta\hat{s}_B^4(\tilde{\kappa}_0\hat{s}_A+2\zeta\tilde{\kappa}_1 s'_A-12\zeta^2\lambda_1 s''_A+2\zeta^3 s'''_A)$	$\frac{\alpha_B^4\beta_A''}{8(1+\alpha_A\alpha_B)^4}$
κ_B	$\alpha^{-4}\zeta\hat{s}_A^4(\tilde{\kappa}_0\hat{s}_B+2\zeta\tilde{\kappa}_1 s'_B-12\zeta^2\lambda_1 s''_B+2\zeta^3 s'''_B)$	$\frac{\alpha_A^4\beta_B''}{8(1+\alpha_A\alpha_B)^4}$
	$\tilde{\kappa}_0\equiv\lambda_3-13\lambda_1\lambda_2+28\lambda_1^3+\zeta(3\lambda_2-13\lambda_1^2)+\lambda_1\zeta^2$ $\tilde{\kappa}_1\equiv 19\lambda_1^2-4\lambda_2-4\lambda_1\zeta$	
	$S_-\equiv-\alpha^{-1/2}(s_A-s_B)$	$\sqrt{\frac{1+\alpha_0^2}{1+\alpha_A\alpha_B}}\frac{(\alpha_A-\alpha_B)}{2\alpha_0}$
	$S_+\equiv\alpha^{-1/2}(1-s_A-s_B)$	$\sqrt{\frac{1+\alpha_0^2}{1+\alpha_A\alpha_B}}\frac{(\alpha_A+\alpha_B)}{2\alpha_0}$
	$\xi\equiv 1+\frac{\gamma}{2}+\frac{\zeta S_+^2}{6}$	$\frac{1}{1+\alpha_A\alpha_B}\left(1+\frac{2}{3}\left(\frac{\alpha_A+\alpha_B}{4}\right)^2\right)$

Table 1.1: Parameters which enter in the gravitational wave emission by binary systems. The subscripts ”0” points that the variable is evaluated at the scalar VEV, ϕ_0 (in Jordan frame), or φ_0 (in Einstein frame). The scalar field transformation is defined by $\phi=A^{-2}(\varphi)$ (see Eq.(1.2.17)).

The body mass, as function of the scalar field[3], can be expansed about the background field, i.e. the VEV ϕ_0. In Jordan frame it reads:

$$m_A(\phi) = m_A^{(0)}\Big[1+s_A\Psi+\frac{1}{2}(s_A^2-s_A+s_A')\Psi^2+\frac{1}{6}\big(s_A^3-3s_A^2+s_A(2+3s_A')-3s_A'+s_A''\big)\Psi^3+O(4)\Big], \tag{1.5.4}$$

where

$$\Psi \equiv \frac{\phi-\phi_0}{\phi_0} \tag{1.5.5}$$

and s_A, s_A', $\cdots$ are defined in Table 1.1, as well as all the ST parameters we will use in the following. $m_A^{(0)}$ denotes the background mass value, i.e. $m_A(\phi_0)$, which correspond to the usual mass in the GR limit.

In Table 1.1 we gathered also the transformation from Jordan frame to Einstein frame, noticing the huge simplification of the PN parameter expressions within the last one.

These conversions are based on the mass function transformation law between the Jordan and Einstein frame. This can be found by imposing the invariance of the "skeletonized" matter action with respect to the change from one frame to the other:

$$\begin{aligned}
-\sum_A \int d\lambda\sqrt{-g_{\mu\nu}\frac{dx_A^\mu}{d\lambda}\frac{dx_A^\nu}{d\lambda}}m_A(\varphi) \overset{\mathrm{JF}}{=} S_m \overset{\mathrm{EF}}{=} &-\sum_A \int d\lambda\sqrt{-g^\star_{\mu\nu}\frac{dx_A^\mu}{d\lambda}\frac{dx_A^\nu}{d\lambda}}m^\star_A(\varphi)\\
&= -\sum_A \int d\lambda\sqrt{-g_{\mu\nu}\frac{dx_A^\mu}{d\lambda}\frac{dx_A^\nu}{d\lambda}}A^{-1}(\varphi)m^\star_A(\varphi),
\end{aligned} \tag{1.5.6}$$

where we used the metric conformal transformation law (1.2.1). Following the previous notation, the mass $m_A^\star$ represents the Einstein frame one, while m_A the Jordan frame one.

Therefore, the transformation law involving the mass functions reads:

$$m_A^\star(\varphi) = A(\varphi)m_A(\varphi). \tag{1.5.7}$$

By taking the EF mass we can define the strong-field coupling parameters by its logarithmic derivatives with respect to the scalar field[4]:

$$\alpha_A \equiv \frac{d\log m_A^\star}{d\varphi}, \qquad \beta_A \equiv \frac{\alpha_A}{d\varphi}, \qquad \beta_A' \equiv \frac{\beta_A}{d\varphi}, \qquad \beta_A'' \equiv \frac{\beta_A'}{d\varphi}, \qquad \cdots \tag{1.5.8}$$

These definitions and transformation laws implie that the sensitivity of the compact body A transforms as (see Table 1.1):

$$\begin{aligned}
s_A \equiv \frac{d\log m_A(\phi)}{d\log\phi}|_{\phi_0} &= \phi_0\left(\frac{d\log m_A^\star(\varphi)}{d\varphi}-\frac{d\log A(\varphi)}{d\varphi}\right)|_{\varphi_0}\left(\frac{d\phi}{d\varphi}\right)^{-1}|_{\varphi_0}\\
&= A_0^{-2}\left(\alpha_A-\alpha_0\right)\left(-2A_0^{-2}\alpha_0\right)^{-1}\\
&= \frac{1}{2}-\frac{\alpha_A}{2\alpha_0},
\end{aligned} \tag{1.5.9}$$

where we used the definitions of the scalar field in Einstein frame in Eq.(1.2.17) and the EF strong-field coupling parameters in Eq. (1.5.8).

[3]The JF "skeletonized" body is described by the matter action introduced in Eq. (1.1.7).
[4]As well as with the JF one done in Table 1.1.

In Table1.1 we combined the notation introduced in Refs. [9, 54, 55]. We updated these parameter tables with λ_3 and s_A''', both in their natural definition in Jordan frame and in the Einstein frame transformation. These allow to define the 3PN parameters $\kappa_{A,B}$ (see Ref. [17]) and we also calculated their Einstein frame version.

We defined the intermediate factors $\hat{s}_{A,B}$, $\tilde{\chi}_0$ and $\tilde{\kappa}_{1,2}$ in order to simplify the notation and, nevertheless, it can be note the massive compactness of Einstein frame parameters.

It can be also defined the bodies-dependent coupling parameters:

$$\tilde{G}_{AB} \equiv G\alpha, \qquad G_{AB} \equiv \frac{\tilde{G}_{AB}}{A_0^2}. \tag{1.5.10}$$

The strong-field base parameter s_A is called *sensitivity* of the body and for test bodies it clearly reads $s_A = 0$. Furthermore, in ref. [56] we can read that, for stationary black holes, $s_A = 1/2$. This makes explicit the important fact, that we mentioned above, for which the gravitational wave emission in binary black hole (BBH) systems is identical to GR. In fact, from Table 1.1, all PN scalar-tensor parameters are proportionals to $\hat{s}_{A,B} = 1 - 2s_{A,B}$, which are both 0 for BBH systems.

Therefore, important difference with respect to GR could be detected by black hole-neutron star (BHNS) and, mostly, BNS systems. As we will discuss below, ST theories include a dipolar radiation, which is gauged away in GR, proportional to $(\alpha_A - \alpha_B)^2$ (in addition to the universal scalar G constant: Table 1.1) [9]. BHNS systems could admit even a more intense dipolar term with respect to BNS due to the zero value of the scalar sensitivity of black hole, i.e. $s_B = 0$. This case is realized by the condition [9]:

$$G_{AB}^2(\alpha_A - \alpha_B)^2 < \alpha_A^2. \tag{1.5.11}$$

1.5.1 DEF gravity

In Sec. 1.4 we introduced the simplest and oldest class of scalar-tensor theories, i.e. the Brans-Dicke model, in which the universal coupling function ($A(\varphi)$ in EF and $\omega(\phi)$ in JF) depends only on one free parameter. However, despite the simplicity of this theory, it can not be used to perform calculations and to put constraints due to its deficiency of PN information. A useful class of theories, which is very used, is the *DEF gravity* or, as sometimes it is called in literature, *quadratic theory*. This was elaborated by T. Damour and G. Esposito-Farèse [9, 12, 36] and it includes higher order effects, with respect to the BD model, and allows nonperturbative phenomena in the NSs, as we will see below.

The DEF theory, in the Einstein frame, is completely defined by

$$A_{DEF}(\varphi) = e^{\frac{1}{2}\beta_{DEF}\varphi^2}, \qquad \beta_{DEF} = const, \tag{1.5.12}$$

while in BD model was $A_{BD}(\varphi) = e^{\alpha_{BD}\varphi}$. Since we will always use this DEF model during all this work, we are making explicit the subscript "DEF" only within this sections, while elsewhere we will omit it.

From Eq. (1.5.2) and Table 1.1 we can observe that, in this particular case, we have only 2 universal (bodies-independent) ST parameters. These are collected in the set $\{\varphi_0, \beta_{DEF}^0\}$ or, equivalently, $\{\alpha_{DEF}^0, \beta_{DEF}^0\}$. The subscript "0" always represents the parameter evaluated at the cosmological scalar field φ_0. The coupling factors, i.e. the logarithmic derivatives of $A(\varphi)$, can be read from Eq. (1.5.12):

$$\alpha_{DEF}(\varphi) = \beta_{DEF}\varphi, \tag{1.5.13a}$$
$$\beta_{DEF}(\varphi) = \beta_{DEF} = const, \tag{1.5.13b}$$
$$\beta'_{DEF}(\varphi) = 0, \tag{1.5.13c}$$
$$\beta''_{DEF}(\varphi) = 0 \tag{1.5.13d}$$

and, clearly, their cosmological values. These could be reinserted into Table 1.1 to simplify the higher order weak-parameters and, consequently, also the higher order strong-ones, which only depend on the universal factors.

1.6 TOV scalar metric

The analysis of the gravitational radiation emitted by BNS systems requires the knowledge of the interior, and exterior, spacetime structure of these compact bodies. In GR, through the Birkhoff theorem [57], the unique spherically symmetric solution of the Einstein field equations in vacuum is the Schwarzschild one [58]. In the interior zone, in which there is the star's matter, the metric tensor is given by the solutions of the Tolman–Oppenheimer–Volkoff (TOV) equations [59], which are not exact in general, even if equipped with a certain equation of state (EOS).

Similarly, we can write the TOV equations in ST theories. The spherically symmetric, static metric (in Einstein frame) of an isolated non-rotating neutron star reads:

$$\begin{aligned} ds_\star^2 &= g^\star_{\mu\nu}dx^\mu dx^\nu \\ &= -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\Omega^2, \end{aligned} \tag{1.6.1}$$

where $d\Omega^2 \equiv d\theta^2 + \sin^2(\theta)d\varphi^2$ is the usual unitary 2-sphere metric. Note that φ in $d\Omega^2$ does not to be confused with the scalar field $\varphi(r)$.

We also define the usual mass function as

$$e^{2\lambda(r)} \equiv \left(1 - \frac{2m(r)}{r}\right)^{-1} \tag{1.6.2}$$

and we will assume a perfect-fluid stress-energy tensor for the physical the neutron star's matter[5]:

$$\begin{aligned} T_{\mu\nu} &= A^{-2}(\varphi)T^\star_{\mu\nu} \\ &= (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu}. \end{aligned} \tag{1.6.3}$$

u_μ and $g_{\mu\nu}$ are respectively the 4-velocity and the metric tensor in the physical Jordan frame, as well as the pressure p and energy density ϵ.

The physical Jordan frame 4-velocity of the fluid matter, in the non-rotating case, can be written as

$$u_\mu = A(\varphi)e^{-\nu}(1,0,0,0), \tag{1.6.4}$$

due to the normalization

$$\begin{aligned} -1 &= g^{\mu\nu}u_\mu u_\nu = A^{-2}(\varphi)g^{\star\mu\nu}u_\mu u_\nu. \\ &= g^{\star\mu\nu}u^\star_\mu u^\star_\nu. \end{aligned} \tag{1.6.5}$$

[5]As usual within this thesis, we identify with "$\star$" the EF quantities and the JF version with the "naked" ones.

Moreover, the relative Einstein frame expression is given by:

$$u^{\star}_{\mu} = A^{-1}(\varphi)u_{\mu} = e^{-\nu}(1,0,0,0). \tag{1.6.6}$$

In order to get an easier set of equations we will work in the Einstein frame, thus we have to use the "$\star$" stress-energy tensor given by its transformation law (1.3.2).

Then, TOV equations can be read from the field equations (1.2.28) and (1.2.23b). We also have to use the transformation law Eq.(1.3.6) for the trace $T^{\star}$:

$$T^{\star} = A^4(\varphi)T = A^4(\varphi)\,(3p - \epsilon)\,. \tag{1.6.7}$$

The metric structure of Eq.(1.6.1) clearly gives the same Ricci tensor components of the GR-Schwarzschild ones (see Ref. [60]).

Eq.(1.6.1) also gives the wave operator acting on the scalar field:

$$\begin{aligned}\Box_{\star}\varphi(r) &= \frac{1}{\sqrt{-g^{\star}}}\partial_{\mu}\left(\sqrt{-g^{\star}}g^{\star\mu\nu}\partial_{\nu}\varphi(r)\right)\\ &= \Gamma^{\star}_{r}e^{-2\lambda}\varphi' - 2\lambda' e^{-2\lambda}\varphi' + e^{-2\lambda}\varphi''\\ &= e^{-2\lambda}\left[\left(\nu' - \lambda' + \frac{2}{r}\right)\varphi' + \varphi''\right]\end{aligned} \tag{1.6.8}$$

or, in terms of the mass function $m(r)$ defined in Eq.(1.6.2):

$$\Box_{\star}\varphi = \left(1 - \frac{2m}{r}\right)\left[\left(\nu' + \frac{m'r - m}{r(r-2m)} + \frac{2}{r}\right)\psi + \psi'\right], \tag{1.6.9}$$

where we defined the scalar field radial derivative $\psi \equiv \varphi' \equiv d\varphi/dr$. In Eq.(1.6.8) we used the "Γ-vector" notation, introduced above in Eq.(1.1.14), for the Einstein frame metric $g^{\star}_{\mu\nu}$.

Field equations can be simply obtained from Eqs.(1.2.23a),(1.2.23b) and the definition (1.6.2). The (tt), (rr) and $(\theta\theta)$ components, together with the scalar equation (1.6.9), give:

$$(tt):\; m' = 4\pi r^2 A^4(\varphi)\epsilon + \frac{1}{2}r(r-2m)\psi^2, \tag{1.6.10a}$$

$$(rr):\; \nu' = \frac{m + 4\pi r^3 A^4(\varphi)p}{r(r-2m)} + \frac{1}{2}r\psi^2, \tag{1.6.10b}$$

$$\begin{aligned}(\theta\theta):\; \nu'' &= \frac{1}{r^2(r-2m)^2}\Big[2m(m-r) + 4\pi r^3 A^4(\varphi)\Big(r(\epsilon+p)\\ &\quad - m(\epsilon-p) - 4\pi r^3 A^4(\varphi)p(p-\epsilon)\Big)\Big] + \frac{\psi^2}{(r-2m)^2}\Big[m(3r-2m)\\ &\quad - r^2 + 4\pi r^3 A^4(\varphi)\Big(\frac{1}{2}r(\epsilon+p) - m(\epsilon-p)\Big)\Big],\end{aligned} \tag{1.6.10c}$$

$$(\varphi):\; 4\pi\frac{rA^4(\varphi)}{r-2m}\Big(\alpha(\varphi)(\epsilon-3p) + r\psi(\epsilon-p)\Big) - \frac{2(r-m)}{r(r-2m)}\psi. \tag{1.6.10d}$$

These are the only equations which enter because $(\varphi\varphi) = \sin^2(\theta)(\theta\theta)$ due to the spherical symmetry of the metric (1.6.1).

In order to obtain the radial derivative of the pressure we could take the r component of $\nabla^{\star\nu}T^{\star}_{\mu\nu} = \alpha T^{\star}\nabla^{\star}_{\mu}\varphi$ (see Eq.(1.3.11)). Otherwise, we can take the radial derivative of the (rr) component and use the $(\theta\theta)$ to remove ν''. These steps give:

$$p' = -(\epsilon+p)\Big(\nu' + \alpha(\varphi)\psi\Big). \tag{1.6.11}$$

Therefore, the TOV equations in scalar-tensor theories can be written as

$$m' = 4\pi r^2 A^4(\varphi)\epsilon + \frac{1}{2}r(r-2m)\psi^2, \tag{1.6.12a}$$
$$\nu' = \frac{m + 4\pi r^3 A^4(\varphi)p}{r(r-2m)} + \frac{1}{2}r\psi^2, \tag{1.6.12b}$$
$$p' = -(\epsilon + p)\left(\frac{m + 4\pi r^3 A^4(\varphi)p}{r(r-2m)} + \frac{1}{2}r\psi^2 + \alpha(\varphi)\psi\right), \tag{1.6.12c}$$
$$\varphi' = \psi, \tag{1.6.12d}$$
$$\psi' = 4\pi\frac{rA^4(\varphi)}{r-2m}\Big(\alpha(\varphi)(\epsilon - 3p) + r\psi(\epsilon - p)\Big) - \frac{2(r-m)}{r(r-2m)}\psi. \tag{1.6.12e}$$

We can check that we correctly get back the general relativistic TOV equations [59] by setting $A(\varphi) = 1$ and $\varphi = const$. Notice that some terms are proportional to $\psi^2 = (\varphi')^2$, therefore proportional to the scalar energy density.

As well as in the general relativistic TOV metric, the solution can be found by integrating Eq.(1.6.12) from the star center $r = 0$ to the surface $r = R_s$. In order to make this integration, one have to match the scalar TOV metric with the vacuum *exact* solution of the field equations, i.e. the spherically symmetric and statically solution of (see Eqs.(1.2.28),(1.2.23b)):

$$R_{\mu\nu} = 2\nabla^\star_\mu\varphi\nabla^\star_\nu\varphi, \tag{1.6.13a}$$
$$\Box\varphi = 0. \tag{1.6.13b}$$

The Einstein frame vacuum metric can be written in Just radial coordinate (see Refs. [9, 61–63]):

$$\begin{aligned} ds^2_{\star,ext} &= g^{\star\,ext}_{\mu\nu}dx^\mu dx^\nu \\ &= -e^{2\nu(\rho)}dt^2 + e^{-2\nu(\rho)}\left(d\rho^2 + e^{-2\lambda(\rho)}\rho^2 d\Omega^2\right), \end{aligned} \tag{1.6.14}$$

where we denotes as ρ the radial coordinate in Just metric in order to distinguish it from r in Eq.(1.6.1).

In Eq.(1.6.14) ν and λ functions are the external metric functions and they must not be confused with the ones defined in Eq.(1.6.1). The metric (1.6.14) allows us to write Eq.(1.6.13b) as

$$\begin{aligned} 0 = \Box\varphi &= \Gamma^{\star\,ext}_{\rho}e^{2\nu}\psi + 2\nu' e^{2\nu}\psi + e^{2\nu}\psi' \\ &= e^{2\nu}\Big[-2\lambda'\psi + \psi'\Big]. \end{aligned} \tag{1.6.15}$$

Here the superscript $'$ denotes the derivative with respect to ρ.

Moreover, the tensor field equations (1.6.13a) give

$$(tt) : \nu'' = 2\lambda'\nu', \tag{1.6.16a}$$
$$(rr) : 2\lambda'' = 2\big(\lambda'^2 + \psi^2 + \lambda'\nu' + \nu'^2\big) - \nu'', \tag{1.6.16b}$$
$$(\theta\theta) : e^{-2\lambda}\big(2\lambda'^2 + 2\lambda'\nu' - \lambda'' - \nu''\big) = 1. \tag{1.6.16c}$$

By using the (tt) equation to remove ν'' into the $(\theta\theta)$ one we get:

$$\begin{aligned} 1 &= e^{-2\lambda}\big(2\lambda'^2 - \lambda''\big) \\ &= \frac{1}{2}\left(e^{-2\lambda}\right)''. \end{aligned} \tag{1.6.17}$$

Therefore, we can write

$$e^{-2\lambda} = \rho^2 + \alpha\rho + \beta, \tag{1.6.18}$$

where α and β are the two real integration constants.

Using this last equation (1.6.18) into the (tt) component we obtain the differential equation for ν

$$\nu'' = -\frac{2\rho + \alpha}{\rho^2 + \alpha\rho + \beta}\nu' \implies \nu' = \gamma\frac{1}{\rho^2 + \alpha\rho + \beta}. \tag{1.6.19}$$

We can now distinguish the cases: $\alpha^2 - 4\beta > 0$ and $\alpha^2 - 4\beta < 0$ in which, respectively, the metric function $e^{-2\lambda}$ admits and does not admits reals roots. This allows us to rewrite λ, and then also ν, in both cases [9]:

$$\alpha^2 - 4\beta > 0 \qquad\qquad \alpha^2 - 4\beta < 0 \tag{1.6.20a}$$

$$e^{-2\lambda} = \rho^2 - a\rho \qquad\qquad e^{-2\lambda} = \rho^2 + a^2 \tag{1.6.20b}$$

$$e^{2\nu} = \left(1 - \frac{a}{\rho}\right)^{\frac{b}{a}} \qquad\qquad 2\nu = \frac{b}{a}\left[\arctan\left(\frac{\rho}{a}\right) - \frac{\pi}{2}\right]. \tag{1.6.20c}$$

The second case represents a wormhole metric that, in this kind of works, it will be ignored.

By taking the first class of solutions in Eq. (1.6.20) we can solve the scalar equation (1.6.13b):

$$\varphi(\rho) = \varphi_0 + \frac{d}{a}\log\left(1 - \frac{a}{\rho}\right), \tag{1.6.21}$$

where the constants a, b, d and φ_0 are due to the integration. The last one is the scalar field value at the spatial infinity and the other ones are related by the relation $a^2 - b^2 = 4d^2$.

The scalar VEV φ_0, within the quadratic DEF gravity (see Sec. 1.5.1), is universally fixed by the first two derivatives of the coupling function $A(\varphi)$. The other integration constants are related with the body ones through:

$$\begin{aligned} b &= 2m_A, \\ \frac{a}{b} &= \sqrt{1 + \alpha_A}, \\ \frac{d}{b} &= \frac{\alpha_A}{2}. \end{aligned} \tag{1.6.22}$$

By comparing the interior and exterior metric, Eqs. (1.6.1), (1.6.14), the radial coordinates transform as:

$$r = \rho\left(1 - \frac{a}{\rho}\right)^{\frac{a-b}{2a}}, \tag{1.6.23a}$$

$$e^{2\lambda(r)} = \left(1 - \frac{a}{\rho}\right)\left(1 - \frac{a+b}{2\rho}\right)^{-2}. \tag{1.6.23b}$$

The relations due to the matching at the neutron star surface can be read in Ref. [12]. In the following section we will briefly discuss the most common equations of state used to describe the NS structure.

1.6.1 Equation of state

The structure of an isolated neutron star can be completely found by imposing an equation of state, i.e. a relation between the NS fluid variables $\{\rho, p, \epsilon\}$, which are respectively the baryonic density, the pressure and the energy density[6]. Within this section, we are referring to Refs. [21, 64–66] for conventions and calculations.

The NS fluid variables $\{\rho, p, \epsilon\}$ corresponds to the "$\sim$" ones in the references we have mentioned. These are given in Jordan frame, which is the physical frame of ST theories (in particular of the mono-scalar-tensor theories we are considering) due to its minimally coupling in the matter action, while we defined the EF quantities with the $\star$ symbols (see Chap. 1 for more details).

The simplest EOS to consider for a NS is a *polytropic* one, i.e. an equation in which the baryonic density and the pressure are linked with a power law:

$$p = K\rho^{\Gamma}, \tag{1.6.24a}$$

$$e = \rho + \frac{p}{\Gamma - 1}, \tag{1.6.24b}$$

where the second one is due to the first law of thermodynamics and the parameters K, Γ are, respectively, the polytropic constant and the adiabatic index. By inserting these equations into the TOV equations, Eq. (1.6.12), one can solve them numerically and extract all the information on the NS structure.

An extension of a simple polytropic EOS is a *piecewise polytropic* one. This stitches together several density region of the NS and each zone is described by a single polytropic equation. Therefore, if we split the NS in n different regions, we have:

$$p_i = K_i\rho_i^{\Gamma_i}, \qquad i = 1, \cdots, n, \tag{1.6.25a}$$

$$e_i = \rho_i + \frac{p_i}{\Gamma_i - 1}, \quad i = 1, \cdots, n. \tag{1.6.25b}$$

For each EOS, the best fit values of the tabulated solutions, [67], are given by the set $\{\log(p), \Gamma_1, \Gamma_2, \Gamma_3\}$. These sets and methods can be elaborated deeply in Refs. [21, 64–66]. Our final analysis on the `TEOBResumS` scalar-tensor waveform will be based on three of these EOSs, for three different gravitational wave events. All the ST and dynamics parameters of these cases will be written in the following sections.

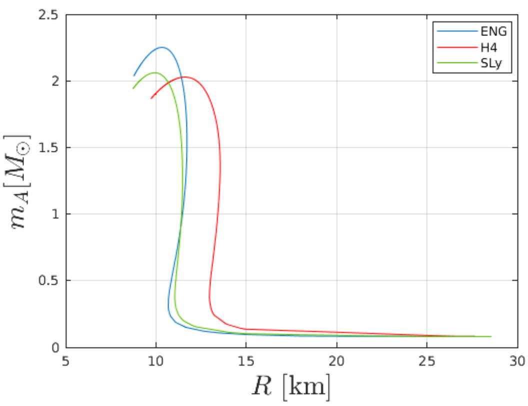

Figure 1.1: General relativistic Mass vs Radius relation of an isolated NS with a certain EOS. The former is given in units of solar masses. The latter, instead, in units of solar masses given in Km, i.e. the NS radius R is corrected by a multiplicative factor $GM_\odot/c^2 \simeq 1.476625061404649Km$.

[6]Do not confuse the baryonic density ρ with the radial coordinate used in Just metric (1.6.14).

In Fig. 1.1 we can read the mass-radius relation, in general relativity, for the two EOSs we have chosen within this work. This plot was realized thanks to the piecewise polytropic's parameters given in Table III of Ref. [64].

In this work we have chosen 3 of the multitude written in Ref. [64]. We taken the EOS: ENG [68], SLy[7] [69] and H4 [70], as they can be read in Fig. 1.1. All the EOSs try to describe the complex structure of its relative NS, which is composed by degenerate matter at very high density. The composition of the inner most NS region is unknown for now and the GW observations could put constraints on its EOS and, consequently, on the aforementioned structure.

The 3 EOSs, that we will use to show the deviation with respect to GR, were strategically chosen. The first two: ENG and SLy, will be used in the "GW170817" event while the latter: H4, in a more massive test BNS "GWtest", in order to increase the potential differences. In Ref. [71] it is shown that "soft" EOSs, such as ENG and SLy, are favored with respect to "stiff" ones, such as H4. In fact, we can see in Fig. 1.1 how the NS dimension with ENG is very close to the SLy one, for GW170817, while is greater for the H4. This last EOS could be interesting because, as opposed to the others, contains strange condensed matter. In particular, in Ref. [70] the authors elaborated on the hypernuclei structure of NSs within a relativistic mean-field theory. This could be relevant in GW signal, with ST correction, due to the large mass allowed and this larger scalar values.

1.7 Nonperturbative scalarization

Mono scalar-tensor theories are simple generalization of general relativity but, nevertheless, they allow phenomena which do not exist within the Einstein theory, as we introduced previously. Is was shown, by T. Damour and G. Esposito-Farèse in 1993, that in mono scalar-tensor theories with quadratic coupling (called DEF gravity, see Sec. 1.5.1) a NS could develops a non perturbative phenomenon called *spontaneous scalarization* [9, 12, 36].

By remembering the DEF scalar coupling from Eq. (1.5.12):

$$A(\varphi) = e^{1/2\beta\varphi^2}, \tag{1.7.1}$$

they proved that, when the universal parameter β satisfies the condition

$$\beta \lesssim -4, \tag{1.7.2}$$

a massive NS could assumes a very high value of the scalar charge α_A, much larger then α_0. This can be obtained by integrating numerically the scalar TOV equations (1.6.12) (see Refs. [12, 36, 66] for more details).

[7]EOS SLy often is called SLy4 in literature.

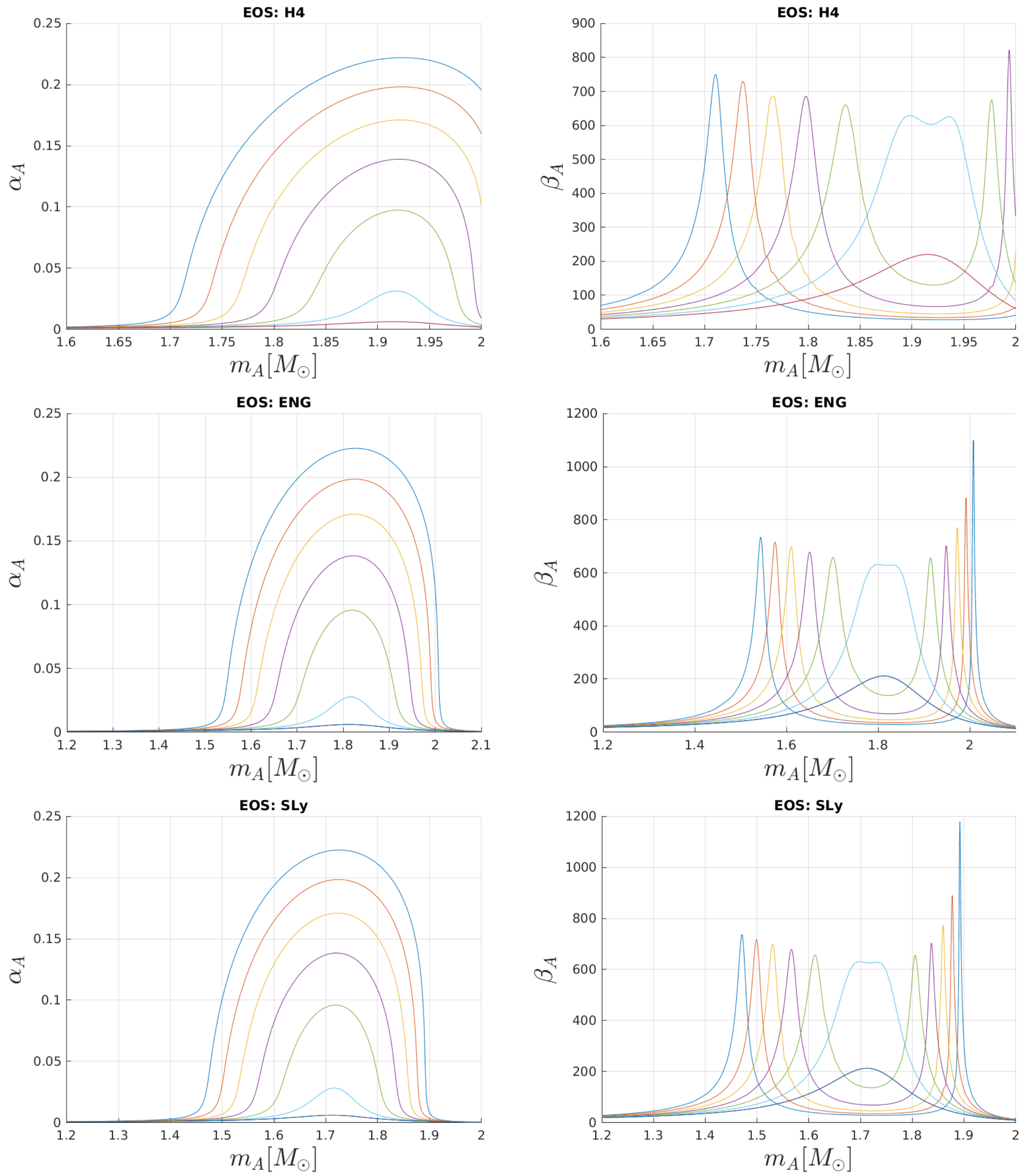

Figure 1.2: ST body-parameters α_A and β_A of an isolated NS with a certain EOS, as function of its mass m_A. These curves correspond to the following set of parameters. $\alpha_0 = 1.2 \cdot 10^{-4}$ in all the three cases. For the EOS ENG: β_0 varies from -4.54 to -4.24. EOS SLy: β_0 varies from -4.54 to -4.24. EOS H4: β_0 varies from -4.54 to -4.28. β_0 varies (from the top to the bottom) with a step of 0.04 in every plot. In DEF theories these are the only two universal parameters and they can be read in Eqs. (1.5.2), (1.5.13).

Fig. 1.2 shows the results of Ref. [66], in particular the scalar charge, and its scalar derivative (see Eq. (1.5.8)), as function of the NS mass[8].

By taking the derivatives with respect to the scalar field in Eq. (1.5.8), one must

[8] All numerical results are tabulated for a wide range of the universal parameter (α_0, β_0) and they can be read in the public repository: https://github.com/eXtremeGravityInstitute/scalar_charges.

keep constant the baryonic mass, defined in EF by [36]:

$$\bar{m}_A^\star = m_b \int_0^{R_{star}} dr 4\pi r^2 n A^3(\varphi(r)) \frac{1}{\sqrt{1 - 2\frac{m(r)}{r}}}. \tag{1.7.3}$$

In Eq. (1.7.3), $m_b \simeq 1.66 \cdot 10^{-24} g$ and n denote respectively the atomic mass unit and the baryonic number density. The integral covers the whole dimension of NS and the mass function $m(r)$ was defined in Eq. (1.6.2), within the scalar TOV discussion.

Within this quadratic DEF theory there is an apparent paradox that can be solved with the spontaneous scalarization phenomenon, that we will discuss in the following. From Eqs. (1.5.12), (1.5.13) we read that the coupling function $\alpha(\varphi)$ is linear in the EF scalar field through the constant β, i.e. $\alpha(\varphi) = \beta\varphi$. By inserting this special case into Eq. (1.2.23b) we can see that the inhomogeneous term is linear in φ. Therefore, the trivial function $\varphi(x) = 0$ is the unique solution when we impose the homogeneous boundary condition, i.e. $\varphi_0 = 0$ (or, equivalently, $\alpha_0 = \beta\varphi_0 = 0$). In Fig. 1 of Ref. [12] we can see that this conclusion is correct for low-mass compact object (such as ordinary astronomic object, white dwarfs (WDs) and low-masses NSs) but, when a NS is sufficiently massive, emerges a scalar charge.

This phenomenon can be understood by making an analogy with the well-known ferromagnetic transition. This last case can be described, within the Landau theory, by imposing a Z_2 symmetry [72]. It can be found that, when the temperature T is below a critical value, $T < T_c$, (T_c denotes the Curie temperature) the ferromagnet assumes a non-zero magnetization m (which is the order parameter of the theory) in absence of external fields. The Landau free density energy under the Z_2 symmetry, near the critical point, can be written as

$$f(T, m) = \frac{1}{2}(T - T_c) m^2 + \frac{1}{4} T m^4, \tag{1.7.4a}$$

$$\frac{\partial f(T, m)}{\partial m} = h_{ext}, \tag{1.7.4b}$$

where h_{ext} is the external field conjugate to the order parameter m. If we turn off h_{ext}, we obtain that the free density energy is minimized when

$$m = \begin{cases} 0 & \text{if} \quad T > T_c \\ \pm\sqrt{\frac{T_c - T}{T}} & \text{if} \quad T < T_c \end{cases} \tag{1.7.5}$$

In a completely analogous way, the spontaneous scalarization of a massive NS can be described within the Landau theory, Z_2 invariant. This is allowed by the structure of the coupling function (1.5.12), which is invariant under the reflection $\varphi \to -\varphi$. Eqs. (1.7.4), (1.7.5) can be totally mirrored into the ST theories by using the baryonic mass $\bar{m}_A^\star$ as the thermodynamic temperature and the pair of conjugate variable $\{\omega_A, \varphi_0\}$ instead of $\{m, h_{ext}\}$[9]. ω_A plays the role of the conjugate variable to the external field φ_0: $\omega_A = \partial m_A / \partial \varphi_0$, where m_A corresponds to the mass-energy function (1.5.4), equivalent to f.

Therefore, the spontaneous scalarization of massive, isolated NS can be interpreted as a first-order transition, completely analogues to the ferromagnetic one.

[9]Note that our convention of the conjugation (1.7.4b) differs with respect to the one used in Ref. [12] by a sign minus.

1.8 Constraints from binary pulsars

Within this last introduction section, we will briefly report the constraints on the ST parameters found by L. Shao, N. Sennett, A. Buonanno, M. Kramer and N. Wex in Ref. [13]. We will base all our future plots and final discussions on these constraints. Their analysis were made within the DEF theory. We defined this special class of scalar-tensor theories in Sec. 1.5.1 and we commented the interesting nonperturbative phenomenon of spontaneous scalarization of massive neutron star in Sec. 1.7.

The choice of the ST theory is crucial in order to fix the number of parameters which are the only $\{\alpha_0, \beta_0\}$ (or, equivalently, $\{\varphi_0, \beta_0\}$) in DEF mono-scalar-tensor gravity. These universal parameters were constrained in Ref. [13] by performing a parameter estimation based on 5 binary pulsars data (see Table I or Ref. [13]), in particular they are NS-WD binary systems: J0348+0432, J1012+5307, J1738+0333, J1909-3744, J2222-0137.

These analysis were realized by picking the $\alpha_0 = \beta_0\varphi_0$ parameter within a range upper bounded from the Cassini spacecraft experiment: $\alpha_0 < 3.4 \cdot 10^{-3}$. The DEF β_0 constant was instead chosen by including the spontaneous spalarization phenomenon, that we described in Sec. 1.7: $\beta_0 \in [-5, -4]$. For more details see, in particular, Table II or Ref. [13].

We will use these limits, together with the choice of a NS EOS as we discussed in Sec. 1.6.1, on ST parameters to construct our sets of parameters that we will use to test our calculations and implementations.

Chapter 2

ST correction at the 3PN order

The purpose of this chapter is to recall the effective-one-body (EOB) approach to the GR two-body problem [15, 73, 74], i.e. mapping the real two-body problem into an one-body of a test particle moving into an effective metric, and its extension to the ST theories.

Moreover we will recall the tidal interaction in GR [21, 22] and, briefly, in ST theories [75].

Finally we will find part of the scalar-tensor 3PN correction to the A potential of the effective metric, which we will discuss below, that generates the non-spinning circular energy found by L. Bernard in [17, 75].

In order to perform this calculation, we will introduce a new term in the A function that, within the EOB circular approximation, gives the effective angular momentum and orbital frequency which allow to compute the circular energy. Then, this ST new term will be found by comparing this generic 3PN energy with the already known Bernard's one.

We also will plot the position of the Last Stable Orbit (LSO), and Light Ring (LR), in 5PN GR with 3PN ST correction for some values of the ST parameters.

Note that all scalar-tensor calculations will be perform for non-spinning bodies and they will be implemented in the public code `TEOBRseumS`. It can be found in the repository:

https://bitbucket.org/eob_ihes/workspace/projects/EOB.

Its complete form is in C, which is the language in which we made our scalar-tensor changes.

2.1 Effective-one-body approach

The Effective-One-Body (EOB) approach was introduced within the general relativistic two-body problem by T. Damour and A. Buonanno [15, 73, 74].

The EOB method maps the real two-body problem into an one-body problem of a test particle moving into an effective metric. General relativistic effective metric is a static and spherically symmetric ν-deformation of the Schwarzschild one, where $\nu \equiv \mu/M \equiv m_1 m_2/M^2$ is the symmetric mass ratio ($0 < \nu \leq 1/4$). $\nu = 1/4$ is the equal-mass case and the limit $\nu = 0$ reduces to Schwarzschild case.

EOB mapping of Hamiltonians required that the variables $\mathcal{N}$ and $\mathcal{J}$, in a quantum analogy, must be the same between real and effective problems:

$$\frac{H_e}{\mu} = \frac{H^2 - m_1^2 - m_2^2}{2m_1 m_2}, \tag{2.1.1}$$

where H_e and H are respectively the relativistic energy of effective and real problem, i.e. with also the rest-mass contribution. This map also holds in scalar-tensor theories of gravity [55, 63].

The unique solution of eq.(2.1.1) is

$$H = M\sqrt{1 + 2\nu\left(\frac{H_e}{\mu} - 1\right)}. \tag{2.1.2}$$

EOB approach is a powerful technique to study the gravitational wave signal emitted by a binary system because of its non-perturbative resummation of Post-Newtonian (PN) expansion of the equations of motion. In the following sections we will summarize the PN theory basics and its ST parameters.

2.2 Dynamics

Let us consider a generic binary system of two objects with masses m_i, $i = A, B$, and spin vectors $\mathbf{S}_i$. The projection of the spin's body along the orbital angular momentum $\mathbf{L}$ is called

$$S_i \equiv \mathbf{L} \cdot \mathbf{S}_i. \tag{2.2.1}$$

The following binary mass variables are respectively the total mass M, the reduced mass μ and the symmetric mass ratio ν:

$$M \equiv m_A + m_B, \qquad \mu \equiv \frac{m_A m_B}{M}, \qquad \nu \equiv \frac{\mu}{M}, \qquad q \equiv \frac{m_A}{m_B}, \tag{2.2.2a}$$

$$X_i \equiv \frac{m_i}{M}, \qquad \psi \equiv X_{AB} \equiv X_A - X_B \tag{2.2.2b}$$

$$= \sqrt{1 - 4X_A X_B} = \sqrt{1 - 4\nu} \quad (m_A > m_B).$$

We use the spherical (Schwarzschild-Droste) coordinates (T, R, θ, φ)[1] and we now focus ourself on non-spinning bodies and ignoring the tidal effects, i.e. avoiding the structure of the bodies.

As well as in GR we restrict ourself to the equatorial plane $\theta = \pi/2$ and we define the *tortoise radial coordinate* $R_\star$ by

$$\frac{dR_*}{dR} = \left(\frac{A(R)}{B(R)}\right)^{-1/2}, \tag{2.2.3}$$

where A and B are the EOB effective potentials we will define in the following.

Furthermore, we will use the dimensionless phase space variables $(r, p_r, \varphi, p_\varphi)$, with [9, 18, 54, 55, 76]

$$t = \frac{T}{\tilde{G}_{AB} M}, \qquad r = \frac{R}{\tilde{G}_{AB} M}, \qquad p_r = \frac{P_R}{\mu}, \qquad p_\varphi = \frac{P_\varphi}{\mu \tilde{G}_{AB} M}. \tag{2.2.4}$$

The ST parameter $\tilde{G}_{AB}$ is defined in Table. 1.1 and Eq. (1.5.10).

[1] Here the azimuthal angle φ is not to be confused with the EF scalar field defined in Eq. (1.2.19).

2.2.1 Effective metric

In Schwarzschild-Droste coordinates, the static and spherically symmetric effective metric in Einstein frame (for $\theta = \pi/2$) reads:

$$\begin{aligned} ds_e^2 &= g^{\star}_{\mu\nu} dx^{\mu} dx^{\nu} \\ &= -A(r)dt^2 + B(r)dr^2 + r^2 d\varphi^2. \end{aligned} \tag{2.2.5}$$

By defining the variable

$$u \equiv \frac{1}{r} \tag{2.2.6}$$

we can remember the GR-Schwarzschild [58] limit

$$A(u)^{GR}_{Schw} = \left(B(u)^{GR}_{Schw}\right)^{-1} = 1 - 2u. \tag{2.2.7}$$

These results are recovered in GR test mass limit ($\nu = 0$) and, in the following sections, also by setting all ST parameters to zero.

In GR-Schwarzschild limit we also could observe that the tortoise-radial-coordinate, defined in Eq.(2.2.3), coincides with the one used in the Schwarzschild metric written in Eddington-Finkelstein coordinates [77–79]:

$$\begin{cases} \frac{dR_*}{dR} = \left|1 - \frac{2M}{R}\right|^{-1} \\ V = T + R_*. \end{cases} \tag{2.2.8}$$

From A and B metric potential definitions in Eq.(2.2.5) we can define the D and $\bar{D}$ potentials as

$$D(u) \equiv A(u)B(u), \tag{2.2.9}$$

$$\bar{D}(u) \equiv \frac{1}{D(u)}. \tag{2.2.10}$$

The general relativistic PN expansion, using $u \ll 1$, of the effective A and $\bar{D}$ potentials read, respectively up to 5PN and 3PN, :

$$\begin{aligned} A^{GR}(u) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \nu(a_5^c + a_5^{log} \log u)u^5 \\ + \nu(a_6^c + a_6^{log} \log u)u^6 + \mathcal{O}(u^7), \end{aligned} \tag{2.2.11}$$

$$\bar{D}^{GR}(u) = 1 + 6\nu u^2 + 2\bar{d}_3 u^3 + \mathcal{O}(u^4), \tag{2.2.12}$$

where [80]

$$\begin{aligned} \bar{d}_3 &= \nu\left(26 - 3\nu\right), \\ a_4 &= \frac{94}{3} - \frac{41}{32}\pi^2, \\ a_5^c &= \frac{2275}{512}\pi^2 - \frac{4237}{60} + \frac{128}{5}\gamma_E + \frac{256}{5}\log 2 \\ &\quad + \nu\left(\frac{41}{32}\pi^2 - \frac{221}{6}\right), \\ a_5^{log} &= \frac{64}{5}, \\ a_6^{log} &= -\frac{7004}{105} - \frac{144}{5}\nu. \end{aligned} \tag{2.2.13}$$

We can note how the PN information, from Eqs.(2.2.9)-(2.2.12), is extremely simplified in the EOB approach with respect to the formulas present in [81, 82].

We used as γ_E the Euler-Mascheroni constant and the following calibration with NR for parameter a_6^c [30]:

$$a_6^c = 3097.3\nu^2 - 1330.6\nu + 81.38. \tag{2.2.14}$$

From Eq.(2.2.11) and Eq.(2.2.12) we can thus write also the general relativistic B and D potentials:

$$D^{GR} = 1 - 6\nu u^2 - 2\bar{d}_3 u^3 + \mathcal{O}(u^4), \tag{2.2.15}$$
$$B^{GR} = 1 + 2u + 2(2-3\nu)u^2 + (8 - \bar{d}_3 - 14\nu)u^3 + \mathcal{O}(u^4). \tag{2.2.16}$$

The scalar-tensor extension up to 2PN order can be read in Ref. [55] and it can be written as

$$A(u) = A^{GR}(u) + \delta A^{ST}(u), \tag{2.2.17}$$
$$B(u) = B^{GR}(u) + \delta B^{ST}(u), \tag{2.2.18}$$

where the ST terms can be decomposed as:

$$\delta A^{ST}(u) = \delta A^{ST}_{1PN}(u) + \delta A^{ST}_{2PN}(u), \tag{2.2.19}$$
$$\delta B^{ST}(u) = \delta B^{ST}_{1PN}(u) + \delta B^{ST}_{2PN}(u), \tag{2.2.20}$$

and

$$\delta A^{ST}_{1PN}(u) = 2\delta a_2 u^2, \qquad \delta A^{ST}_{2PN}(u) = \delta a_3 u^3, \tag{2.2.21}$$
$$\delta B^{ST}_{1PN}(u) = 2\delta b_1 u, \qquad \delta B^{ST}_{2PN}(u) = \delta b_2 u^2. \tag{2.2.22}$$

Ref. [55] gives these 2PN ST parameters in terms of the derivatives of coupling function and body mass, in Einstein frame:

$$\delta a_2 = \bar{\beta} - \gamma, \tag{2.2.23a}$$
$$\begin{aligned}\delta a_3 = {} & -\frac{5}{3}\gamma\Big(1 + \frac{7}{4}\gamma\Big) - 2\bar{\beta}(1-2\gamma) + \frac{1}{3}(\bar{\delta} - \bar{\epsilon}) \\ & + \nu\Big(-6\beta_+ + \frac{1}{3}\gamma(10+\gamma) - \frac{8}{3}\chi_+ + \frac{4}{3}\delta_+ - 2\mathcal{Z}\Big),\end{aligned} \tag{2.2.23b}$$

$$\delta b_1 = \gamma, \tag{2.2.24a}$$
$$\delta b_2 = 4\bar{\beta} - \bar{\delta} + \gamma(9 + \frac{19}{4}\gamma) + \nu\Big(2\bar{\beta} - 4\gamma\Big). \tag{2.2.24b}$$

Here we combined the Julié's [55] and Sennett's [18] notations. Thus, from Table 1.1 and Eq. (2.2.27) these parameters read:

$$\bar{\beta} \equiv -\beta_- X_{AB} + \beta_+, \qquad \bar{\delta} \equiv \delta_- X_{AB} + \delta_+, \qquad \bar{\epsilon} \equiv -\epsilon_- X_{AB} + \epsilon_+, \tag{2.2.25a}$$
$$\bar{\kappa} \equiv -\kappa - X_{AB} + \kappa_+, \qquad \bar{\chi} \equiv -\chi_- X_{AB} + \chi_+ \equiv -\frac{\bar{\epsilon}}{4}, \tag{2.2.25b}$$

$$\mathcal{Z} = -8\frac{\beta_+^2 - \beta_-^2}{\gamma}, \tag{2.2.26}$$

with, for all ST parameters "$\mathcal{K}$", the "$\mathcal{K}_\pm$" are defined by:

$$\mathcal{K}_\pm \equiv \frac{\mathcal{K}_A \pm \mathcal{K}_B}{2}. \tag{2.2.27}$$

We can observe that the parameter $\mathcal{Z}$ has no problem in GR limit, where both $\beta_\pm$ and γ go to zero, because they are enter both at 1PN order. From Table 1.1 we can see that the numerator $\beta_+^2 - \beta_-^2$ is a 3PN order factor, then $\mathcal{Z}$ enters as a 2PN term. In fact, in Einstein frame, it reads

$$\mathcal{Z} = \frac{\alpha_A \beta_A \alpha_B \beta_B}{(1+\alpha_A\alpha_B)^3} \tag{2.2.28}$$

Therefore, from Eqs.(2.2.9),(2.2.10) and using the same structure introduced in Eqs.(2.2.17), (2.2.18), we can read the 2PN order correction to the D and $\bar{D}$ potentials:

$$\bar{D}(u) = \bar{D}^{GR}(u) + 2\delta\bar{d}_1 u + \delta\bar{d}_2 u^2, \tag{2.2.29}$$
$$D(u) = D^{GR}(u) - 2\delta\bar{d}_1 u + (4\delta\bar{d}_1{}^2 - \delta\bar{d}_2 - 6\nu)u^2. \tag{2.2.30}$$

The $\bar{D}$ scalar-tensor 2PN parameters $\delta\bar{d}_1$ and $\delta\bar{d}_2$ can be obtained by comparing Eqs.(2.2.17) - (2.2.22) with Eqs.(2.2.23), (2.2.24):

$$\delta\bar{d}_1 = -\gamma, \tag{2.2.31a}$$
$$\delta\bar{d}_2 = -3\gamma\left(1+\frac{\gamma}{4}\right) + \bar{\delta} - 6\bar{\beta} + \nu\left(4\gamma - 2\bar{\beta}\right), \tag{2.2.31b}$$

where the "bar" notation was introduced in Eq.(2.2.25).

2.2.2 Effective Hamiltonian

The effective metric is written in term of the effective potentials A and B from Eq.(2.2.5). The geodesic dynamics of a test-particle, with mass μ, moving into this effective metric is described by the action

$$\begin{aligned} S_e = \int dT L_e &= -\mu \int dT \sqrt{-g^\star_{\mu\nu}\frac{dx^\mu}{dT}\frac{dx^\nu}{dT}} \\ &= -\mu \int dT \sqrt{A(R) - B(R)\dot{R}^2 - R^2\dot{\varphi}^2}, \end{aligned} \tag{2.2.32}$$

where $\dot{R}$ and $\dot{\varphi}$ are the derivative with respect to the time T. Applying a Legendre transformation on this lagrangian we get the (dimensionless) Hamiltonian in term of the dimensionless variables, defined in Eq.(2.2.4).

The (dimensionless) conjugate momenta read, from Eq.(2.2.32):

$$p_r = \frac{P_R}{\mu} = \frac{1}{\mu}\frac{\partial L_e}{\partial \dot{R}} = \tilde{G}_{AB} M \frac{B\dot{r}}{L_e}, \tag{2.2.33}$$
$$p_\varphi = \frac{P_\varphi}{\mu \tilde{G}_{AB} M} = \frac{1}{\mu \tilde{G}_{AB} M}\frac{\partial L_e}{\partial \dot{\varphi}} = \tilde{G}_{AB} M \frac{r^2\dot{\varphi}}{L_e}. \tag{2.2.34}$$

Hence, solving for $\dot{r}$ and $\dot{\varphi}$, we get

$$\dot{r} = \frac{\sqrt{A}p_r}{B\sqrt{1+\frac{p_r^2}{B}+\frac{p_\varphi^2}{r^2}}}, \tag{2.2.35}$$

$$\dot{\varphi} = \frac{\sqrt{A} p_\varphi}{\tilde{G}_{AB} M r^2 \sqrt{1 + \frac{p_r^2}{B} + \frac{p_\varphi^2}{r^2}}}. \tag{2.2.36}$$

Therefore, this Legendre transformation gives the orbital (μ-rescaled) effective Hamiltonian:

$$\begin{aligned} \hat{H}_e \equiv \frac{H_e}{\mu} &= \frac{1}{\mu}\left(\dot{r} p_r + \dot{\varphi} p_\varphi - L_e\right) \\ &= \sqrt{A\left(1 + \frac{p_r^2}{B} + \frac{p_\varphi^2}{r^2}\right)}. \end{aligned} \tag{2.2.37}$$

By remembering the tortoise radial coordinate (2.2.3) we can also read the tortoise radial conjugate momentum:

$$p_{r_\star} = \frac{\partial L_e}{\partial r}\frac{dr}{dr_\star} = \left(\frac{A}{B}\right)^{1/2} p_r. \tag{2.2.38}$$

Thus we get

$$\hat{H}_e = \sqrt{p_{r_\star}^2 + A\left(1 + \frac{p_\varphi^2}{r^2}\right)}. \tag{2.2.39}$$

This Hamiltonian describe the geodesic motion of a test-particle moving into the effective metric (2.2.5). The non-geodesic contribution adds a function of r and $p_{r_\star}$ to $\hat{H}_e$ [74, 80]. Here we focus ourself to the 3PN general relativistic term:

$$\hat{H}_e = \sqrt{p_{r_\star}^2 + A\left(1 + \frac{p_\varphi^2}{r^2} + z_3 \frac{p_{r_\star}^4}{r^2}\right)}, \tag{2.2.40}$$

where

$$z_3 = 2\nu(4 - 3\nu). \tag{2.2.41}$$

2.2.3 Spin interactions

Our calculations and discussions are all based on the non-spinning case because the ST corrections are only given on spherically symmetric bodies. In this section we then briefly report the spin structure of the effective metric and effective Hamiltonian only within the general relativistic theory of gravity.

In `TEOBResumS` model the spin terms are limited to the spin-aligned cases and to the even spin power driven. Let us now define the dimensionless spin variables:

$$a_i \equiv \frac{S_i}{m_i}, \qquad \chi_i \equiv \frac{a_i}{m_i}, \qquad \tilde{a}_i \equiv \frac{a_i}{M} = X_i \chi_i, \tag{2.2.42a}$$

$$\tilde{a}_0 \equiv \tilde{a}_A + \tilde{a}_B, \qquad \tilde{a}_{AB} \equiv \tilde{a}_A - \tilde{a}_B, \tag{2.2.42b}$$

where the mass variables were defined in Eq. (2.2.2) and S_i denotes the body spin projection along the orbital angular momentum (2.2.1). All spin effects are collected in the centrifugal radius [27], r_c, that can be generically written as

$$r_c^2 = r^2 + \tilde{a}_Q^2\left(1 + \frac{2}{r}\right) + \delta\tilde{a}^2, \tag{2.2.43}$$

where

$$\tilde{a}_Q^2 \equiv C_{QA}\tilde{a}_A^2 + 2\tilde{a}_A\tilde{a}_B + C_{QB}\tilde{a}_B^2 \tag{2.2.44}$$

contains the deformation parameters $C_{QA,QB}$ due to the NSs rotation. $\delta\tilde{a}^2$ includes all the nexts to leading order spin terms, the NLO and NNLO can be read in Ref. [34].

The non-spinning effective metric was defined in Eq. (2.2.5), where the A and B potentials denote the orbital ones. We will identify them as a subscript "orb" within this section. In terms of the centrifugal radius r_c (2.2.43) the spinning metric functions read as [27, 83]:

$$A(u; S_i) = \frac{1+2u_c}{1+2u} A_{orb}(u_c), \tag{2.2.45a}$$

$$D(u; S_i) = D_{orb}(u_c), \tag{2.2.45b}$$

$$B(u; S_i) = \frac{u_c^2}{u^2} \frac{D(u; S_i)}{A(u; S_i)}, \tag{2.2.45c}$$

where $u = 1/r$ and $u_c = 1/r_c$.

In the spinning case the Hamiltonian (2.2.40) becomes:

$$\hat{H}_e^{orb} = \sqrt{p_{r_\star}^2 + A\left(1 + \frac{p_\varphi^2}{r_c^2} + z_3 \frac{p_{r_\star}^4}{r_c^2}\right)}. \tag{2.2.46}$$

The EOB Hamiltonian (2.1.2) involves the effective energy $\hat{H}_e$ which is the sum of a orbital (even in spin), Eq. (2.2.46), and a spin-orbit (odd in spin) term. This is realized through:

$$\hat{H}_e = \hat{H}_e^{orb} + p_\varphi \tilde{G}. \tag{2.2.47}$$

This last spin term reads

$$\tilde{G} = G_S \hat{S} + G_{S_*} \hat{S}_*, \tag{2.2.48}$$

$$\hat{S} = \frac{S_A + S_B}{M^2}, \qquad \hat{S}_* = \frac{q^{-1}S_A + qS_B}{M^2}. \tag{2.2.49}$$

The functions (G_S, G_{S_*}) are given in Damour-Jaranowski-Schäfer gauge [84, 85], in which the dependence on angular momentum p_φ is gauged away, and they can be found in Refs. [26, 85].

2.2.4 Tidal interactions

As we commented in TOV section, Sec. 1.6, one of the main differences of a NS with respect to a BH is the tidal deformability. BHs are very compact objects and, even in binary systems, they are not deformed by the companion. The NSs, instead, are highly deformed if they belong to a binary system and, in general, when they are in an external gravitational field [21, 22, 86]. These can be classified as

- deformation of the shape of the NS surface,
- gravitoelectric deformation induced by a gravitoelectric external field,

- gravitomagnetic deformation induced by a gravitomagnetic external field.

The multipolar quantities that describe these different tidal effects are, respectively, h_l, k_l and j_l, and they are called *Love numbers*. They can be found by considering the tensorial perturbation to the stationary NS metric, i.e. the TOV metric introduced in ST theories in Sec. 1.6. At the linearized level, the TOV perturbation can be written as

$$g_{\mu\nu} = g^0_{\mu\nu} + h_{\mu\nu}, \tag{2.2.50}$$

where with $g^0_{\mu\nu}$ we identify the TOV metric, in EF, defined in Eq. (1.6.1) as $g^\star_{\mu\nu}$. The linear perturbation $h_{\mu\nu}$ is splitted into even and odd-parity contribution, and the multipolar Love numbers can be found by matching the interior and exterior solution of the perturbed equations (see Refs. [21, 22, 86] for more details). The tensorial perturbations were studied in ST theories in Refs. [87, 88] but we do not have the scalar-tensorial Love number yet and the dipolar perturbation is completely unknown. This could be an interesting theoretical future work.

By limiting to the GR theory, if we have a TOV perturbation (2.2.50), the NS develops a multipolar moment (mass-type or spin-type respectively for gravitoelectric-type or gravitomagnetic-type perturbation), proportional to the perturbation, at linear order. The multipolar tidal number that describe this inducted moment are called $\mu_l \sim [length]^{2l+1}$ for the gravitoelectric-type and $\sigma_l \sim [length]^{2l+1}$ for the gravitomagnetic-type[2]. By limiting to the former type of perturbations, the response number μ_l and Love number k_l are linked through the relation

$$k_l = \frac{(2l-1)!!}{2}\frac{\mu_l}{R^{2l+1}}, \tag{2.2.51}$$

where R is the dimensional NS radius. In literature, the *tidal deformability* is used as

$$\Lambda_l = \frac{2}{(2l-1)!!}\frac{k_l}{C^{2l+1}}, \tag{2.2.52}$$

where

$$C \equiv m/R \tag{2.2.53}$$

represents the *compactness* of the NS.

Within the `TEOBResumS` model, the tidal deformation induces an additive term to the point-mass A potential in the form:

$$A = A_0 + A_T^{(+)}, \tag{2.2.54}$$

where A_0 is the point-mass term defined in Eq. (2.2.5). A_T^+ models the gravitoelectric deformations of the NS an it reads [22, 35]

$$A_T^{(+)} = -\sum_{l=2}^{4}\sum_{i=A}^{B}\left[\kappa_l^{(i)} u^{2l+2}\hat{A}_i^{(l+)}\right]. \tag{2.2.55}$$

[2] We always work with the Einstein frame Newtonian constant $G_\star = 1$. Otherwise, these response numbers must be corrected by a $G_\star$ multiplicative factor.

Here, l denotes the multipolar index and $i = A, B$ differentiates the compact body. The tidal coupling multipolar constants $\kappa_l^{(i)}$ are defined by:

$$\kappa_l^{(i)} = 2\frac{X_i}{1 - X_i}\left(\frac{X_i}{C_i}\right)^{2l+1} k_l^{(i)}, \tag{2.2.56}$$

where the X_i mass variable was defined in Eq. (2.2.2b), from Eq. (2.2.53) C_i is the i-th compactness and the gravitoelectric multipolar number $k_l^{(i)}$, for each body, can be reads in Eq. (2.2.51). The last terms $A_i^{(l+)}$ denote the PN-expanded correction factors. See Ref. [21] for more details.

The ST tidal interaction, at LO, can be found in Ref. [75] in terms of the dipolar scalar Love numbers and it follows the same strategy of the quadrupolar, and higher l-values, ones in GR. Indeed, in ST theories the NS develops a dipolar moment from a deformation which does not exist in GR. This, since it enters at dipolar order instead of the first not trivial in GR, that is the quadrupolar, appears within the conserved quantities at 3PN order instead of at 5PN in GR (see Eq. (2.2.55)).

2.2.5 Hamilton's equations

The (circular) dynamics of binary system is driven by the EOB Hamilton's equations:

$$\frac{d\varphi}{dt} \equiv \Omega = \frac{\partial \hat{H}_{EOB}}{\partial p_\varphi}, \tag{2.2.57a}$$

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{EOB}}{\partial p_{r_\star}}, \tag{2.2.57b}$$

$$\frac{dp_\varphi}{dt} = -\frac{\partial \hat{H}_{EOB}}{\partial \varphi} = 0, \tag{2.2.57c}$$

$$\frac{dp_{r_\star}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{EOB}}{\partial r}, \tag{2.2.57d}$$

where the effective metric potentials A and B were defined in Eq. (2.2.5), while the μ-rescaled EOB Hamiltonian can be read in Eq.(2.1.2):

$$\hat{H}_{EOB} = \frac{H_{EOB}}{\mu} = \frac{1}{\nu}\sqrt{1 + 2\nu\left(\hat{H}_e - 1\right)}. \tag{2.2.58}$$

We can also insert, by hand, the back-reaction on the system in the form of a μ-rescaled reaction terms denoted by $\hat{\mathcal{F}} \equiv \mathcal{F}/\mu$. Thus, Hamilton's equations become:

$$\frac{d\varphi}{dt} \equiv \Omega = \frac{\partial \hat{H}_{EOB}}{\partial p_\varphi}, \tag{2.2.59a}$$

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{EOB}}{\partial p_{r_\star}}, \tag{2.2.59b}$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi, \tag{2.2.59c}$$

$$\frac{dp_{r_\star}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{\partial \hat{H}_{EOB}}{\partial r} + \hat{\mathcal{F}}_{r_\star}. \tag{2.2.59d}$$

From Eqs. (2.2.58) and (2.2.47), the spinning Hamilton's equations (2.2.59) explicitly read:

$$\frac{d\varphi}{dt} \equiv \Omega = \frac{1}{\nu \hat{H}_{EOB} \hat{H}_e} \left[A p_\varphi r_c^2 + \hat{H}_e^{orb} \tilde{G} \right], \tag{2.2.60a}$$

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{1}{\nu \hat{H}_{EOB} \hat{H}_e} \left[p_{r_\star} \left(1 + 2 z_3 \frac{A}{r_c^2} p_{r_\star}^2 \right) + \hat{H}_e^{orb} p_\varphi \left(\frac{\partial \tilde{G}}{\partial p_{r_\star}} \right) \right], \tag{2.2.60b}$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi, \tag{2.2.60c}$$

$$\frac{dp_{r_\star}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{1}{2\nu \hat{H}_{EOB} \hat{H}_e} \left[A' + p_\varphi^2 \left(\frac{A}{r_c^2} \right)' + z_3 p_{r_\star}^4 \left(\frac{A}{r_c^2} \right)' + 2 \hat{H}_e^{orb} p_\varphi \tilde{G}' \right] + \hat{\mathcal{F}}_{r_\star}, \tag{2.2.60d}$$

where, as usual, the prime represents the radial derivative, i.e. $(\cdots)' \equiv \partial_r(\cdots)$. The centrifugal radius r_c (2.2.43) collects the spin interaction terms, while the spin-orbit function $\tilde{G}$ was defined in Eq. (2.2.48).

In the following sections we limit ourself to the non-spinning case to compute and discuss the ST correction. Therefore, we make explicit the Hamilton's equations (2.2.60) for non-rotating bodies:

$$\frac{d\varphi}{dt} \equiv \Omega = \frac{1}{\nu \hat{H}_{EOB} \hat{H}_e} A \frac{p_\varphi}{r^2}, \tag{2.2.61a}$$

$$\frac{dr}{dt} = \left(\frac{A}{B}\right)^{1/2} \frac{1}{\nu \hat{H}_{EOB} \hat{H}_e} p_{r_\star} \left(1 + 2 z_3 \frac{A}{r^2} p_{r_\star}^2 \right), \tag{2.2.61b}$$

$$\frac{dp_\varphi}{dt} = \hat{\mathcal{F}}_\varphi, \tag{2.2.61c}$$

$$\frac{dp_{r_\star}}{dt} = -\left(\frac{A}{B}\right)^{1/2} \frac{1}{2\nu \hat{H}_{EOB} \hat{H}_e} \left[A' + p_\varphi^2 \left(\frac{A}{r^2} \right)' + z_3 p_{r_\star}^4 \left(\frac{A}{r^2} \right)' \right] + \hat{\mathcal{F}}_{r_\star}. \tag{2.2.61d}$$

2.2.6 Circular approximation

Within this work we limit our attention to the ST correction on circular orbits of non-spinning binary systems. This consist in neglecting the loss of energy of the system, i.e. imposing $\hat{\mathcal{F}} = 0$, and to set

$$p_{r_\star} = 0, \qquad \frac{\partial \hat{H}_{EOB}}{\partial p_{r_\star}} = 0 \tag{2.2.62}$$

in the Hamilton's equations (2.2.57). These conditions, from the equation for the time derivative of $p_{r_\star}$ in Eq. (2.2.61), give the following expression for the angular momentum, of non-rotating objects, on circular orbits $j \equiv p_\varphi$:

$$j^2 = -\frac{A'}{(Au^2)'}, \tag{2.2.63}$$

where the dimensionless variable u is defined as $u \equiv 1/r$.

The effective energy Eq.(2.2.40) on circular orbits, therefore, can be read by using the circular angular momentum, $j = p_\varphi$, from Eq.(2.2.63):

$$\hat{E}_e = \sqrt{A\left(1 + \frac{p_\varphi^2}{r^2}\right)} = \sqrt{A\left(1 - \frac{A' u^2}{(Au^2)'}\right)} = \sqrt{\frac{2uA^2}{(Au^2)'}}. \tag{2.2.64}$$

We can note that, in the general relativistic Schwarzschild limit ($\nu = 0$), the circular angular momentum and the effective energy read respectively:

$$j(u) = \sqrt{\frac{1}{u}}\frac{1}{\sqrt{1-3u}} \equiv j_N(u)\frac{1}{\sqrt{1-3u}}, \tag{2.2.65}$$

$$\hat{E}_e = \frac{1-2u}{\sqrt{1-3u}}, \tag{2.2.66}$$

where the A function is taken as the Schwarzschild one from Eq. (2.2.7) and $j_N(u) \equiv \sqrt{1/u}$ is the Newtonian angular momentum.

As we will see below, it is also useful to use the dimensionless variable $x \equiv \Omega^{2/3}$, with Ω can be read from Eq. (2.2.61). This gives the GR-Schwarzschild relations:

$$\Omega(u) = u^{3/2}, \qquad x(u) = u. \tag{2.2.67}$$

This shows that they present a coordinate singularity $\sim 1/\sqrt{1-3u}$ at the light ring $u = x = 1/3$.

2.3 Effective A potential at the 3PN order

In this section, our aim is to compute the 3PN order ST correction to the effective A potential of the metric (2.2.5). This result can be completely obtained by post-Newtonian expression of the binding energy on circular orbits. In fact, from Eq. (2.2.64), we can see that it depends only on the A function of the effective metric, and not on B. Therefore, by taking the Bernard's results from Refs. [17, 75], we have to compute the (gauge invariant) EOB energy (see Eq. (2.1.2)) on circular orbits and to compare it with the 3PN Bernard's formula (Eq. 5.4 of Ref. [17]). In order to make this comparison we have introduce a new ST term into the A function, compute the circular angular momentum, from Eq. (2.2.63), the orbital frequency and finally the gauge-invariant energy (all of these expanded up to 3PN order). Therefore, this allows us to find the new ST term of the effective metric.

The effective A potential in GR, up to 3PN order, is given by [74]

$$A^{GR}(u) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4, \tag{2.3.1}$$

where $u = 1/r$ and

$$a_4 = \frac{94}{3} - \frac{41}{32}\pi^2. \tag{2.3.2}$$

The ST new correction can be parametrized in the form [55]:

$$A(u) = A^{GR}(u) + \delta A^{ST}(u), \tag{2.3.3}$$

where

$$\delta A^{ST}(u) = \delta A^{ST}_{1PN} + \delta A^{ST}_{2PN} + \delta A^{ST}_{3PN}. \tag{2.3.4}$$

and we can read the 1PN and 2PN terms from Eqs.(2.2.17) - (2.2.24).

The 3PN ST term δA^{ST}_{3PN} is, thus, the new one we introduce to generate the 3PN non-spinning, circular energy, and the tidal energy correction, found by L. Bernard in Refs. [17, 75], as we introduced above.

We could express this 3PN ST term in the following form:

$$\delta A^{ST}_{3PN} = \delta A^{ST}_{3PN,inst} + \delta A^{ST}_{3PN,tail} + \delta A^{ST}_{3PN,tidal}, \tag{2.3.5}$$

with

$$\delta A^{ST}_{3PN,inst} = \delta a_4 u^4, \tag{2.3.6a}$$
$$\delta A^{ST}_{3PN,tail} = (\delta a^0_{4,tail} + \delta a^{log}_{4,tail} log\, u)u^4, \tag{2.3.6b}$$
$$\delta A^{ST}_{3PN,tidal} = \delta at_4 u^4. \tag{2.3.6c}$$

It is useful to split it in order to associate $\delta A^{ST}_{3PN,inst}$ and $\delta A^{ST}_{3PN,tail}$ to E^{ST}_{3PN} (instantaneous plus tail contributions) and $\delta A^{ST}_{3PN,tidal}$ to $\Delta E_{(f\,s)}$. This last one represents the LO scalar tidal energy found in Eq. (8) of Ref. [75]. This, in turn, is expressed in term of the LO scalar Love numbers, which do not exist yet, thus we will do not use them into the ST corrections to the `TEOBResumS` model in this work.

We now focus our attention to the 3PN expansion for the variables which are needed to found the 3PN energy in terms of the ST parameters defined above.

First, we have to compute the 3PN order angular momentum,

$$j^2 = -\frac{A'}{(Au^2)'}, \tag{2.3.7}$$

in circular approximation (2.2.63). Hence, from the definition of A with our conventions for the new ST term in Eqs. (2.3.3), (2.3.4), its 3PN order expression is given by

$$\begin{aligned}\frac{j^2(u)}{1/u} &= 1 + u\Big(3 - 2\delta a_2\Big) + u^2\Big(9 - \frac{3\delta a_3}{2} - 10\delta a_2 - 3\nu\Big)\\ &+ u^3\Big(27 - 7\delta a_3 - 2\delta a_4 - 2\delta at_4 - 42\delta a_2 + 8\delta {a_2}^2\\ &+ \Big(-\frac{230}{3} + \frac{41}{16}\pi^2\Big)\nu\\ &- 2\delta a^0_{4,tail} - 2\delta a^{log}_{4,tail}\big(\frac{1}{4} + log\, u\big)\Big) + \mathcal{O}(u^4)\end{aligned} \tag{2.3.8}$$

and, by taking its square root expansion:

$$\begin{aligned}\frac{j(u)}{1/\sqrt{u}} &= 1 + u\Big(\frac{3}{2} - \delta a_2\Big) + u^2\Big(\frac{27}{8} - \frac{3\delta a_3}{4} - \frac{7\delta a_2}{2} - \frac{\delta {a_2}^2}{2} - \frac{3\nu}{2}\Big)\\ &+ u^3\Big(\frac{135}{16} - \delta a_4 - \delta at_4 + \frac{1}{8}\delta a_3\big(-19 - 6\delta a_2\big) - \frac{99\delta a_2}{8} + \frac{5\delta {a_2}^2}{4}\\ &- \frac{\delta {a_2}^3}{2} + \Big(-\frac{433}{12} + \frac{41}{32}\pi^2 - \frac{3\delta a_2}{2}\Big)\nu - \delta a^0_{4,tail} - \delta a^{log}_{4,tail}\big(\frac{1}{4} + log\, u\big)\Big)\\ &+ \mathcal{O}(u^4).\end{aligned} \tag{2.3.9}$$

In this section we will show all the PN variable divided by their Newtonian term, in order to make explicit the perturbed behaviour as $1 + (PN)$.

Let us now invert $j(u)$ in order to find the variable u as function of j, up to 3PN order:

$$\begin{aligned}\frac{u(j)}{1/j^2} &= 1 + \frac{1}{j^2}\Big(3 - 2\delta a_2\Big) + \frac{1}{j^4}\Big(18 - \frac{3\delta a_3}{2} - 22\delta a_2 + 4\delta a_2{}^2 - 3\nu\Big) \\ &+ \frac{1}{j^6}\Big(135 - \frac{41\delta a_3}{2} - 2\delta a_4 - 2\delta at_4 - 240\delta a_2 + 9\delta a_3\delta a_2 + 104\delta a_2{}^2 \\ &- 8\delta a_2{}^3 + \Big(-\frac{311}{3} + \frac{41}{16}\pi^2 + 18\delta a_2\Big)\nu \\ &- 2\delta a^0_{4,tail} - 2\delta a^{log}_{4,tail}\big(\frac{1}{4} + \frac{1}{2}log(\frac{1}{j^2})\big)\Big) + \mathcal{O}\left(\frac{1}{j^8}\right).\end{aligned} \tag{2.3.10}$$

We can now express the circular EOB Hamiltonian (2.1.2), as function of u, by using the 3PN formula of $p_\varphi = j(u)$ found above:

$$\begin{aligned}\frac{E(u) - M}{-\mu u/2} &= 1 + u\Big(-\frac{3}{4} + \frac{\nu}{4}\Big) + u^2\Big(-\frac{27}{8} + \frac{\delta a_3}{2} + 4\delta a_2 + \frac{5\nu}{8} + \frac{\nu^2}{8}\Big) \\ &+ u^3\Big(-\frac{675}{64} + \frac{13\delta a_3}{4} + \delta a_4 + \delta at_4 + 18\delta a_2 - 4\delta a_2{}^2 \\ &+ \big(\frac{6967}{192} - \frac{41}{32}\pi^2 + \frac{\delta a_3}{4} + 2\delta a_2\big)\nu + \frac{7\nu^2}{32} + \frac{5\nu^3}{64} \\ &+ \delta a^0_{4,tail} + \delta a^{log}_{4,tail}(\frac{1}{2} + log\, u)\Big) + \mathcal{O}(u^4)\end{aligned} \tag{2.3.11}$$

and, removing the u variable using the 3PN order $j(u)$, the gauge-invariant EOB binding energy as function of the circular angular momentum reads:

$$\begin{aligned}\frac{E(j) - M}{-\mu/(2j^2)} &= 1 + \frac{1}{j^2}\left(\frac{9}{4} - 2\delta a_2 + \frac{\nu}{4}\right) + \frac{1}{j^4}\Big(-\delta a_3 + 4\delta a_2{}^2 - 15\delta a_2 + \frac{81}{8} \\ &+ \nu\Big(-\delta a_2 - \frac{7}{8}\Big) + \frac{\nu^2}{8}\Big) + \frac{1}{j^6}\Big(6\delta a_3\delta a_2 - \frac{21\delta a_3}{2} - \delta a_4 - \delta at_4 \\ &- 8\delta a_2{}^3 + 67\delta a_2{}^2 - \frac{495\delta a_2}{4} + \frac{3861}{64} + \nu\Big(-\frac{\delta a_3}{2} + 3\delta a_2{}^2 + \frac{9\delta a_2}{4} \\ &+ \frac{41\pi^2}{32} - \frac{8833}{192}\Big) + \nu^2\Big(-\frac{3\delta a_2}{4} - \frac{5}{32}\Big) + \frac{5\nu^3}{64} \\ &- \delta a^0_{4,tail} - \delta a^{log}_{4,tail} log(\frac{1}{j^2})\Big) + \mathcal{O}\left(\frac{1}{j^8}\right).\end{aligned} \tag{2.3.12}$$

It also needed to compute the orbital frequency Ω and

$$x = \Omega^{2/3} \tag{2.3.13}$$

as function of u and then x as function of the angular momentum j. The orbital frequency is given by (2.2.59):

$$\Omega(u, p_\varphi) = \frac{\partial \hat{H}_{EOB}(u, p_\varphi)}{\partial p_\varphi}. \tag{2.3.14}$$

These variables are dimensionless due to the space phase definition in Eq. (2.2.4). Ω is related to the dimension one, Ω_{phys}, through the scaling:

$$\Omega = \tilde{G}_{AB} M \Omega_{phys}, \tag{2.3.15}$$

Where the bodies-dependent ST parameter $\tilde{G}_{AB}$ is written in Eq. (1.5.10).

Now, by using the identity $p_\varphi = j(u)$, the Eq.(2.3.14) up to 3PN order as function of u becomes:

$$\begin{aligned}\frac{\Omega(u)}{u^{3/2}} &= 1 + u\Big(-\delta a_2 + \frac{\nu}{2}\Big) + u^2\Big(-\frac{3\delta a_3}{4} - \frac{\delta {a_2}^2}{2} + \Big(-\frac{15}{8} - \frac{\delta a_2}{2}\Big)\nu + \frac{3\nu^2}{8}\Big)\\ &+ u^3\Big(-\delta a_4 - \delta at_4 - \frac{3\delta a_3 \delta a_2}{4} - \frac{\delta {a_2}^3}{2} + \Big(-\frac{1585}{48} + \frac{41}{32}\pi^2 - \frac{\delta a_3}{8}\\ &+ \frac{7\delta a_2}{8} - \frac{\delta {a_2}^2}{4}\Big)\nu + \Big(-\frac{13}{16} - \frac{3\delta a_2}{8}\Big)\nu^2 + \frac{5\nu^3}{16}\\ &- \delta a^0_{4,tail} - \delta a^{log}_{4,tail}\big(\frac{1}{4} + log\, u\big)\Big) + \mathcal{O}(u^4)\end{aligned} \tag{2.3.16}$$

and, consequently:

$$\begin{aligned}\frac{\Omega^2(u)}{u^3} &= 1 + u\Big(-2\delta a_2 + \nu\Big) + u^2\Big(-\frac{3\delta a_3}{2} + \Big(-\frac{15}{4} - 2\delta a_2\Big)\nu + \nu^2\Big)\\ &+ u^3\Big(-2\delta a_4 - 2\delta at_4 + \Big(-\frac{1585}{24} + \frac{41}{16}\pi^2 - \delta a_3 + \frac{11\delta a_2}{2}\Big)\nu\\ &+ \Big(-\frac{7}{2} - 2\delta a_2\Big)\nu^2 + \nu^3\\ &- 2\delta a^0_{4,tail} - 2\delta a^{log}_{4,tail}\Big(\frac{1}{4} + log\, u\Big)\Big) + \mathcal{O}(u^4).\end{aligned} \tag{2.3.17}$$

Hence, the variable $x = \Omega^{2/3}$ as function of u is given by

$$\begin{aligned}\frac{x(u)}{u} &= 1 + u\Big(-\frac{2\delta a_2}{3} + \frac{\nu}{3}\Big) + u^2\Big(-\frac{\delta a_3}{2} - \frac{4\delta {a_2}^2}{9} + \Big(-\frac{5}{4} - \frac{2\delta a_2}{9}\Big)\nu + \frac{2\nu^2}{9}\Big)\\ &+ u^3\Big(-\frac{2\delta a_4}{3} - \frac{2\delta at_4}{3} - \frac{2\delta a_3 \delta a_2}{3} - \frac{40\delta {a_2}^3}{81} + \Big(-\frac{1585}{72} + \frac{41}{48}\pi^2 + \frac{\delta a_2}{6}\\ &- \frac{4\delta {a_2}^2}{27}\Big)\nu + \Big(-\frac{1}{3} - \frac{4\delta a_2}{27}\Big)\nu^2 + \frac{14\nu^3}{81}\\ &- \frac{2}{3}\delta a^0_{4,tail} - \frac{2}{3}\delta a^{log}_{4,tail}\big(\frac{1}{4} + log\, u\big)\Big) + \mathcal{O}(u^4).\end{aligned} \tag{2.3.18}$$

By inverting $x(u)$ we get

$$\begin{aligned}\frac{u(x)}{x} &= 1 + x\Big(\frac{2\delta a_2}{3} - \frac{\nu}{3}\Big) + x^2\Big(\frac{\delta a_3}{2} + \frac{4\delta {a_2}^2}{3} + \big(\frac{5}{4} - \frac{2\delta a_2}{3}\big)\nu\Big)\\ &+ x^3\Big(\frac{2\delta a_4}{3} + \frac{2\delta at_4}{3} + \frac{7\delta a_3 \delta a_2}{3} + \frac{280\delta {a_2}^3}{81} + \Big(\frac{1585}{72} + \frac{41}{48}\pi^2\\ &- \frac{5\delta a_3}{6} + 4\delta a_2 - \frac{56\delta {a_2}^2}{27}\Big)\nu + \Big(-\frac{7}{4} + \frac{4\delta a_2}{27}\Big)\nu^2 + \frac{\nu^3}{81}\\ &+ \frac{2}{3}\delta a^0_{4,tail} + \frac{2}{3}\delta a^{log}_{4,tail}\big(\frac{1}{4} + log\, u\big)\Big) + \mathcal{O}(x^4).\end{aligned} \tag{2.3.19}$$

We can now write the the x variable as function of j up to 3PN order by using the $j(x)$ and $u(x)$ formulas founded above:

$$\begin{aligned}\frac{x(j)}{1/j^2} &= 1 + \frac{1}{j^2}\Big(- \frac{8\delta a_2}{3} + 3 + \frac{\nu}{3}\Big) + \frac{1}{j^4}\Big(- 2\delta a_3 + \frac{56\delta {a_2}^2}{9} - 26\delta a_2 + 18 \\ &+ \nu\Big(- \frac{14\delta a_2}{9} - \frac{9}{4}\Big) + \frac{2\nu^2}{9}\Big) + \frac{1}{j^6}\Big(\frac{40\delta a_3\delta a_2}{3} - 25\delta a_3 - \frac{8\delta a_4}{3} - \frac{8\delta at_4}{3} \\ &- \frac{1120\delta {a_2}^3}{81} + \frac{412\delta {a_2}^2}{3} - 270\delta a_2 + 135 + \nu\Big(- \delta a_3 + \frac{140\delta {a_2}^2}{27} + 9\delta a_2 \\ &+ \frac{41\pi^2}{12} - \frac{8779}{72}\Big) + \nu^2\Big(- \frac{40\delta a_2}{27} - \frac{1}{3}\Big) + \frac{14\nu^3}{81} \\ &- \frac{8}{3}\delta a^0_{4,tail} - \frac{8}{3}\delta a^{log}_{4,tail}\big(\frac{1}{4} + \frac{5}{8}log(\frac{1}{j^2})\big)\Big) + \mathcal{O}\left(\frac{1}{j^8}\right).\end{aligned} \tag{2.3.20}$$

This expression allows us to invert it to find j as function of x and then also the useful $1/j^2$:

$$\begin{aligned}\frac{j(x)}{1/\sqrt{x}} &= 1 + x\Big(\frac{3}{2} - \frac{4\delta a_2}{3} + \frac{\nu}{6}\Big) + x^2\Big(\frac{27}{8} - \delta a_3 - 3\delta a_2 - \frac{4\delta {a_2}^2}{3} + \nu\Big(- \frac{19}{8} \\ &+ \frac{\delta a_2}{3}\Big) + \frac{\nu^2}{24}\Big) + x^3\Big(\frac{135}{16} - 2\delta a_3 - \frac{4\delta a_4}{3} - \frac{4\delta at_4}{3} - 9\delta a_2 - \frac{8\delta a_3\delta a_2}{3} \\ &- \frac{4\delta {a_2}^2}{3} - \frac{224\delta {a_2}^3}{81} + \nu\Big(- \frac{6889}{144} + \frac{41}{24}\pi^2 + \frac{2\delta a_3}{3} - \frac{11\delta a_2}{3} + \frac{28\delta {a_2}^2}{27}\Big) \\ &+ \nu^2\Big(\frac{31}{24} + \frac{\delta a_2}{27}\Big) + \frac{7\nu^3}{1296} \\ &- \frac{4}{3}\delta a^0_{4,tail} - \frac{4}{3}\delta a^{log}_{4,tail}\big(\frac{1}{4} + \frac{5}{8}log\, x\big)\Big) + \mathcal{O}(x^4),\end{aligned} \tag{2.3.21}$$

$$\begin{aligned}\frac{1/j^2(x)}{x} &= 1 + x\Big(\frac{8\delta a_2}{3} - 3 - \frac{\nu}{3}\Big) + x^2\Big(2\delta a_3 + 8\delta {a_2}^2 - 6\delta a_2 + \nu\Big(\frac{25}{4} - 2\delta a_2\Big)\Big) \\ &+ x^3\Big(\frac{40\delta a_3\delta a_2}{3} - 5\delta a_3 + \frac{8\delta a_4}{3} + \frac{8\delta at_4}{3} + \frac{2080\delta {a_2}^3}{81} - \frac{52\delta {a_2}^2}{3} \\ &+ \nu\Big(- \frac{7\delta a_3}{3} - \frac{260\delta {a_2}^2}{27} + \frac{103\delta a_2}{3} - \frac{41\pi^2}{12} + \frac{5269}{72}\Big) \\ &+ \nu^2\Big(\frac{10\delta a_2}{27} - \frac{61}{12}\Big) + \frac{\nu^3}{81} \\ &+ \frac{8}{3}\delta a^0_{4,tail} + \frac{8}{3}\delta a^{log}_{4,tail}\big(\frac{1}{4} + \frac{5}{8}log\, x\big)\Big) + \mathcal{O}(x^4).\end{aligned} \tag{2.3.22}$$

We can then finally write the gauge-invariant circular EOB binding energy as function of x:

$$\begin{aligned}\frac{E(x)-M}{-\mu x/2} &= 1 + x\left(\frac{2\delta a_2}{3} - \frac{3}{4} - \frac{\nu}{12}\right) + x^2\left(\delta a_3 + \frac{4\delta a_2{}^2}{3} + 3\delta a_2 - \frac{27}{8}\right.\\ &+ \left.\nu\left(\frac{19}{8} - \frac{\delta a_2}{3}\right) - \frac{\nu^2}{24}\right) + x^3\left(\frac{10\delta a_3\delta a_2}{3} + \frac{5\delta a_3}{2} + \frac{5\delta a_4}{3} + \frac{5\delta at_4}{3}\right.\\ &+ \frac{280\delta a_2{}^3}{81} + \frac{5\delta a_2{}^2}{3} + \frac{45\delta a_2}{4} - \frac{675}{64} + \nu\left(-\frac{5\delta a_3}{6} - \frac{35\delta a_2{}^2}{27}\right.\\ &+ \left.\frac{55\delta a_2}{12} - \frac{205\pi^2}{96} + \frac{34445}{576}\right) + \nu^2\left(-\frac{5\delta a_2}{108} - \frac{155}{96}\right) - \frac{35\nu^3}{5184}\\ &+ \left.\frac{5}{3}\delta a^0_{4,tail} + \frac{2}{3}\delta a^{log}_{4,tail}(1 + log\, x)\right) + \mathcal{O}(x^4).\end{aligned} \tag{2.3.23}$$

This result correctly reduced to the GR one [82,89] when we set $\delta a_2 = \delta a_3 = \delta a_4 = \delta at_4 = \delta a^0_{4,tail} = \delta a^{log}_{4,tail} = 0$.

By remembering the 1PN (δa_2) and 2PN (δa_3) parameter definitions (see Eq.(2.2.23)), the binding energy (2.3.23), truncated at the 2PN order, is in full agreement with the result of [17,18]. We can now compare this general scalar-tensor 3PN information with the already known one and find the new δA^{ST}_{3PN} term (see Eq. (2.3.5)).

2.3.1 Tidal term

Moreover, we can find the tidal parameter δat_4 in terms of the tidal energy [75] by setting to zero the other parameters in order to obtain the GR energy plus the LO 3PN tidal correction $\Delta E_{(fs)}$ (see Eq. (8) of Ref. [75]).

The 3PN ST tidal correction (Newtonian-normalized) is given by

$$\frac{5}{3}\delta at_4 x^3 \tag{2.3.24}$$

from Eq.(2.3.23).

We can then write the following result:

$$\delta at_4 = \frac{3}{5}\Delta E_{(fs)}. \tag{2.3.25}$$

As we discussed above at the end of Sec. 2.2.4, we will do not implement this tidal contribution because the scalar dipolar Love numbers do not exist yet. These could be found from the dipolar perturbation to the scalar TOV equations, explored in Sec. 1.6. We leave these studies to other works.

2.3.2 Instantaneous term

The strategy to find the parameter δa_4 is similar but its expression is more complicated. Let us focus on the instantaneous part, then, let us set for now $\delta a^0_{4,tail} = \delta a^{log}_{4,tail} = 0$. Now have also to set $\delta at_4 = 0$ and compare the energy written above in Eq. (2.3.23) with the Bernard's one [17].

The 1PN (δa_2), and the 2PN (δa_3), parameters can be read in Eq. (2.2.23).

Let us define as C_n the coefficient, at nPN order, of the (Newtonian normalized) non-spinning circular energy, as function of x, written in Eq. (2.3.23):

$$C^{inst}_n \equiv C_n\big|_{\delta at_4, \delta a^0_{4,tail}, \delta a^{log}_{4,tail}=0}. \tag{2.3.26}$$

In order to find the instantaneous 3PN parameter of the A potential (2.3.3), δa_4, we have to compare our C_3^{inst} with the one found in [17].

The C_3^{inst} coefficient is given by:

$$\begin{aligned} C_3^{inst} &= \frac{10\delta a_3 \delta a_2}{3} + \frac{5\delta a_3}{2} + \frac{5\delta a_4}{3} + \frac{280\delta {a_2}^3}{81} + \frac{5\delta {a_2}^2}{3} + \frac{45\delta a_2}{4} - \frac{675}{64} \\ &+ \nu\Big(- \frac{5\delta a_3}{6} - \frac{35\delta {a_2}^2}{27} + \frac{55\delta a_2}{12} - \frac{205\pi^2}{96} + \frac{34445}{576}\Big) \\ &+ \nu^2\Big(- \frac{5\delta a_2}{108} - \frac{155}{96}\Big) - \frac{35\nu^3}{5184}. \end{aligned} \tag{2.3.27}$$

Or, writing the 1PN and 2PN parameters in terms of their definitions:

$$C_3^{inst} = \frac{5\delta a_4}{3} + C_{3,1}^{inst} + C_{3,\nu}^{inst}\nu + C_{3,\nu^2}^{inst}\nu^2 + C_{3,\nu^3}^{inst}\nu^3, \tag{2.3.28}$$

where the coefficients of this ν-polynomial read:

$$\begin{aligned} C_{3,1}^{inst} &= -\frac{175}{18}\gamma^2 \left(\bar\beta - \gamma\right) + \frac{10}{9}\left(\bar\beta - \gamma\right)\left(\bar\delta + 4\bar\chi\right) + \frac{280}{81}\left(\bar\beta - \gamma\right)^3 \\ &+ \frac{5}{3}\left(\bar\beta - \gamma\right)^2 - 5(1-2\gamma)\bar\beta - \frac{50}{9}\gamma\left(\bar\beta - \gamma\right) - \frac{20}{3}(1-2\gamma)\bar\beta\left(\bar\beta - \gamma\right) \\ &+ \frac{45}{4}\left(\bar\beta - \gamma\right) - \frac{175\gamma^2}{24} - \frac{25\gamma}{6} + \frac{5}{6}\left(\bar\delta + 4\bar\chi\right) - \frac{675}{64}, \end{aligned} \tag{2.3.29}$$

$$\begin{aligned} C_{3,\nu}^{inst} &= \frac{10}{3}\left(\bar\beta - \gamma\right)\left(\frac{16\left({\beta_+}^2 - {\beta_-}^2\right)}{\gamma} - 6\beta_+ + \frac{1}{3}\gamma(\gamma+10) \right. \\ &\left. + \frac{4\delta_+}{3} - \frac{8\chi_+}{3}\right) + \frac{5}{2}\Big(\frac{16\left({\beta_+}^2 - {\beta_-}^2\right)}{\gamma} - 6\beta_+ + \frac{1}{3}\gamma(\gamma+10) \\ &+ \frac{4\delta_+}{3} - \frac{8\chi_+}{3}\Big) - \frac{35}{27}\left(\bar\beta - \gamma\right)^2 + \frac{5}{3}(1-2\gamma)\bar\beta + \frac{55}{12}(\bar\beta - \gamma) \\ &+ \frac{175\gamma^2}{72} + \frac{25\gamma}{18} - \frac{5}{18}\left(\bar\delta + 4\bar\chi\right) - \frac{205\pi^2}{96} + \frac{34445}{576}, \end{aligned} \tag{2.3.30}$$

$$\begin{aligned} C_{3,\nu^2}^{inst} &= -\frac{5}{6}\Big(\frac{16\big({\beta_+}^2 - {\beta_-}^2\big)}{\gamma} - 6\beta_+ + \frac{1}{3}\gamma(\gamma+10) + \frac{4\delta_+}{3} - \frac{8\chi_+}{3}\Big) \\ &- \frac{5}{108}(\bar\beta - \gamma) - \frac{155}{96}, \end{aligned} \tag{2.3.31}$$

$$C_{3,\nu^3}^{inst} = -\frac{35}{5184}. \tag{2.3.32}$$

Here, all the ST PN parameters can be read in Table 1.1 and the auxiliary "bar" terms in Eq. (2.2.25).

These coefficients must be compared with the following ones in order to find the instantaneous 3PN term δa_4.

We denote as $\tilde C$ the same coefficient of Bernard's calculation [17]:

$$\tilde C_3^{inst} = \tilde C_{3,1}^{inst} + \tilde C_{3,\nu}^{inst}\nu + \tilde C_{3,\nu^2}^{inst}\nu^2 + \tilde C_{3,\nu^3}^{inst}\nu^3, \tag{2.3.33}$$

where

$$\tilde{C}^{inst}_{3,1} = \gamma^2\left(\frac{10\bar{\beta}}{27} - \frac{785}{72}\right) + \gamma\left(-\frac{160}{27}\beta_+\beta_- X_{AB} + \frac{80\beta_-^2}{27} + \frac{80\beta_+^2}{27} + \frac{10\bar{\delta}}{9} + \frac{20\bar{\chi}}{9} - \frac{75}{4}\right)$$
$$+ \beta_+\left(-10\beta_- X_{AB} + \frac{280\beta_-^2}{27} + \frac{40\bar{\chi}}{9} - \frac{55}{36}\right) + \beta_-\left(\frac{55}{36}X_{AB} - \frac{40\bar{\chi}}{9}\right)$$
$$- \frac{280}{81}\beta_-^3 X_{AB} + \beta_+^2\left(5 - \frac{280\beta_- X_{AB}}{27}\right) + \frac{280\beta_+^3}{81} + 5\beta_-^2 - \frac{335\gamma^3}{162}$$
$$+ \frac{35\bar{\delta}}{18} + \frac{10\bar{\kappa}}{9} + \frac{10\bar{\chi}}{3} - \frac{675}{64} \tag{2.3.34}$$

$$\tilde{C}^{inst}_{3,\nu} = \frac{1}{\alpha(\gamma+2)}\left(\gamma\Big(-\frac{5\bar{\delta}}{3} + \frac{55}{3}\Big) + \frac{55\gamma^2}{3}\left(1 + \frac{\gamma}{4}\right)\right) + \frac{1}{\gamma^2}\left(\frac{160}{3}\left(\beta_-^3 X_{AB} + \beta_+^3\right)\right.$$
$$\left. - \frac{160}{3}\beta_+\beta_-\left(\beta_- + \beta_+ X_{AB}\right)\right) + \gamma\left(\frac{200\beta_- X_{AB}}{27} + \frac{5\delta_- X_{AB}}{6} - \frac{320\beta_-^2}{27} - \frac{560\beta_+}{27}\right.$$
$$\left. + \pi^2\Big(\frac{35\delta_+}{96} - \frac{125}{64}\Big) - \frac{25\delta_+}{18} - \frac{40\chi_+}{9} + \frac{6365}{108}\right) + \frac{1}{\gamma}\left(\beta_-\left(X_{AB}\Big(\frac{80\chi_+}{3} - \frac{80\delta_+}{9}\Big)\right.\right.$$
$$\left. + \frac{160\delta_-}{9} - \frac{80\chi_-}{3}\right) + \beta_+\left(X_{AB}\Big(-\frac{80\delta_-}{9} - \frac{80\chi_-}{3}\Big) - \frac{160\beta_-^2}{3} + \frac{160\delta_+}{9} + \frac{80\chi_+}{3}\right)$$
$$\left. + \frac{160}{3}\beta_-^3 X_{AB} - \frac{160}{3}\beta_+^2\beta_- X_{AB} + \frac{160\beta_+^3}{3}\right) + \beta_+\Big(\frac{70\beta_- X_{AB}}{27} - \frac{1120\beta_-^2}{27}$$
$$- \frac{80\chi_+}{9} - \frac{415}{36}\Big) + \beta_-\left(\Big(\frac{80\chi_+}{9} - \frac{35}{36}\Big)X_{AB} - \frac{160\chi_-}{9}\right) + \frac{1120}{81}\beta_-^3 X_{AB}$$
$$+ X_{AB}\left(\frac{5\delta_-}{18} + \frac{10\kappa_-}{9} + \frac{70\chi_-}{9}\right) - \frac{1205\beta_-^2}{27} + \frac{595\beta_+^2}{27} + \left(\frac{35\pi^2}{384} - \frac{125}{72}\right)\gamma^3$$
$$+ \left(\frac{935}{72} - \frac{25\pi^2}{96}\right)\gamma^2 + \pi^2\left(\frac{35\delta_+}{48} - \frac{205}{96}\right) - \frac{755\delta_+}{54} - \frac{10\kappa_+}{3} - \frac{130\chi_+}{9} + \frac{34445}{576}, \tag{2.3.35}$$

$$\tilde{C}^{inst}_{3,\nu^2} = \frac{40}{3}\frac{\left(\beta_-^2 - \beta_+^2\right)}{\gamma} + \frac{40\beta_-}{27}\Big(\frac{X_{AB}}{32} - \beta_-\Big) - \frac{5\gamma}{18}\Big(\gamma + \frac{59}{6}\Big)$$
$$+ \frac{535\beta_+}{108} - \frac{10\delta_+}{9} + \frac{20\chi_+}{9} - \frac{155}{96}, \tag{2.3.36}$$

$$\tilde{C}^{inst}_{3,\nu^3} = -\frac{35}{5184}. \tag{2.3.37}$$

In order to find easily the parameter δa_4 it useful to write it as:

$$\delta a_4 = \sum_{n=0}^{N} \delta a_{4,\nu^n}\nu^n. \tag{2.3.38}$$

This allows to adapt the new ST 3PN parameter to the C_3^{inst} structure defined in Eq. (2.3.28), i.e. polynomial in ν. The comparison, then, must be realized order by order.

$N = 3$ because the greatest exponent of ν in $\tilde{C}_3^{inst}$ is 3.

Moreover, we can note that the ν^3 term has not been modified at 2PN order:

$$C^{inst}_{3,\nu^3} = \tilde{C}^{inst}_{3,\nu^3}. \tag{2.3.39}$$

Thus, in Eq. (2.3.38) we read

$$\delta a_{4,\nu^3} = 0. \tag{2.3.40}$$

Finally, the non-trivial conditions

$$\frac{5}{3}\delta a^{ST}_{4,\nu^i} + C^{inst}_{3,\nu^i} = \tilde{C}^{inst}_{3,\nu^i}, \qquad i = 0,1,2, \tag{2.3.41}$$

give

$$\delta a_4 = \delta a_{4,1} + \delta a_{4,\nu}\nu + \delta a_{4,\nu^2}\nu^2, \tag{2.3.42}$$

with

$$\begin{aligned}
\delta a_{4,1} &= -5\gamma^3 + \frac{\gamma^2}{2}\left(\frac{47\bar{\beta}}{3} - 13\right) + 2\gamma\bigg(-\frac{7\bar{\beta}}{3} + \frac{2\bar{\delta}}{3} + 2\bar{\chi} - 1\bigg) && \text{(2.3.43a)}\\
&\quad - 12\beta_+\beta_- X_{AB} - \frac{14\bar{\beta}}{3} + \frac{2\bar{\delta}}{3}\Big(\beta_- - \beta_+\Big) + 6\Big(\beta_-^2 + \beta_+^2\Big) + \frac{2}{3}\Big(\bar{\delta} + \bar{\kappa}\Big),\\
\delta a_{4,\nu} &= \frac{\gamma}{\alpha(\gamma+2)}\bigg(11\Big(\frac{\gamma^2}{4} + \gamma + 1\Big) - \bar{\delta}\bigg) + \gamma^3\bigg(\frac{7}{128}\pi^2 - \frac{3}{8}\bigg) && \text{(2.3.43b)}\\
&\quad + \gamma^2\bigg(-\frac{5\pi^2}{32} + \frac{239}{18} - \frac{2\bar{\beta}}{3}\bigg) + \frac{32}{\gamma^2}\Big(\beta_-^3 X_{AB} + \beta_+^3 - \beta_-\beta_+\big(\beta_+ X_{AB} + \beta_-\big)\Big)\\
&\quad + \gamma\bigg(\frac{32}{3}\big(\beta_- X_{AB} - \frac{23\beta_+}{8}\big) + \frac{\delta_- X_{AB}}{2} + \frac{\pi^2}{32}\Big(7\delta_+ - \frac{75}{2}\Big) + \frac{11\delta_+}{6} - 8\chi_+ + \frac{581}{18}\bigg)\\
&\quad + \frac{16}{\gamma}\bigg(\beta_+\Big(-X_{AB}\frac{\delta_-}{3} + \frac{2\delta_+}{3} + \bar{\chi}\Big) + \beta_-\Big(X_{AB}\big(\chi_+ - \frac{\delta_+}{3}\big) - \chi_-\Big)\\
&\quad + \frac{3}{2}\big(\beta_-^2 - \beta_+^2\big) + \frac{(2+\gamma)\delta_-\beta_-}{3}\bigg) + \beta_-\bigg(\frac{8\psi}{3}\Big(\delta_+ + \frac{19}{16}\Big) - 8\delta_-\bigg)\\
&\quad + \beta_+\bigg(-12\beta_- X_{AB} - \frac{8\delta_+}{3} - \frac{5}{3}\bigg) - 70\beta_-^2 + 58\beta_+^2 + \frac{X_{AB}}{3}\big(\delta_- + 2\kappa_-\big)\\
&\quad + \delta_+\Big(\frac{7}{16}\pi^2 - \frac{92}{9}\Big) - 2\kappa_+ - 4\bar{\chi},\\
\delta a_{4,\nu^2} &= -4\beta_-^2. && \text{(2.3.43c)}
\end{aligned}$$

Note that, collecting for ν, we did use the identity

$$X_{AB}^2 = \left(\frac{m_A - m_B}{M}\right)^2 = 1 - 4\nu. \tag{2.3.44}$$

All the ST parameters were defined in Table 1.1 and the "bar" notation in Eq.(2.2.25). We also observe that the $32/\gamma^2(\cdots)$ and $16/\gamma(\cdots)$ terms in $\delta a_{4,\nu}$, Eq.(2.3.43b), have no problems in the GR limit (in which every ST PN parameters go to zero). In fact, from the Table 1.1 we can rewrite them as:

$$T_1\left[\delta a_{4,\nu}\right] \equiv \frac{32}{\gamma^2}\Big(\beta_-^3 X_{AB} + \beta_+^3 - \beta_-\beta_+\big(\beta_+ X_{AB} + \beta_-\big)\Big) \tag{2.3.45a}$$
$$= \frac{\beta_A\beta_B}{2(1+\alpha_A\alpha_B)^4}\Big({\alpha_A}^2\beta_B(1+X_{AB}) + {\alpha_B}^2\beta_A(1-X_{AB})\Big),$$
$$T_2\left[\delta a_{4,\nu}\right] \equiv \frac{16}{\gamma}\Big(\beta_+\Big(-X_{AB}\frac{\delta_-}{3} + \frac{2\delta_+}{3} + \bar{\chi}\Big) + \beta_-\Big(X_{AB}\big(\chi_+ - \frac{\delta_+}{3}\big) - \chi_-\Big)$$
$$+ \frac{3}{2}\big(\beta_-^2 - \beta_+^2\big) + \frac{(2+\gamma)\delta_-\beta_-}{3}\Big) \tag{2.3.45b}$$
$$= \frac{1}{6(1+\alpha_A\alpha_B)^4}\Big(-4{\alpha_B}^4\beta_A - 4{\alpha_A}^4\beta_B + \alpha_A\alpha_B\Big(-3\alpha_B\beta_B\beta'_A(X_{AB}-1)$$
$$-4\beta_B(X_{AB}+2) + 2\beta_A(-4+9\beta_B+2\psi)\Big) + {\alpha_A}^2\alpha_B\Big(-4\alpha_A\beta_B(X_{AB}+1)$$
$$+3\beta_A\beta'_B(X_{AB}+1) + 2\alpha_B\beta_A(-2+9\beta_B+2\psi)\Big)\Big),$$

which do not present singularities in the general relativistic limit because the βs, δs and χs parameters enter, at least, at the same PN order of γ, thus each terms in Eq.(2.3.45) have no problems when all ST parameters go to zero.

Moreover, we can observe that in the test-mass limit ($\nu = 0$) the ST parameters of the PN expansion *do not* go to zero, in contrast to the GR expansion. In fact, in the general relativistic expansion of the effective metric we reduce to the Schwarzschild one for $\nu = 0$ for each PN order. On the contrary, the scalar-tensor nPN parameters are polynomials in ν of degree $n-1$, with a non-zero ν^0 terms. Therefore, the test-mass limit does not reduce only to Schwarzschild metric as we expected.

2.3.3 Tail term

Let us focus now on the tail part of the 3PN ST correction.

The tail contribution to the energy can be read in (2.3.23):

$$\frac{E(x)_{tail}}{-\mu x/2} = \Big(\frac{5}{3}\delta a^0_{4,tail} + \frac{2}{3}\delta a^{log}_{4,tail}(1+log\, x)\Big)x^3. \tag{2.3.46}$$

Comparing with the following Bernard's result [75]:

$$\frac{\tilde{E}(x)_{tail}}{-\mu x/2} = \frac{20}{9}\nu\left(2\delta_+ + \frac{\gamma(2+\gamma)}{2}\right)\left(log(4x) + 2\gamma_E + \frac{2}{5}\right)x^3, \tag{2.3.47}$$

we can find our tail coefficients:

$$\delta a^0_{4,tail} = \frac{4}{15}\nu\left(2\delta_+ + \frac{\gamma(2+\gamma)}{2}\right)\Big(10\left(\log 2 + \gamma_E\right) - 3\Big), \tag{2.3.48a}$$
$$\delta a^{log}_{4,tail} = \frac{10}{3}\nu\left(2\delta_+ + \frac{\gamma(2+\gamma)}{2}\right). \tag{2.3.48b}$$

The full tail term $\delta A^{ST}_{3PN,tail}$ introduced in Eqs. (2.3.5),(2.3.6b) can be written by using Eqs. (2.3.48a),(2.3.48b) and by collecting the $\delta a^{log}_{4,tail}$ factor:

$$\delta A^{ST}_{3PN,tail}(u) = \frac{10}{3}\nu\left(2\delta_+ + \frac{\gamma(2+\gamma)}{2}\right)\left(\frac{4}{5}\left(\log 2 + \gamma_E - \frac{3}{10}\right) + log\, u\right)u^4. \tag{2.3.49}$$

We can note that, from Table 1.1, the common factor of Eq.(2.3.49) can be written as:

$$\begin{aligned}\left(2\delta_+ + \frac{\gamma(2+\gamma)}{2}\right) &= \frac{1}{(1+\alpha_A\alpha_B)^2}\left[\alpha_A^2 + \alpha_B^2 - \alpha_A\alpha_B\left(2(1+\alpha_A\alpha_B) - 2\alpha_A\alpha_B\right)\right] \\ &= \frac{(\alpha_A-\alpha_B)^2}{(1+\alpha_A\alpha_B)^2}.\end{aligned}$$

Therefore, when $\nu > 0$, the scalar-tensor tail terms (both $\delta a^0_{4,tail}$ and $\delta a^{log}_{4,tail}$) assume zero values only if $\alpha_A = \alpha_B$, i.e. if both the compact bodies assume the same scalar charge. This condition is automatically verified for a BBH system in which $\alpha_A = \alpha_B = 0$.

2.3.4 Resummation of the A and D functions

By merging all 3PN ST terms we can write the correction to the A potential as

$$A^{ST}(u) = A^{GR}(u) + \delta A^{ST}(u), \tag{2.3.50}$$

where

$$\delta A^{ST}(u) = \left(\delta a_{4,tot} + \delta a^{log}_{4,tail}\log u\right)u^4 \tag{2.3.51}$$

and we collected

$$\delta a_{4,tot} = \delta a_4 + \delta a^0_{4,tail} + \delta at_4, \tag{2.3.52}$$

found respectively in (2.3.42),(2.3.25),(2.3.48a) and (2.3.48b).

By implementing these corrections, for instance in the calculations of the position of the Last Stable Orbit in the following section, we have to use the best available A^{GR} function, i.e. the (resummed) 5PN-NR version. This can be realized by taking Eq. (2.3.50), where the GR terms are taking by Eq.(2.2.11):

$$A^{GR}(u) = 1 - 2u + 2\nu u^3 + \nu a_4 u^4 + \nu(a_5^c + a_5^{log} log\, u)u^5 + \nu(a_6^c + a_6^{log} log\, u)u^6. \tag{2.3.53}$$

This 5PN-NR $A^{GR}(u)$ function (2.2.11) mixed with our $\delta A^{ST}(u)$ up to 3PN order (2.3.51) is thus the best available version and we will use in every application.

Here we will discuss the need of a non-perturbative resummation of the effective metric components. As in every perturbation theory, also in PN theory is presents a dimensionaless expansion parameter u which must satisfy the condition $u \ll 1$, otherwise the theory breaks down, i.e. the limit to infinity of the perturbative series does not converge to the real result.

We could decide to use the PN series as they are but, during the merger phase in binary systems evolution, PN parameter u assumes values that does not satisfy $u \ll 1$ (In Schwarzschild black holes, $u = \frac{1}{2}$ on event horizon). Therefore, in order to combine the pre-merger phase with the post-merger and ringdown, it is needed a non-perturbative resummation to avoid perturbation problems even if u assumes high values.

In `TEOBResumS` the effective metric functions are resummed with a Padé approximant. The choice of this resummation is arbitrary and the Padés are commonly selected because they do not highly modify the PN polynomial with strong oscillations or poles for reasonable values of u.

Padé approximants P_n^m is a rational function in the form

$$P_n^m[A](u) = \frac{\sum_{k=0}^{m} n_k u^k}{\sum_{k=0}^{n} d_k u^k}. \tag{2.3.54}$$

The condition $A(u=0)=1$ and the choice freedom of one of the $m+n+2$ parameters, allow us to fix $n_0 = d_0 = 1$. All the others $m+n$ are fixed by imposing that the $(m+n)$-Taylor polynomial must be equal to the same-order expansion of A function.

`TEOBResumS` model uses a Padé P_n^1 for nPN order A function and inverse P_n^0 approximant for nPN D function. In Fig.2.1 we plot A potential up to nPN order, and their relative Padés $P_n^1[A]$, together with the absolute and relative difference between GR and ST cases. Similarly, in Fig. 2.2 we will show the same plots for D potential and their Padés $P_n^0[D]$.

Choice of parameters. In order to plot the differences with respect to GR we have to set the mass and ST parameters. Our 2PN D, and 3PN A, functions require the coupling-derivatives ST parameters up to $\beta''_{A,B}$, Eq. (1.5.8). In Secs. 1.6.1, 1.7 we plotted the (spontaneous scalarized) relations between α_A, β_A and the isolated neutron-star mass m_A, for a given EOS. We will discuss the higher derivatives $\beta'_{A,B}$ and $\beta''_{A,B}$ below.

We have chosen three sets of parameters to be used, especially in the final waveform chapter, Chap. 3. The first two are based on the well known event GW170817 [16], in which the LIGO-Virgo collaboration detected a GW signal from a BNS system while the last one will be a test BNS, following the recent constraints given in Ref. [13].

Therefore, we will fix some EOSs to get the relative $\alpha_{A,B}$ and $\beta_{A,B}$ of the bodies, once fixed their two masses. To complete the set of ST parameters we have to estimate the β'_s and β''_s ones, from the α_s and β_s. We could approx $\beta'_{A,B}$, which enters at 2PN order, by imposing equality between the 1PN and 2PN ST factors in A potential (see Eq. (2.2.23)) in the equal-masses limit ($\nu = 1/4$), [55], i.e.:

$$\delta a_2 = \delta a_3, \qquad \nu = \frac{1}{4}. \tag{2.3.55}$$

Eq.(2.3.55) has 2 variables for 1 equation alone. We thus solve it for

$$\beta'_{AB} \equiv \beta'_A + \left(\frac{\alpha_A}{\alpha_B}\right)^3 \beta'_B, \tag{2.3.56}$$

getting one solution alone. This is allowed by the equal-mass approximation, otherwise we could not solve for this linear combination of $\beta'_{A,B}$.

In order to obtain $\beta'_A = \beta'_B$ when $\alpha_A = \alpha_B$ (which follows from $\nu = 1/4$) we can write:

$$\beta'_A = \left(1 - \frac{\alpha_B}{2\alpha_A}\right) \beta'_{AB} \tag{2.3.57a}$$

$$\beta'_B = \left(\frac{\alpha_B}{\alpha_A}\right)^3 \left(\frac{\alpha_B}{2\alpha_A}\right) \beta'_{AB}. \tag{2.3.57b}$$

Hence, by combining all these definitions, we get

$$\beta'_A = -\frac{2\alpha_A - \alpha_B}{\alpha_A\alpha_B^3}\Big[\big(-2+\frac{27}{4}\beta_A\big)\alpha_B^2 + \big(-2+72\alpha_B^2+\frac{27}{4}\beta_B\big)\alpha_A^2 \tag{2.3.58a}$$
$$+\big(-2+70\alpha_B^2+\frac{75}{4}\beta_B\big)\alpha_A^3\alpha_B + \big(2+(-2+\frac{75}{4}\beta_A)\alpha_B^2+3\beta_A\beta_B\big)\alpha_A\alpha_B\Big],$$
$$\beta'_B = -\frac{\alpha_B}{\alpha_A^4}\Big[\big(-2+\frac{27}{4}\beta_A\big)\alpha_B^2 + \big(-2+72\alpha_B^2+\frac{27}{4}\beta_B\big)\alpha_A^2 \tag{2.3.58b}$$
$$+\big(-2+70\alpha_B^2+\frac{75}{4}\beta_B\big)\alpha_A^3\alpha_B + \big(2+(-2+\frac{75}{4}\beta_A)\alpha_B^2+3\beta_A\beta_B\big)\alpha_A\alpha_B\Big].$$

Similarly, we can assume the same approximation for the new 3PN parameters $\beta''_{A,B}$, i.e. find them by imposing the equality between the 1PN (or 2PN) and 3PN factors, Eq. (2.3.52), once removed the tidal δat_4. This gives the condition[3]:

$$\delta a_3 = \delta a_4, \qquad \nu = \frac{1}{4}. \tag{2.3.59}$$

Analogously to what done in Eqs. (2.3.56), we can solve this condition by defining:

$$\beta''_{AB} \equiv \beta''_A + \left(\frac{\alpha_A}{\alpha_B}\right)^4 \beta''_B \tag{2.3.60}$$

and

$$\beta''_A = \left(1 - \frac{\alpha_B}{2\alpha_A}\right)\beta''_{AB}, \tag{2.3.61a}$$
$$\beta''_B = \left(\frac{\alpha_B}{\alpha_A}\right)^4\left(\frac{\alpha_B}{2\alpha_A}\right)\beta''_{AB}. \tag{2.3.61b}$$

Solutions of Eq. (2.3.59) are then given by:

$$\beta''_A = \frac{2\alpha_A - \alpha_B}{\alpha_A^3\alpha_B^5}\Big[\big(-12+\frac{81}{2}\beta_A\big)\alpha_B^5\beta_A + \big(\mathfrak{c}_1 + (142+440\alpha_B^2)\beta_A \tag{2.3.62a}$$
$$+(-\frac{105}{4}\alpha_B^2 - 6\beta_B)\beta_A^2 + 12\beta_B\big)\alpha_A^2\alpha_B^3 + \big(12 + (-12+\frac{225}{2}\beta_A)\alpha_B^2$$
$$+18\beta_A\beta_B\big)\alpha_A\alpha_B^4\beta_A + \big(\mathfrak{c}_2 + (132-6\alpha_B^2)\alpha_0^2 + (16+420\alpha_B^2+81\beta_B-18\beta_B^2)\beta_A$$
$$+(\mathfrak{c}_3 + 468\beta_A + 12\beta_B)\alpha_B^2 - 20\beta_B\big)\alpha_A^3\alpha_B^2 + \big(\mathfrak{c}_4\alpha_B^4 + (-16+\frac{795}{4}\beta_B)\beta_B$$
$$+(\frac{16}{3} - 6\alpha_0^2 + 1374\beta_B)\alpha_B^2\big)\alpha_A^6\alpha_B + \big(\mathfrak{c}_1 + (\frac{16}{3} - 6\alpha_0^2 + 534\beta_A)\alpha_B^4 + 190\beta_B$$
$$+(-\frac{81}{2} + 30\beta_A)\beta_B^2 + (\mathfrak{c}_5 + 132\alpha_0^2 + 24\beta_A - 432\beta_B + \frac{351}{2}\beta_A\beta_B)\alpha_B^2\big)\alpha_A^4\alpha_B$$
$$+\big((\mathfrak{c}_6 - 420\beta_B)\alpha_B^4 + (-24+81\beta_B)\beta_B + (\mathfrak{c}_3 - 6\alpha_0^2 + 1356\beta_B - \frac{225}{2}\beta_B^2)\alpha_B^2\big)\alpha_A^5\Big],$$

[3]The tail terms $\delta a_{4,tail}^{0,log}$ are both zero in the equal-masses case $\nu = 1/4$ because it implies that $\alpha_A = \alpha_B$.

$$
\begin{aligned}
\beta''_B = \frac{1}{\alpha_A^7}\bigg[&\big(-12+\frac{81}{2}\beta_A\big)\alpha_B^5\beta_A + \big(\mathfrak{c}_1 + (142+440\alpha_B^2)\beta_A \\
&+ (-\frac{105}{4}\alpha_B^2 - 6\beta_B)\beta_A^2 + 12\beta_B\big)\alpha_A^2\alpha_B^3 + \big(12 + (-12+\frac{225}{2}\beta_A)\alpha_B^2 \\
&+ 18\beta_A\beta_B\big)\alpha_A\alpha_B^4\beta_A + \big(\mathfrak{c}_2 + (132-6\alpha_B^2)\alpha_0^2 + (16+420\alpha_B^2+81\beta_B-18\beta_B^2)\beta_A \\
&+ (\mathfrak{c}_3 + 468\beta_A + 12\beta_B)\alpha_B^2 - 20\beta_B\big)\alpha_A^3\alpha_B^2 + \big(\mathfrak{c}_4\alpha_B^4 + (-16+\frac{795}{4}\beta_B)\beta_B \\
&+ (\frac{16}{3} - 6\alpha_0^2 + 1374\beta_B)\alpha_B^2\big)\alpha_A^6\alpha_B + \big(\mathfrak{c}_1 + (\frac{16}{3} - 6\alpha_0^2 + 534\beta_A)\alpha_B^4 + 190\beta_B \\
&+ (-\frac{81}{2} + 30\beta_A)\beta_B^2 + (\mathfrak{c}_5 + 132\alpha_0^2 + 24\beta_A - 432\beta_B + \frac{351}{2}\beta_A\beta_B)\alpha_B^2\big)\alpha_A^4\alpha_B \\
&+ \big((\mathfrak{c}_6 - 420\beta_B)\alpha_B^4 + (-24+81\beta_B)\beta_B + (\mathfrak{c}_3 - 6\alpha_0^2 + 1356\beta_B - \frac{225}{2}\beta_B^2)\alpha_B^2\big)\alpha_A^5\bigg],
\end{aligned}
\tag{2.3.62b}
$$

where

$$
\begin{aligned}
&\mathfrak{c}_1 \simeq -4.574378219526231, && \mathfrak{c}_2 \simeq 557.0840428860286, \\
&\mathfrak{c}_3 \simeq 0.7589551138071045, && \mathfrak{c}_4 \simeq 3381.5881656665847, \\
&\mathfrak{c}_5 \simeq 3003.9408283329217, && \mathfrak{c}_6 \simeq 5828.444951113477.
\end{aligned}
\tag{2.3.63}
$$

These numerical factors can not be written in a rational form due to presence of π in Eqs. (2.3.42)-(2.3.43c).

Now we can set all the ST parameters, approximately for the 2PN and 3PN ones. As we introduced above, we will base our first set on the GW170817 event, simply choosing two EOSs (see Secs. 1.6.1, 1.7). The second one is made to push up to the constraint limit the scalar-tensor parameters in order to get the greater differences with respect to GR possible[4].

In the former case we choose, for the GW170817 event, the EOSs: ENG and SLy, which are intermediate ones and we have discussed them in (see Secs. 1.6.1, 1.7), and the BNS masses: $m_A = 1.46M_\odot$, $m_B = 1.27M_\odot$[5]. In the GWtest case, instead, we will use the EOS: H4 which allows an higher value of the scalar charges $\alpha_{A,B}$ but, clearly, within the perturbative structure of the ST theories.

Here, in order to test high ST values, we choose $m_A = 1.84M_\odot$ and $m_B = 1.70M_\odot$. We will refer to these events respectively as GW170817:ENG, GW170817:SLy and GWtest:H4, in order to clarify immediately which EOS we are using.

We have also turn off the neutron star spins in the GWtest event because the ST corrections (that we discussed previously in this chapter) are given only to the orbital part of dynamics, as well as for the waveform that we will analyze in the following.

These two sets of masses, from Fig. 1.2, give the $\alpha_{A,B}$ and $\beta_{A,B}$. Our approximated $\beta'_{A,B}$, $\beta''_{A,B}$ parameters are given in Eqs. (2.3.58),(2.3.62) and we read the universal ones α_0 and β_0 in Ref. [13].

We can collect them in Table 2.1.

[4] Keeping in mind that the ST theory must be a perturbation of GR, therefore the correction must not break the GR structure.

[5] $M_\odot = 1.9891 \cdot 10^{30}$Kg always represents the solar mass.

Parameters	GW170817:ENG	GW170817:SLy	GWtest:H4
$m_A[M_\odot]$	1.46	1.46	1.84
$m_B[M_\odot]$	1.27	1.27	1.70
χ_A	0.1	0.1	0
χ_B	0.1	0.1	0
$\alpha_0[10^{-4}]$	1.2	1.2	1.2
β_0	-4.30	-4.32	-4.32
q	1.14961	1.14961	1.08235
ν	0.24879	0.24879	0.249609
$\alpha_A[10^{-4}]$	9.76377	14.0971	46.63990
$\alpha_B[10^{-4}]$	5.57138	6.53206	14.07751
$\beta_A[10^1]$	3.498646500	5.074684120	16.74751996
$\beta_B[10^1]$	1.996408983	2.351534543	5.067647952
$\beta'_A[10^7]$	-1.107711107	-2.070384498	-10.76291321
$\beta'_B[10^5]$	-8.215974314	-6.210991980	-5.260446959
$\beta''_A[10^{12}]$	5.546086151	12.87213672	106.3158712
$\beta''_B[10^{11}]$	2.347277383	1.789291052	1.568408343

Table 2.1: Mass, spin and ST parameters (see Eqs. (2.2.2), (2.2.42) and Sec. 2.3.4) for all the three GW events that we are considering, with their relative EOSs (see Sec. 1.6.1).

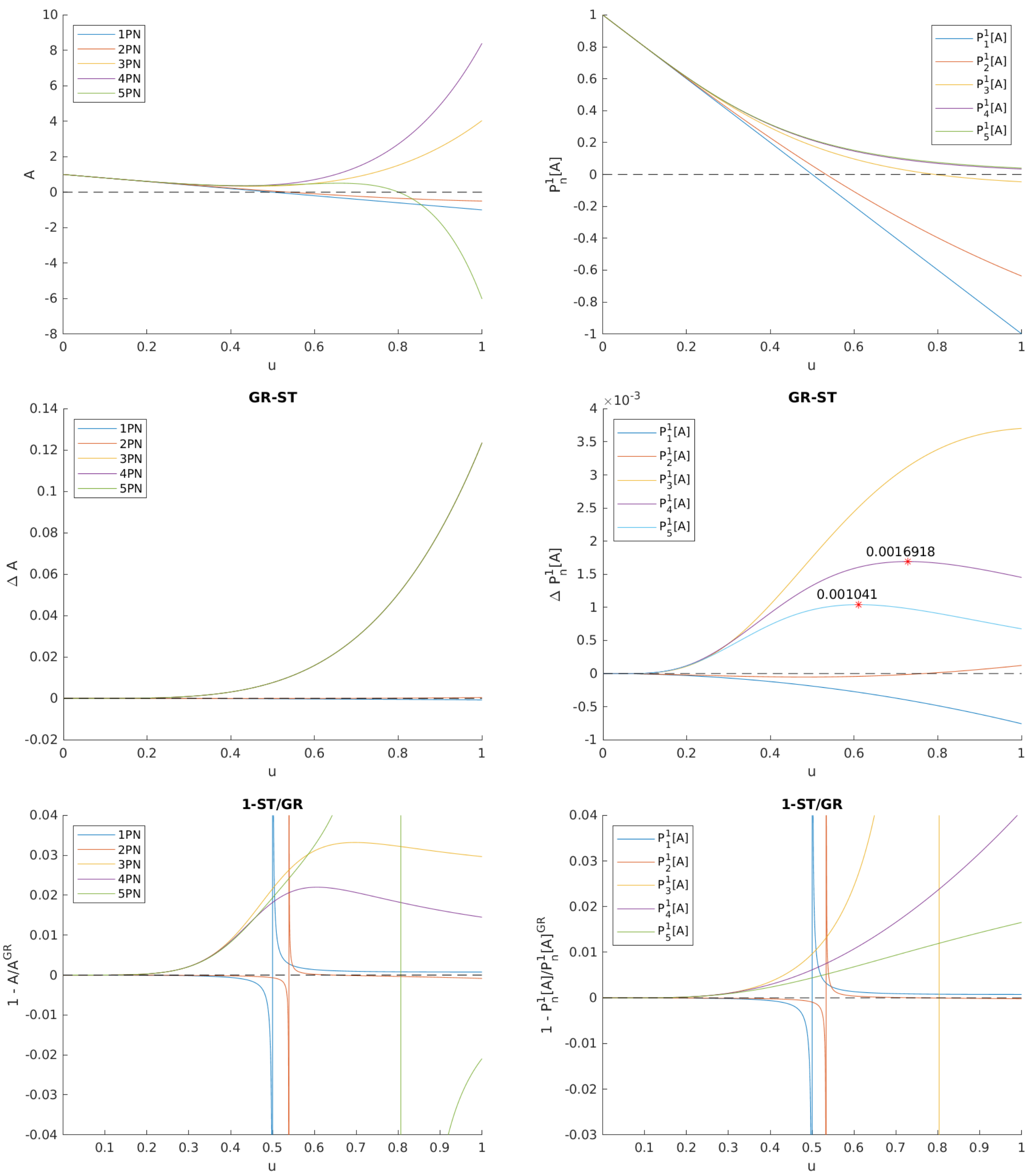

Figure 2.1: PN expansions of the A potential and $P_n^1[A]$ up to nPN order. The mass and scalar-tensor parameters are fixed in Table 2.1 within the "GWtest:H4" case. It can be seen that Padé resummation stabilizes the PN oscillations, both in ST A functions and in the differences with respect to GR.

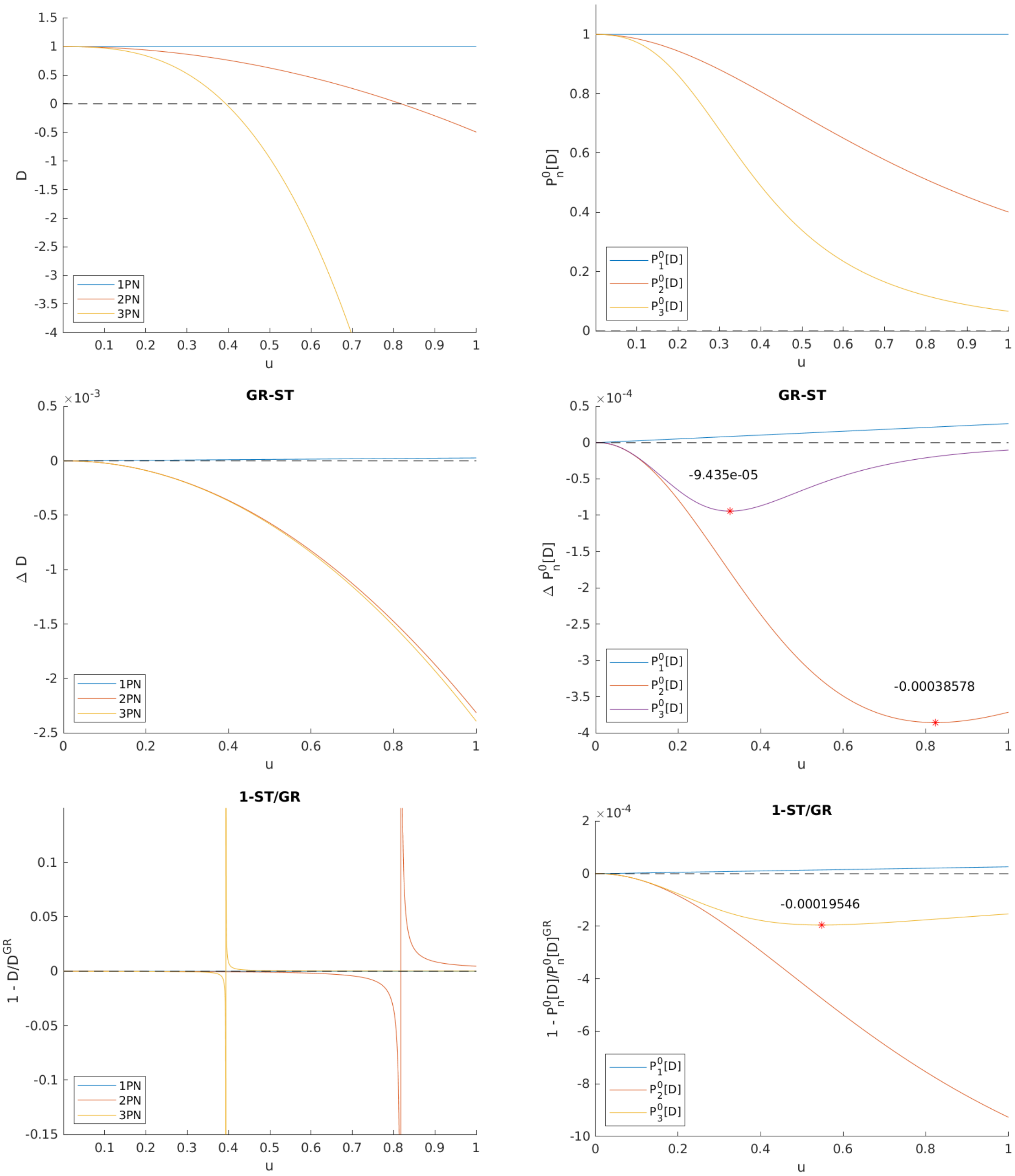

Figure 2.2: PN expansions of the D potential and $P^0_n[D]$ up to nPN order. The mass and scalar-tensor parameters are fixed in Table 2.1 within the "GWtest:H4" case. It can be seen that Padé resummation stabilizes the PN oscillations, both in ST D functions and in the differences with respect to GR, as well as for the A potential figured in Fig. 2.1.

We can see in Fig. 2.1, and Fig. 2.2 that, with the parameters chosen above, increasing PN order and u expansion variable the series does not converge, presenting highly oscillations. Padé approximants are, instead, more convergent and they have a more predictable behaviour.

The divergences in the relative difference plots are clearly dues to the zeros of the functions in GR limit.

In these figures we inserted the (PN and Padé-resummed) effective potential A and D for the event "GWtest:H4". This because of its higher value of ST parameters, as we discussed above, and then to obtain the bigger difference as possible.

Event	$\delta a_2[10^{-6}]$ (2.2.23a)	$\delta a_3[10^{-6}]$ (2.2.23b)	$\delta a_{4,tot}[10^{-5}]$ (2.3.42)	$\delta a^{log}_{4,tail}[10^{-7}]$ (2.3.48)
GWtest:H4	379.3070130	−1180.245389	−12313.68472	88.21974737
GW170817:ENG	8.703108927	−3.006160458	−3.320377712	1.457581704
GW170817:SLy	19.37403395	−22.84425533	−45.81808956	4.746043184

Table 2.2: ST parameters which enter in effective potential A for each event we considered in Table. 2.1.

Table 2.2 shows all the ST parameters of A function by varying the GW event. In particular, the $\delta a^{log}_{4,tail}$ is the only 3PN term in ST theories, thus is identically zero in GR (as well as for the 1PN term, which is made only by δa_2) and it goes as $\sim (\alpha_A - \alpha_B)^2$, similarly to the dipolar -radiation term.

We can also observe that the 3PN ST tail term $\delta a^{log}_{4,tail}$ (see Eqs. (2.3.48), (2.3.50)) vanishes in the equal-scalar-charge limit: $\alpha_A = \alpha_B$. This can be obtained from the equal-mass limit: $m_A = m_B$. Note that $\alpha_A = \alpha_B$ is also reached when $m_A \neq m_B$ (see Fig. 1.2) but we strategically ignore this case in order to get a non-trivial 3PN ST tail term, which does not exist in GR (see Table 1.1).

2.4 Effective Dynamics

In this section we will comment the LSO and LR position in ST theories by using our new 3PN A effective potential (2.3.50).

By staticity and spherical symmetry of the metric $g^{\star}_{\mu\nu}$ in Einstein frame, we have the following Killing symmetries [1,3]:

$$g^{\star}_{\mu\nu} U^{\mu} K^{\nu} = C(K), \tag{2.4.1}$$

where U^{μ} is the 4-velocity with respect to an affine parameter λ, K^{ν} is the Killing field of some symmetry and $C(K)$ is the constant of motion associated with the Killing field K^{ν}.

By using $K^{\nu} = (1,0,0,0)$ and $K^{\nu} = (0,0,0,1)$ we have respectively the statical and the rotational symmetry:

$$-A\frac{dt}{d\lambda} = -E, \tag{2.4.2}$$

$$r^2\frac{d\phi}{d\lambda} = j, \tag{2.4.3}$$

with E and j are the conserved energy and angular momentum, and λ is an affine parameter.

We define as ϵ the normalization of the 4-velocity:

$$\begin{aligned} g^{\star}_{\mu\nu}\frac{dx^{\star\mu}}{d\lambda}\frac{dx^{\star\nu}}{d\lambda} &= -\epsilon \\ &= -A\left(\frac{dt}{d\lambda}\right)^2 + B\left(\frac{dr}{d\lambda}\right)^2 + r^2\left(\frac{d\phi}{d\lambda}\right)^2, \end{aligned} \tag{2.4.4}$$

where $\epsilon = 1$ is the timelike geodetics normalization and $\epsilon = 0$ is the nulllike geodetics normalization.

By combining the Killing symmetries and the 4-velocity normalization we get:

$$\frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \bar{D}(r)V(r) = \frac{\bar{D}(r)}{2}E^2, \tag{2.4.5}$$

with

$$V(u) = \frac{A}{2}(\epsilon + j^2u^2), \tag{2.4.6}$$

is the modified potential and j the angular momentum, per unit of mass, given above for circular orbit, from Eq. (2.2.63). The $\bar{D}(r)$ function was defined in Eq. (2.2.10) and it is the only structural difference of Eq. (2.4.5) with respect to the GR-test-mass one, in which $\bar{D}^{GR-Schw}(r) = 1$.

We used the same notation of [3] and we can note that Eq. (2.4.5) reduces to the GR Schwarzschild limit when we turn off the ST parameters and when we set $\nu = 0$.

2.4.1 Last Stable Orbit

When we set $\epsilon = 1$ in Eq. (2.4.6) we describe the massive bodies case, i.e. timelike geodetics. The circular orbit is given by the relative expression of j^2 in (2.2.63). This gives (2.2.64):

$$\begin{aligned} \sqrt{2V(u)} = \sqrt{A(1+j^2u^2)} &= \hat{H}_e(u, 0, j(u)) \\ &= \sqrt{\frac{2uA^2}{(Au^2)'}}. \end{aligned} \tag{2.4.7}$$

The last (or innermost) stable (circular) orbit ("LSO" or "ISCO") required also the inflection condition

$$\frac{\partial^2 V(u)}{\partial u^2}|_{u=u_{LSO}} = 0 \tag{2.4.8}$$

or, equivalently,

$$\frac{\partial^2 \hat{H}_e(u)}{\partial u^2}|_{u=u_{LSO}} = 0. \tag{2.4.9}$$

The general relativistic Schwarzschild limit gives the well known result:

$$u_{LSO}^{GR-Schw} = \frac{1}{6}. \tag{2.4.10}$$

In the following Figs. 2.3 - 2.5 we will show the (absolute and relative) difference of the LSO position u_{LSO}, the angular momentum j_{LSO} (see (2.2.63)) and the orbital frequency Ω_{LSO} (see (2.2.61)) for each PN order, up to the 5th one. We obtained these results by numerically solve the LSO condition (2.4.9), with Eq. (2.2.64), using the resummed A effective potential (2.3.54) up to the nPN order into the non expanded circular energy. For the ST parameters we used the "GWtest:H4" ones (see Table 1.1) in order to get the larger differences as possible.

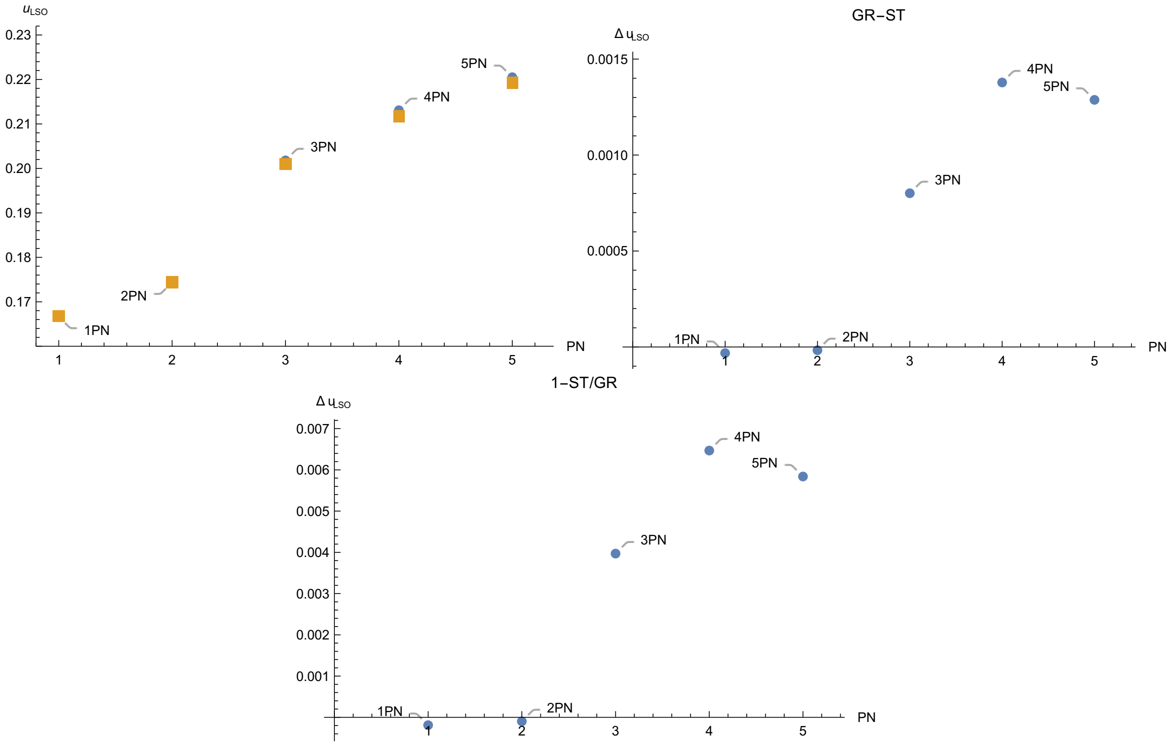

Figure 2.3: LSO position, u_{LSO}, for each PN order (left panel), its differences (right panel) and relative differences (bottom panel) with respect to GR . In the left panel blue points are the general relativistic LSO variables, while the orange squares are the scalar-tensor LSO ones. The ST parameters have been fixed in "GWtest:H4" case of Table 2.1.

PN	u_{LSO}^{GR}	u_{LSO}^{ST}	Δu_{LSO}	$\Delta u_{LSO}^{(rel)}$
1	1/6	0.1666982816	−0.0000316149	−0.000189689
2	0.1743778790	0.1743948243	−0.0000169453	−0.000097176
3	0.2017954137	0.2009943165	0.0008010972	0.003969848
4	0.2130792899	0.2117011120	0.0013781779	0.006467911
5	0.2205067236	0.2192192429	0.0012874807	0.005838737

Table 2.3: LSO position u_{LSO} in GR and ST theories with the 3PN correction to the A potential. We defined the difference between GR and ST as Δu_{LSO}, while $\Delta u_{LSO}^{(rel)}$ represents the relative one: (GR-ST)/GR.

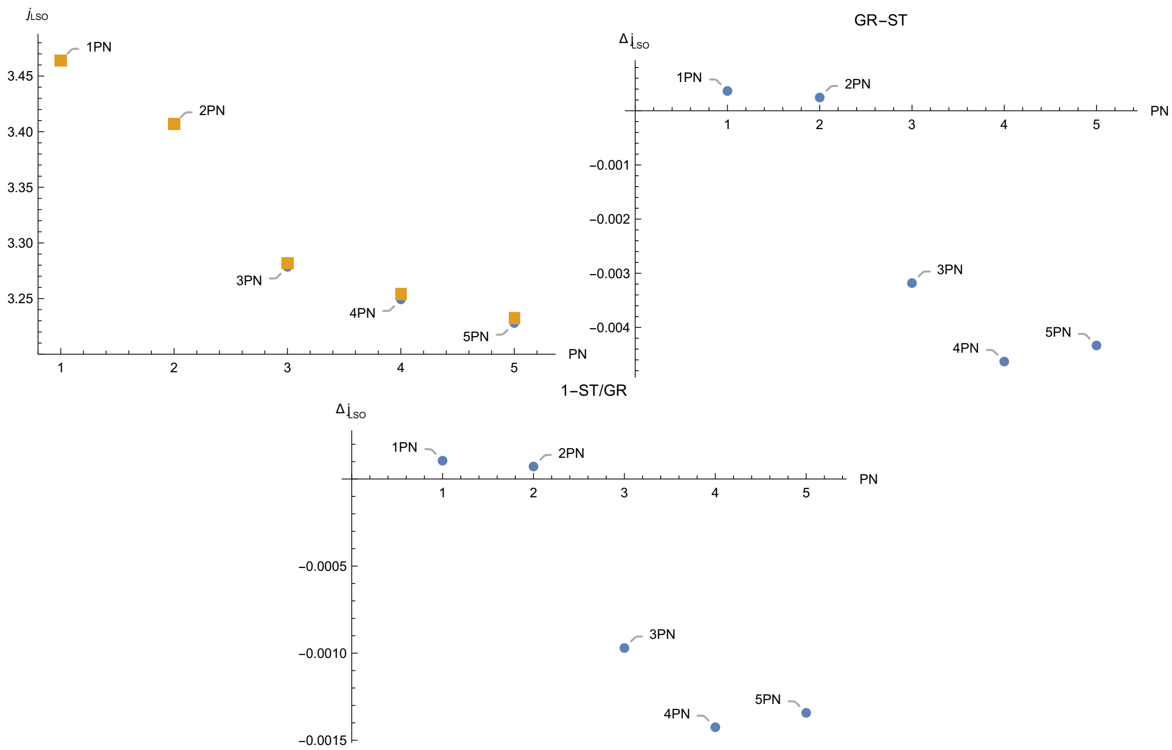

Figure 2.4: LSO angular momentum, j_{LSO}, for each PN order (left panel), its differences (right panel) and relative differences (bottom panel) with respect to GR . In the left panel blue points are the general relativistic LSO variables, while the orange squares are the scalar-tensor LSO ones. The ST parameters have been fixed in "GWtest:H4" case of Table 2.1.

PN	j_{LSO}^{GR}	j_{LSO}^{ST}	Δj_{LSO}	$\Delta j_{LSO}^{(rel)}$
1	3.464101615	3.463736608	0.000365007	0.000105368
2	3.406984207	3.406739407	0.000244799	0.000071852
3	3.278521086	3.281702816	−0.003181730	−0.000970477
4	3.249290717	3.253921453	−0.004630735	−0.001425153
5	3.227973506	3.232307144	−0.004333638	−0.001342526

Table 2.4: LSO angular momentum j_{LSO} in GR and ST theories with the 3PN correction to the A potential. We defined the difference between GR and ST as Δj_{LSO}, while $\Delta j_{LSO}^{(rel)}$ represents the relative one: (GR-ST)/GR.

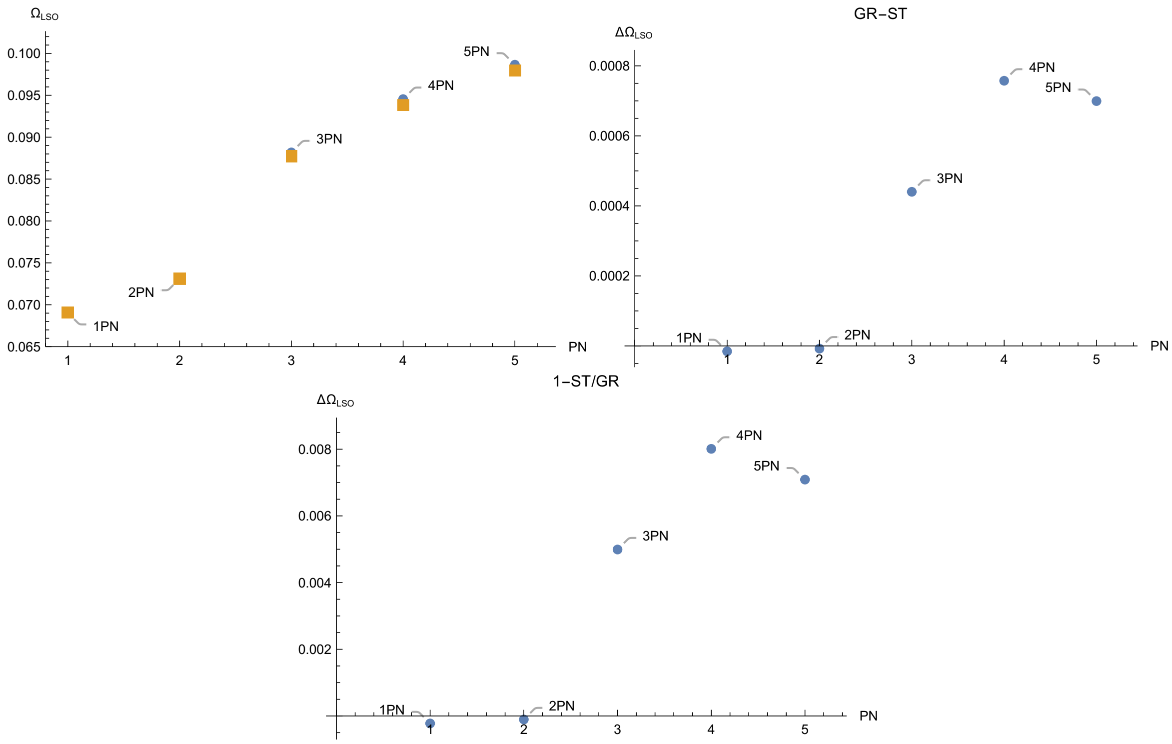

Figure 2.5: LSO orbital frequency, Ω_{LSO}, for each PN order (left panel), its differences (right panel) and relative differences (bottom panel) with respect to GR . In the left panel blue points are the general relativistic LSO variables, while the orange squares are the scalar-tensor LSO ones. The ST parameters have been fixed in "GWtest:H4" case of Table 2.1.

PN	Ω^{GR}_{LSO}	Ω^{ST}_{LSO}	$\Delta\Omega_{LSO}$	$\Delta\Omega^{(rel)}_{LSO}$
1	0.06903400454	0.06904945831	−0.0000154538	−0.000223857
2	0.07311113753	0.07311880107	−0.0000076635	−0.000104820
3	0.08818223724	0.08774201917	0.0004402181	0.004992140
4	0.09454579632	0.09378839180	0.0007574045	0.008010980
5	0.09864989421	0.09795059142	0.0006993028	0.007088733

Table 2.5: LSO orbital frequency Ω_{LSO} in GR and ST theories with the 3PN correction to the A potential. We defined the difference between GR and ST as $\Delta\Omega_{LSO}$, while $\Delta\Omega^{(rel)}_{LSO}$ represents the relative one: (GR-ST)/GR.

2.4.2 Light Ring

On the contrary with the LSO position (see Sec. 2.4.1), when $\epsilon = 0$ in Eq. (2.4.6) we describe the nulllike geodetics. Therefore the light ring (LR) location is defined by the conditions

$$\frac{\partial V(u)}{\partial u}|_{u=u_{LR}} = 0 \iff \frac{\partial^2 \hat{H}_{orb}(u)}{\partial u^2}|_{u=u_{LR}} = 0 \iff \frac{\partial (Au^2)}{\partial u}|_{u=u_{LR}} = 0, \tag{2.4.11}$$

which corresponds to the *unstable* innermost circular orbit admitted.

The GR-Schwarzschild limit now gives the exact result:

$$u^{GR-Schw}_{LR} = \frac{1}{3}. \tag{2.4.12}$$

In the following Fig. 2.6 we will show the (absolute and relative) difference of the LR position u_{LR} for each PN order, up to the 5th one. We obtained these results by numerically solve the LR condition (2.4.11), with Eq. (2.2.64), using the resummed $(1n)$ A effective potential (2.3.54) up to the nPN order into the non expanded circular energy. For the ST parameters we used the "GWtest:H4" ones (see Table 1.1) in order to get the larger differences as possible.

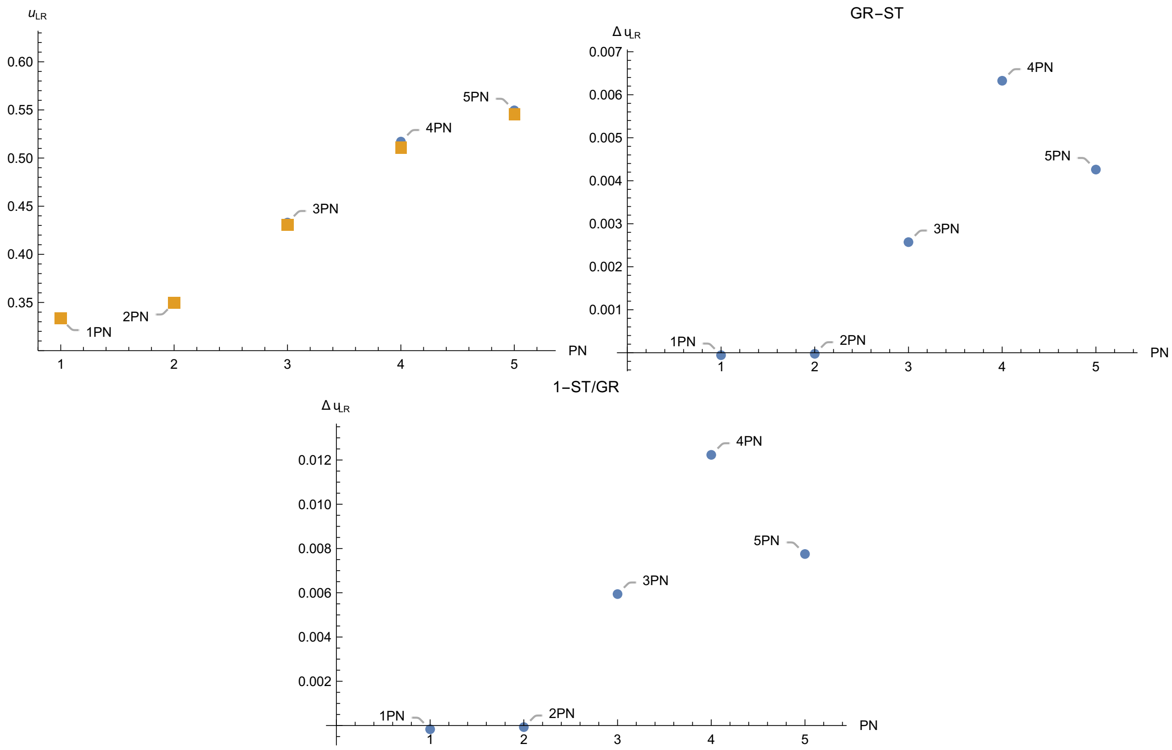

Figure 2.6: Light ring position, u_{LR}, for each PN order (left panel), and its difference with respect to GR (right panel). In the left panel blue points are the general relativistic LR positions, while the orange squares are the scalar-tensor LR positions ones. The ST parameters have been fixed in "GWtest:H4" case of Table 2.1.

PN	u_{LR}^{GR}	u_{LR}^{ST}	Δu_{LR}	$\Delta u_{LR}^{(rel)}$
1	$1/3$	0.3333895370	−0.0000562037	−0.000168611
2	0.3495533234	0.3495782219	−0.0000248985	−0.000071230
3	0.4329899370	0.4304177552	0.0025721818	0.005940512
4	0.5170909964	0.5107659761	0.006325020	0.012231929
5	0.5495528907	0.5452931306	0.004259760	0.007751320

Table 2.6: LR position u_{LR} in GR and ST theories with the 3PN correction to the A potential. We defined the difference between GR and ST as Δu_{LR}, while $\Delta u_{LR}^{(rel)}$ represents the relative one: (GR-ST)/GR.

Chapter 3

Scalar-tensor waveform

In the previous sections we focused on the dynamics of binary systems. Here we will move on the GW waveform, in particular on its factorization into a general relativistic part times a scalar-tensor correction.

We now briefly recall the relaxed field equations in which perturbation theory can be used to find the perturbed metric and its multipolar decomposition. We will also summarize the resummation basics, applied to the full waveform, within the `TEOBResumS` model.

Then, we will collect the 2PN ST information on the waveform, given by N. Sennett, S. Marsat and A. Buonanno in Ref. [18], in order to analyze it within the EOB framework, i.e. in a GR-factorized form.

After that, we will focus on the main feature of this chapter, i.e. the implementation of the ST correction to the waveform. Therefore, we will discuss its GR-factorization and some resummation choices.

3.1 `TEOBResumS` waveform

3.1.1 Multipolar expansion

As well as in GR, field equations are treated in a reduced form. Here we consider the physical Jordan frame equations (1.1.44),(1.1.51b). Moreover, we assume that, far away from the binary system, the metric $g_{\mu\nu}$ reduces to the Minkowski one $\eta_{\mu\nu}$, and that the scalar field ϕ reduces to a constant value ϕ_0.

In order to make this reformulation we define (see Refs. [90, 91])

$$\tilde{h}^{\mu\nu} \equiv \eta^{\mu\nu} - \tilde{\mathfrak{g}}^{\mu\nu}, \tag{3.1.1}$$

where the "gothic" metric $\tilde{\mathfrak{g}}^{\mu\nu}$ is defined by

$$\tilde{\mathfrak{g}}^{\mu\nu} \equiv \sqrt{-\tilde{g}}\tilde{g}^{\mu\nu}, \qquad \tilde{g} \equiv \det(\tilde{g}). \tag{3.1.2}$$

These "$\sim$" definitions differ with respect to the general relativistic ones by:

$$\tilde{g}_{\mu\nu} \equiv \frac{\phi}{\phi_0} g_{\mu\nu}. \tag{3.1.3}$$

In order to write the solutions of the field equations as a Green's integral to expand them into multipolar modes, it must be use the $\tilde{h}^{\mu\nu}$ metric definition.

Therefore, field equations (1.1.44),(1.1.51b) in terms of $\tilde{h}^{\mu\nu}$, in the Lorenz gauge

$$\partial_\nu \tilde{h}^{\mu\nu} = 0 \tag{3.1.4}$$

read [90]:

$$\Box_\eta \tilde{h}^{\mu\nu} = 16\pi\tilde{g}T^{\mu\nu} - (\Lambda^{\mu\nu} + \Lambda_s^{\mu\nu}), \tag{3.1.5a}$$

$$\Box_\eta \phi = -8\pi\phi_0\tau_s. \tag{3.1.5b}$$

The wave operator is the flat-spacetime one, $\Box_\eta \equiv \eta^{\mu\nu}\partial_\mu\partial_\nu$, and the source terms $\Lambda^{\mu\nu}$, $\Lambda_s^{\mu\nu}$ and τ_s are written in Ref. [90].

By introducing the spherical coordinates (R, Θ, Φ), he solutions of Eqs.(3.1.5) can be decomposed into spin-weighted spherical harmonics $_{-2}Y_{lm}(\Theta, \Phi)$.

$_sY_{lm}(\Theta, \Phi)$ represents the s-weighted spherical harmonics and $_0Y_{lm}(\Theta, \Phi) \equiv Y_{lm}(\Theta, \Phi)$ the ordinary ones.

Therefore, the plus and cross polarization of the waveform transverse-traceless (TT) projection, h^{ij}_{TT} can be written as (see Refs. [18, 35, 90–93]):

$$h_+ - ih_\times = \sum_{l\geq 2}\sum_{m=1}^{l} {}_{-2}Y_{lm}(\Theta, \Phi)h_{lm}, \tag{3.1.6}$$

where the coefficients h_{lm} are the spin-weighted modes of which we will write the scalar-tensor factorization[1]. Clearly we must stop the l-sum in Eq. (3.1.6) up to some l_{max}. The current state-of-the-art `TEOBResumS` model gathers multipoles up to $l_{max} = 8$ and the l_{max}-dependent total energy flux, in terms of the multipolar waveform h_{lm}, reads

$$\begin{aligned} F^{l_{max}} = \sum_{l=2}^{l_{max}}\sum_{m=1}^{l} F_{lm} &= \frac{2}{16\pi}\sum_{l=2}^{l_{max}}\sum_{m=1}^{l}\left|R\dot{h}_{lm}\right|^2, \\ &= \frac{2}{16\pi}\sum_{l=2}^{l_{max}}\sum_{m=1}^{l}(m\Omega)^2|Rh_{lm}|^2. \end{aligned} \tag{3.1.7}$$

The general relativistic Newtonian (or quadrupolar) contribution is the well-known result [7]

$$F_{22}^{GR} = \frac{32}{5}\nu^2 x^5, \tag{3.1.8}$$

where the variable x was defined in Eq. (2.3.13). Its ST correction, instead, reads [90, 91]

$$F_{22}^{ST} = \frac{F_{22}^{GR}}{\tilde{G}_{AB}}, \tag{3.1.9}$$

with the bodies-dependent $\tilde{G}_{AB}$ term can be read in Eq. (1.5.10).

[1]Here we do not shown the non-oscillating modes $m = 0$ because they do not contribute for circular orbits.

3.1.2 GR factorization

Here we will recall the factorization technique for the general relativistic multipolar waveform from quasi-circular BNS, h_{lm} from Eq. (3.1.6), introduced by T. Damour, B. Iyer and A. Nagar in Ref. [19].

This factorization method differs with respect to the PN-approach, which is an additive technique, because it is a multiplicative technique. Within the PN approach, as well as all the perturbation theories, each variable is written as a sum of some terms starting from the leading order (LO), or Newtonian order. The factorization approach, used in the `TEOBResumS` model, is instead based on a multiplicative writing of the gravitational waveform, in order to makes explicit the physical meaning of each term.

Following the notation introduced in Ref. [19], we factorize out the Newtonian contribution of h_{lm} in the form:

$$h_{lm} = h_{lm}^{(N,\epsilon)} \hat{h}_{lm}^{(\epsilon)}, \tag{3.1.10}$$

where $h_{lm}^{(N,\epsilon)}$ is the Newtonian term and $\hat{h}_{lm}^{(\epsilon)}$ collects all the other factors which are, in turn, PN-expanded in the form $\hat{h}_{lm}^{(\epsilon)}(x) = 1 + \mathcal{O}(x)$. The x variable was defined in Eq. (2.3.13) and ϵ represents the parity of $l+m$:

$$\epsilon \equiv \pi(l+m) \equiv \begin{cases} 0 & \text{if} \quad l+m \ \text{ is even} \\ 1 & \text{if} \quad l+m \ \text{ is odd} \end{cases} \tag{3.1.11}$$

The general relativistic Newtonian contribution for circular orbits can be written for each pair (l,m) and as function of x and $X_{1,2}$ (see Eqs. (2.3.13),(2.2.2b)) [19,35,93–95]:

$$h_{lm}^{(N,\epsilon)} = \frac{M\nu}{R} n_{lm}^{(\epsilon)} c_{l+\epsilon}(\nu) x^{(l+\epsilon)/2} Y_{l-\epsilon,-m}\left(\Theta = \frac{\pi}{2}, \Phi\right). \tag{3.1.12}$$

Here, ϵ denotes the parity of $l+m$, Eq. (3.1.11). The ϵ-dependent numerical coefficients $n_{lm}^{(\epsilon)}$ are

$$n_{lm}^{(0)} = (im)^l \frac{8\pi}{(2l+1)!!} \sqrt{\frac{(l+1)(l+2)}{l(l-1)}}, \tag{3.1.13a}$$

$$n_{lm}^{(1)} = -(im)^l \frac{16\pi i}{(2l+1)!!} \sqrt{\frac{(2l+1)(l+2)(l^2-m^2)}{(2l-1)(l+1)l(l-1)}} \tag{3.1.13b}$$

and the ν-dependent $c_{l+\epsilon}(\nu)$ can be expressed as

$$c_{l+\epsilon}(\nu) = X_2^{l+\epsilon-1} + (-1)^{l+\epsilon} X_1^{l+\epsilon-1}. \tag{3.1.14}$$

The $Y_{lm}(\Theta,\Phi)$ represents the usual spherical harmonics which, following the previous notation, correspond to ${}_{s=0}Y_{lm}(\Theta,\Phi) \equiv Y_{lm}(\Theta,\Phi)$.

We now write some of the first modes of

$$\begin{aligned} \tilde{h}_{lm}^{(N,\epsilon)} &\equiv h_{lm}^{(N,\epsilon)} / (\frac{M\nu}{R} x) \\ &\equiv \left|\tilde{h}_{lm}^{(N,\epsilon)}\right| \exp(-im\Phi + i c_{lm}^{(N)}) \end{aligned} \tag{3.1.15}$$

(see Eq. (3.1.12))[2]:

$$\left|\tilde{h}_{22}^{(N,\epsilon)}\right| = 8\sqrt{\frac{\pi}{5}}, \qquad \left|\tilde{h}_{21}^{(N,\epsilon)}\right| = \frac{8}{3}\sqrt{\frac{\pi}{5}}X_{AB}\sqrt{x}, \qquad \left|\tilde{h}_{33}^{(N,\epsilon)}\right| = 3\sqrt{\frac{6\pi}{7}}X_{AB}\sqrt{x}, \tag{3.1.16a}$$
$$\left|\tilde{h}_{32}^{(N,\epsilon)}\right| = \frac{8}{3}\sqrt{\frac{\pi}{7}}(1-3\nu)x, \qquad \left|\tilde{h}_{31}^{(N,\epsilon)}\right| = \frac{1}{3}\sqrt{\frac{2\pi}{35}}X_{AB}\sqrt{x},$$
$$\left|\tilde{h}_{44}^{(N,\epsilon)}\right| = \frac{64}{9}\sqrt{\frac{\pi}{7}}(1-3\nu)x, \qquad \left|\tilde{h}_{42}^{(N,\epsilon)}\right| = \frac{8}{63}\sqrt{\pi}(1-3\nu)x,$$
$$c_{22}^{(N)} = \pi, \qquad c_{21}^{(N)} = \frac{3\pi}{2}, \qquad c_{33}^{(N)} = \frac{\pi}{2}, \tag{3.1.16b}$$
$$c_{32}^{(N)} = \pi, \qquad c_{31}^{(N)} = \frac{3\pi}{2}, \qquad c_{44}^{(N)} = 0, \qquad c_{42}^{(N)} = \pi.$$

The two-body variable X_{AB} was defined in Eq. (2.2.2b) and we used the identity: $(X_{AB})^2 = 1 - 4\nu$.

This for the GR Newtonian contribution. The PN correction $\hat{h}_{lm}^{(\epsilon)}$ can be written (without the next-to-quasi-circular term, Ref. [35]) as [19]

$$\hat{h}_{lm}^{(\epsilon)} = \hat{S}_{eff}^{(\epsilon)} T_{lm} e^{i\delta_{lm}} \left(\rho_{lm}\right)^l, \tag{3.1.17}$$

where each factor is separately resummed.

The first term $\hat{S}_{eff}^{(\epsilon)}$ represents the effective source, parity-dependent. This choice is motivated by the test-mass limit ($\nu = 0$), in which $\hat{h}_{lm}^{(\epsilon)}$ is the spatial-asymptotic value of a Regge-Wheeler-Zerilli (RWZ) equation's solution (see Refs. [96–98]). According to the parity of the multipole this effective source contribution reads:

$$\hat{S}_{eff}^{(\epsilon)}(x) \equiv \begin{cases} \hat{E}_e(x) & \text{if} \quad \epsilon = 0 \\ \hat{j}(x) & \text{if} \quad \epsilon = 1 \end{cases} \tag{3.1.18}$$

Here $\hat{E}_e$ is the effective energy (see Eq. (2.2.40)) and $\hat{j}(x)$ is the EOB angular momentum, $j(x) \equiv p_\varphi(x)$, divided by the Newtonian term (this last one was defined in Eq. (2.2.65) as $j_N(x)$):

$$\hat{j}(x) \equiv \frac{j(x)}{j_N(x)}. \tag{3.1.19}$$

Their circular restrictions can be read in Eqs. (2.2.63),(2.2.64), both within GR and ST, as function of $u = 1/r$[3].

The T_{lm} contribution is the tail one. It is a resummation of an infinite sum of "leading logarithms", due to the back-scattering of the GWs against the Schwarzschild background of mass H_{EOB}.

This tail factor reads:

$$T_{lm} = \frac{\Gamma(l+1-2i\hat{\hat{k}}_m)}{\Gamma(l+1)} \exp(\pi\hat{\hat{k}}_m + 2i\hat{\hat{k}}_m \log(2k_m r_0)). \tag{3.1.20}$$

Here $r_0 = 2M$, $k_m = m\Omega$ and $\hat{\hat{k}}_m = H_{EOB} k_m$.

The residual phase δ_{lm} and the $\left(\rho_{lm}\right)^l = f_{lm}$ factors take into account the subleading contributions, in contrast with the tail ones, and they can be computed from the ratio of the PN-expansion $\hat{h}_{lm}^{(\epsilon)}$ and the source and tail terms defined above (see Ref. [19]).

[2] Here Φ is the orbital phase (φ in Eq. (2.2.57)) and we will discuss it below.

[3] The conversion between u and x (on circular orbits) is written in Sec. 2.3 up to the 3PN order in ST theories.

3.1.3 ST 2PN correction

In this section we will follow the work of N. Sennett, S. Marsat and A. Buonanno (see Ref. [18]), as we introduced above. We will take their 2PN ST information in order to implement them into the factorized structure of the `TEOBResumS` waveform. We will add a multiplicative factor, factorizing out the known GR part, both for the amplitude and the phase of the multipolar waveform. From now, we will denote the waveform's quantities of the previous Sec. 3.1.2 with the label "GR", in order to do not confuse them with the ST ones.

The scalar-tensor Newtonian order differs from the GR one only by a multiplicative constant. Following the notation introduced in Eqs. (3.1.12),(3.1.15) the ST $h_{lm}^{(N,\epsilon)}$ is equal to the GR one with $M \to GM(1-\zeta)$ or, in other words:

$$h_{lm}^{(N,\epsilon)} = h_{Newt}^{ST} h_{lm}^{(N,\epsilon),GR}, \tag{3.1.21}$$

where the ST universal factor h_{Newt}^{ST} is given by

$$h_{Newt}^{ST} \equiv G(1-\zeta), \tag{3.1.22}$$

$h_{lm}^{(N,\epsilon),GR}$ is the leading order contribution written in Eq. (3.1.12) while the ST parameters G and ζ can be read in Table 1.1. In the Einstein frame this numerical factor, once fixed the ST theory, simply reads as:

$$h_{Newt}^{ST} = A_0^2. \tag{3.1.23}$$

In our work we use the ST special theory called DEF gravity, in which the coupling function reads: $A(\varphi) = e^{\beta_0 \varphi^2/2}$ (see Sec. 1.5.1). The ST correction to the Newtonian waveform thus reads: $h_{Newt}^{ST} = A_0^2 = e^{\beta_0 \varphi_0^2} = e^{\alpha_0^2/\beta_0}$.

Event	$(1 - h_{Newt}^{ST})[10^{-9}]$
GWtest:H4	3.33333327606766
GW170817:SLy	"
GW170817:ENG	3.34883720753965

Table 3.1: ST correction to the Newtonian waveform (see Eqs. (3.1.21), (3.1.22)) for each GW event reported in Table 2.1.

In Table 3.1 we can make explicit see the correction to the Newtonian waveform (3.1.21), (3.1.22) for the 3 sets of parameters we have chosen in Table 2.1. These values are clearly the differences of ST with respect to GR , because the GR limit of h_{Newt}^{ST} is: $\left(h_{Newt}^{ST}\right)|_{GR} = 1$.

We will now focus on the corrections to the waveform amplitude and phase, separately.

Factorized Amplitude. By following the 2PN calculations in Eq. (67) of Ref. [18], we factorize the ST terms of the mode amplitudes in the form:

$$\begin{aligned} |h_{lm}| = \left|h_{lm}^{GR} + \delta h_{lm}^{ST}\right| &= \left|h_{lm}^{GR}\right| \left|1 + \frac{\delta h_{lm}^{ST}}{h_{lm}^{GR}}\right| \\ &\equiv \left|h_{lm}^{GR}\right| |\delta \mathcal{F}_{lm}|, \end{aligned} \tag{3.1.24}$$

where $\left|h_{lm}^{GR}\right|$ represents the full known (quasi-circular and non-spinning) waveform amplitude in general relativity (see Sec. 3.1.2) and

$$\delta\mathcal{F}_{lm} \equiv 1 + \frac{\delta h_{lm}^{ST}}{h_{lm}^{GR}} \tag{3.1.25}$$

was defined in order to have $|\delta\mathcal{F}_{lm}(x=0)| = 1$ and, clearly, $(\delta\mathcal{F}_{lm})\,|_{GR} = 1$.

These factorized ST quasi-circular corrections to the waveform amplitude, $|\delta\mathcal{F}_{lm}|$, up to the 2PN order, read[4]:

$$\begin{aligned}
|\delta\mathcal{F}_{22}| = 1 &+ x\left(-\frac{2\gamma}{3} - \frac{4\bar{\beta}}{3}\right) + x^{3/2}\left(2\pi\frac{1-\zeta-\alpha}{\alpha}\right) \\
&+ x^2\Bigg(5\gamma\Big(\frac{\gamma}{12} - \frac{22}{63}\Big) - \frac{4}{3}\left({\beta_+}^2 + {\beta_-}^2\right) + X_{AB}\frac{8\beta_+\beta_-}{3} - \frac{101\bar{\beta}}{63} \\
&+ \frac{\bar{\delta}}{3} - \frac{4\bar{\chi}}{3} + \frac{317}{63}\beta_+ + \nu\Big(+ \frac{16}{\gamma}\left({\beta_-}^2 - {\beta_+}^2\right) - \frac{4\delta_+}{3} + \frac{8\chi_+}{3} \\
&- \frac{\gamma}{3}\Big(\gamma + \frac{167}{21}\Big) + \beta_-\Big(\frac{16}{3}\beta_- + X_{AB}\frac{61}{63}\Big)\Big)\Bigg),
\end{aligned} \tag{3.1.26a}$$

$$|\delta\mathcal{F}_{21}| = 1 + x\left(\frac{\gamma}{2} - 2\bar{\beta}\right) + x^{3/2}\left(\pi\frac{1-\zeta-\alpha}{\alpha}\right), \tag{3.1.26b}$$

$$|\delta\mathcal{F}_{33}| = 1 + x\left(-\gamma - 2\bar{\beta}\right) + x^{3/2}\left(3\pi\frac{1-\zeta-\alpha}{\alpha}\right), \tag{3.1.26c}$$

$$|\delta\mathcal{F}_{32}| = 1 + x\left(-\frac{8\bar{\beta}}{3}\right), \tag{3.1.26d}$$

$$|\delta\mathcal{F}_{31}| = 1 + x\left(-\gamma - 2\bar{\beta}\right) + x^{3/2}\left(\pi\frac{1-\zeta-\alpha}{\alpha}\right), \tag{3.1.26e}$$

$$|\delta\mathcal{F}_{44}| = 1 + x\left(-\frac{4\gamma}{3} - \frac{8\bar{\beta}}{3}\right), \tag{3.1.26f}$$

$$|\delta\mathcal{F}_{42}| = 1 + x\left(-\frac{4\gamma}{3} - \frac{8\bar{\beta}}{3}\right). \tag{3.1.26g}$$

All the ST parameters were defined in Table 1.1 and the "bar" notation in Eq.(2.2.25).

As well as in the $\delta a_{4,\nu}$ term in Eq.(2.3.43b), here we note that the factor $16/\gamma(\cdots)$ does not presents any problem in the GR limit. In fact, following the ST definition in Table 1.1, we have:

$$\frac{16}{\gamma}\left({\beta_-}^2 - {\beta_+}^2\right) = \frac{2\alpha_A\alpha_B\beta_A\beta_B}{(1+\alpha_A\alpha_B)^3}. \tag{3.1.27}$$

Within this work, we will use the PN-expansion of the factorized ST amplitude correction (3.1.26) into the `TEOBResumS` model. We could try to use an (inverse) Padé resummation, $(0n)$, instead of the raw PN information that we have found above. However, if we compute the inverse resummation (02) of the (22) mode (3.1.26a) for the "GWtest:H4" event we can note the very limited difference between this resummation and the PN expression. The 2PN (22) multipole (3.1.26a) can be written in the generic form:

$$|\delta\mathcal{F}_{22}| \equiv |\delta\mathcal{F}_{22}|^{2PN} \equiv 1 + \delta\mathfrak{f}_1^{(22)}x + \delta\mathfrak{f}_{3/2}^{(22)}x^{3/2} + \delta\mathfrak{f}_2^{(22)}x^2. \tag{3.1.28}$$

[4]The multipolar modes that are not shown do not differ with respect to GR up to 2PN order.

These PN ST terms, within the "GWtest:H4" event (see Table 1.1), reads:

$$\delta\mathfrak{f}_1^{(22)} \simeq -4.7948 \cdot 10^{-4},$$
$$\delta\mathfrak{f}_{3/2}^{(22)} \simeq -4.1253 \cdot 10^{-5}, \tag{3.1.29}$$
$$\delta\mathfrak{f}_2^{(22)} \simeq -3.4318 \cdot 10^{-3}.$$

We have chosen this GW event in order to obtain the larger values as possible. The inverse Padé of the general expression (3.1.28) can be written as

$$P_2^0\Big[|\delta\mathcal{F}_{22}|\Big] = \Big[1 - \delta\mathfrak{f}_1^{(22)}x - \delta\mathfrak{f}_{3/2}^{(22)}x^{3/2} + \Big(\Big(\delta\mathfrak{f}_1^{(22)}\Big)^2 - \delta\mathfrak{f}_2^{(22)}\Big)x^2\Big]^{-1}. \tag{3.1.30}$$

$1 - \lvert\delta\mathcal{F}_{22}\rvert^{1PN}$	$1.02369 \cdot 10^{-4}$
$1 - \lvert\delta\mathcal{F}_{22}\rvert^{1.5PN}$	$1.06439 \cdot 10^{-4}$
$1 - \lvert\delta\mathcal{F}_{22}\rvert^{2PN}$	$2.62868 \cdot 10^{-4}$
$\lvert\delta\mathcal{F}_{22}\rvert^{2PN} - P_2^0\Big[\lvert\delta\mathcal{F}_{22}\rvert\Big]$	$-5.86073 \cdot 10^{-8}$
$1 - P_2^0\Big[\lvert\delta\mathcal{F}_{22}\rvert\Big]/\lvert\delta\mathcal{F}_{22}\rvert^{2PN}$	$-5.86227 \cdot 10^{-8}$

Table 3.2: Differences between the raw PN and inverse Padé of the (22) ST factorized multipole (3.1.26a), within the "GWtest:H4" event (see Table 1.1), evaluated at the LSO: $x_{LSO} \simeq 0.21350$ (see Sec. 2.4.1 and Eq. (2.2.61)). The first two values denote the 1PN and 1.5PN truncation, i.e. respectively given by setting $\delta\mathfrak{f}_1^{(22)} = 0$ and $\delta\mathfrak{f}_{3/2}^{(22)} = 0$.

In Table 3.2 we inserted the differences between the PN information and the inverse Padé one evaluated at the LSO: $x_{LSO} \simeq 0.21350$ (see Sec. 2.4.1 and Eq. (2.2.61)). As we introduced above, we can note that, even at the LSO, the differences could be treated as negligible.

Factorized Phase. By focusing on the phase of $\delta\mathcal{F}_{lm}$, we could note that, once scaled out a common term, in Eq. (67) of Ref. [18] the imaginary part enters only in the 1.5PN factor (i.e. $x^{3/2}(\cdots)$) while the real one does not have the 0.5PN contribution. This allows us to conclude that, up to the 2PN order, the phase factors correspond to the imaginary ones, which are at last at 1.5PN order.

From the definition of the factorized waveform (3.1.25), we can express it as:

$$\delta\mathcal{F}_{lm} \equiv |\delta\mathcal{F}_{lm}|\exp\Big[i\mathcal{C}_{lm} + i\pi\mathcal{H}\big(-\mathcal{R}_{lm}\big)\Big(1 - 2\mathcal{H}\big(-\mathcal{I}_{lm}\big)\Big)\Big]. \tag{3.1.31}$$

Here $\mathcal{H}$ is the Heaviside step function[5], which is associated to the negativity of the waveform's real part while we define the phase $\mathcal{C}_{lm}$, and we defined $\mathcal{R}_{lm} \equiv Re(\delta\mathcal{F}_{lm})$, $\mathcal{I}_{lm} \equiv Im(\delta\mathcal{F}_{lm})$. In fact, it is defined as:

$$\mathcal{C}_{lm} \equiv \arctan\left(\frac{\mathcal{I}_{lm}}{\mathcal{R}_{lm}}\right). \tag{3.1.32}$$

We have chosen this Heaviside-given convention, in Eq. (3.1.31), in order to conform to the `Mathematica`'s one, in which the total phase of a complex number goes

[5]Here we use the convention: $\mathcal{H}(0) = 0$.

from $-\pi$ to $+\pi$ and it impose the 2π-discontinuity on the $\{\mathcal{R}_{lm} < 0, \mathcal{I}_{lm} = 0\}$ branch. The formal singularity of Eq. (3.1.32) for $\mathcal{R}_{lm} \to 0$ is clearly treated as[6]:

$$\mathcal{C}_{lm}|_{\mathcal{R}_{lm}=0} \equiv \lim_{y\to+\infty} \arctan\left(\mathcal{I}_{lm} y\right) = \text{sign}\big(\mathcal{I}_{lm}\big)\frac{\pi}{2}. \tag{3.1.33}$$

Once given these definitions, however, we can observe that all multipoles in Eq. (67) of Ref. [18] are written in the form (excluding the Newtonian factor): $1 + \mathcal{O}(x)$. Therefore, unless we push the x variable up to unrealistically high values, The real part of the factorized ST waveform will be always near to 1, and then positive. This fact allows us to ignore the Heaviside terms in Eq. (3.1.31).

We can then write the 2PN expressions of $\mathcal{C}_{lm}$ by truncating the only Eq. (3.1.32):

$$\tilde{\mathcal{C}}_{lm} \equiv \frac{\mathcal{C}_{lm}}{x^{3/2}}, \tag{3.1.34}$$

$$\tilde{\mathcal{C}}_{22} = \frac{\zeta}{3}\left(S_+^2 - S_-^2\right) - \frac{3}{2}\zeta\nu S_-^2, \tag{3.1.35a}$$

$$\tilde{\mathcal{C}}_{21} = \frac{1}{2\alpha}\left(\zeta + \alpha - 1\right)\big(1 + 4\log(2)\big) + \nu\Big(-\frac{4}{3}\zeta S_-^2 - \frac{4\zeta S_- S_+}{3X_{AB}}\Big), \tag{3.1.35b}$$

$$\tilde{\mathcal{C}}_{33} = \frac{3}{5\alpha}\left(\zeta + \alpha - 1\right)\left(7 - 10\log(\frac{3}{2})\right) + \frac{3\zeta}{10}\left(S_+^2 - S_-^2\right) + \nu\Big(-\frac{8}{9}\zeta S_-^2 - \frac{8\zeta S_- S_+}{9X_{AB}}\Big), \tag{3.1.35c}$$

$$\tilde{\mathcal{C}}_{32} = 0, \tag{3.1.35d}$$

$$\tilde{\mathcal{C}}_{31} = \frac{1}{5\alpha}\left(\zeta + \alpha - 1\right)\big(7 + 10\log(2)\big) + \frac{\zeta}{10}\left(S_+^2 - S_-^2\right) + \nu\Big(-\frac{40}{3}\zeta S_-^2 + \frac{8\zeta S_- S_+}{3X_{AB}}\Big), \tag{3.1.35e}$$

$$\tilde{\mathcal{C}}_{44} = 0, \tag{3.1.35f}$$

$$\tilde{\mathcal{C}}_{42} = 0. \tag{3.1.35g}$$

All the ST parameters, including $S_\pm$, can be read in Table 1.1.

These $\mathcal{C}_{lm}$ phase factors, and consequently the $\tilde{\mathcal{C}}_{lm}$ ones, are the factorized phases due to the definition of $\delta\mathcal{F}_{lm}$ in Eq. (3.1.25). They were found by subtracting the full ST versions with the GR ones, following, as usual in this section, the notation used in Ref. [18]:

$$\mathcal{C}_{lm} \equiv \mathcal{C}_{lm}^{FULL,ST} - \mathcal{C}_{lm}^{GR}. \tag{3.1.36}$$

These (factorized) phase difference affects the m-odd multipoles simply as $(\zeta - 1) \to (\zeta + \alpha - 1)$. Clearly, by definition, we have $\left(\mathcal{C}_{lm}\right)|_{GR} = 0$. We also recognize the bodies-dependent factor $(\zeta + \alpha - 1)/\alpha$ which enters at 1.5PN order in Eqs. (3.1.26),(3.1.35). Within the Einstein frame, this term reads (see Table 1.1):

$$\frac{\zeta + \alpha - 1}{\alpha} = \frac{\alpha_A\alpha_B}{1 + \alpha_A\alpha_B} = -\frac{\gamma}{2}. \tag{3.1.37}$$

[6]The sign function is defined as: $\text{sign}(f) \equiv \mathcal{H}(f) - \mathcal{H}(-f)$.

Following the same JF-EF conversion of Eq. (1.5.9), we can observe that also the parameter $\zeta(S_+^2 - S_-^2)$ reads, in Einstein frame, as the one in Eq. (3.1.37).

We can also note that the m-odd factorized phasing terms contain a formal singularity in the equal-masses case. In fact, from its definition in Eq. (2.2.2b), X_{AB} goes to zero when $m_A = m_B$. This singularity is not physical because, once fixed an EOS, if the two bodies have the same mass they have also the same scalar charge $\alpha_A = \alpha_B$ (see Ref. 1.6). This, from the ST parameters Table 1.1, implies that also the numerator

$$\zeta S_- S_+ = \frac{1}{1+\alpha_A\alpha_B}\frac{\alpha_A^2 - \alpha_B^2}{4} \tag{3.1.38}$$

goes to zero. Then, before splitting the ST correction in Eq. (67) of Ref. [18] into amplitude and phase through Eq. (3.1.31), this factor disappears in the equal-masses case due to the X_{AB} factor in the Newtonian waveform (see Eqs. (3.1.10)-(3.1.16), (3.1.21)). Therefore, it is not a real singularity when $m_A = m_B$.

3.2 2PN correction to the orbital phase

Let us now focus on the orbital phase, which in this chapter we identify it as Φ from Eq. (3.1.15). In `TEOBResumS` model the orbital phase is found by solving numerically the first Hamilton's equation (see Eqs. (2.2.57),(2.2.61)). Here we want to make explicit the ST corrections, up to the 2PN order, with respect the GR solution, also by plotting them as function of x. The results we will use are given in Refs. [18, 90, 91].

In order to make explicit these analytic PN differences we have to distinguish two cases: the dipole-driven (DD) and the quadrupole-driven (QD) regime.

The former one can be reached by spontaneously scalarized binary systems (see Sec. 1.7), where the scalar charges effects could be not negligible. This can be implemented by factorizing the LO dipolar emitted energy flux, written up to the 2PN relative order as:

$$\mathcal{F}^{2PN} = \mathcal{F}_{-1} + \mathcal{F}_0 + \mathcal{F}_{0.5} + \mathcal{F}_1, \tag{3.2.1}$$

[7]where the "-1PN" is conventionally the dipolar contribution, which does not exists in GR. The leading-order dipolar emission is given by [18, 91]:

$$\mathcal{F}_{LO}^{DD}(x) = \frac{4}{3}\frac{S_-^2\zeta}{\tilde{G}_{AB}}\nu x^4. \tag{3.2.2}$$

As mentioned above, the dipolar emission is driven by the scalar-tensor factor $S_-^2 \propto (\alpha_A - \alpha_B)^2$ (see Table 1.1), where α_A and α_B are the scalar charges of the bodies. The whole bodies-dependent ST factor, in Einstein frame, reads:

$$\frac{4}{3}\frac{S_-^2\zeta}{\tilde{G}_{AB}} = \frac{1}{3A_0^2}\frac{(\alpha_A-\alpha_B)^2}{(1+\alpha_A\alpha_B)^2}, \tag{3.2.3}$$

which is zero if and only if $\alpha_1 = \alpha_2$. The (relative) 2PN expression of $\mathcal{F}^{DD}(x)$ can be read in Eq. (48) of Ref. [18]. The dimensionless orbital phase, Φ, can be also obtained, as function of $x = \Omega^{2/3}$ (see Eq. (2.3.14)), from the energy-balance equation

$$\frac{dE}{dt} = -\mathcal{F} \tag{3.2.4}$$

[7]Here $\mathcal{F}$ is not to be confused with the factorized waveform in Eq. (3.1.25).

and by using (2.2.57)

$$\frac{d\Phi}{dt} = \Omega \tag{3.2.5}$$

to get:

$$\frac{d\Phi}{dx} = -x^{3/2}\frac{dE/dx}{\mathcal{F}(x)}, \tag{3.2.6}$$

where, in this section, the EOB energy $E(x)$ is taken as the already known 2PN version of Eq. (2.3.23), i.e. truncated up to the x^2 term.

Note that, in the DD regime, Φ is the non-factorized orbital phase because we can not collect the GR one, which does not exist. In the QD regime, contrariwise, we will can do it because we have a general relativistic one. Therefore, we allow ourself this abuse of the definition in Eq. (3.1.31) for Φ. Anyhow, we always factorize the amplitude $|\delta\mathcal{F}_{lm}|$ in both cases.

In the DD regime, the non-factorized orbital phase Φ up to the 2PN relative order reads:

$$\frac{\Phi^{DD}(x)}{\Phi^{DD}_{LO}(x)} = 1 + 3x\rho_2^{DD} - \frac{3}{2}x^{3/2}\log x\rho_3^{DD} - 3x^2\rho_4^{DD}, \tag{3.2.7}$$

where the LO term is given by

$$\Phi^{DD}_{LO}(x) = -\frac{1}{4S_-^2\zeta\nu x^{3/2}}, \tag{3.2.8}$$

and the PN factors ρ_n^{DD} are written in Appendix B of Ref. [18].

The dimensionless variable ρ is defined as

$$\rho(x) \equiv -\frac{dE/dx}{\mathcal{F}(x)} \equiv x^{-3/2}\frac{d\Phi}{dx}. \tag{3.2.9}$$

In the QD regime, instead, the dominant term in Eq. (3.2.1) is the quadrupolar $\mathcal{F}_0$, i.e. the (ST-corrected) Newtonian one. We remember that the GR limit is $(\mathcal{F}_0)|_{GR} = 32\nu^2x^5/5$ [7]. The factorization is made by splitting the flux into scalar dipole-dependent and independent terms: $\mathcal{F}^{QD} \equiv \mathcal{F}^{dip} + \mathcal{F}^{non-dip}$ and by expanding the phasing (3.2.6) up to the first order in $\mathcal{F}^{dip}/\mathcal{F}^{non-dip}$, which is a perturbative parameter within this QD regime. This gives the ST correction to the non-dipolar LO flux:

$$\mathcal{F}^{non-dip}_{LO}(x) = \frac{32\xi\nu^2x^5}{5\tilde{G}_{AB}}, \tag{3.2.10}$$

while the LO dipolar contribution (which corresponds to the DD one) reads:

$$\mathcal{F}^{dip}_{LO}(x) = \frac{4S_-^2\zeta\nu^2x^4}{3\tilde{G}_{AB}}. \tag{3.2.11}$$

The ST parameters S_-, ξ, ζ and $\tilde{G}_{AB}$ can be read in Table 1.1 and Eq. (1.5.10).

As well as for the quadrupolar LO term (3.2.10), we can observe the vector spin factor in Eq. (3.2.11), which corresponds to the $l = 1$ contribution.

The PN corrections to Eqs. (3.2.10),(3.2.11) are also given in Appendix B of Ref. [18], paying attention to the fact that, following the flux PN decomposition in Eq. (3.2.1), the "-1PN" term does not exist within the QD regime. Therefore, $\mathcal{F}^{dip}$ is known up to 2PN relative order but $\mathcal{F}^{non-dip}$ only up to 1PN, its 1.5PN and 2PN terms are free for now. We must have a 3PN relative order flux in order to complete the 2PN information.

Keeping this in mind, the (factorized) 2PN phasing Φ within the QD regime reads:

$$\Phi^{QD}(x) = \Phi^{non-dip}(x) + \Phi^{dip}(x) - \Phi^{non-dip}_{GR}(x), \tag{3.2.12}$$

where

$$\frac{\Phi^{non-dip}(x)}{\Phi^{non-dip}_{LO}(x)} = 1 + \frac{5}{3}\rho_2^{nd}x + \frac{5}{2}\rho_3^{nd}x^{3/2} + 5\rho_4^{nd}x^2, \tag{3.2.13a}$$

$$\frac{\Phi^{dip}(x)}{\Phi^{dip}_{LO}(x)} = 1 + \frac{7}{5}\rho_2^{d}x + \frac{7}{4}\rho_3^{d}x^{3/2} + \frac{7}{3}\rho_4^{d}x^2 \tag{3.2.13b}$$

and

$$\Phi^{non-dip}_{LO}(x) = -\frac{1}{32x^{5/2}\nu\xi}, \qquad \Phi^{dip}_{LO}(x) = \frac{25S_-^2\zeta}{5376x^{7/2}\nu\xi^2}. \tag{3.2.14}$$

All these $\rho_n^{nd,d}$ parameters are written in Appendix B of Ref. [18], remembering that ρ_3^{nd} and ρ_4^{nd} are not completely known.

Thus, the full-known 2PN GR term $\Phi^{non-dip}_{GR}(x)$ can be simply written as:

$$\begin{aligned}\left(-\frac{1}{32x^{5/2}\nu}\right)^{-1}\Phi^{non-dip}_{GR}(x) = 1 &+ \frac{5}{3}x\left(\frac{743}{336} + \frac{11}{4}\nu\right) + \frac{5}{2}x^{3/2}\Big(-4\pi\Big)\\ &+ 5x^2\left(\frac{3058673}{1016064} + \frac{5429}{1008}\nu + \frac{617}{144}\nu^2\right).\end{aligned} \tag{3.2.15}$$

3.3 Results: scalar-tensor waveform

In this final section, before the "conclusions" one, we will gather all the scalar-tensor corrections by plotting the waveform's multipoles. These are given up to the 2PN relative order in the GR-factorized, as we explained in Sec. 3.1.3. This new improvement of the `TEOBResumS` was further updated by the 3PN terms in the A effective potential. This new PN order does not enter yet at the waveform.

We will plot all the multipoles corrected by the scalar-tensor theories up to the 2PN order. In particular we will focus on the modes: (22), (21), (33), (32), (31), (44), (42), and also on the complete scalar-tensor waveform (3.1.6). Their GR-factorized corrections were written, in terms of the results of N. Sennett in Ref. [18], in Sec. 3.1.3 by splitting them into amplitude and phase. We will show all the differences with respect to GR, in particular the phase ones, in order to get the variation of the number of cycles that our BNS systems realize up to the merger.

Therefore, we will analyze this ST-corrected waveform by starting from the "GWtest:H4" event, in order to show the greater differences with respect to GR as possible. Subsequently we will show the "GW170817:Eng" and "GW170817:SLy" events, which are realistically based on the EOS constraints. We gathered all the dynamics and ST parameters of these 3 GW events in Table 2.1 and we discuss their EOS in Secs. 1.6 - 1.7.

By plotting all the multipoles we will adopt the Regge-Wheeler-Zerilli [98] normalization

$$\Psi_{lm} = \frac{h_{lm}}{\sqrt{(l+2)(l+1)l(l-1)}} \tag{3.3.1}$$

to the waveform that we discussed in Sec. 3.1.

All the waveform's plots are realized by interpolating the raw data in order to have the opportunity of see clearly the oscillations. For low m-index we will focus also on the merger region while, for every modes we are considering, we will show the amplitude and phase for both the state-of-the-art-GR and the state-of-the-art-ST correction. Within this section we will identify this split as

$$\Psi_{lm} = |\Psi|_{lm} e^{i\phi_{lm}}. \tag{3.3.2}$$

The initial condition for each GW event run are given in geometric units, as well as the results. We can obtain the physical evolution time, in seconds, by using the solar mass expressed in seconds: $M_\odot[s] \equiv G_\star M_\odot[Kg]/c^3 \simeq 4.925491025543576 \cdot 10^{-6} s$.

The ST time conversion to the physical one, as well as for the initial frequency, is given by:

$$t \longrightarrow T[s] = t \cdot T_{fact}, \qquad T_{fact} \equiv \frac{1}{\tilde{G}_{AB} M M_\odot[s]}, \tag{3.3.3}$$

where M is the total mass and extra ST factor $\tilde{G}_{AB}$ was defined in Eq. (1.5.10).

Once setted the initial frequency, f_0, for each GW event, the initial radius r_0 in `TEOBResumS` is simply computed from the Kepler's law, i.e.

$$r_0 = (\pi f_0)^{-2/3}. \tag{3.3.4}$$

In ST theories, the geometric units conversion for the initial frequency f_0 follows the one written in Eq. (3.3.3), with the same $\tilde{G}_{AB}$ correction with respect to GR.

3.3.1 GWtest:H4

Here we will gather all the waveform's plots associated with the event "GWtest:H4", its parameters can be read in Table 2.1 and we started this `TEOBResumS` simulation with a initial frequency, in geometric units, of $f_0 = 7$Hz (see Eqs. (3.3.3), (3.3.4)).

The main numerical results are collected in Figs. 3.3 - 3.5, especially the difference of merger time and of the number of evolution cycles. The latest, for this "GWtest:H4" event, reads

$$\Delta\mathcal{N}_{rel} \equiv \frac{\mathcal{N}_{GR} - \mathcal{N}_{ST}}{\mathcal{N}_{GR}} \simeq -8.3347772171 \cdot 10^{-6}. \tag{3.3.5}$$

In fact, from Figs. 3.1 - 3.8, we can see that the ST correction extends the inspiral up to merger, which is postponed by the quantity written in Table 3.3.

As well as for the others events, we will comment all results in the final chapter, Chap. 4.

−	Merger time
$GR[10^8]$	1.0194439744
$ST[10^8]$	1.0194524625
$(GR - ST)[10^2]$	−8.4880693901
$(GR - ST)/GR[10^{-6}]$	−8.3261754480

Table 3.3: Difference of the merger time, in "GWtest:H4" event, between GR and ST with also the relative one.

In Table 3.3 we can see the difference of the merger time for the "GWtest:H4" run that we made in `TEOBResumS` in geometric units.

From Eq. (3.3.3) the physical merger time difference between GR and ST reads: $\Delta T^{merger} \simeq -26.46499811ms$, on a evolution, respectively, of $T^{merger}_{GR} \simeq 1777.526800s$ and $T^{merger}_{ST} \simeq 1777.553265s$.

$\max(\|\Psi\|_{lm})/\nu$				
(lm)	$GR[10^{-4}]$	$ST[10^{-4}]$	$(GR - ST)[10^{-8}]$	$(GR - ST)/GR[10^{-4}]$
(22)	8610.8467355	8608.1252347	27215.008319	3.1605496132
(21)	60.822595587	60.672121825	1504.7376199	24.739779770
(33)	49.315198457	49.303029841	121.68616249	2.4675184588
(32)	64.880328741	64.719117884	1612.1085711	24.847416811
(31)	1.1406997518	1.1411860159	−4.8626416253	−4.2628584934
(44)	68.413868914	68.347485448	663.83465687	9.7032175991
(42)	2.4220832476	2.4231070200	−10.237723701	−4.2268256929
$\max(\|h\|)$				
−	6782.3395992	6779.9319814	2407.6177770	3.5498337141

Table 3.4: Peak value of each amplitude multipole, for the "GWtest:H4" event, with its absolute and relative difference.

$\Delta\phi_{lm}/2\pi$				
(lm)	$GR[10^3]$	$ST[10^3]$	$(GR - ST)[10^{-2}]$	$(GR - ST)/GR[10^{-6}]$
(22)	19.935447888	19.935537594	−8.9705198945	−4.4997834735
(21)	9.9677442645	9.9677890756	−4.4811101157	−4.4956110398
(33)	29.903189885	29.903324407	−13.452163116	−4.4985712786
(32)	19.935426316	19.935516079	−8.9763144115	−4.5026949860
(31)	9.9677516402	9.9677964364	−4.4796261531	−4.4941189496
(44)	39.870854035	39.871033568	−17.953323649	−4.5028690967
(42)	19.935440400	19.935530132	−8.9731445853	−4.5011017591
$\mathcal{N}$				
−	8867.6985543	8867.7724646	−7391.0291879	−8.3347772171

Table 3.5: Multipolar phase difference between the merger point and the initial one, divided by 2π, for the "GWtest:H4" event, with its absolute and relative difference between GR and ST.

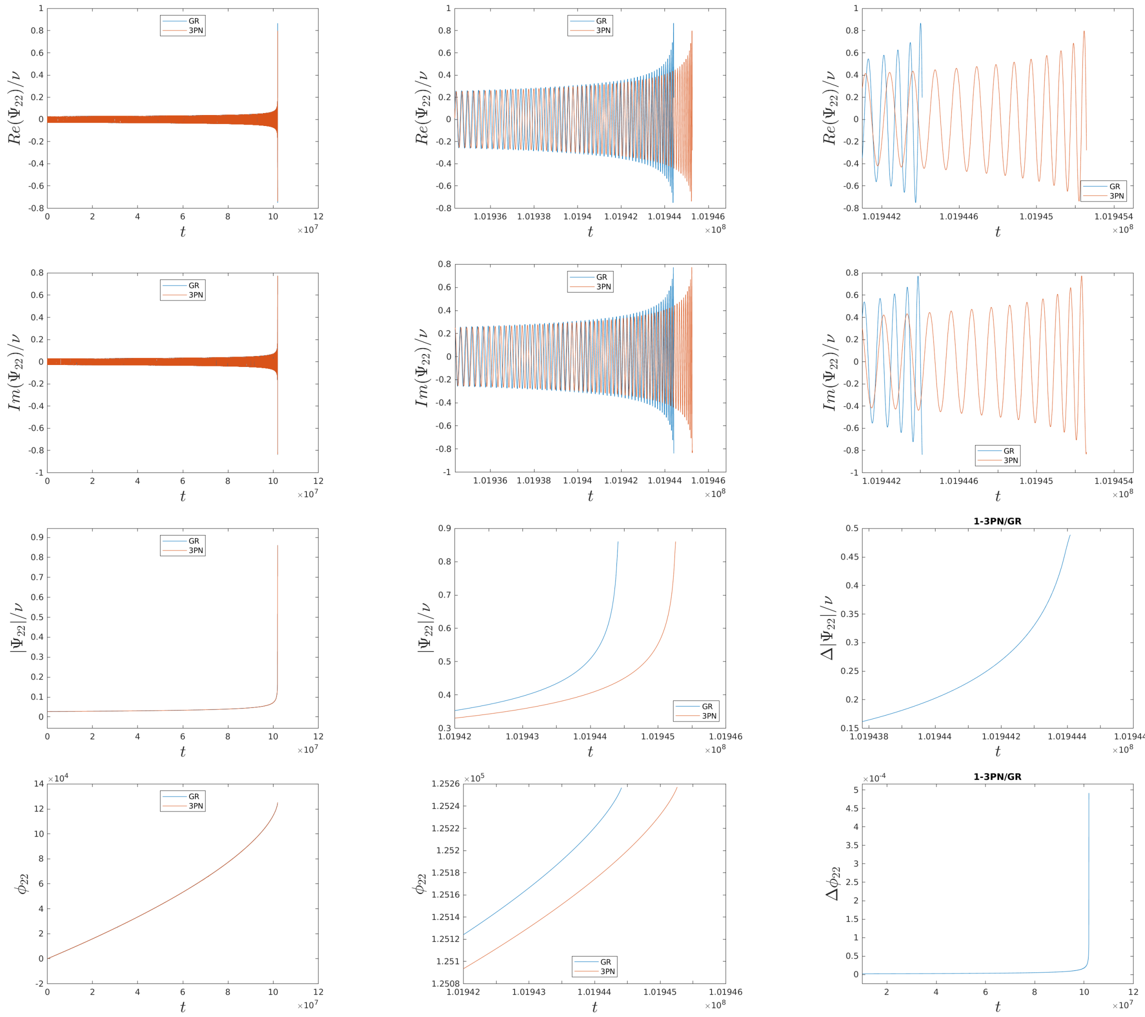

Figure 3.1: Multipole (22) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{22}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

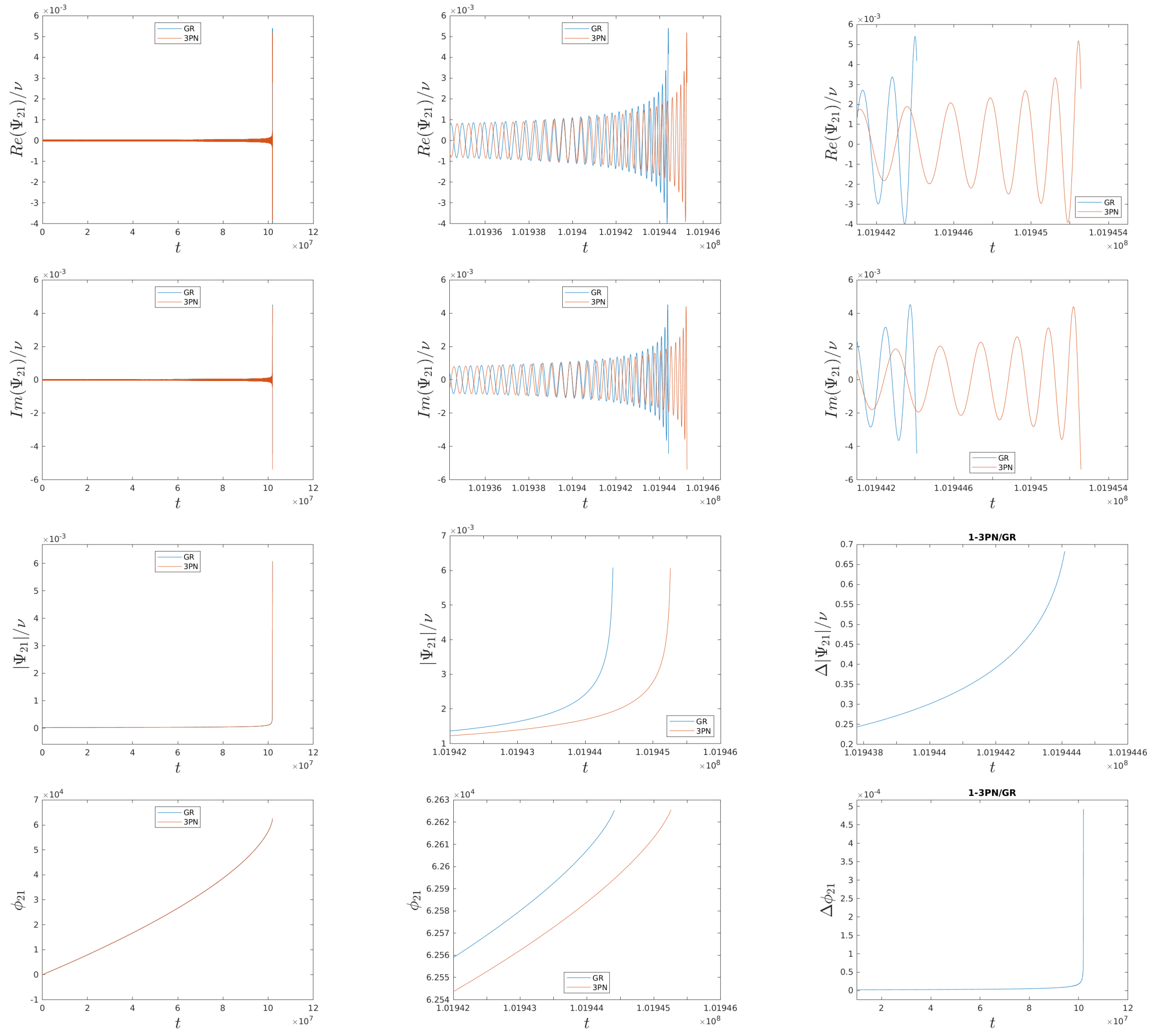

Figure 3.2: Multipole (21) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{21}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

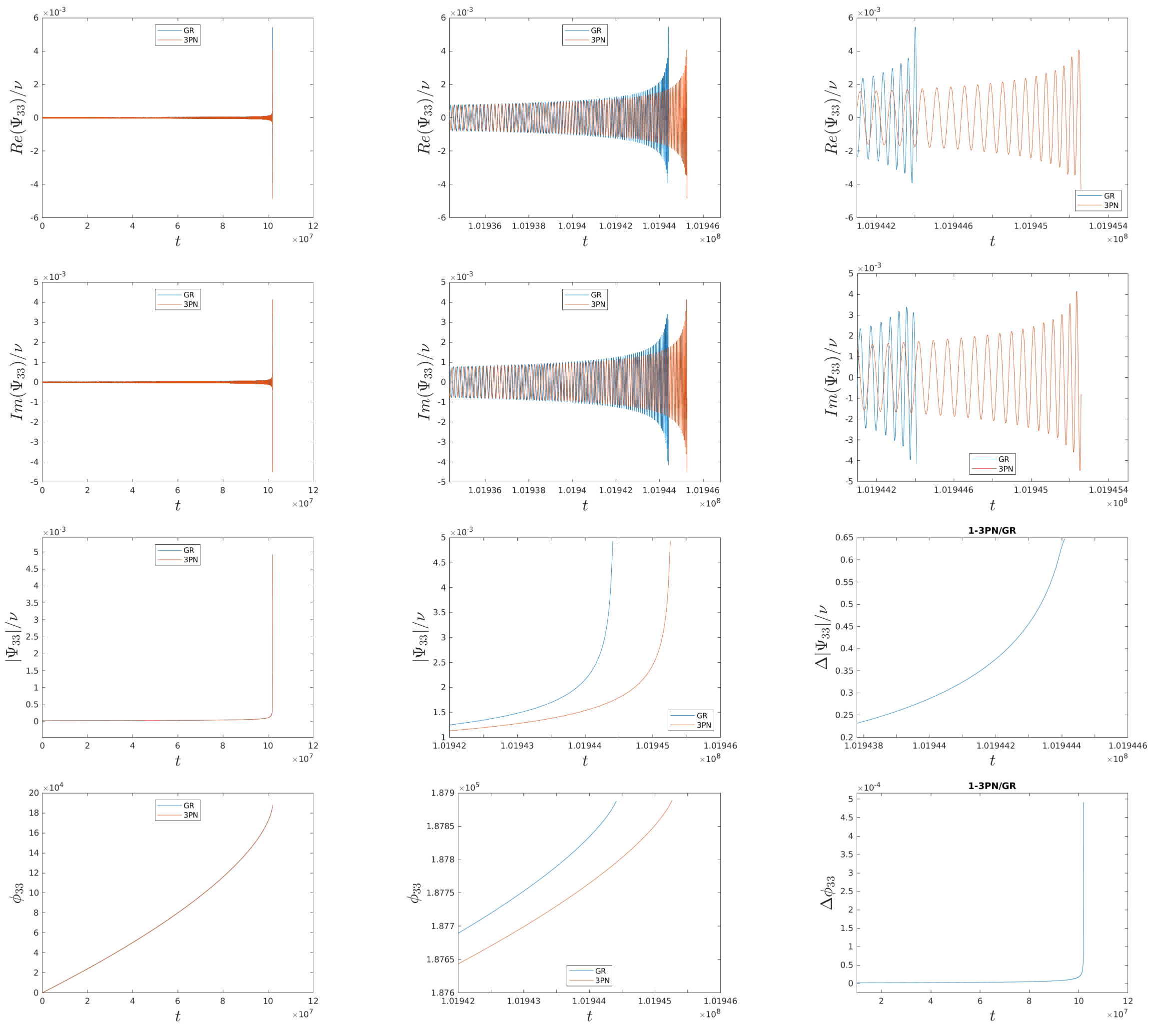

Figure 3.3: Multipole (33) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{33}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

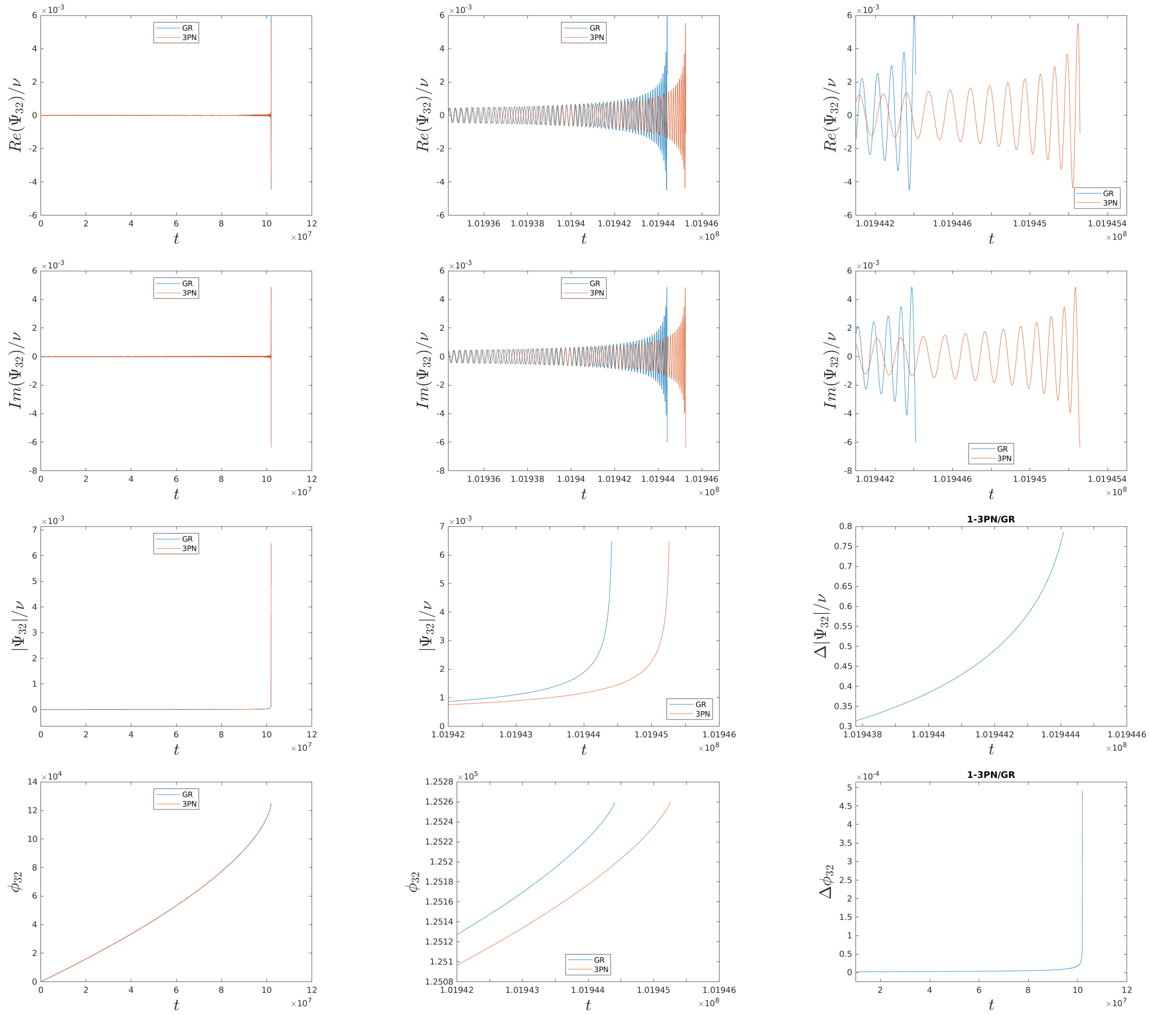

Figure 3.4: Multipole (32) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{32}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

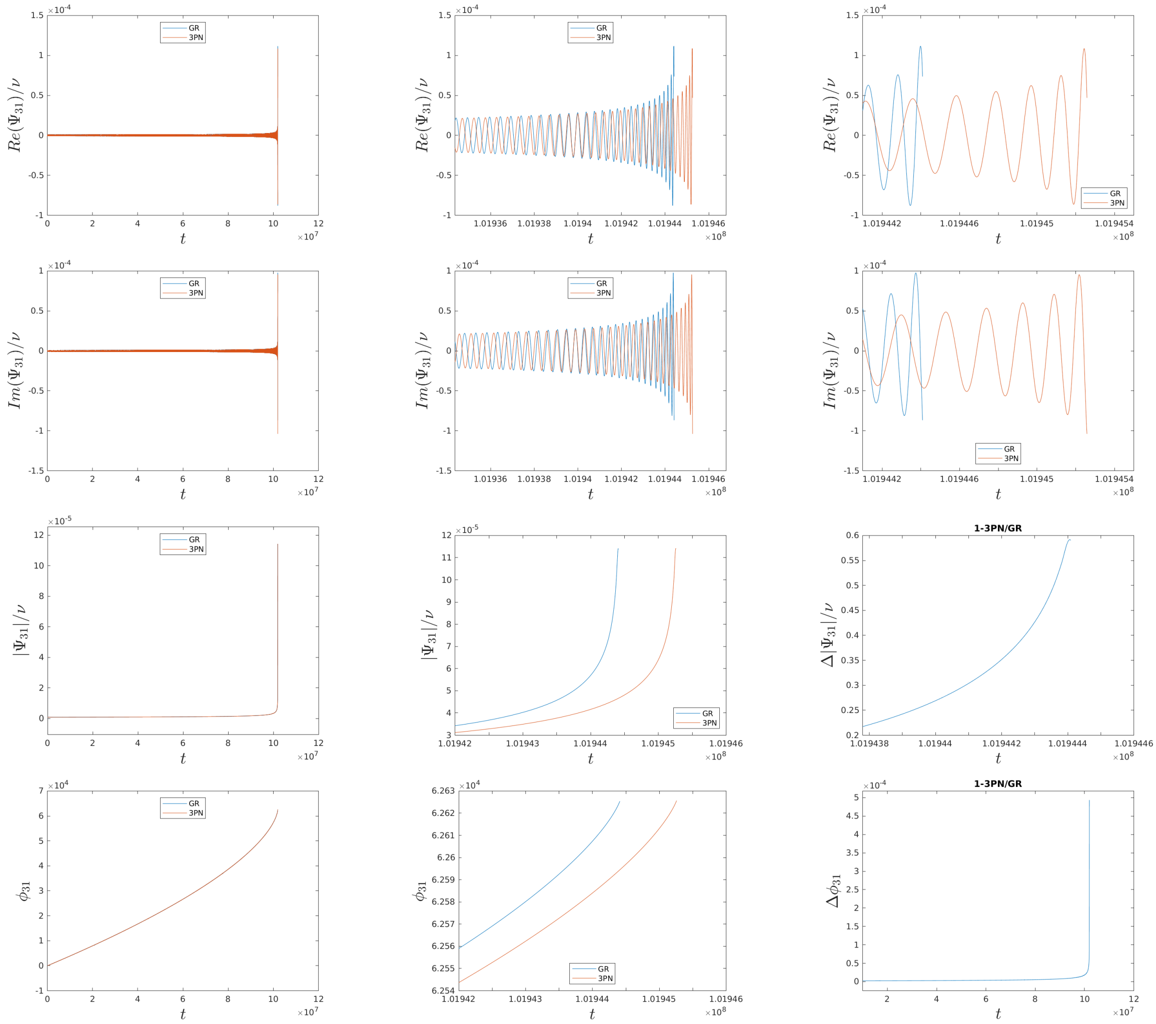

Figure 3.5: Multipole (31) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{31}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

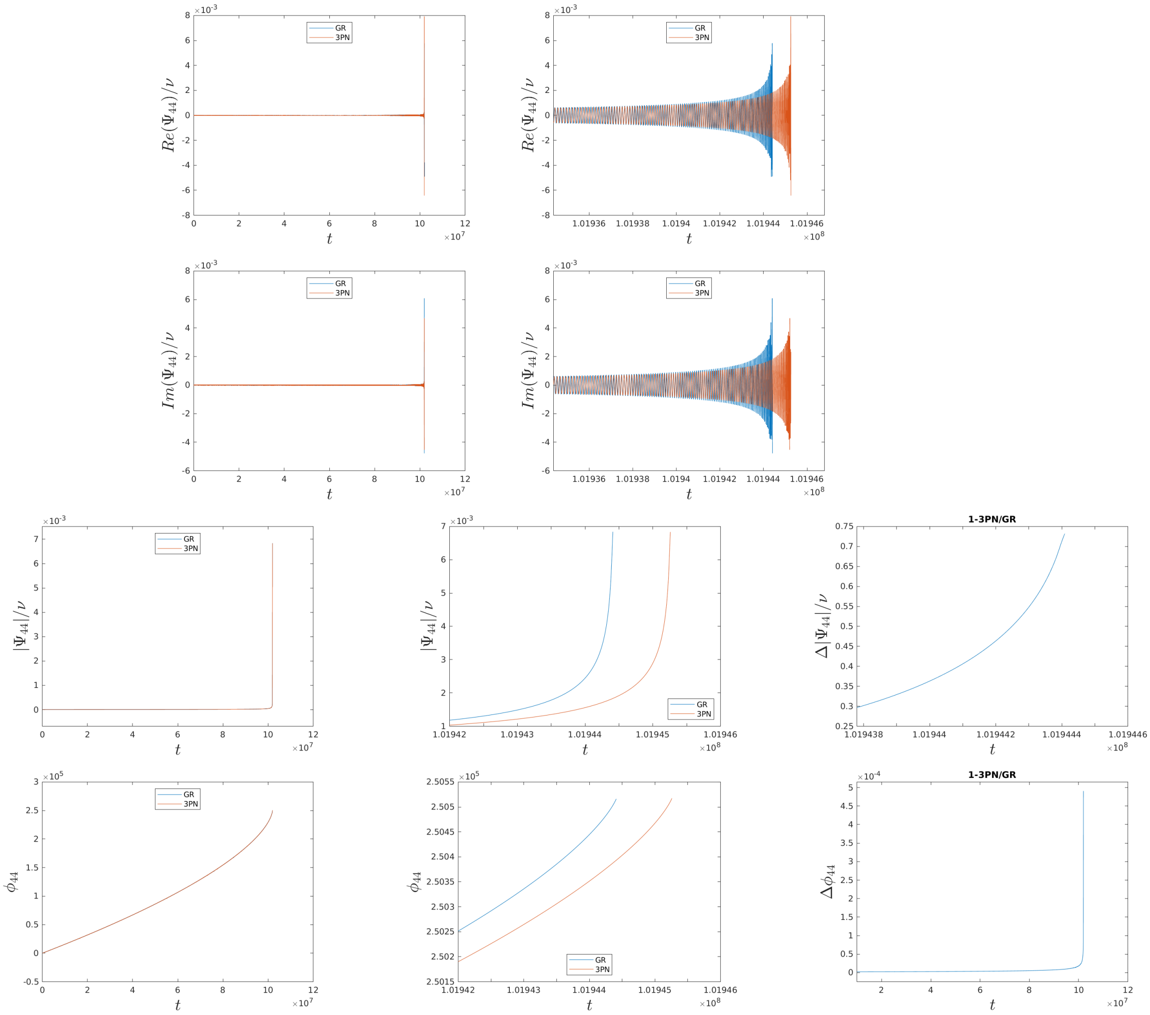

Figure 3.6: Multipole (44) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{44}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

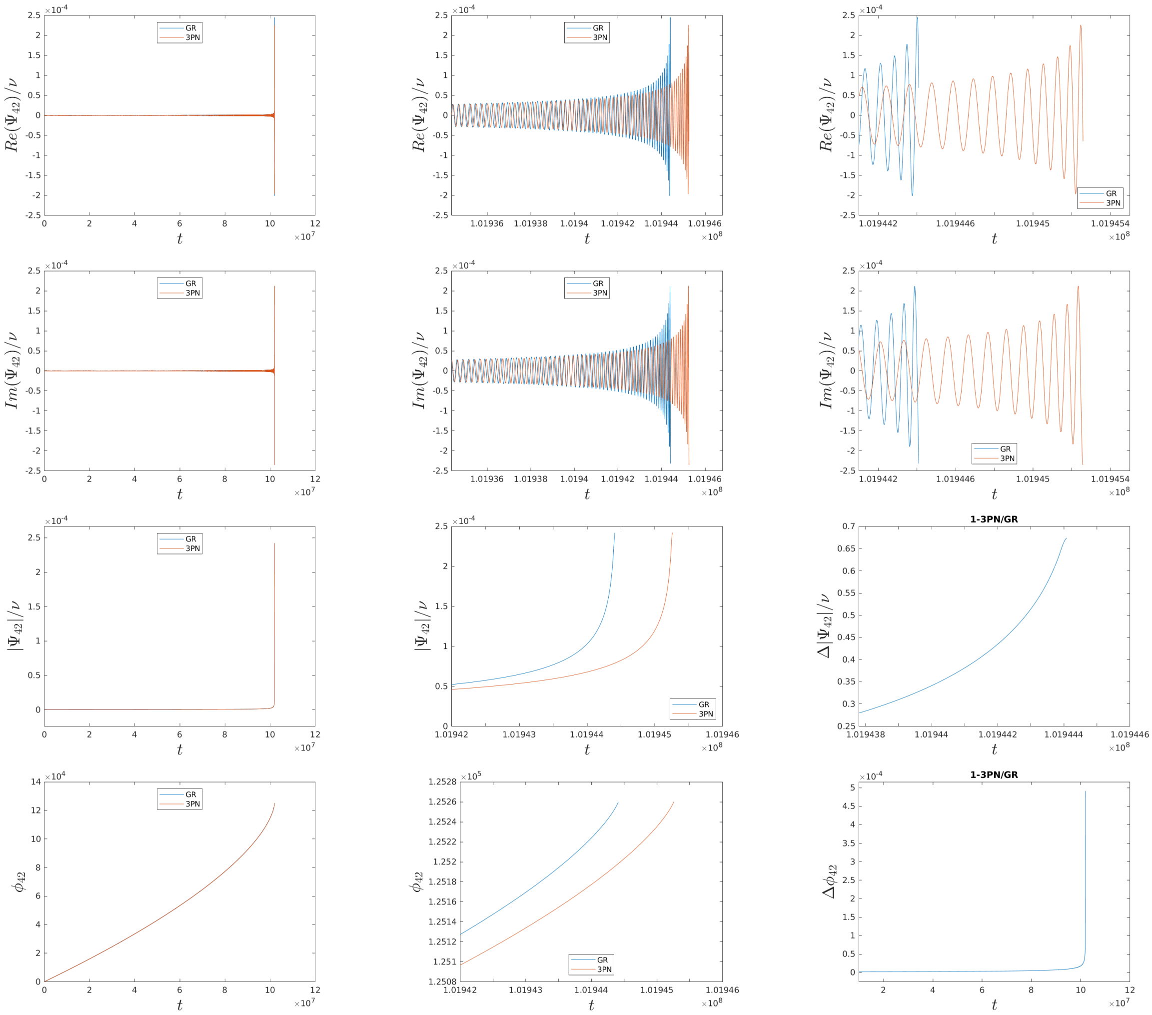

Figure 3.7: Multipole (42) of "GWTEST:H4" event. In the top panels we plot the real and imaginary part of Ψ_{42}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

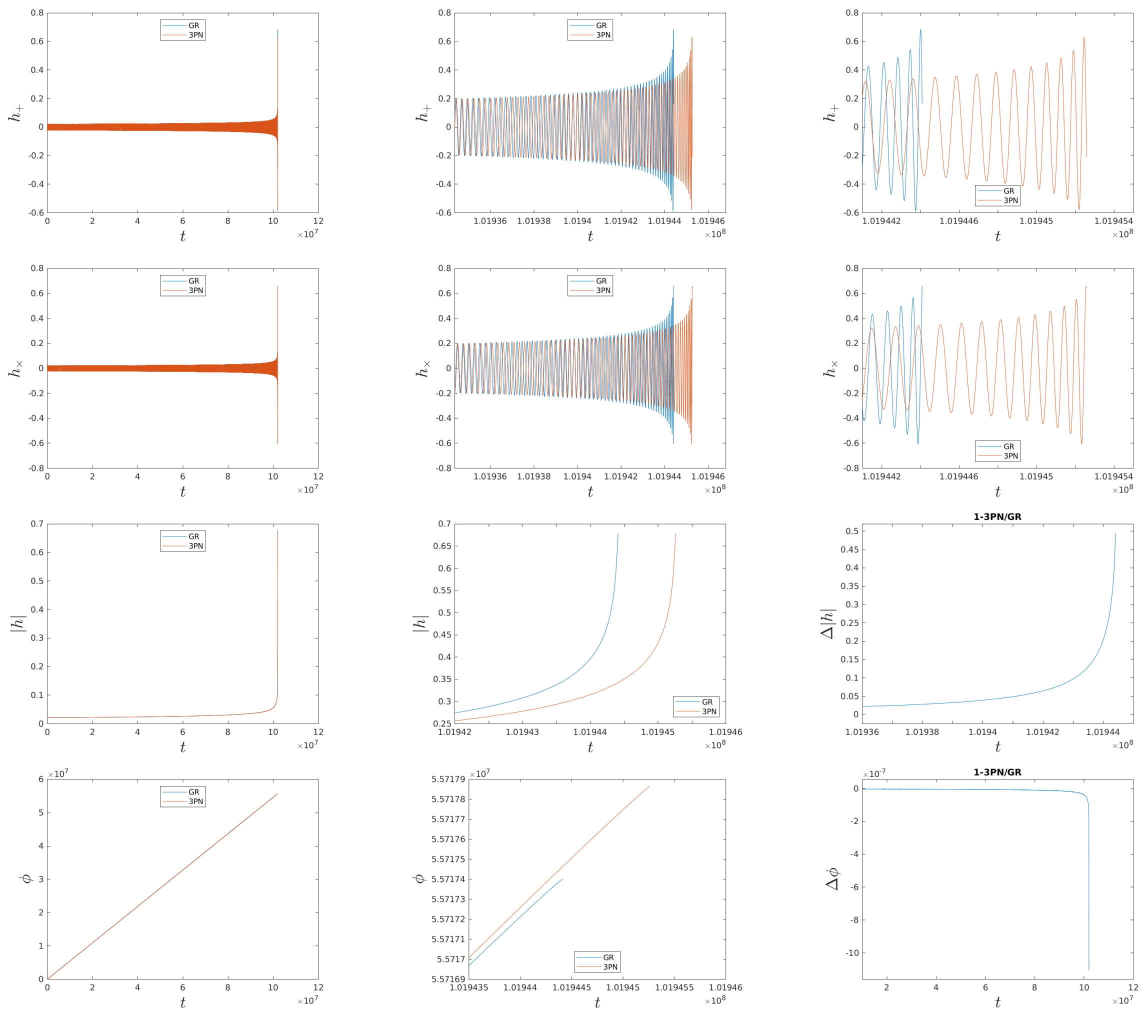

Figure 3.8: Complete waveform generated by the event "GWtest:H4". In the top panels we plot the plus and cross polarization (3.1.6) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR. The phase data was manipulated in order to keep in consideration its negative starting value.

3.3.2 GW170817:ENG

Here we will gather all the waveform's plots associated with the event "GW170817:ENG", its parameters can be read in Table 2.1 and we started this `TEOBResumS` simulation with a initial frequency, in geometric units, of $f_0 = 8\text{Hz}$ (see Eqs. (3.3.3), (3.3.4)).

The main numerical results are collected in Figs. 3.6 - 3.8, especially the difference of merger time and of the number of evolution cycles. The latest, for this "GW170817:ENG" event, reads

$$\Delta\mathcal{N}_{rel} \equiv \frac{\mathcal{N}_{GR} - \mathcal{N}_{ST}}{\mathcal{N}_{GR}} \simeq -2.0346680691 \cdot 10^{-7}. \tag{3.3.6}$$

In fact, from Figs. 3.9 - 3.16, we can see that the ST correction extends the inspiral up to merger, which is postponed by the quantity written in Table 3.6.

As well as for the others events, we will comment all results in the final chapter, Chap. 4.

−	Merger time
$GR[10^8]$	1.4322347364
$ST[10^8]$	1.4322350339
$(GR-ST)[10^1]$	−2.9747317791
$(GR-ST)/GR[10^{-7}]$	−2.0769861986

Table 3.6: Difference of the merger time, in "GWtest:H4" event, between GR and ST with also the relative one.

In Table 3.6 we can see the difference of the merger time for the "GW170817:ENG" run that we made in `TEOBResumS` in geometric units.

From Eq. (3.3.3) the physical merger time difference between GR and ST reads: $\Delta T^{merger} \simeq -1.441213913ms$ on a evolution, respectively, of $T^{merger}_{GR} \simeq 1925.867400s$ and $T^{merger}_{ST} \simeq 1925.868841s$.

$\max(\lvert\Psi\rvert_{lm})/\nu$				
(lm)	$GR[10^{-4}]$	$ST[10^{-4}]$	$(GR-ST)[10^{-7}]$	$(GR-ST)/GR[10^{-4}]$
(22)	7695.8865935	7721.9354094	−26048.815925	−33.847712813
(21)	84.285210100	86.454487048	−2169.2769474	−257.37338078
(33)	71.773714150	72.095501453	−321.78730340	−44.833586671
(32)	59.538768003	61.041212192	−1502.4441892	−252.34720831
(31)	1.7557418088	1.7426534443	13.088364465	74.546065943
(44)	53.080060447	53.589612008	−509.55156119	−95.996793692
(42)	2.0206166737	2.0185993515	2.0173222075	9.9836957389
$\max(\lvert h\rvert)$				
−	6044.9506351	6067.9411720	−22990.536962	−38.032629793

Table 3.7: Peak value of each amplitude multipole, for the "GW170817:ENG" event, with its absolute and relative difference.

$\Delta\phi_{lm}/2\pi$				
(lm)	$GR[10^{4}]$	$ST[10^{4}]$	$(GR-ST)[10^{-2}]$	$(GR-ST)/GR[10^{-7}]$
(22)	2.4689597566	2.4689643347	−4.5781331974	−18.542761522
(21)	1.2344815740	1.2344838868	−2.3127714310	−18.734758620
(33)	3.7034411484	3.7034480363	−6.8879095910	−18.598674355
(32)	2.4689580970	2.4689626454	−4.5484146176	−18.422405075
(31)	1.2344822027	1.2344845232	−2.3205854150	−18.798046744
(44)	4.9379163354	4.9379254294	−9.0940360024	−18.416747844
(42)	2.4689592546	2.4689638205	−4.5658259536	−18.492917390
$\mathcal{N}$				
−	960.46474049	960.46493592	−195.42269390	−2.0346680691

Table 3.8: Multipolar phase difference between the merger point and the initial one, divided by 2π, for the "GW170817:ENG" event, with its absolute and relative difference between GR and ST.

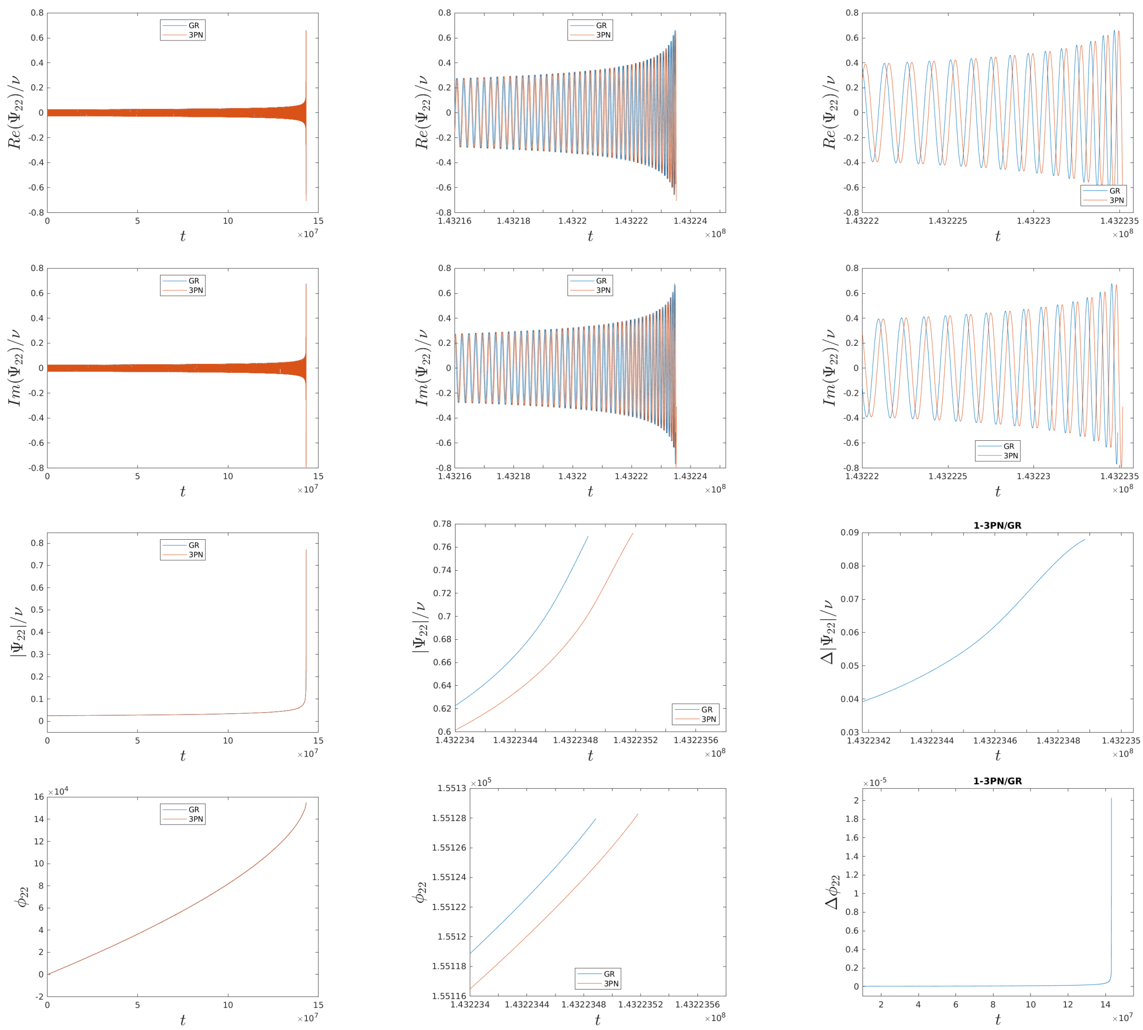

Figure 3.9: Multipole (22) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{22}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

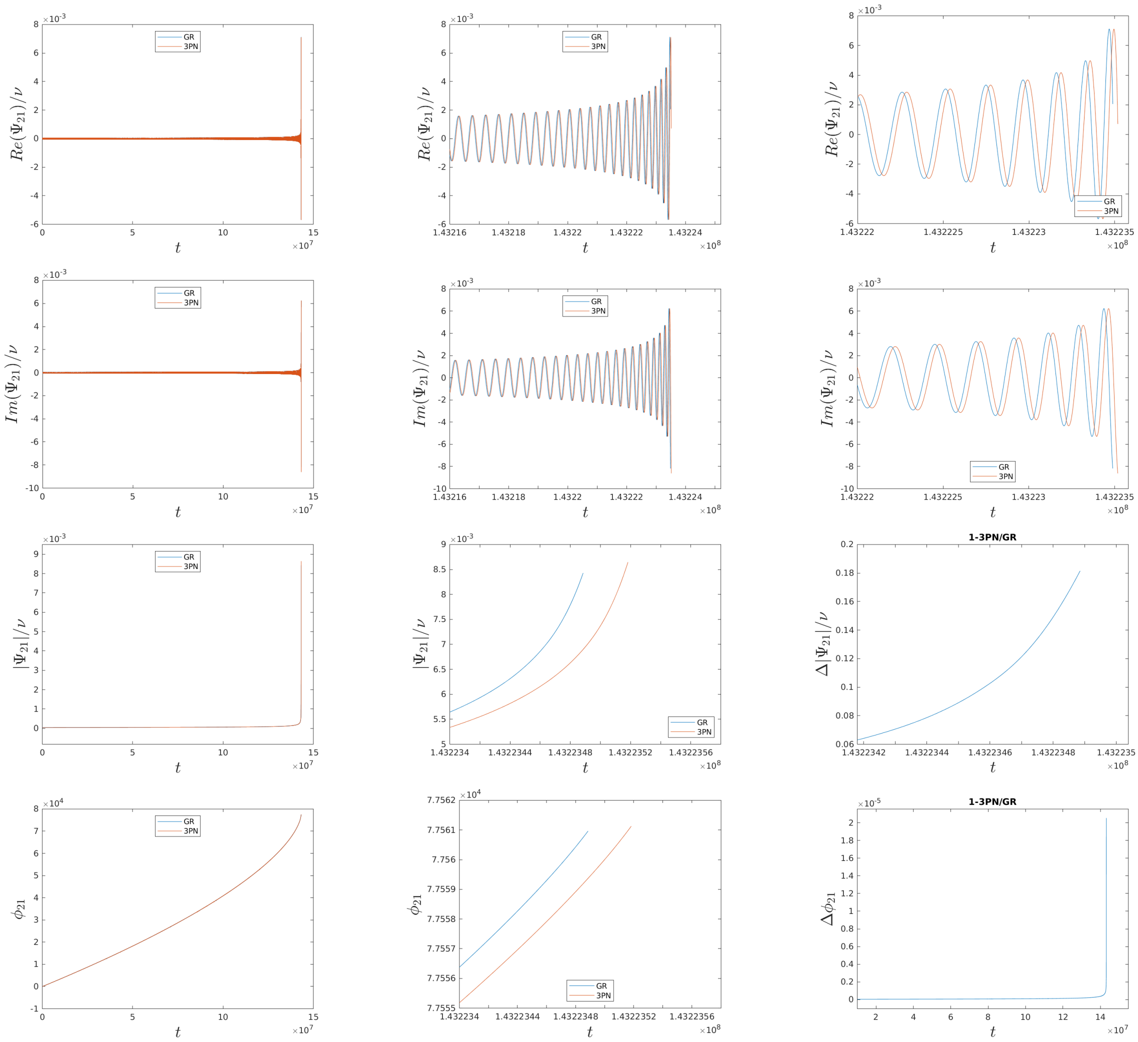

Figure 3.10: Multipole (21) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{21}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

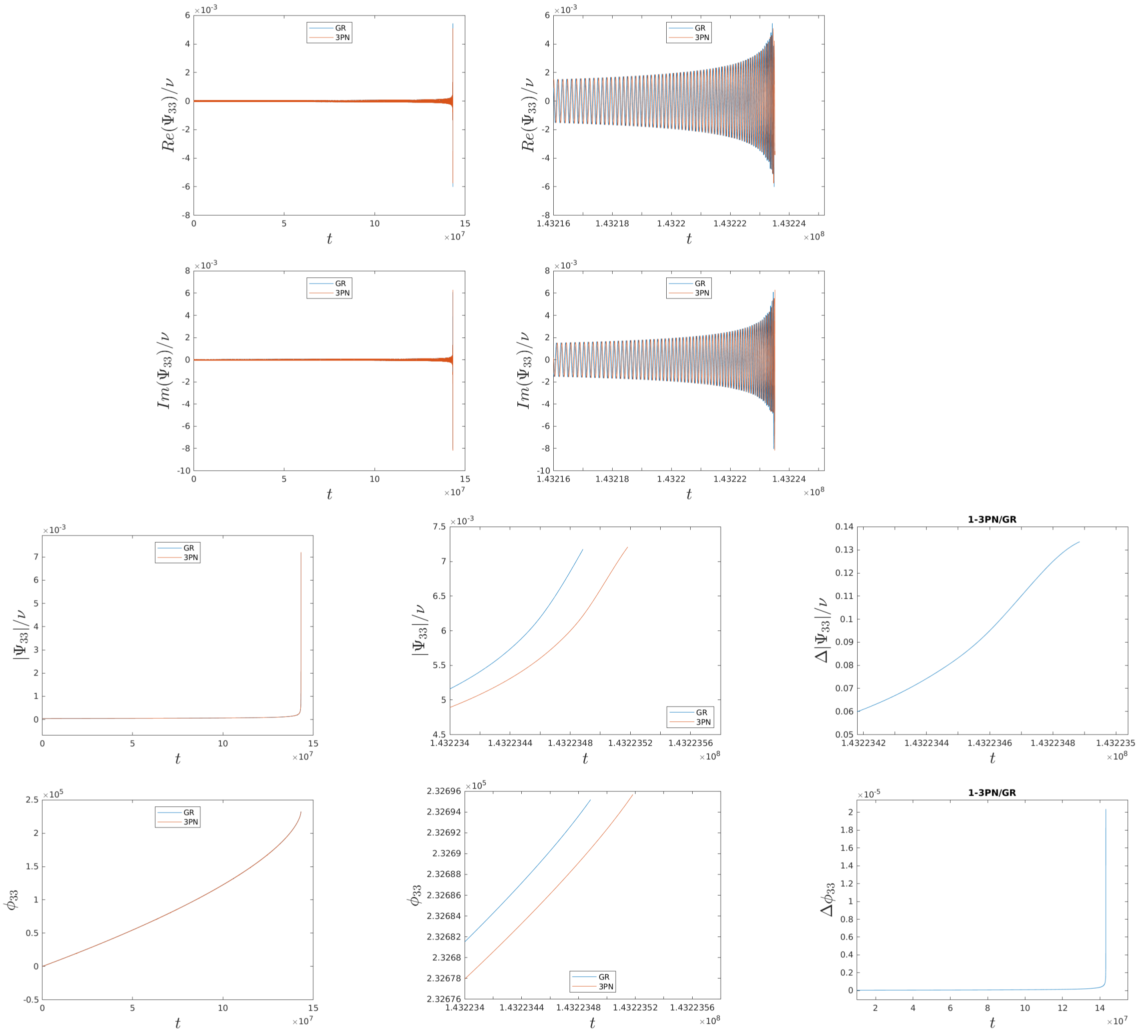

Figure 3.11: Multipole (33) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{33}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

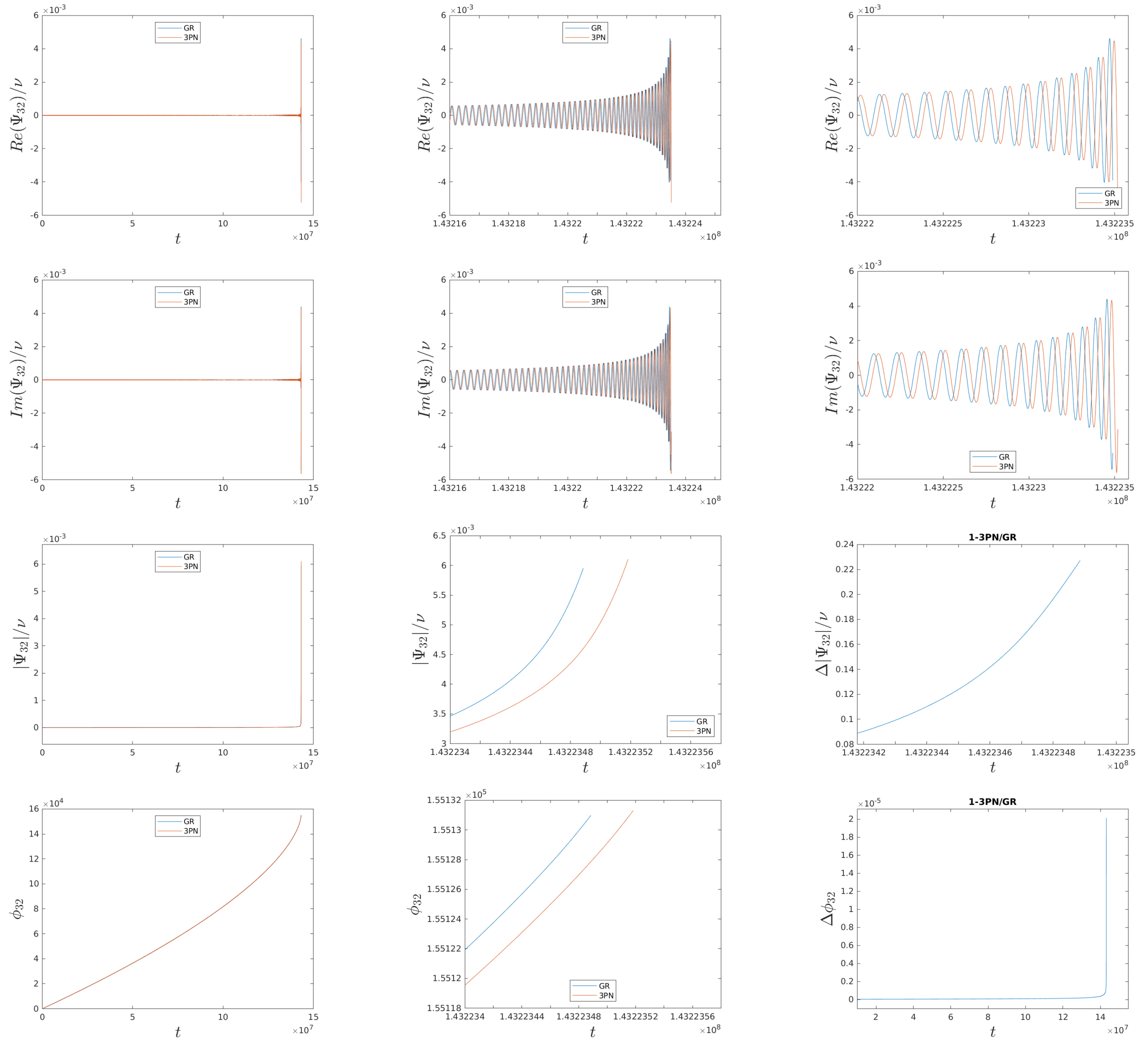

Figure 3.12: Multipole (32) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{32}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

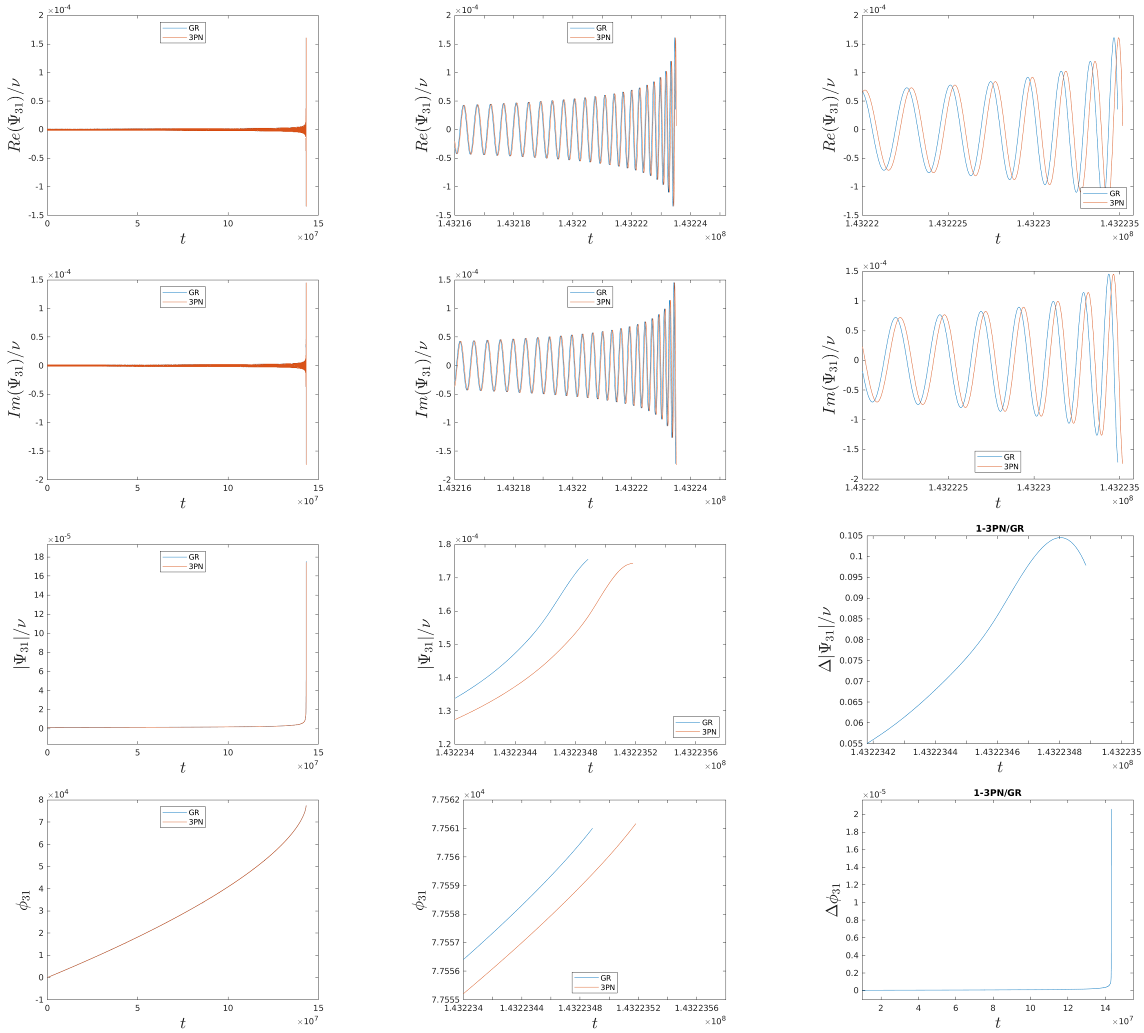

Figure 3.13: Multipole (31) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{31}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

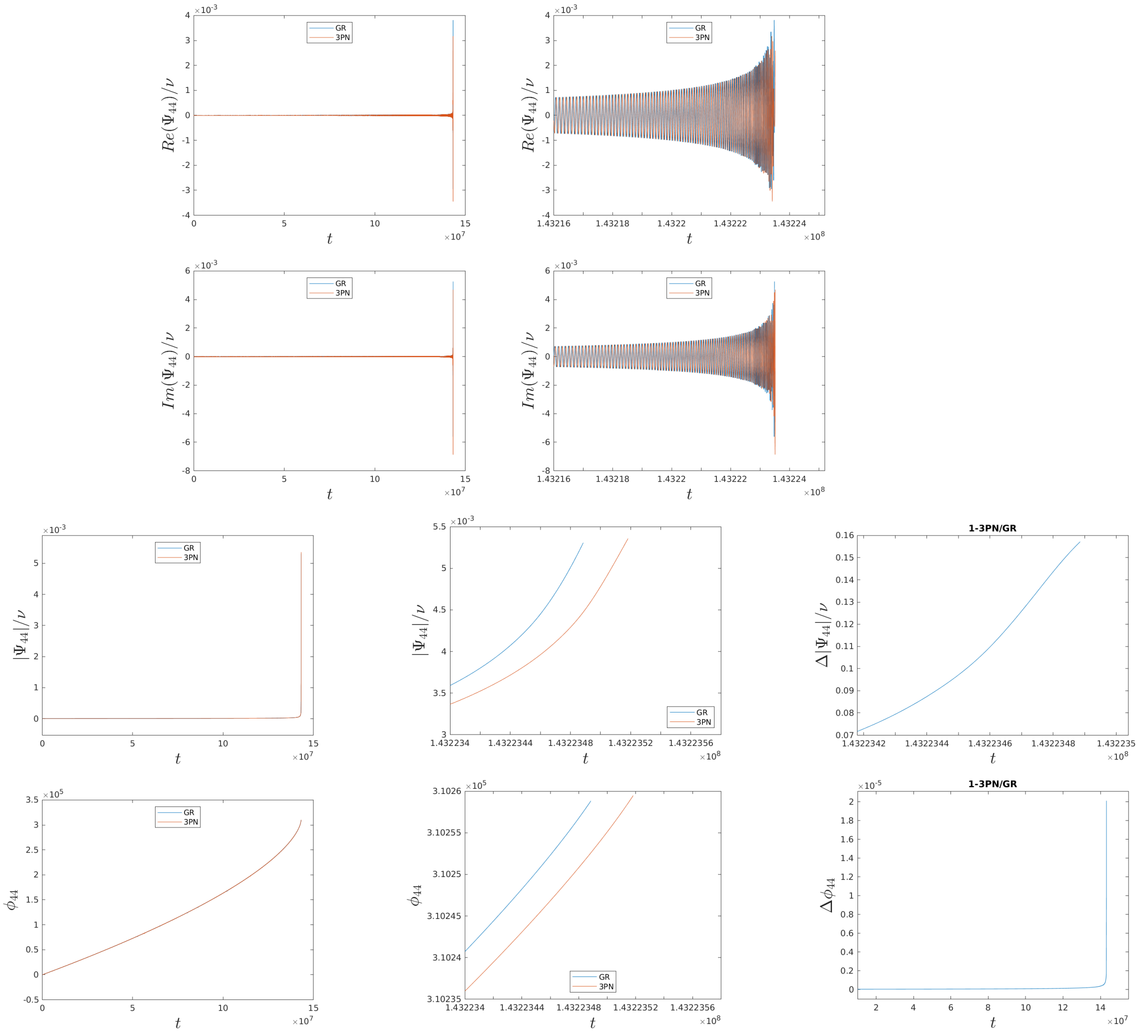

Figure 3.14: Multipole (44) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{44}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

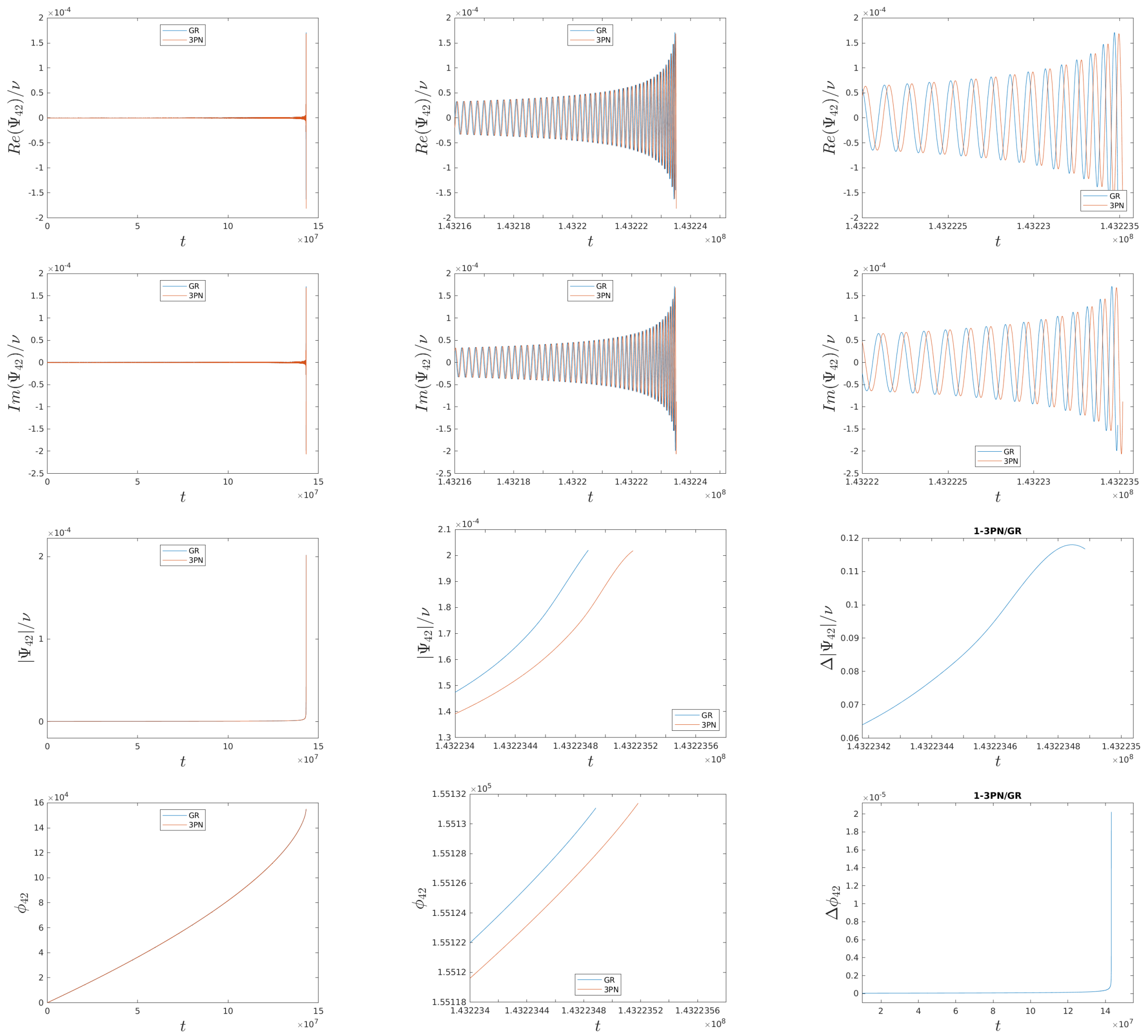

Figure 3.15: Multipole (42) of "GW170817:ENG" event. In the top panels we plot the real and imaginary part of Ψ_{42}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

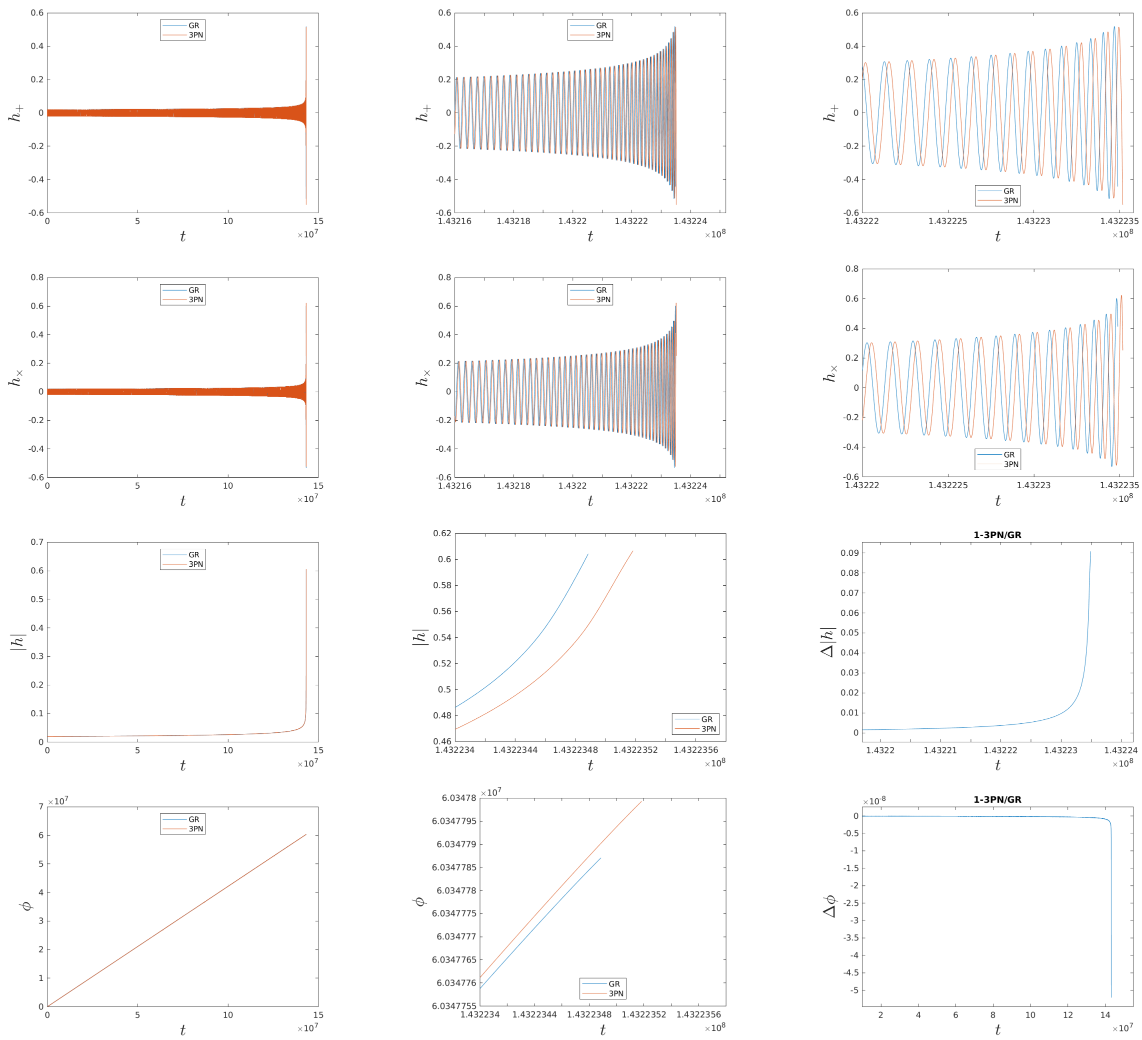

Figure 3.16: Complete waveform generated by the event "GW170817:ENG". In the top panels we plot the plus and cross polarization (3.1.6) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR. The phase data was manipulated in order to keep in consideration its negative starting value.

3.3.3 GW170817:SLy

Here we will gather all the waveform's plots associated with the event "GW170817:SLy", its parameters can be read in Table 2.1 and we started this `TEOBResumS` simulation with a initial frequency, in geometric units, of $f_0 = 8$Hz (see Eqs. (3.3.3), (3.3.4)).

The main numerical results are collected in Figs. 3.9 - 3.11, especially the difference of merger time and of the number of evolution cycles. The latest, for this "GW170817:SLy" event, reads

$$\Delta\mathcal{N}_{rel} \equiv \frac{\mathcal{N}_{GR} - \mathcal{N}_{ST}}{\mathcal{N}_{GR}} \simeq -4.1752576573 \cdot 10^{-7}. \tag{3.3.7}$$

In fact, from Figs. 3.17 - 3.24, we can see that the ST correction extends the inspiral up to merger, which is postponed by the quantity written in Table 3.9.

As well as for the others events, we will comment all results in the final chapter, Chap. 4.

$-$	Merger time
$GR[10^8]$	1.4322357776
$ST[10^8]$	1.4322363725
$(GR-ST)[10^1]$	-5.9494635403
$(GR-ST)/GR[10^{-7}]$	-4.1539693618

Table 3.9: Difference of the merger time, in "GWtest:H4" event, between GR and ST with also the relative one.

In Table 3.9 we can see the difference of the merger time for the "GW170817:SLy" run that we made in `TEOBResumS` in geometric units.

From Eq. (3.3.3) the physical merger time difference between GR and ST reads: $\Delta T^{merger} \simeq -2.566522207ms$ on a evolution, respectively, of $T^{merger}_{GR} \simeq 1925.868800s$ and $T^{merger}_{ST} \simeq 1925.871367s$.

$\max(\lvert\Psi\rvert_{lm})/\nu$				
(lm)	$GR[10^{-4}]$	$ST[10^{-4}]$	$(GR-ST)[10^{-7}]$	$(GR-ST)/GR[10^{-4}]$
(22)	7886.7274386	7877.2795180	9447.9205959	11.979519604
(21)	89.107973465	88.142111962	965.86150251	108.39226446
(33)	74.783889207	74.680701609	103.18759833	13.798105370
(32)	63.789414707	63.123975009	665.43969798	104.31820092
(31)	1.8007587849	1.8083963815	-7.6375965275	-42.413212649
(44)	56.222958251	56.013538082	209.42016894	37.248159018
(42)	2.1047521487	2.1071361046	-2.3839558531	-11.326539586
$\max(\lvert h\rvert)$				
$-$	6200.4447097	6191.8791648	8565.5449010	13.814404131

Table 3.10: Peak value of each amplitude multipole, for the "GW170817:SLy" event, with its absolute and relative difference.

$\Delta\phi_{lm}/2\pi$				
(lm)	$GR[10^{4}]$	$ST[10^{4}]$	$(GR-ST)[10^{-3}]$	$(GR-ST)/GR[10^{-7}]$
(22)	2.4690054235	2.4690044041	10.194223909	4.1288787023
(21)	1.2345044806	1.2345039599	5.2069238955	4.2178250279
(33)	3.7035097126	3.7035081739	15.387211199	4.1547646402
(32)	2.4690036717	2.4690026662	10.054702838	4.0723725741
(31)	1.2345051333	1.2345046090	5.2430485466	4.2470852531
(44)	4.9380074757	4.9380054662	20.095082145	4.0694717949
(42)	2.4690048831	2.4690038696	10.135766635	4.1052031548
$\mathcal{N}$				
$-$	960.46539483	960.46579585	-4010.1904944	-4.1752576573

Table 3.11: Multipolar phase difference between the merger point and the initial one, divided by 2π, for the "GW170817:SLy" event, with its absolute and relative difference between GR and ST.

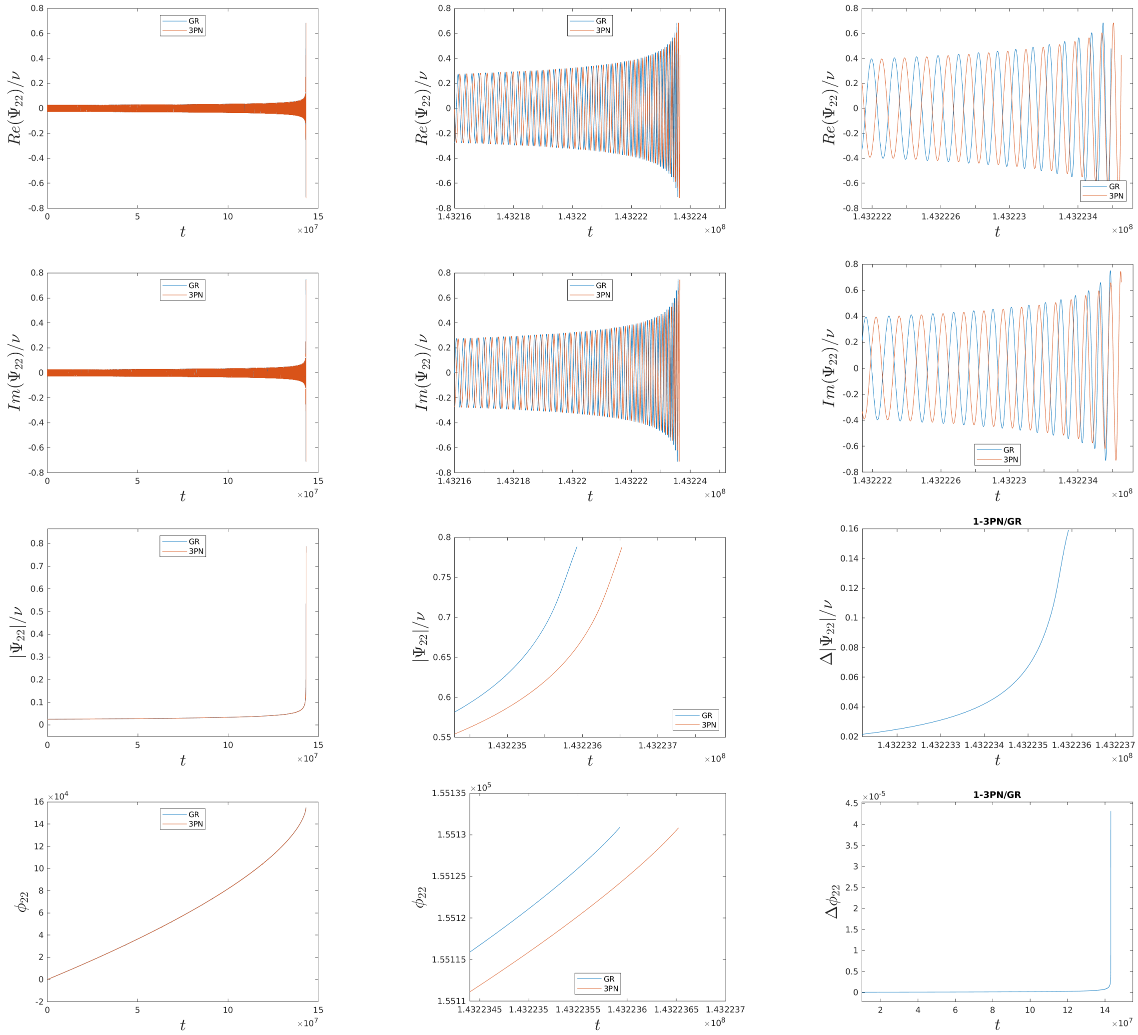

Figure 3.17: Multipole (22) of "GW170817:SLy" event. In the top panels we plot the real and imaginary part of Ψ_{22}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

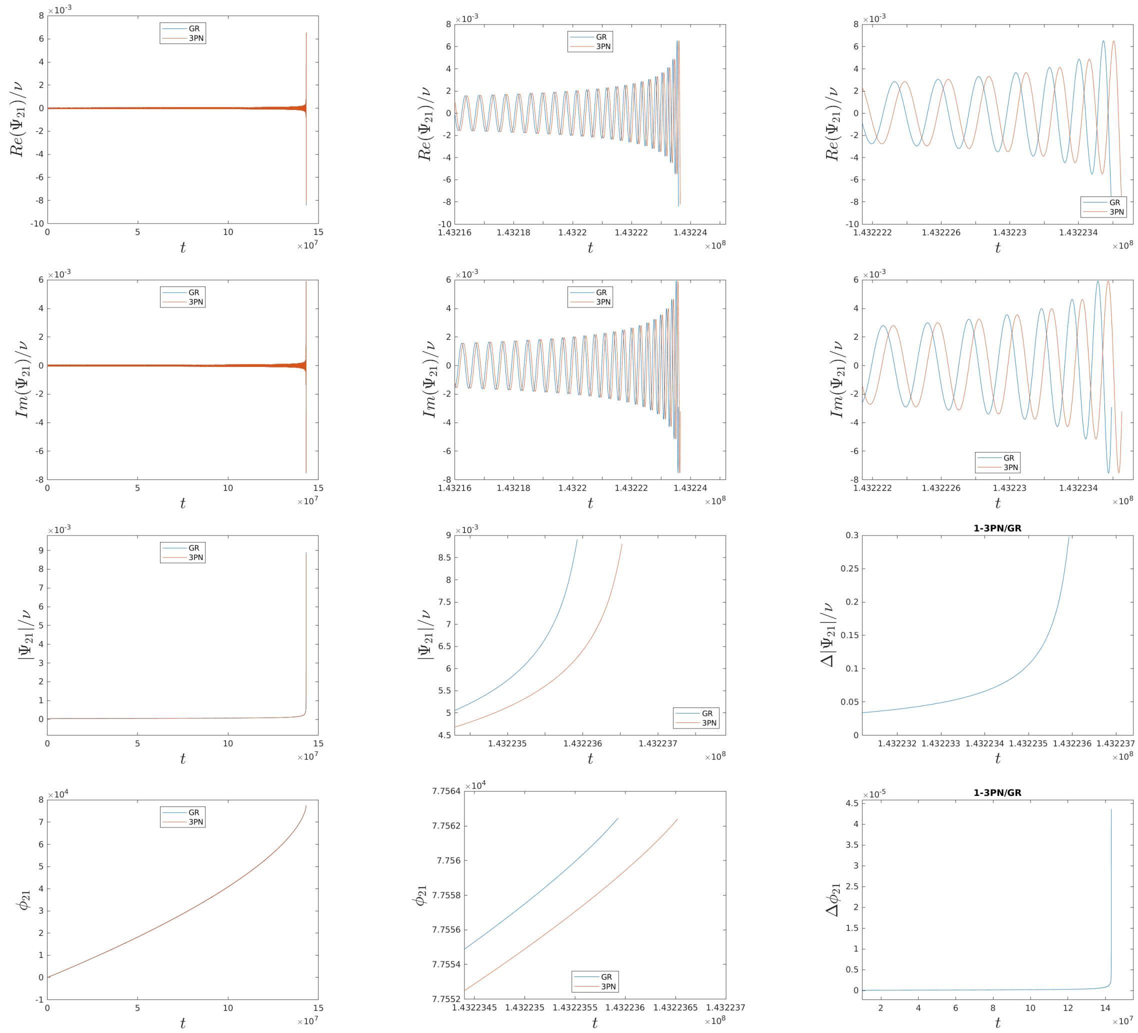

Figure 3.18: Multipole (21) of "GW170817:SLy" event. In the top panels we plot the real and imaginary part of Ψ_{21}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

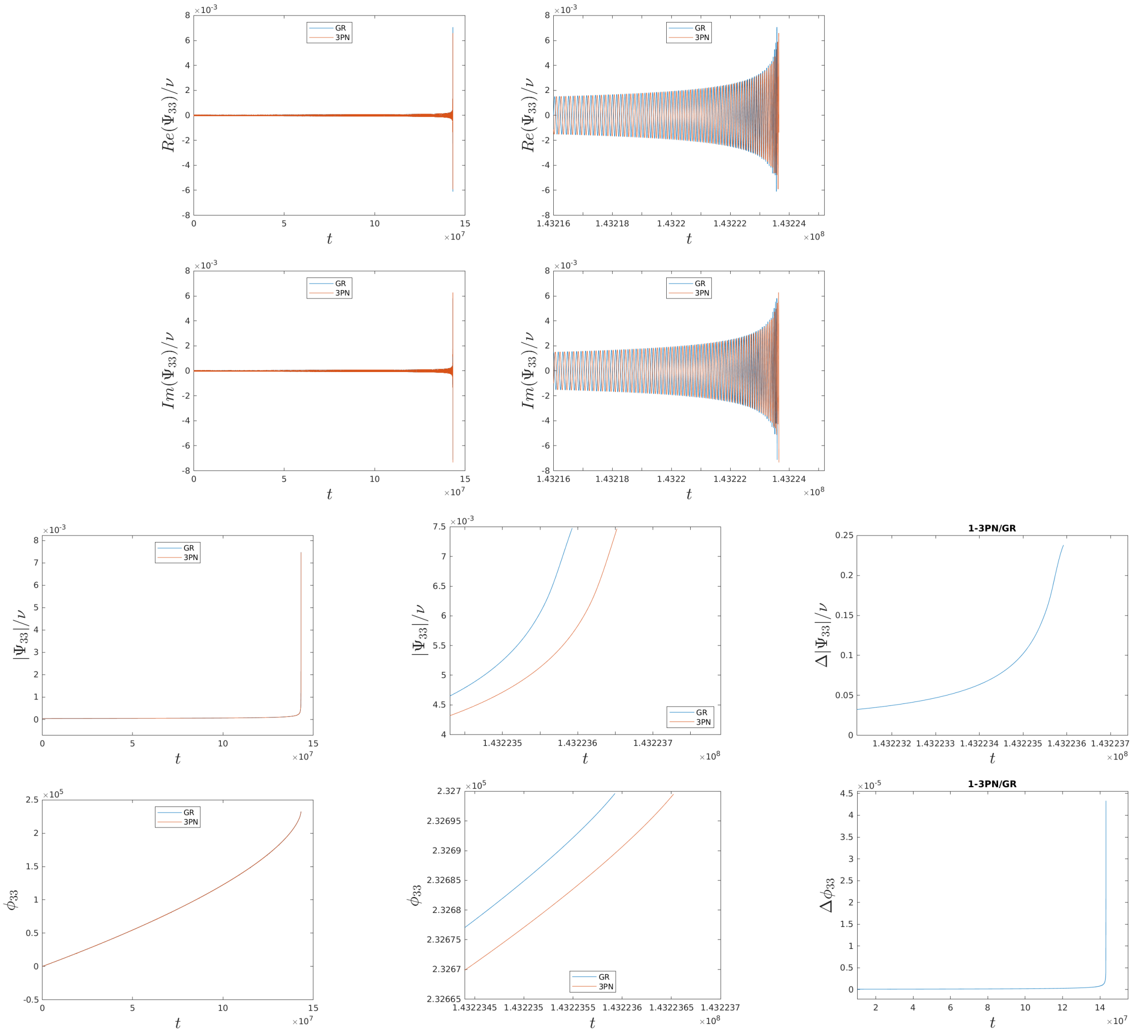

Figure 3.19: Multipole (33) of "GW170817:SLy" event. In the top panels we plot the real and imaginary part of Ψ_{33}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

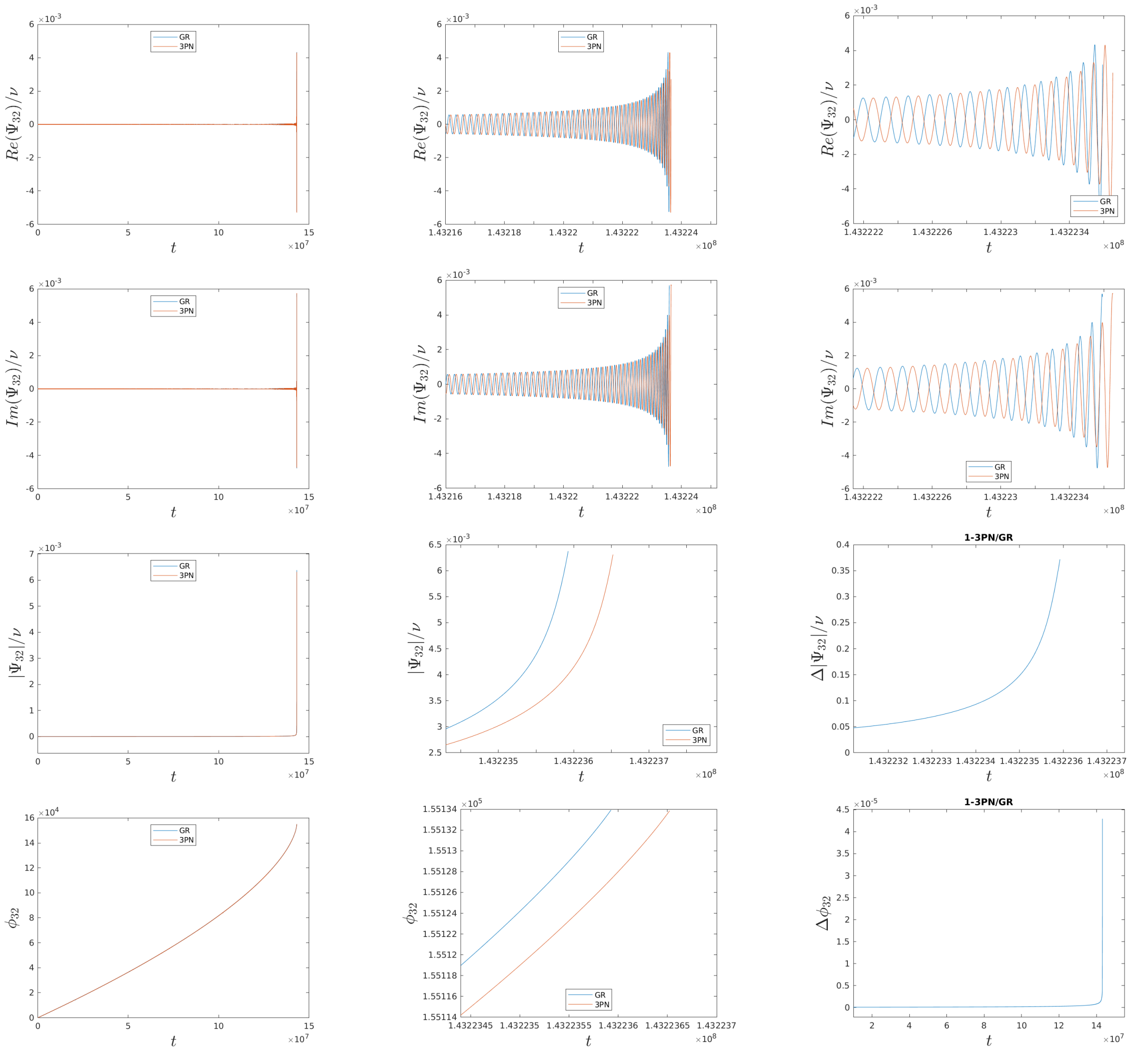

Figure 3.20: Multipole (32) of "GW170817:SLy" event. In the top panels we plot the real and imaginary part of Ψ_{32}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

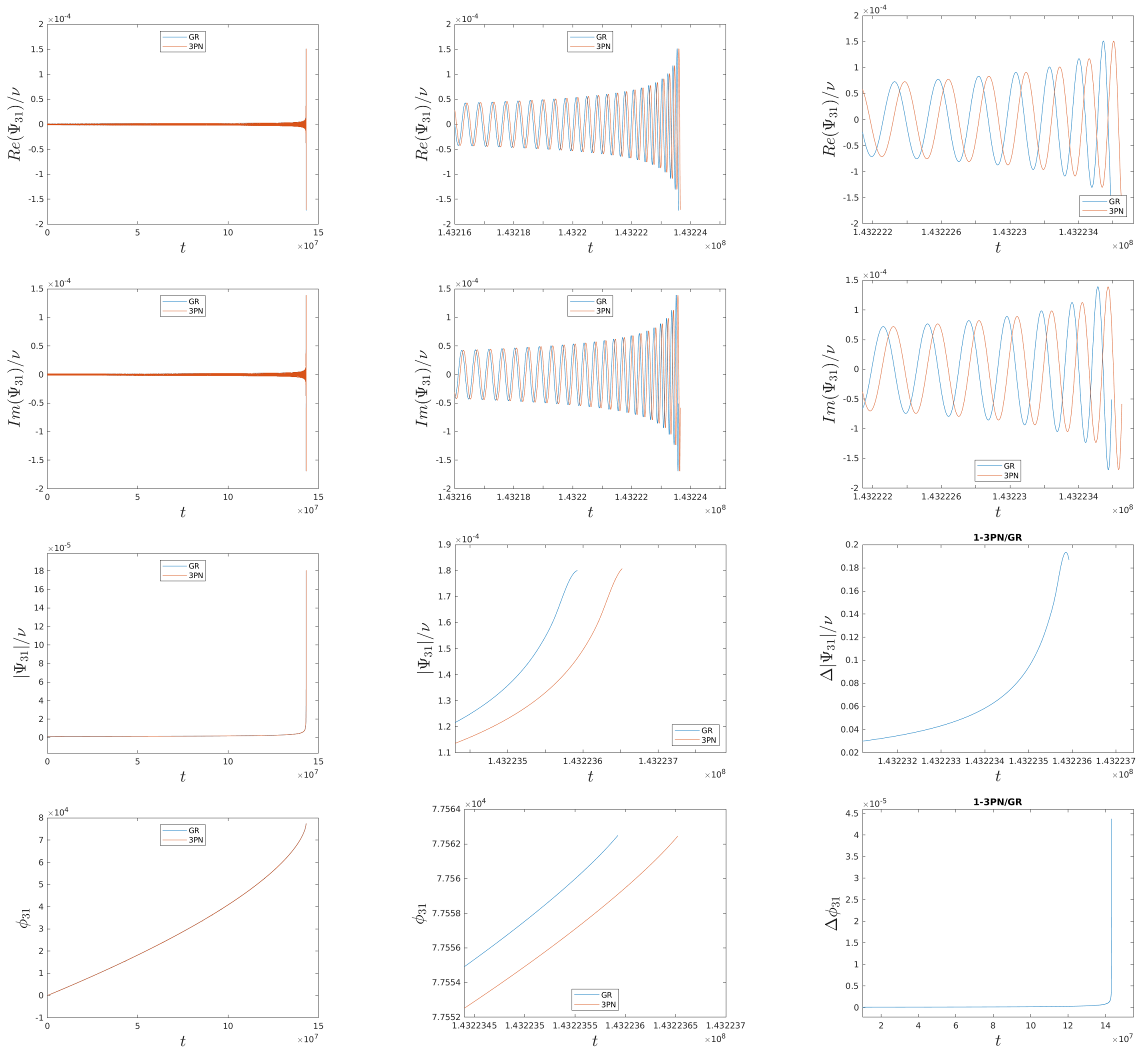

Figure 3.21: Multipole (31) of "GW170817:SLy" event. In the top panels we plot the real and imaginary part of Ψ_{31}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

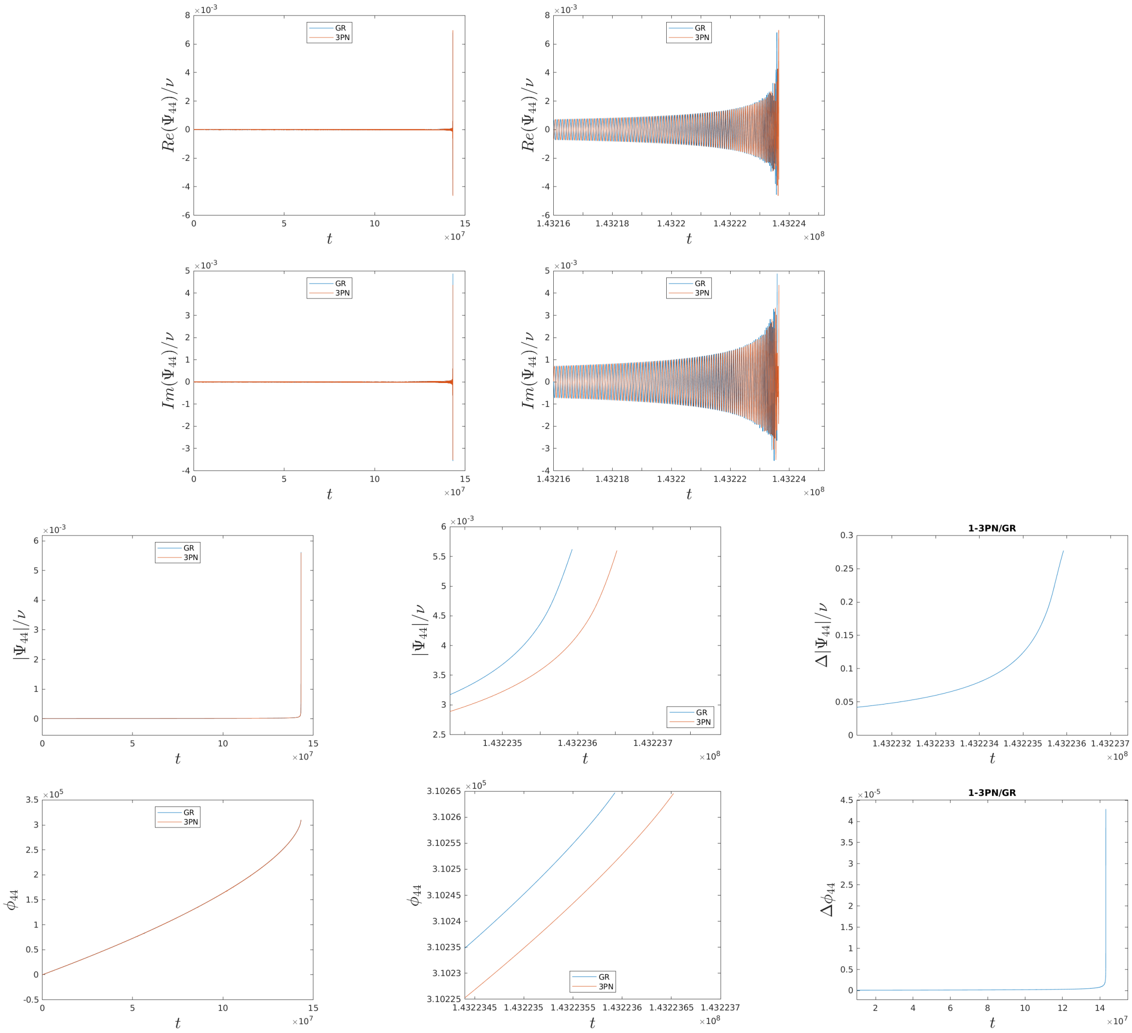

Figure 3.22: Multipole (44) of "GW170817:SLy" event. In the top panels we plot the real and imaginary part of Ψ_{44}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

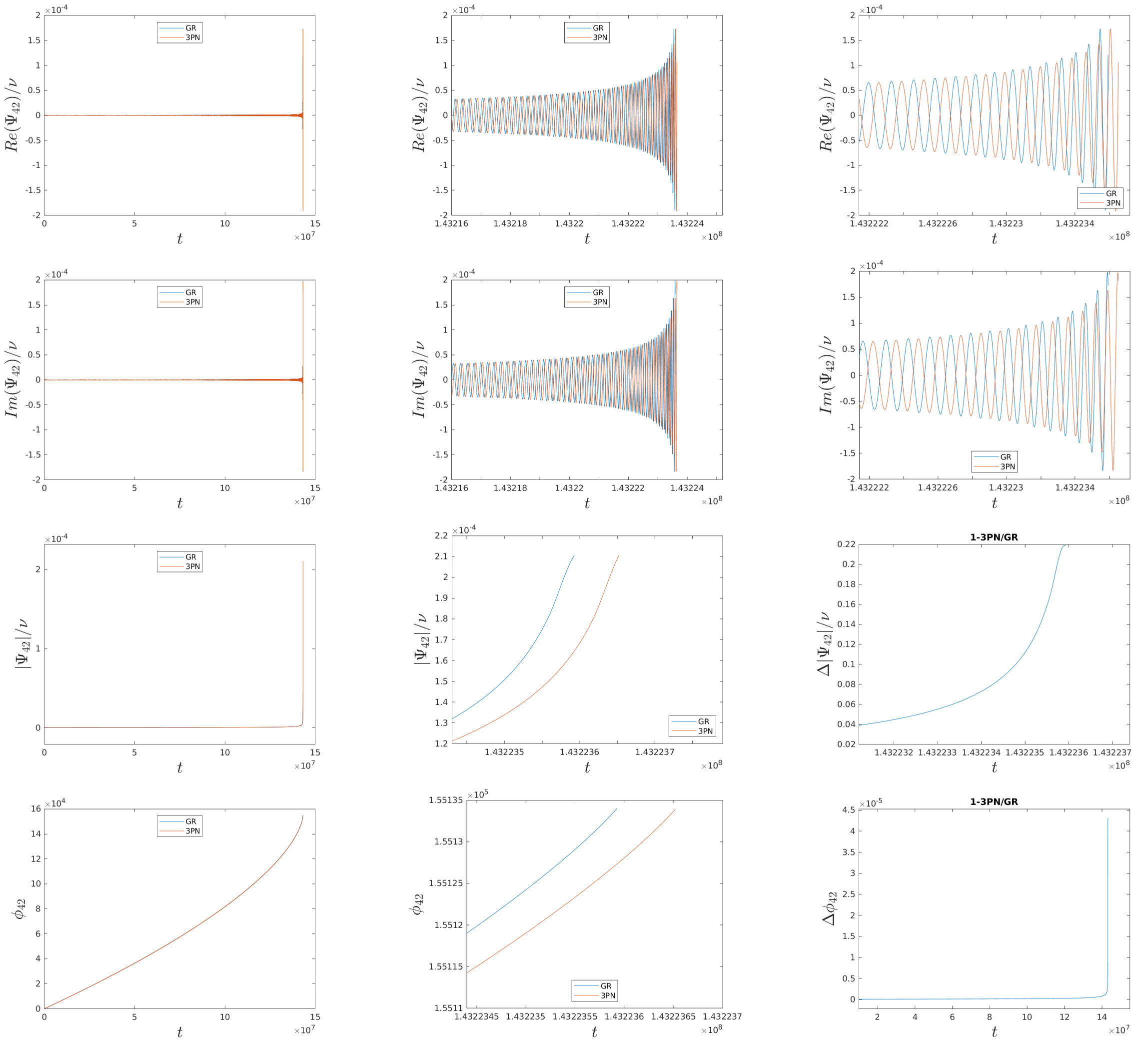

Figure 3.23: Multipole (42) of ”GW170817:SLy” event. In the top panels we plot the real and imaginary part of Ψ_{42}/ν (3.3.2) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR.

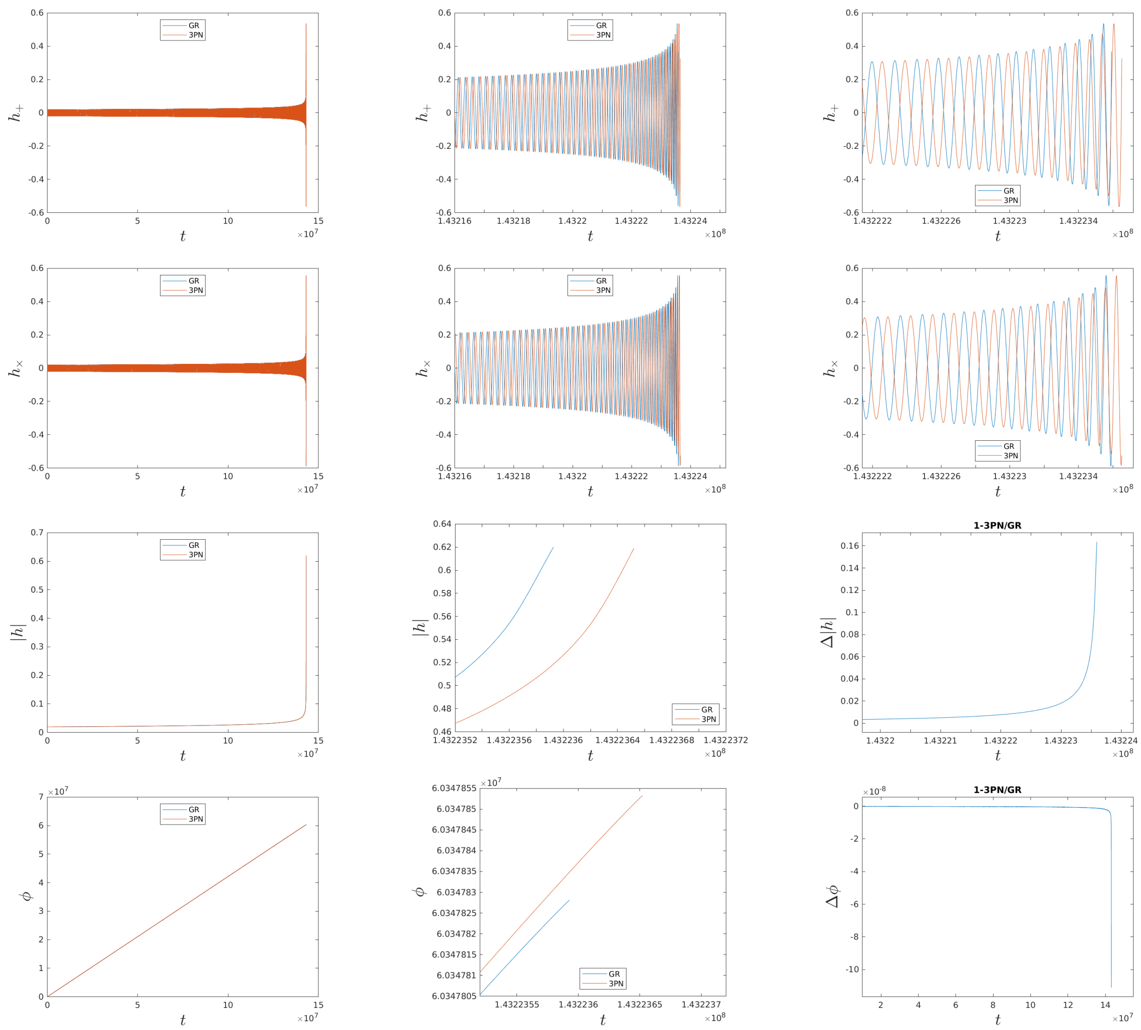

Figure 3.24: Complete waveform generated by the event "GW170817:SLy". In the top panels we plot the plus and cross polarization (3.1.6) while, in the bottom panels, the amplitude and the phase with their relative differences with respect to GR, i.e. (GR-ST)/GR. The phase data was manipulated in order to keep in consideration its negative starting value.

Chapter 4

Conclusions and future prospects

In this thesis we updated the `TEOBResumS` model for BNS coalescence with the recent PN scalar-tensor information.

In Chap. 1 we first concentrated on the theoretical aspects of ST theories, in particular on the difference between the (physical) Jordan and Einstein frame within the mono-scalar-tensor theories framework. In Secs. 1.5 - 1.8 we discussed the nonperturbative phenomenon of spontaneous scalarization, which describe the possibility of the development of large value for NS scalar charge. This phenomenon for scalarized NSs is interesting because BNS systems could manifests visible differences in GW radiation with respect to GR, within the constraints given by the binary pulsars.

The second part of this work, Chap. 2, was based on the elaboration of the EOB dynamics in ST theories. We computed the 3PN point-mass effective A potential, at the 3PN order, from the already known circular energy. At the end of the chapter, we discussed the resummation technique applied to the metric potentials within the `TEOBResumS` model. In order to plot these functions, and to make the final simulation running, we have chosen 3 GW event to be tested, with 3 different EOSs. Two of these were based on the real GW170817 event while the last one is a purely test one with very high scalar couplings.

The final aim of this thesis, in Chap 3, is to update the `TEOBResumS` waveform with 2PN scalar-tensor information, with the new 3PN term in the point-mass effective A function, in order to be used in the future to testing GR with ever greater precision. We factorized the quasi-circular and non-spinning general relativistic multipolar waveform and we PN-expanded this correction, by splitting into amplitude and phase terms. At the end, we plotted all the ST-corrected multipoles, by focusing on the differences of the amplitude peaks and phases at the merger.

Event	$\Delta\mathcal{N}_{rel}[10^{-7}]$
GWtest:H4(Sec.3.3.1)	-83.347772171
GW170817:ENG(Sec.3.3.2)	-2.0346680691
GW170817:SLy(Sec.3.3.3)	-4.1752576573

Table 4.1: Relative difference of the number of cycles for each GW event we considered (see Table. 1.1)

In Table 4.1 we summarized the number of GW cycles (relative) difference between GR and ST theory within the 3 sets of parameters we chosen in this work.

We can note that, in all of these cases, the merger was postponed in ST theory and this implies a repulsive scalar effect. However, we have to underline that the ST

3PN information is *not* complete yet. We have the only the new point-mass term within the A potential, but we do not have the "rr" component B (or D) and the 3PN contribution to the waveform is totally unknown. Clearly, a complete and exact structure can be implemented only with the whole 3PN information, which do not exist yet. Anyhow, the ST 3PN, non-tidal, A function term is exact, because is given only from the already known circular energy, and this allows to compute the LSO and LR behaviour by varying the ST parameters.

Future works could elaborate on several aspects of ST implementation within the `TEOBResumS` model.

One of these is the exact determination of all the derivatives of the scalar charge of the NS. Within this work we used an approximation structure in which $\beta'_{A,B}$ and $\beta''_{A,B}$ were given in terms of the first two coupling parameters by imposing that the 1PN, 2PN and 3PN ST term of A are equal to each other in the equal-mass limit, $\nu = 1/4$. This preserve the perturbative structure of ST theories with respect to GR, except for very high values of ST parameters. The numerically exact values can be found by solving the scalar TOV equations, as well as for the $\alpha_{A,B}$ and $\beta_{A,B}$ ones. Unfortunately, they are not yet tabulated in literature and it could be a useful improvement.

A more physical and interesting improvement could be given by the scalar tidal deformation, in particular, the dipolar perturbation to the scalar TOV equations. As we discussed in this thesis, the GR tidal effects on NSs are collected in the Love numbers and they can be found by studying the tensorial perturbation to the TOV metric. In ST theories there is also the scalar perturbation to the scalar TOV metric and the tensorial one starts from the dipolar term, instead of the quadrupolar one in GR. The scalar tensorial-Love numbers do not exist yet, while, for the dipolar part, does not even exist its perturbed equation. This is an interesting theoretical study in order to implement also the scalar deformation to the NSs.

Future developments of `TEOBResumS` model could include the analysis of the ST information, partial or PN-complete, with different combination of spins and eccentricity.

The analytical, or semi-analytical, models are fundamental for GW detection because they are a good agreement between analytical accuracy and evaluation velocity. An ever greater precision is crucial to extract more GW signals from the detector noise and, the development of the modified theories of gravity, such as ST theories, applied to an ever better implementation in these models, could be a powerful tool to tell us where and how the GR needs to be updated.

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