HOMEWORK 1:
BACKGROUND TEST

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Minimum Background Test [80 pts]

1 Vectors and Matrices [20 pts]

Consider the matrix X and the vectors $\mathbf{y}$ and $\mathbf{z}$ below:

$$X = \begin{pmatrix} 9 & 8 \\ 7 & 6 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} 9 \\ 8 \end{pmatrix} \qquad \mathbf{z} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

1. What is the inner product of the vectors $\mathbf{y}$ and $\mathbf{z}$? (this is also sometimes called the *dot product*, and is sometimes written as $\mathbf{y}^T\mathbf{z}$)
   $\mathbf{y}^T\mathbf{z} = 111$

2. What is the product $X\mathbf{y}$?
   $\begin{pmatrix} 145 \\ 111 \end{pmatrix}$

3. Is X invertible? If so, give the inverse, and if no, explain why not.
   Yes
   $\begin{vmatrix} 9 & 8 \\ 7 & 6 \end{vmatrix} = -2 \neq 0$
   $\begin{pmatrix} -3 & 4 \\ 3.5 & 4.5 \end{pmatrix}$

4. What is the rank of X?
   2

2 Calculus [20 pts]

1. If $y = 4x^3 - x^2 + 7$ then what is the derivative of y with respect to x?
   $12x^2 - 2x$

2. If $y = \tan(z)x^{6z} - \ln(\frac{7x+z}{x^4})$, what is the partial derivative of y with respect to x?
   $6z\tan(z)x^{6z-1} - \frac{7}{7x+z} + \frac{4}{x}$

3 Probability and Statistics [20 pts]

Consider a sample of data $S = \{0, 1, 1, 0, 0, 1, 1\}$ created by flipping a coin x seven times, where 0 denotes that the coin turned up heads and 1 denotes that it turned up tails.

1. What is the sample mean for this data?
   $\frac{4}{7}$

2. What is the sample variance for this data?
   $\frac{12}{49}$

3. What is the probability of observing this data, assuming it was generated by flipping a biased coin with $p(x = 1) = 0.7, p(x = 0) = 0.3$.
   0.0065

4. Note that the probability of this data sample would be greater if the value of $p(x = 1)$ was not 0.7, but instead some other value. What is the value that maximizes the probability of the sample S? Please justify your answer.
   $\frac{4}{7}$
   The expression of the probability to achieve this sequence can be written as $f(p) = (1-p)^3 \times p^4$, and its derivative is $f'(p) = 4(1-p)^3p^3 - 3(1-p)^2p^4$.
   The local maximum can be found at the zero point of f'(p) within [0,1]. In this case, the value is $p = \frac{4}{7}$

5. Consider the following joint probability table where both A and B are binary random variables:

A	B	$P(A, B)$
0	0	0.1
0	1	0.4
1	0	0.2
1	1	0.3

   (a) What is $P(A = 0, B = 0)$?
   0.1
   (b) What is $P(A = 1)$?
   0.5
   (c) What is $P(A = 0|B = 1)$?
   $\frac{4}{7}$
   (d) What is $P(A = 0 \vee B = 0)$?
   0.7

4 Big-O Notation [20 pts]

For each pair (f, g) of functions below, list which of the following are true: $f(n) = O(g(n))$, $g(n) = O(f(n))$, both, or neither. Briefly justify your answers.

1. $f(n) = \frac{n}{2}, g(n) = \log_2(n)$.
   $g(n) = O(f(n))$ is true;
   For $n > 2$, we can find $c = 1$ rendering $g(n) <= cf(n)$
   But we cannot find such c to prove $f(n) = O(g(n))$

2. $f(n) = \ln(n), g(n) = \log_2(n)$.
   Both equations are correct.
   We can see $f(n) = \frac{g(n)}{\log_2 e}$
   So for $n >= 1$, we can always find $c = \log_2 e$ to satisfy these both equations.

3. $f(n) = n^{100}, g(n) = 100^n$.
   For $n >= 100$, we can see $f(n) <= g(n)$.
   Thus $f(n) = O(g(n))$ is true.

Medium Background Test [20 pts]

5 Algorithm [5 pts]

Divide and Conquer: Assume that you are given a sorted array with n integers in the range $[-10, +10]$. Note that some integer values may appear multiple times in the array. Additionally, you are told that somewhere in the array the integer 0 appears exactly once. Provide an algorithm to locate the 0 which runs in $O(\log(n))$. Explain your algorithm in words, describe why the algorithm is correct, and justify its running time.

Because this array is exactly sorted, we can use "Binary Search" to find this 0. First, we define two variables, left and right, with initial index of the first and last element of this array. Then we get the third variable, mid, which is calculated by int(left + right) / 2. If array[mid] is greater than 0, then set right = mid, and do the next loop. Similarly, if array[mid] is smaller than 0, set left = mid + 1 and do the next loop. Keep on doing that until we find the 0.
The running time should be no more than O(log(n)).

6 Probability and Random Variables [5 pts]

6.1 Probability

State true or false. Here Ω denotes the sample space and A^c denotes the complement of the event A.

1. For any $A, B \subseteq \Omega$, $P(A|B)P(B) = P(B|A)P(A)$.
   True. They both equal $P(AB)$
2. For any $A, B \subseteq \Omega$, $P(A \cup B) = P(A) + P(B) - P(A|B)$.
   False.
3. For any $A, B, C \subseteq \Omega$ such that $P(B \cup C) > 0$, $\frac{P(A \cup B \cup C)}{P(B \cup C)} \geq P(A|B \cup C)P(B \cup C)$.
   True.
4. For any $A, B \subseteq \Omega$ such that $P(B) > 0, P(A^c) > 0$, $P(B|A^C) + P(B|A) = 1$.
   False.
5. For any n events $\{A_i\}_{i=1}^n$, if $P(\bigcap_{i=1}^n A_i) = \sum_{i=1}^n P(A_i)$, then $\{A_i\}_{i=1}^n$ are mutually independent.
   True.

6.2 Discrete and Continuous Distributions

Match the distribution name to its probability density / mass function. Below, $|\boldsymbol{x}| = k$.

(a) Laplace (h).

(b) Multinomial (i).

(c) Poisson (l).

(d) Dirichlet (k).

(e) Gamma (j).

(f) $f(\boldsymbol{x}; \boldsymbol{\Sigma}, \boldsymbol{\mu}) = \frac{1}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\right)$

(g) $f(x; n, \alpha) = \binom{n}{x}\alpha^x(1-\alpha)^{n-x}$ for $x \in \{0, \ldots, n\}$; 0 otherwise

(h) $f(x; b, \mu) = \frac{1}{2b}\exp\left(-\frac{|x-\mu|}{b}\right)$

(i) $f(\boldsymbol{x}; n, \boldsymbol{\alpha}) = \frac{n!}{\Pi_{i=1}^k x_i!}\Pi_{i=1}^k \alpha_i^{x_i}$ for $x_i \in \{0, \ldots, n\}$ and $\sum_{i=1}^k x_i = n$; 0 otherwise

(j) $f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ for $x \in (0, +\infty)$; 0 otherwise

(k) $f(\boldsymbol{x}; \boldsymbol{\alpha}) = \frac{\Gamma(\sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(\alpha_i)} \prod_{i=1}^k x_i^{\alpha_i - 1}$ for $x_i \in (0, 1)$ and $\sum_{i=1}^k x_i = 1$; 0 otherwise

(l) $f(x; \lambda) = \lambda^x \frac{e^{-\lambda}}{x!}$ for all $x \in Z^+$; 0 otherwise

6.3 Mean and Variance

1. Consider a random variable which follows a Binomial distribution: $X \sim \text{Binomial}(n, p)$.

   (a) What is the mean of the random variable?
   np

   (b) What is the variance of the random variable?
   $n(1-p)p$

2. Let X be a random variable and $\mathbb{E}[X] = 1, \text{Var}(X) = 1$. Compute the following values:

   (a) $\mathbb{E}[3X]$
   3

   (b) $\text{Var}(3X)$
   9

   (c) $\text{Var}(X + 3)$
   1

6.4 Mutual and Conditional Independence

1. If X and Y are independent random variables, show that $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$.

   $\mathbb{E}[XY] = \sum_{j=1}^{k_y}\sum_{i=1}^{k_x} p_{ij}x_iy_j = \sum_{i=1}^{k_x} a_ix_i \sum_{i=1}^{k_y} b_iy_i = \mathbb{E}[X]\mathbb{E}[Y]$

2. If X and Y are independent random variables, show that $\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y)$.
   Hint: $\text{Var}(X + Y) = \text{Var}(X) + 2\text{Cov}(X, Y) + \text{Var}(Y)$

   $\text{Var}(X+Y) = \mathbb{E}[(X+Y-\mathbb{E}[X+Y])^2]$
   $= \mathbb{E}[(X-\mathbb{E}[X]+Y-\mathbb{E}[Y])^2] = \mathbb{E}[(X-\mathbb{E}[X])^2 + (Y-\mathbb{E}[Y])^2 + 2(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$
   $= \text{Var}(X) + \text{Var}(Y) + 2\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$
   if X and Y are independent:
   $\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$
   Thus$\text{Var}(X+Y) = \text{Var}(X) + \text{Var}(Y)$

3. If we roll two dice that behave independently of each other, will the result of the first die tell us something about the result of the second die?

   No.

   If, however, the first die's result is a 1, and someone tells you about a third event — that the sum of the two results is even — then given this information is the result of the second die independent of the first die?

   No.

6.5 Law of Large Numbers and the Central Limit Theorem

Provide one line justifications.

1. Suppose we simultaneously flip two independent fair coins (i.e., the probability of heads is $1/2$ for each coin) and record the result. After 40,000 repetitions, the number of times the result was two heads is close to 10,000. (Hint: calculate how close.)

   $p(bothhead) = 0.25, P((\frac{10000}{40000} - 0.25) < \epsilon) = 1$

2. Let $X_i \sim \mathcal{N}(0, 1)$ and $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then the distribution of $\bar{X}$ satisfies

$$\sqrt{n}\bar{X} \overset{n\to\infty}{\longrightarrow} \mathcal{N}(0, 1)$$

   $nX = \sum_{i=1}^{n} X_i \sim \mathcal{N}(0, n), \sqrt{n}X \sim \mathcal{N}(0, \frac{n}{\sqrt{n}^2}) = \mathcal{N}(0, 1)$

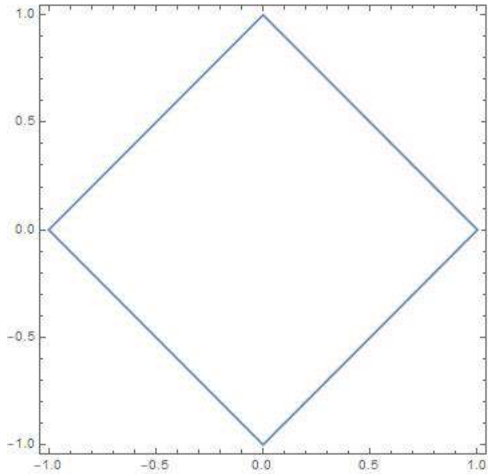
7 Linear algebra [5 pts]

7.1 Norm-enclature

Draw the regions corresponding to vectors $\mathbf{x} \in \mathbb{R}^2$ with the following norms:

1. $||\mathbf{x}||_1 \leq 1$ (Recall that $||\mathbf{x}||_1 = \sum_i |x_i|$)
2. $||\mathbf{x}||_2 \leq 1$ (Recall that $||\mathbf{x}||_2 = \sqrt{\sum_i x_i^2}$)
3. $||\mathbf{x}||_\infty \leq 1$ (Recall that $||\mathbf{x}||_\infty = \max_i |x_i|$)

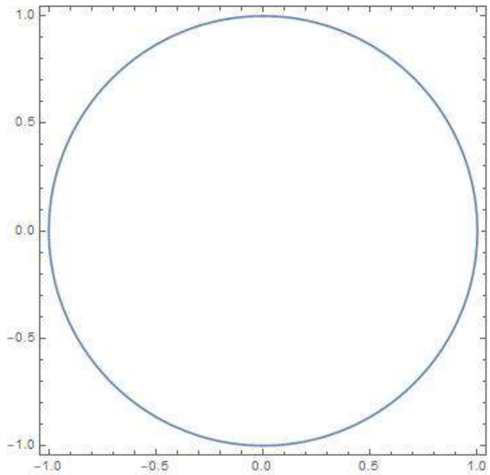
1.



$||x||_1 = 1$

The corresponding area is surrounded by this square.

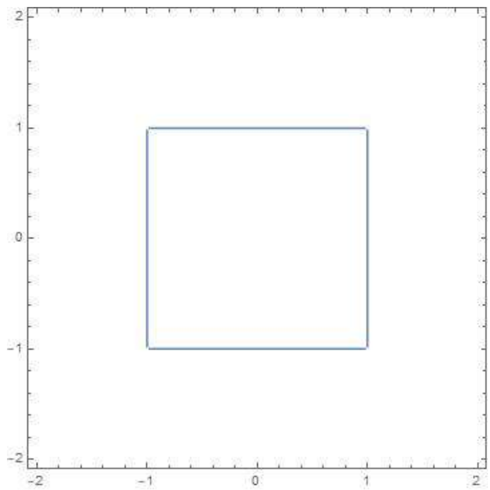
2.



$||x||_2 = 1$

The corresponding area is surrounded by this circle.

3.



$$||x||_\infty = 1$$

The corresponding area is surrounded by this square.

7.2 Geometry

Prove that these are true or false. Provide all steps.

1. The smallest Euclidean distance from the origin to some point $\mathbf{x}$ in the hyperplane $\mathbf{w}^T\mathbf{x} + b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$.
   True.
   The unit vector perpendicular to this plane is $\mathbf{v} = \frac{\mathbf{w}}{||\mathbf{w}||_2}$. Let's assume point, $P = \lambda\mathbf{v}$, right at this plane. So we can see the distance from origin to this plane is $|\lambda|$. Plug the coordinate of P in to this plane, and we have $\lambda = \frac{b}{||\mathbf{w}||_2}$. Thus, the distance from the origin to this plane is $\frac{|b|}{||\mathbf{w}||_2}$.

2. The Euclidean distance between two parallel hyperplane $\mathbf{w}^T\mathbf{x} + b_1 = 0$ and $\mathbf{w}^T\mathbf{x} + b_2 = 0$ is $\frac{|b_1 - b_2|}{||\mathbf{w}||_2}$ (Hint: you can use the result from the last question to help you prove this one).

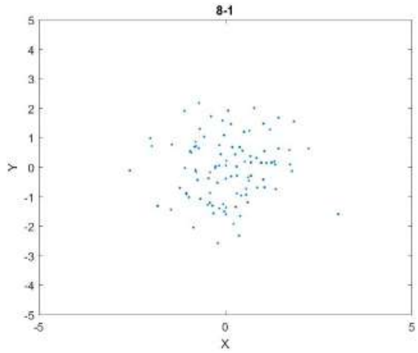
   From the last question, we see the distance from the origin to a hyperplane, $\mathbf{w}^T\mathbf{x}+b = 0$ is $\frac{|b|}{||\mathbf{w}||_2}$. So for two hyperplanes with same normal vector $\mathbf{v} = \frac{\mathbf{w}}{||\mathbf{w}||_2}$, let's do the similar work. Now we have P_1 and P_2 right at the first and second plane, respectively. $\overrightarrow{OP_1} = \lambda_1\mathbf{v}$ and $\overrightarrow{OP_2} = \lambda_2\mathbf{v}$. Then we have $\overrightarrow{P_1P_2} = (\lambda_2 - \lambda_1)\mathbf{v}$. As we see in last question, we have $\lambda_1 = \frac{b1}{||\mathbf{w}||_2}$ and $\lambda_2 = \frac{b2}{||\mathbf{w}||_2}$. Now $\overrightarrow{P_1P_2} = \frac{b2-b1}{||\mathbf{w}||_2}$. Thus the distance is $\frac{|b2-b1|}{||\mathbf{w}||_2}$.

8 Programming Skills - Matlab [5pts]

Sampling from a distribution. For each question, submit a scatter plot (you will have 5 plots in total). Make sure the axes for all plots have the same limits. (Hint: You can save a Matlab figure as a pdf, and then use includegraphics to include the pdf in your latex file.)

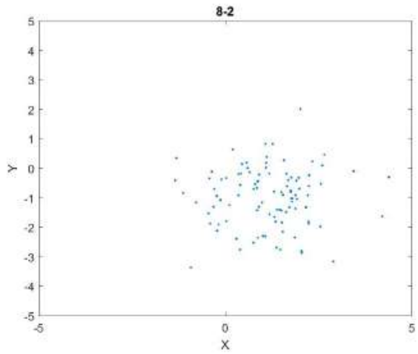
1. Draw 100 samples $\mathbf{x} = [x_1, x_2]^T$ from a 2-dimensional Gaussian distribution with mean $(0, 0)^T$ and identity covariance matrix, i.e., $p(\mathbf{x}) = \frac{1}{2\pi}\exp\left(-\frac{||\mathbf{x}||^2}{2}\right)$, and make a scatter plot (x_1 vs. x_2). For each question below, make each change separately to this distribution.

   Solution figure goes here.



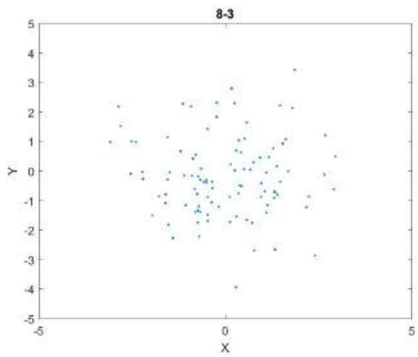
8-1

2. Make a scatter plot with a changed mean of $(1, -1)^T$.
R Solution figure goes here.



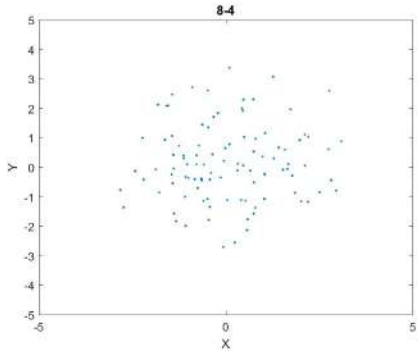
8-2

3. Make a scatter plot with a changed covariance matrix of $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.
Solution figure goes here.



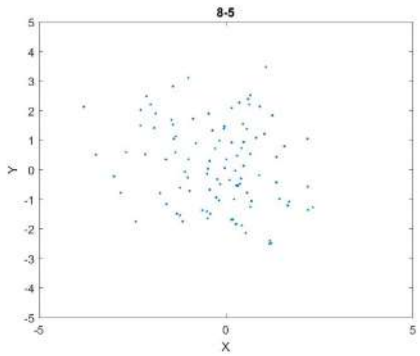
8-3

4. Make a scatter plot with a changed covariance matrix of $\begin{pmatrix} 2 & 0.2 \\ 0.2 & 2 \end{pmatrix}$.
Solution figure goes here.



8-4

5. Make a scatter plot with a changed covariance matrix of $\begin{pmatrix} 2 & -0.2 \\ -0.2 & 2 \end{pmatrix}$.
   Solution figure goes here.



8-5