hw5

>>Lorenzo Lu<<
>>ylu289<<

March 13, 2020

1 HW5 Least Squares

1.1 Q1. Spline fitting

[1]:
```
using CSV;
data = CSV.read("xy_data.csv");
```

1.1.1 a). Polynomial

$$\begin{aligned} \underset{a_1,a_2,a_3,a_4}{\text{minimize}} \quad & \sum_i (x_i^3 a_1 + x_i^2 a_2 + x_i a_3 + a_4 - y_i)^2 && (1) \\ \text{subject to:} \quad & a_4 = 0 \qquad \text{(constraint 1)} && (2) \\ & && (3) \end{aligned}$$

[2]:
```
#using Pkg
#Pkg.add("Gurobi")
```

[3]:
```
#Pkg.build("Gurobi")
using JuMP, LinearAlgebra, Ipopt;
```

[4]:
```
(r,c) = size(data);

highest_order = 3;
Xc = highest_order + 1;
X = zeros((r,Xc));
for i = 1:Xc
    X[:,Xc-i+1] = data[:,1].^(i-1);
end

m1 = Model(with_optimizer(Ipopt.Optimizer))
set_silent(m1);
a = @variable(m1, [1:Xc]);
residual1 = @variable(m1, [1:r,1]);
residual2 = @variable(m1, [1:r,1]);
@constraint(m1, a[Xc] == 0);
```

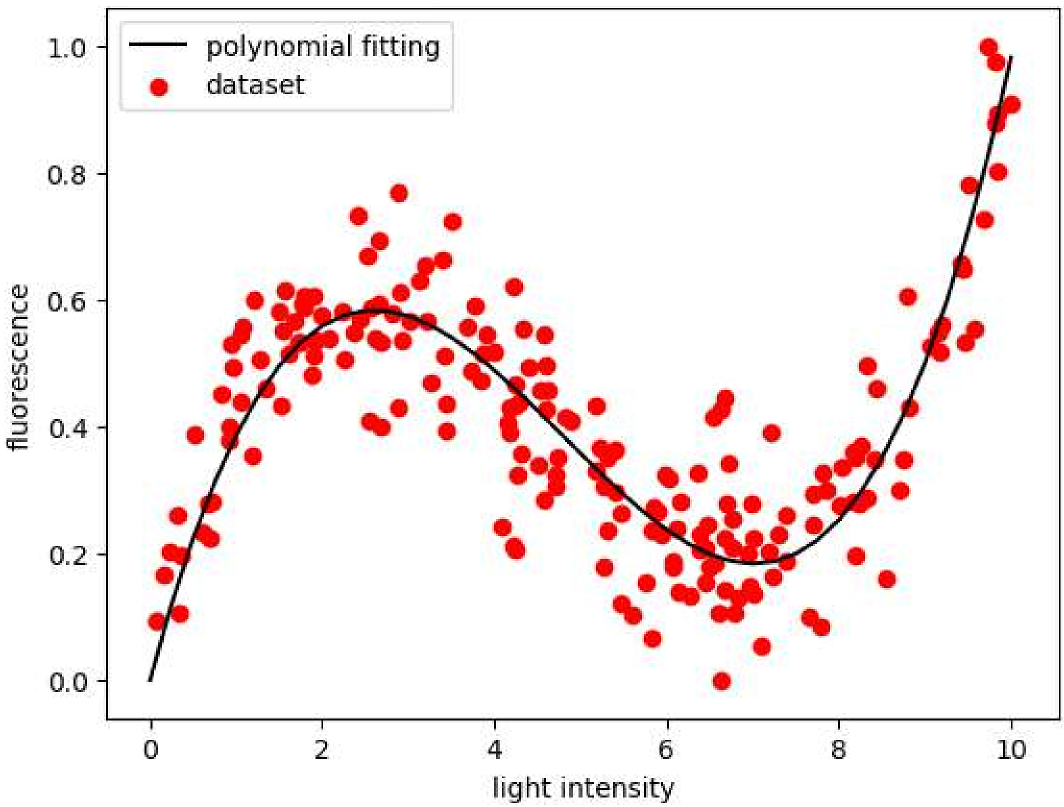
```
for i = 1:r
    @constraint(m1, residual1[i] - ((X[i,:]' * a - data[i,2])).^2 == 0);


end

@objective(m1, Min, sum(residual1));
optimize!(m1);
```

```
******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.10, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:     4176
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian...:     1990

Total number of variables...:      402
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:      200
Total number of inequality constraints...:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  0.0000000e+00 1.00e+00 1.72e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1 -1.2881730e+02 1.79e+00 2.14e+04  -1.0 1.45e+00   0.0 1.00e+00 1.00e+00f  1
   2 -3.9057527e+02 3.01e+01 8.19e+04  -1.0 4.34e+00  -0.5 1.00e+00 1.00e+00f  1
   3 -9.3383951e+02 1.56e+01 1.36e+03  -1.0 1.08e+01  -1.0 1.00e+00 1.00e+00f  1
   4 -5.1247151e+02 1.15e+01 1.31e+04  -1.0 1.29e+01  -1.4 1.00e+00 1.00e+00h  1
   5 -2.8262756e+02 4.15e+00 1.35e+02  -1.0 3.52e+00  -1.9 1.00e+00 1.00e+00h  1
   6 -3.4124977e+00 7.81e-02 6.63e+01  -1.0 4.09e+00  -2.4 1.00e+00 1.00e+00h  1
   7  1.7728084e+00 6.25e-06 1.86e-01  -1.0 7.82e-02  -2.9 1.00e+00 1.00e+00h  1
   8  1.7733305e+00 9.00e-10 9.01e-06  -2.5 2.62e-05  -3.3 1.00e+00 1.00e+00h  1
   9  1.7733306e+00 8.64e-14 3.09e-11  -8.6 2.35e-09  -3.8 1.00e+00 1.00e+00h  1

Number of Iterations...: 9
```

```
                                   (scaled)                 (unscaled)
Objective...:   1.7733305533111241e+00    1.7733305533111241e+00
Dual infeasibility...:   3.0922819860279560e-11    3.0922819860279560e-11
Constraint violation...:   1.6618650899857812e-14    8.6375351315837179e-14
Complementarity...:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error...:   3.0922819860279560e-11    3.0922819860279560e-11


Number of objective function evaluations             = 10
Number of objective gradient evaluations             = 10
Number of equality constraint evaluations            = 10
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 10
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 9
Total CPU secs in IPOPT (w/o function evaluations)   =      1.840
Total CPU secs in NLP function evaluations           =      0.530

EXIT: Optimal Solution Found.
```

[16]:

```
println("The best fitted 'a' values are ->", round.(value.(a),digits = 6));
## plot
using PyPlot;
Xcordinate = 0:0.25:10;
X_plot = zeros(length(Xcordinate),Xc);
for i = 1:Xc
    X_plot[:,Xc - i + 1] = Xcordinate.^(i-1);
end
Y_plot = X_plot * value.(a);

figure();
plot(Xcordinate,Y_plot, label = "polynomial fitting", c = "k");
scatter(data[:,1], data[:,2], label = "dataset", c = "r");
xlabel("light intensity");
ylabel("fluorescence");
legend();
```




```
The best fitted 'a' values are ->[0.009324, -0.13453, 0.5111, -0.0]
```

1.1.2 b). Spline

$$\underset{p_1,p_2,p_3,q_1,q_2,q_3}{\text{minimize}} \quad \sum_{i}^{0\le x_i<4} (x_i^2 p_1 + x_i p_2 + p_3 - y_i)^2 + \sum_{i}^{4\le x_i\le 10} (x_i^2 q_1 + x_i q_2 + q_3 - y_i)^2 \tag{4}$$

$$\text{subject to:} \quad 16p_1 + 4p_2 + p_3 = 16q_1 + 4q_2 + q_3 \qquad \text{(constraint 1)} \tag{5}$$

$$\text{subject to:} \quad 8p_1 + p_2 = 8q_1 + q_2 \qquad \text{(constraint 2)} \tag{6}$$

$$\text{subject to:} \quad p_3 = 0 \qquad \text{(constraint 3)} \tag{7}$$

[6]:
```
X_le4 = [];
Y_le4 = [];
X_ge4 = [];
Y_ge4 = [];

for i = 1:r
```

```
    if data[i,1] <4
        X_le4 = [X_le4; data[i,1]];
        Y_le4 = [Y_le4; data[i,2]];
    else
        X_ge4 = [X_ge4;data[i,1]];
        Y_ge4 = [Y_ge4; data[i,2]];
    end
end
```

[7]:

```
K = 3;
X_quad_le4 = zeros((length(X_le4),K));
X_quad_ge4 = zeros((length(X_ge4),K));

for i = 1:K
    X_quad_le4[:,K-i+1] = X_le4.^(i-1);
    X_quad_ge4[:,K-i+1] = X_ge4.^(i-1);
end
```

[8]:

```
m1b = Model(with_optimizer(Ipopt.Optimizer));
p = @variable(m1b, [1:K]);
q = @variable(m1b, [1:K]);

@constraint(m1b, 16*p[1] + 4 * p[2] + p[3] == 16*q[1] + 4 * q[2] + q[3]);
@constraint(m1b, 8*p[1] + p[2] == 8 * q[1] + q[2]);
@constraint(m1b, p[3] == 0);

@objective(m1b, Min, sum((X_quad_le4 * p - Y_le4).^2) + sum((X_quad_ge4 * q -
 ↪Y_ge4).^2));
optimize!(m1b)
```

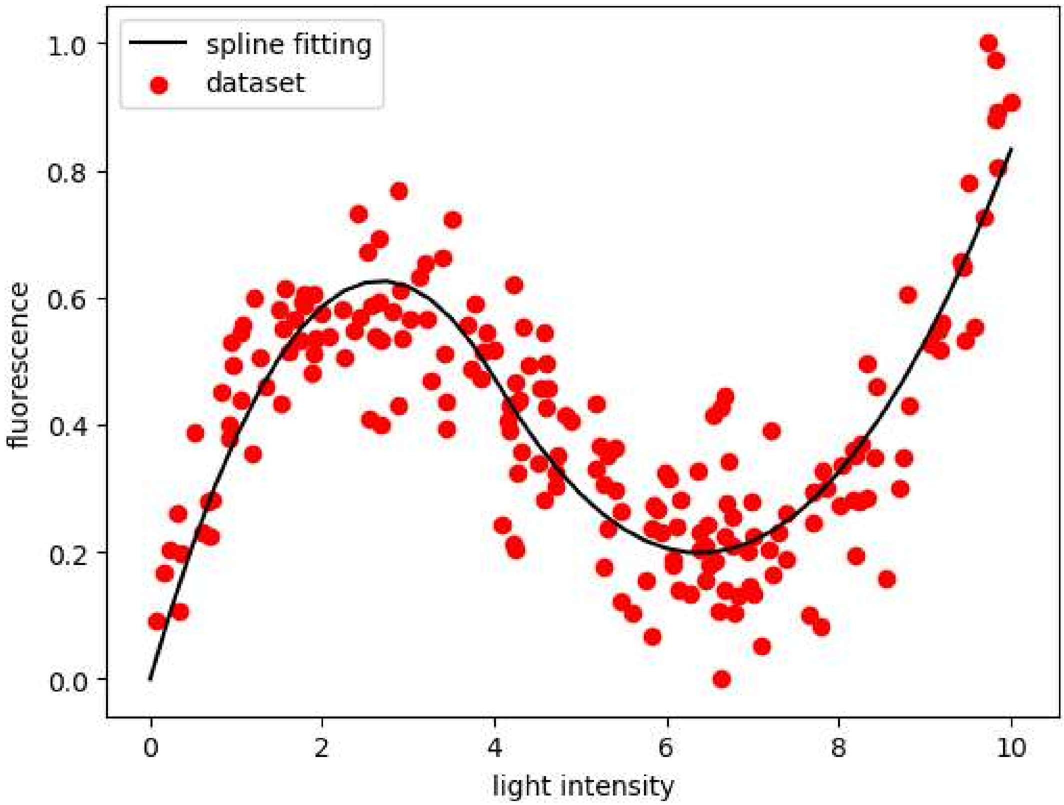
```
This is Ipopt version 3.12.10, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:       11
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:       12

Total number of variables............................:        6
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        3
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0
```

```
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.9316735e+01 0.00e+00 5.49e+01  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.9507600e+00 4.44e-16 1.75e-13  -1.0 2.17e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations....: 1

                                   (scaled)                 (unscaled)
Objective...:   4.2324566241679205e-02    1.9507600232172990e+00
Dual infeasibility...:   1.7519319328584970e-13    8.0747402312482086e-12
Constraint violation...:   4.4408920985006262e-16    4.4408920985006262e-16
Complementarity...:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error...:   1.7519319328584970e-13    8.0747402312482086e-12


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 2
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 1
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.220
Total CPU secs in NLP function evaluations           =      0.070

EXIT: Optimal Solution Found.
```

[9]:

```
println("The optimal p values are->",round.(value.(p),digits = 6));
println("The optimal q values are->",round.(value.(q),digits = 6));
## plot
using PyPlot;
Xcoordinate = 0:0.25:10;
X_plot = zeros(length(Xcoordinate),K);

piece = 0; ## mark the index where there is a piece wise
for i = 1:(length(Xcoordinate)-1)
    if Xcoordinate[i] < 4 && Xcoordinate[i+1] >= 4
        piece = i;
    end
end

for i = 1:K
    X_plot[:,K - i + 1] = Xcoordinate.^(i-1);
end

Y_plot = zeros((length(Xcoordinate),1));
Y_plot[1:piece] = X_plot[1:piece,:] * value.(p);
```

```
Y_plot[piece+1:length(Xcoordinate)] = X_plot[piece+1:length(Xcoordinate),:] *␣
↪value.(q);

figure();
plot(Xcoordinate,Y_plot, label = "spline fitting", c = "k");
scatter(data[:,1], data[:,2], label = "dataset", c = "r");
xlabel("light intensity");
ylabel("fluorescence");
legend();
```




```
The optimal p values are->[-0.087316, 0.467636, 0.0]
The optimal q values are->[0.048463, -0.618597, 2.172466]
```

1.2 Q2. Voltage smoothing

$$\underset{V_{smooth}}{\text{minimize}} \qquad ||V_{raw} - V_{smooth}||^2 + \lambda R(V_{smooth}) \tag{8}$$

$$\tag{9}$$

The $R()$ is a function calcuating the smoothness of a voltage $R(x) = \sum_i^{n-1}(x_{i+1} - x_i)^2$
The λ is the regularization factor.

[10]:
```
using CSV
data2 = CSV.read("voltages.csv");
```

[11]:
```
raw_volt = data2[:,1];
```

[12]:
```
function smoothness(volt)
    L = length(volt);
    R = 0;
    for i = 1:(L-1)
        R = R + (volt[i+1] - volt[i])^2;
    end
    return R;
end
```

[12]:
```
smoothness (generic function with 1 method)
```

[13]:
```
using Ipopt, JuMP;
m2 = Model(with_optimizer(Ipopt.Optimizer));
set_silent(m2);
lambda = [10.0^i for i = -1:2];

opt_volt = zeros((length(raw_volt), length(lambda)));
new_volt = @variable(m2, [1:length(raw_volt)]);

for i = 1:length(lambda)
    @objective(m2, Min, sum((new_volt - raw_volt).^2) + lambda[i]*␣
 ↪smoothness(new_volt));
    optimize!(m2);
    opt_volt[:,i] = value.(new_volt);
end
```

```
This is Ipopt version 3.12.10, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:      397

Total number of variables............................:      199
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:        0
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0
```

```
iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.9000000e+02 0.00e+00 4.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  2.7044936e+00 0.00e+00 1.55e-15  -1.0 2.00e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations...: 1

                                   (scaled)                 (unscaled)
Objective...:   2.7044936151370731e+00    2.7044936151370731e+00
Dual infeasibility...:   1.5543122344752192e-15    1.5543122344752192e-15
Constraint violation...:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity...:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error...:   1.5543122344752192e-15    1.5543122344752192e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.000
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.10, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian...:      397

Total number of variables...:      199
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:        0
Total number of inequality constraints...:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.9000000e+02 0.00e+00 4.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.4310953e+01 0.00e+00 2.22e-15  -1.0 2.00e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations...: 1
```

```
                                   (scaled)                 (unscaled)
Objective...:   1.4310953235786460e+01    1.4310953235786460e+01
Dual infeasibility...:   2.2204460492503131e-15    2.2204460492503131e-15
Constraint violation...:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity...:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error...:   2.2204460492503131e-15    2.2204460492503131e-15


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.000
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.10, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian...:      397

Total number of variables...:      199
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:       0
Total number of inequality constraints...:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.9000000e+02 0.00e+00 4.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  5.0143390e+01 0.00e+00 1.42e-14  -1.0 1.94e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations...: 1

                                   (scaled)                 (unscaled)
Objective...:   5.0143389693362174e+01    5.0143389693362174e+01
Dual infeasibility...:   1.4210854715202004e-14    1.4210854715202004e-14
Constraint violation...:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity...:   0.0000000000000000e+00    0.0000000000000000e+00
```

```
Overall NLP error...:   1.4210854715202004e-14    1.4210854715202004e-14


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.000
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
This is Ipopt version 3.12.10, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:        0
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian...:      397

Total number of variables...:      199
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints...:        0
Total number of inequality constraints...:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  3.9000000e+02 0.00e+00 4.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.4085581e+02 0.00e+00 1.14e-13  -1.0 1.39e+00    -  1.00e+00 1.00e+00f  1

Number of Iterations...: 1

                                   (scaled)                 (unscaled)
Objective...:   1.4085580642066014e+02    1.4085580642066014e+02
Dual infeasibility...:   1.1368683772161603e-13    1.1368683772161603e-13
Constraint violation...:   0.0000000000000000e+00    0.0000000000000000e+00
Complementarity...:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error...:   1.1368683772161603e-13    1.1368683772161603e-13


Number of objective function evaluations             = 2
Number of objective gradient evaluations             = 2
Number of equality constraint evaluations            = 0
```

```
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 0
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 1
Total CPU secs in IPOPT (w/o function evaluations)   =      0.000
Total CPU secs in NLP function evaluations           =      0.000

EXIT: Optimal Solution Found.
```

[14]:
```
using PyPlot;
figure();
plot(raw_volt, label = "raw volt", linewidth = 3);
for i = 1:length(lambda)
    #LABEL = [""regularization =""]
    plot(opt_volt[:,i], label = ("regularization = "*string(lambda[i])));
end

legend();
xlabel("time");
ylabel("voltage");
```