[Lorenzo Yizhou Lu, 21 Apr 2020 4:07pm]:

Function series and manual

To optimize this question, we have to get the data first. In the next cell, I listed the functions for simulating. This is not hard to implement but requires an understanding in this process.

1. The function 1 just returns a idct(inverse discrete cosine transform) matrix, which is just like MATLAB. This function has not to be called directly by programmer!
2. The function 2 returns a matrix with $r \times c$ length in both rows and columns. This matrix is aimed to convert an process, where we idct a image(2D) and then reshape it as a vector, into an equivalent process, where we reshape this image first(1D) and then tranform it with this returned vector. They both give the same final vector. Attention: the reshape function in Julia traverse along columns then rows, thus, `reshape(image', (:,1))` in Julia is exact same as `reshape(image,r_image * c_image,1)` in MATLAB. The **transposing** is necessary! This function is not for data aquiring. Recalling the compressed sensing formula:
   $$y_{m\times1} = \phi_{m\times n}\psi_{n\times n}a_{n\times1}$$
   We are optimizing the a, and this function returns ψ. We know $\psi a = x$, and the simulated intensity measurements $y = \phi x$. When considering the **Poisson noise**, the exact measured signal $\hat{y} \sim \mathcal{P}(X = y)$.
   <u>Anyways, function 2 should be called only in optimization process!</u>
3. Function 3 is just returning $\hat{y} \sim \mathcal{P}(X = y)$.
4. Do Not Call it directly! If necessary: Function 4 (please use the second version, **DMD_freq_pattern(r,c, cycles)**).
5. The function 5 returns ϕ. <u>Before the simulation, save this returned matrix, and upload it to function 6</u>. It is a good way to save time, if you would like to do muitiple simulations based on the same DMD_basis. <u>This saved ϕ will still be used during the optimization process</u>!
6. Function 6 returns the final simulated result $\hat{y}$. Parameters: photon_number(how many photons per pixel during each measurement), cycles(how many measurements), upload_image, upload_basis(DMD_basis, or ϕ).

[Lorenzo Yizhou Lu, 21 Apr 2020 10:32pm]:

Objective and Constraints

I designed three parts for this project. For compressed sensing, the **Lasso regression** is also required. After we optimize a, we can reconstruct the image by ψa.

(A).

Variables:

1. The vector, a, standing for the coefficents in DCT domain. $a \in \mathbb{R}^n$.
2. The number of measurements, m, or cycles in these functions. $m \in \mathbb{N}$.

Objective: Find the best reconstructed image under Poisson Noise.

Constraints:

1. Assuming our device has a constraint that it could only detect 10^6 photons per second. For example, if we finish 10^4 measurements during a second, each one cannot have more than 100 photons.

(B).

variables:

1. The DMD_basis, ψ.
2. ϕ.

Objective: Find the smallest coherence;

Constraints:

1. $\psi \in \{0, 1\}^{(m \times n)}$.
2. $rank(\psi) = m$.
3. $\phi \in \mathbb{R}^{(n \times n)}$.
4. $rank(\phi) = n$.

   The literature says **Noiselets basis** and **Haar wavelets matrix** have low coherence.

(C).

Variables:

Acquisition time, t.

ψ modes: random, Walsh and raster scan

Assuming $\epsilon(t)$ is the error as a function of t, but we would like to find a experiment time t^* where we use the least time to reconstruct the best image.

Objective: minimize $\epsilon(t) + \lambda \|t\|_1$.

Constraints:

$\int_{t_0}^{t_0+1} n(t)dt \leq 10^6, \forall t_0 \in R$

In [1]:

```julia
using FFTW, LinearAlgebra, Distributions, Random, Images, TestImages, ImageMagick, PyPlot;
## 1.
function idctmtx(dims)
    ## produce a DCT matrix like matlab
    diag = I(dims);
    D = zeros(dims,dims);
    for i = 1:dims
        D[:,i] = idct(diag[:,i]);
    end
    return D;
end

## 2.
function IDCT_basis(r_tot,c_tot)
    ## produce a hyper resolution DMD measurement basis series
    ##verified by Python;
    # making a idct2 matrix in a vector form
    D1 = idctmtx(r_tot);
    D2 = idctmtx(c_tot);
    N = r_tot * c_tot;
    Basis = zeros(N,N);

    i = 1;
    for ir = 1:r_tot
        for ic = 1:c_tot

            Basis_Vector = D1[ir:ir,:]' * D2[ic:ic,:];
            Basis[i,:] = reshape(Basis_Vector', (1,:));
            i += 1;
        end
    end
    return Basis;
end

## 3.
function Poisson_noise(signal)
    return rand.(Poisson.(signal));
end

## 4. This two functions are genrating a 0-1 Walsh Matrix
## for compressed sensing, just select part of this rows
#=
function DMD_freq_pattern_old(r,c, cycles = false)
    if cycles == false || cycles < 0 || cycles > r*c;
        cycles = r*c;
    end

    freq_pattern = zeros(r,c);
    ratio = (cycles/r/c)^0.5;

    r1 = Int(ceil(ratio * r))-1;
    c1 = Int(ceil(ratio * c))-1;

    freq_pattern[1:r1, 1:c1] .= 1;

    n_res = cycles - r1 * c1;
    res_index = findall(x->x == 0, freq_pattern[1:r1+1, 1:c1+1]);

    index_distance = zeros(length(res_index), 2);
```

```

    for i = 1:length(res_index)
      index_distance[i,1] = i;
      # res_index[i] is a tuple
      index_distance[i,2] = res_index[i][1]/(r1+0.0) + res_index[i][2]/(c1+0.0);
    end

    sorted_index_distance = index_distance[sortperm(index_distance[:,2]),:];
    selected_index = sorted_index_distance[:,1][1:n_res];

    for i = 1:n_res
      rc_cartesian = res_index[Int(selected_index[i])];
      freq_pattern[rc_cartesian[1], rc_cartesian[2]] = 1;
    end

    return freq_pattern;

end
=#

function DMD_freq_pattern(r,c, cycles = false)
     if cycles == false || cycles < 0 || cycles > r*c;
      cycles = r*c;
    end

    freq_pattern = zeros(r,c);
    n_res = cycles;
    res_index = findall(x->x == 0, freq_pattern);


    index_distance = zeros(length(res_index), 2);

    for i = 1:length(res_index)
      index_distance[i,1] = i;
      # res_index[i] is a tuple
      index_distance[i,2] = res_index[i][1]/(r+0.0) + res_index[i][2]/(c+0.0);
    end

    sorted_index_distance = index_distance[sortperm(index_distance[:,2]),:];
    selected_index = sorted_index_distance[:,1][1:n_res];

    for i = 1:n_res
      rc_cartesian = res_index[Int(selected_index[i])];
      freq_pattern[rc_cartesian[1], rc_cartesian[2]] = 1;
    end

    return freq_pattern;
end

## 5
function DMD_measure_basis(r, c, cycles = false, tag = "Walsh")
    ## it returns a DMD 0-1 matrix as measure basis
    ## this matrix has "cycles" rows and "r*c" columns
    if cycles == false
      cycles = r*c;
    end

    if tag == "Walsh"
      DMD_basis = zeros(cycles, r*c);
      #freq_pattern = DMD_freq_pattern(r,c, cycles);
      freq_pattern = ones(r, c);
```

```
    #rows = 1:r*c;
    #rows_selected = randperm(rows)[1:cycles];

    index_total = findall(x->x==1.0, freq_pattern);
    index = index_total[randperm(cycles)];
    FT_domain = zeros(r,c);
    for i = 1:length(index)
      current_index = index[i];
      FT_domain[current_index[1], current_index[2]] = 1;
      DMD_zero_one = ceil.(idct(FT_domain)); ## 0-1 pattern on DMD

      DMD_basis[i,:] = reshape(DMD_zero_one',(1,:));
      FT_domain[current_index[1], current_index[2]] = 0;
    end
    return DMD_basis;
  elseif tag == "HyperReso"
    DMD_basis = zeros(cycles, r*c);

    freq_pattern = DMD_freq_pattern(r,c, cycles);

    index_total = findall(x->x==1.0, freq_pattern);
    index = index_total[randperm(cycles)];
    FT_domain = zeros(r,c);
    for i = 1:length(index)
      current_index = index[i];
      FT_domain[current_index[1], current_index[2]] = 1;
      DMD_zero_one = ceil.(idct(FT_domain)); ## 0-1 pattern on DMD

      DMD_basis[i,:] = reshape(DMD_zero_one',(1,:));
      FT_domain[current_index[1], current_index[2]] = 0;
    end
    return DMD_basis;

  elseif tag == "Random"
    DMD_basis = round.(rand(cycles, r*c))
    return DMD_basis;
  else
    print("Wrong Tag");
  end
end



## 6 direct upload a black white image and get the measurement
function PMT_measure_simu(photon_number, cycles, upload_image, upload_basis, Poisson = true)
  ### photon - number is the photons of each pixel during each measurement
  ### cycles : the compression rate;
  ### Poisson : true, adding poisson noise
  ### upload_image : Black white image (matrix)
  image = upload_image / maximum(upload_image);
  r_tot, c_tot = size(upload_image);
  #=
  if upload_basis == "null"
    DMD_basis = DMD_measure_basis(r_tot, c_tot, cycles);
  else
    DMD_basis = upload_basis[1:cycles,:];
  end
  =#
  DMD_basis = upload_basis[1:cycles,:];

```

```
    image_vector = reshape(image', (:, 1));
    image_vector *= photon_number; ## here you are light up a image

    measurements = zeros(cycles,1);
    ### if you want to do RGB image later, set zeros(cycles,3)

    for i = 1:cycles
        val = DMD_basis[i:i,:] * image_vector;

        if Poisson == true
            val = Poisson_noise(val);
        end
        measurements[i,:] = val;
    end

    measurements /= photon_number;

    return measurements;
end

## 7 load image as matrix
function Img2matrix(pathname, r = false, c = false, RGB = true)
    img = channelview(load(pathname));
    r0, c0 = size(img[1,:,:]);

    if r == false
        r = r0;
    end
    if c == false
        c = c0;
    end

    red = imresize(img[1,:,:], (r,c));
    green = imresize(img[2,:,:], (r,c));
    blue = imresize(img[3,:,:], (r,c));

    if RGB == true
        matrix = zeros(r,c,3);
        matrix[:,:,1] = red;
        matrix[:,:,2] = green;
        matrix[:,:,3] = blue;

    else
        matrix = zeros(r,c);
        matrix = 0.21 * red  + 0.72 * green + 0.07 * blue;
    end

    return matrix;
end
```

Out[1]:

```
Img2matrix (generic function with 4 methods)
```

make a Haar

In [2]:

```
function Kron(mat1, mat2)
   r1,c1 = size(mat1);
   r2,c2 = size(mat2);
   prod = zeros(r1*r2, c1*c2);
   for r = 1:r1
      for c = 1:c1
         prod[(r-1)*r2 + 1 : r*r2, (c-1)*c2+1, c*c2] = mat1[r1,c1] .* mat2;
      end
   end
   return prod;
end

function inv_Haar_matrix(R)
   #R = edge1*edge2;
   n = ceil(log2(R));
   H = [1 1;1 -1] .* (0.5)^0.5;
   for i = 1:(n-1)
      N = Int(round(2^i));
      top = H;
      bottom = I(N);
      H = [kron(top,[1 1] .* (0.5)^0.5) ; kron(bottom, [1 -1].* (0.5)^0.5)];
   end
   return imresize(H', (R,R));
end

function Haar_basis(r_tot, c_tot)
   D1 = inv_Haar_matrix(r_tot);
   D2 = inv_Haar_matrix(c_tot);
   N = r_tot * c_tot;
   Basis = zeros(N,N);

   i = 1;
   for ir = 1:r_tot
      for ic = 1:c_tot

         Basis_Vector = D1[ir:ir,:]' * D2[ic:ic,:];
         Basis[i,:] = reshape(Basis_Vector', (1,:));
         i += 1;
      end
   end
   return Basis;
end
```

Out[2]:

```
Haar_basis (generic function with 1 method)
```

In [3]:

```
function Resize(mat, new_size)
  r,c = new_size;
  r0, c0 = size(mat);
  mat2 = zeros(r,c)
  for i = 1:r
    y = Int(ceil(i * r0 / r));
    for j = 1:c
      x = Int(ceil(j * c0 / c));
      mat2[i,j] = mat[y,x];
    end
  end
  return mat2;
end

function Haar_tree(seed_raw)
  #if sum(abs.(seed_raw)) == 0
    #print("wrong");
  #end

  r,c = size(seed_raw);
  r2 = Int(round(r/2));
  c2 = Int(round(c/2));
  seed = Resize(seed_raw, (r2,c2));
  k1 = [1 0; 0 0];
  k2 = [0 1; 0 0];
  k3 = [0 0; 1 0];
  k4 = [0 0; 0 1];

  out = zeros(2*r, 2*c);

  out[1:r,1:c] = kron(k1, seed);
  out[1:r,c+1:2*c] = kron(k2, seed);
  out[r+1:2*r,1:c] = kron(k3, seed);
  out[r+1:2*r, c+1:2*c] = kron(k4, seed);

  return out;
end

function make_tensor(n, seed, row,col)
  # r is the edge
  r = 2^(n);

  #r0,c0 = size(matrix);
  #n = Int(round(r0/r));
  tensor = zeros(row,col,2^(2*n));
  layer = 1;

  sub = zeros(r,r);
  for i = 1:2^n
    for j = 1:2^n

      #sub = matrix[(i-1)*r+1: i*r, (j-1)*r+1:j*r];
      sub[i,j] = 1;

      #if sum(abs.(sub)) == 0
        #print("wrong");
      #end
      tensor[:,:,layer] = Resize(kron(sub,seed), (row,col));
      sub[i,j] = 0;
```

```
      layer+=1;
    end
  end
  return tensor;
end

function Haar_recursion(row,col)
  ## currently please let row = col = 2^n
  haar_tensor = zeros(row,col,(row*col));

  seed1= [1 1; 1 1];
  seed2= [1 -1; 1 -1];
  seed3 = [1 1; -1 -1];
  seed4 = [1 -1; -1 1];

  kernel1 = Resize([1 1; 1 1], (row,col));
  kernel2 = Resize([1 -1; 1 -1], (row,col));
  kernel3 = Resize([1 1; -1 -1],(row,col));
  kernel4 = Resize([1 -1; -1 1],(row,col));
  haar_tensor[:,:,1] = kernel1;
  haar_tensor[:,:,2] = kernel2;
  haar_tensor[:,:,3] = kernel3;
  haar_tensor[:,:,4] = kernel4;

  layer = 4;
  kernel_edge = row;
  N = 1;
  while kernel_edge >= 4

    #kernel2 = Haar_tree(kernel2);

    t1 = make_tensor( N, seed2, row,col);
    L = length(t1[1,1,:]);
    haar_tensor[:,:, layer+1:layer+L] = t1;
    layer += L;

    #kernel3 = Haar_tree(kernel3);
    t1 = make_tensor( N, seed3, row,col);
    L = length(t1[1,1,:]);
    haar_tensor[:,:, layer+1:layer+L] = t1;
    layer += L;

    #kernel4 = Haar_tree(kernel4);
    t1 = make_tensor( N, seed4, row,col);
    L = length(t1[1,1,:]);
    haar_tensor[:,:, layer+1:layer+L] = t1;
    layer += L;

    kernel_edge /= 2;
    N += 1;
  end

  haar_2d = zeros(row*col,row*col);
  for i = 1:row*col
    vec = reshape(haar_tensor[:,:,i]', (1,:));
    abs_sum = sum(abs.(vec));
    haar_2d[i,:] = vec ./ abs_sum^0.5;
  end
  return haar_2d;
end
```

Out[3]:

Haar_recursion (generic function with 1 method)

In [21]:

```
r = 64;
c = 64;
cycles = convert(Int, round(r * c * 0.25));
#cycles = 1000;

photon_number = 100;

raw_img = Img2matrix("ECE.jpg", r,c, false);
img_vector = reshape(raw_img', (:,1));
DMD_basis = DMD_measure_basis(r,c, cycles, "HyperReso");
#DMD_basis = DMD_measure_basis(r,c, cycles, "Random");
#DMD_basis = DMD_measure_basis(r,c, cycles, "Walsh");

measure = PMT_measure_simu(photon_number, cycles, raw_img, DMD_basis, false);
```

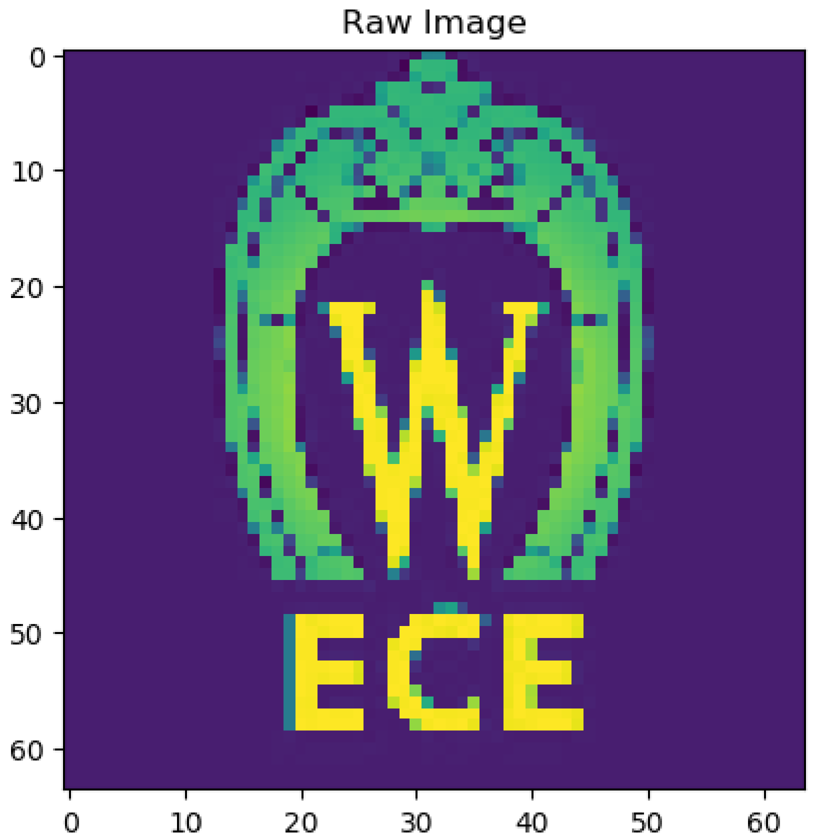
In [5]:

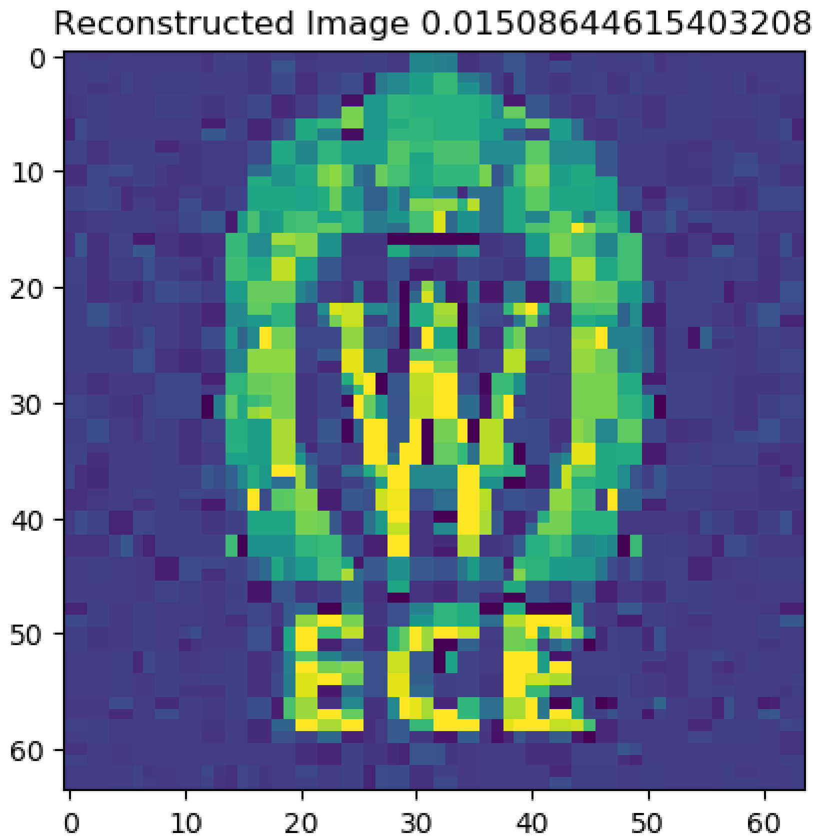
```
## reconstruction
using JuMP, Gurobi;
```

Hard constriant!

In [22]:

```

m = Model(Gurobi.Optimizer);
set_silent(m);
#_phi = IDCT_basis(r,c);
#_phi = Haar_basis(r,c);
#_phi = inv_Haar_matrix(r*c)
_phi = Haar_recursion(r,c)';
tot_mat = DMD_basis*_phi;

a = @variable(m, [1:r*c, 1:1]);
a_abs = @variable(m, [1:r*c, 1:1]);
#@constraint(m, tot_mat*(a) .== measure);
@constraint(m, tot_mat*a - measure .<= 1e-1);
@constraint(m, tot_mat*a - measure .>= -1e-1);
@constraint(m, a_abs .>= a);
@constraint(m, a_abs .>= -a);

@objective(m, Min, sum(a_abs));
optimize!(m)

img_recons_vec = _phi * value.(a);
img_recons = reshape(img_recons_vec, (r,c))';

img_recons[findall(x->x>1, img_recons)] .= 1;
img_recons[findall(x->x<0, img_recons)] .= 0;

figure();
imshow(raw_img);
title("Raw Image");

figure();
imshow(img_recons);
comments = "Reconstructed Image $(sum((raw_img .- img_recons).^2) ./ (r*c))";
title(comments);

println("Objective value ->", objective_value(m));
```






```
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Objective value ->328.36409476111083
```
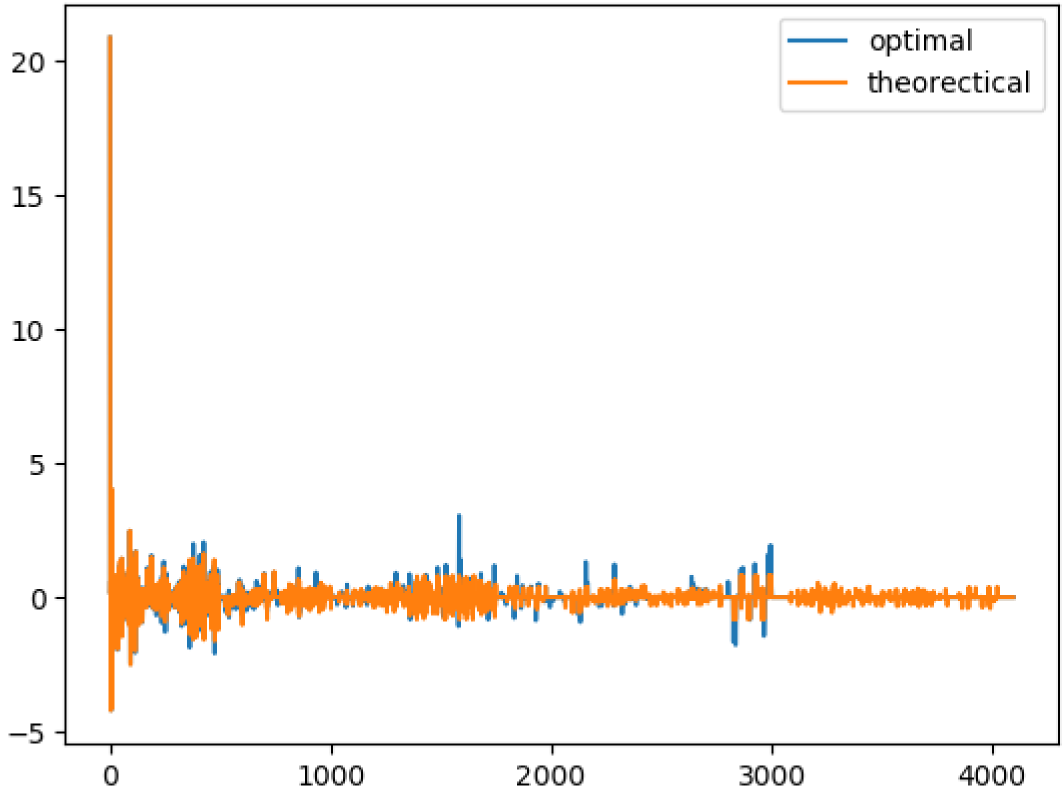

In [ ]:

```
1
```

In [7]:

```
#prod1 = inv_Haar_matrix(r*c)' * img_vector;
#prod = Haar_basis(r,c) * img_vector;
prod1 = Haar_recursion(r,c) * img_vector;
figure();
plot(value.(a), label = "optimal");
plot(prod1, label = "theorectical");
legend();


```



In [ ]:

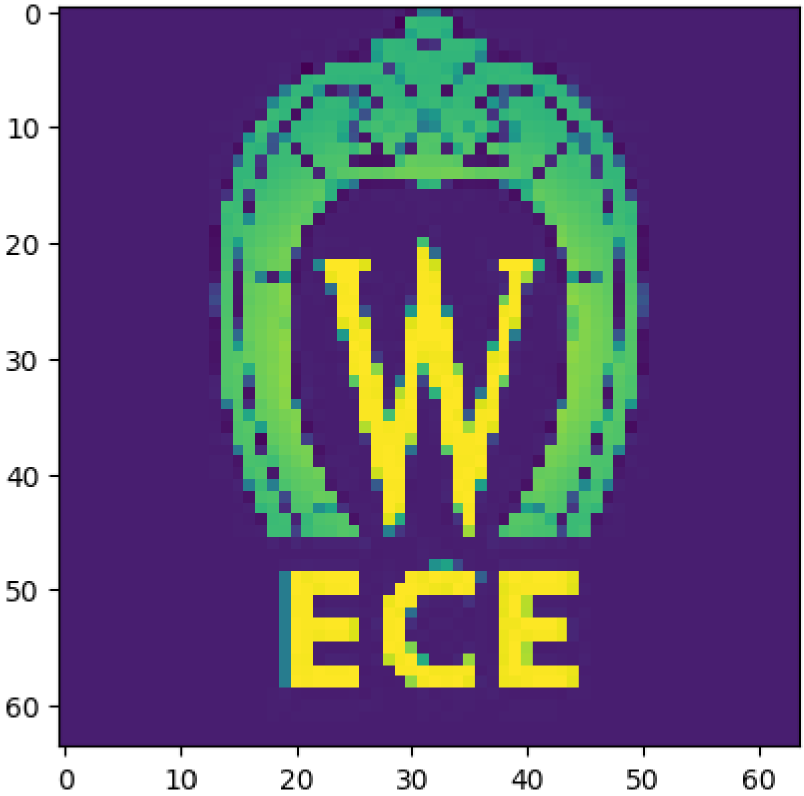
```

```

Compare DCT basis and Haar basis

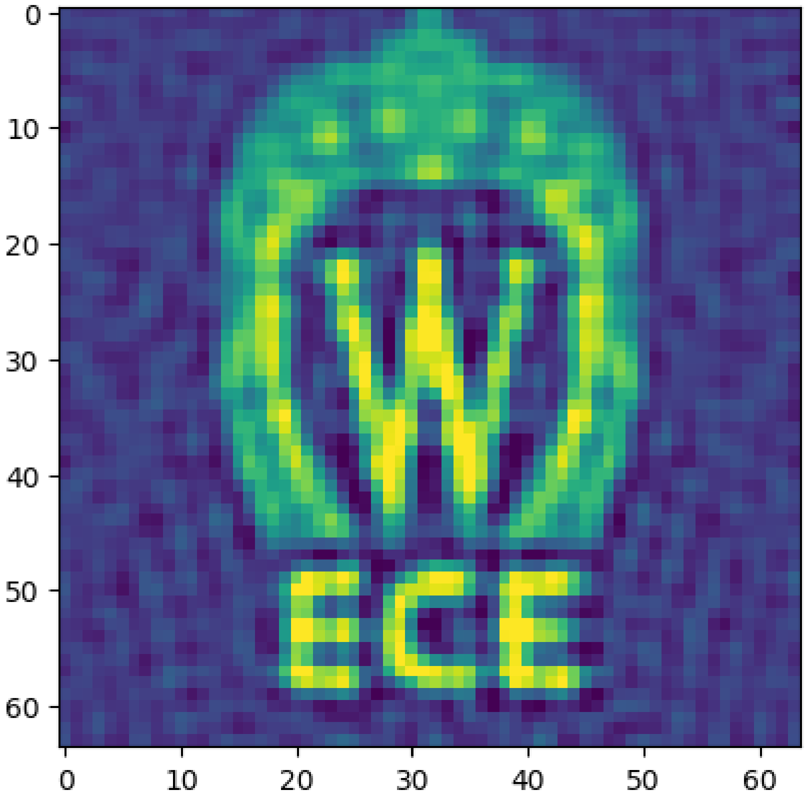
In [23]:

```

m = Model(Gurobi.Optimizer);
set_silent(m);
_phi = (IDCT_basis(r,c));
#_phi = Haar_basis(r,c);
tot_mat = DMD_basis*_phi;

a = @variable(m, [1:r*c, 1:1]);
a_abs = @variable(m, [1:r*c, 1:1]);
#@constraint(m, tot_mat*(a) .== measure);
@constraint(m, tot_mat*a - measure .<= 1e-1);
@constraint(m, tot_mat*a - measure .>= -1e-1);
@constraint(m, a_abs .>= a);
@constraint(m, a_abs .>= -a);

@objective(m, Min, sum(a_abs));
optimize!(m);


img_recons_vec = _phi * value.(a);
img_recons = reshape(img_recons_vec, (r,c))';

img_recons[findall(x->x>1, img_recons)] .= 1;
img_recons[findall(x->x<0, img_recons)] .= 0;

figure();
imshow(raw_img);
title("Raw Image");

figure();
imshow(img_recons);
comments = "Reconstructed Image $(sum((raw_img .- img_recons).^2) ./ (r*c))";
title(comments);

println("Objective value ->", objective_value(m));
```

Raw Image



Reconstructed Image 0.013627139693928666



```
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Objective value ->286.9597210000526
```
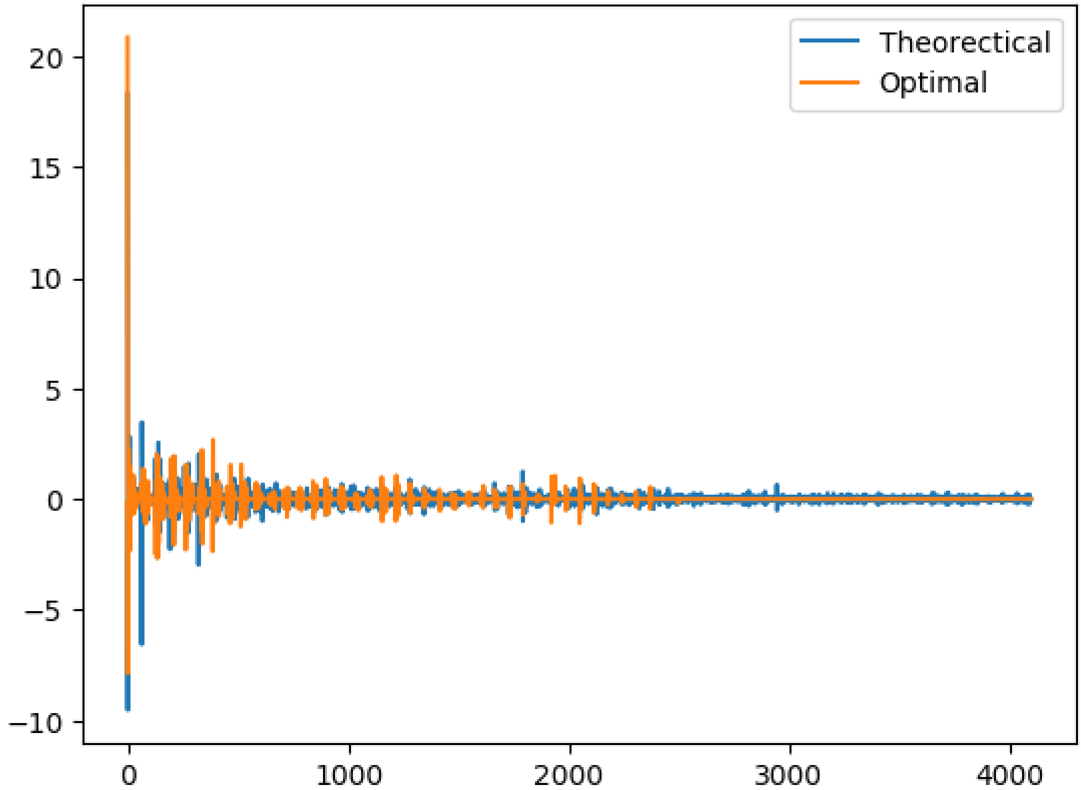
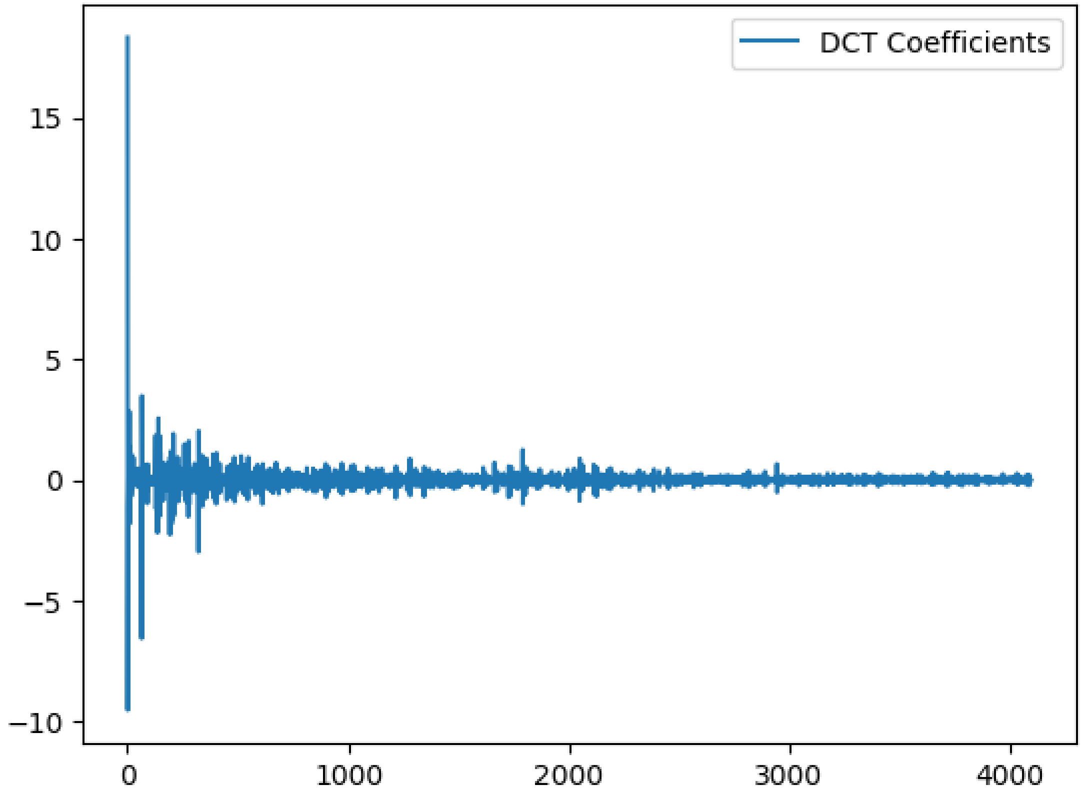

In [9]:

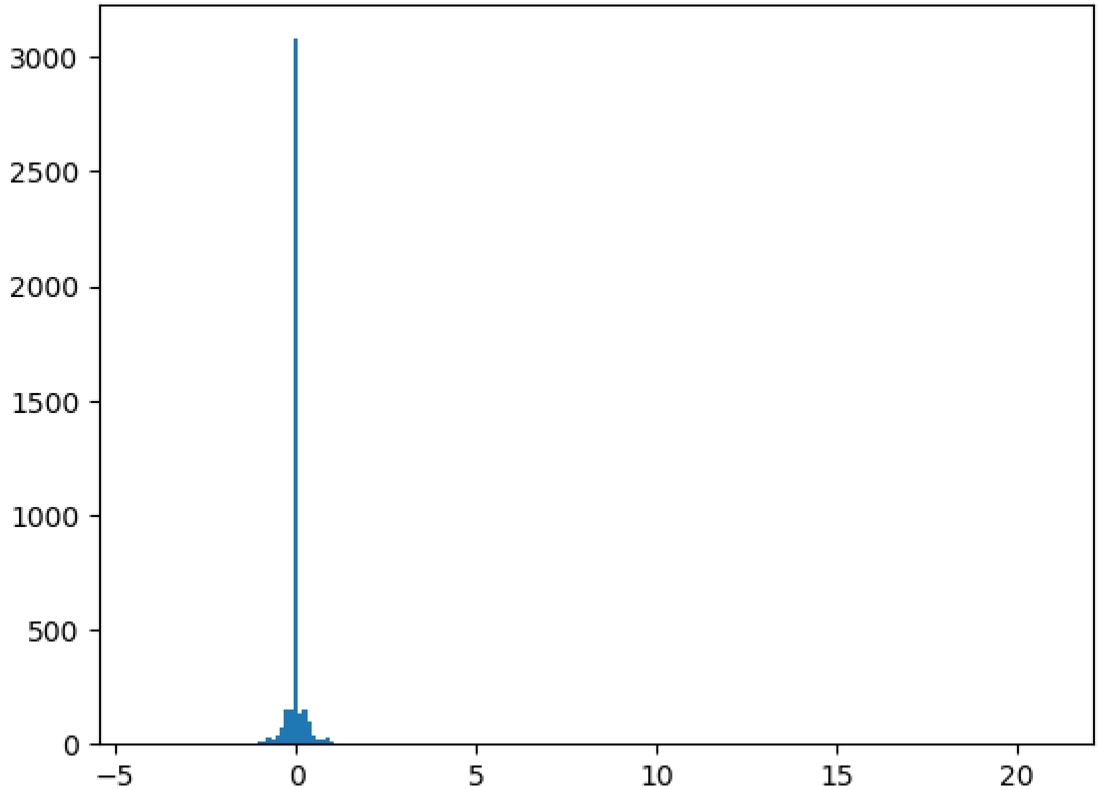
```
prod2 = (IDCT_basis(r,c)) * img_vector;
figure();
plot(prod2, label = "Theorectical");
plot(value.(a), label = "Optimal");
legend();
```

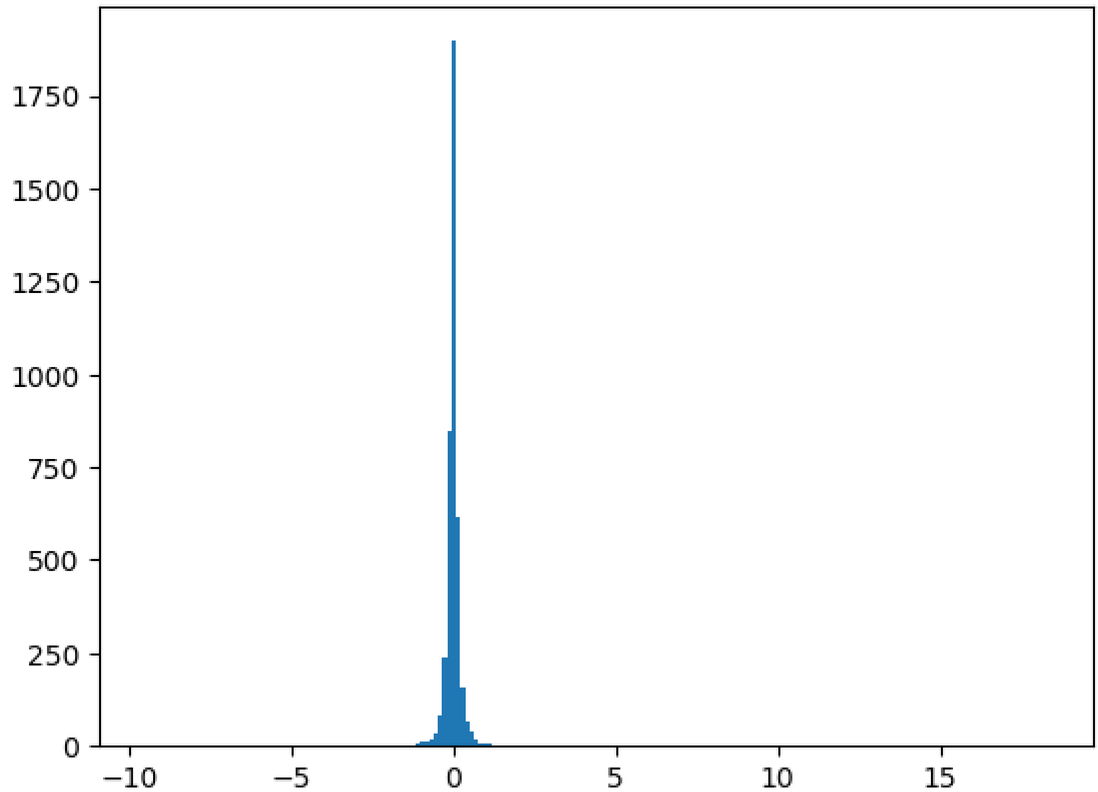


In [28]:

```
figure();
#plot((1:length(prod1)), prod1, label = "Haar Coefficients");
plot((1:length(prod2)), prod2, label = "DCT Coefficients");
legend();
bins = 200;
figure();
hist(prod1, bins);
figure();
hist(prod2,bins);
```




3000
2500
2000
1500
1000
500
0
−5
0
5
10
15
20


1750
1500
1250
1000
750
500
250
0
−10
−5
0
5
10
15

Soft constraint

[Lorenzo Yizhou Lu 1:41am 4/26/2020]
I found the running time when we set the direct Lasso regression objective is **Extremly Long**. It could be a barrier for us to think about a strategy to reconstruct this image during much shorter time.

In [11]:

```
#=
#using Ipopt;
lasso = 1e-4;
m = Model(Gurobi.Optimizer);
set_silent(m);
_phi = IDCT_basis(r,c);
tot_mat = DMD_basis*_phi;

a = @variable(m, [1:r*c, 1:1]);
a_abs = @variable(m, [1:r*c, 1:1]);

@constraint(m, a_abs .>= a);
@constraint(m, a_abs .>= -a);

@objective(m, Min, lasso * sum(a_abs) + sum((tot_mat*a - measure).* (tot_mat*a - measure))/(cyc
optimize!(m);
=#
```

In [12]:

```
#=
img_recons_vec = _phi * value.(a);
img_recons = reshape(img_recons_vec, (r,c))';

img_recons[findall(x->x>1, img_recons)] .= 1;
img_recons[findall(x->x<0, img_recons)] .= 0;

figure();
imshow(raw_img);
title("Raw Image");

figure();
imshow(img_recons);
title("Reconstructed Image");
=#
```

In [13]:

```
#=
img_vector = reshape(raw_img', (:,1));
#prod = Haar_basis(r,c) * img_vector;

figure();
plot(prod[1:2:end,:]);
#figure();
plot(prod[2:2:end,:]);
plot(prod);
#plot(inv_Haar_matrix(r*c)' * img_vector);
=#
```

In [ ]:

```

```

In [ ]:

```

```

In [ ]:

```


```

In [14]:

```
#=
DMD_basis = DMD_measure_basis(r,c, r*c, "Walsh");
haar = Haar_basis(r,c);
dct = IDCT_basis(r,c);
coh1 = 0;
coh2 = 0;
for row = 1:r*c
   v1 = (DMD_basis[row:row, :] * haar');
   max_v1 = maximum(abs.(v1)) * (r*c)^0.5;
   if coh1 < max_v1
      coh1 = max_v1;
   end

   v2 = DMD_basis[row:row, :] * dct';
   max_v2 = maximum(abs.(v2)) * (r*c)^0.5;
   if coh2 < max_v2
      coh2 = max_v2;
   end
end
=#
```

In [15]:

```
#print(coh1, '\n', coh2)
```

For Random DMD basis,
Haar coherence is 256.27;
DCT coherence is 1807.13;

For dct DMD basis,
Haar coherence is 320.39;
DCT coherence is 3353.59;

In [16]:

```
#hh = inv_Haar_matrix(64*64);
#hist(hh * img_vector, 100);
```

In [17]:

```
#hh = Haar_basis(10,10)
##imshow(ceil.(hh))
```

In [ ]:

```

```

In [ ]:

```

```

In [ ]:

```

```

In [ ]:

```


```

In [ ]:

```

```

In [18]:

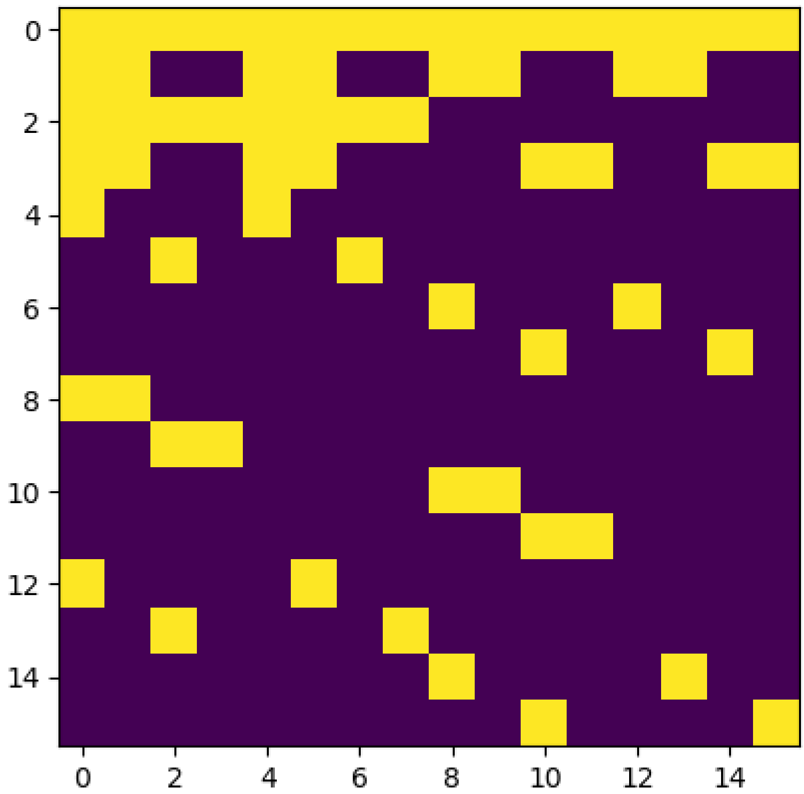
```
sum(abs.(Haar_recursion(4,4) * Haar_recursion(4,4)' - I(16)))
```

Out[18]:

0.0

In [20]:

```
imshow(ceil.(Haar_recursion(4,4)))
```



Out[20]:

PyObject <matplotlib.image.AxesImage object at 0x0000000000EBF2C8>

In [ ]:

```



```

In [ ]:

```
```

In [ ]:

```
```