Homework 7: Convex Programming

1.Enclosing circle.

Variables: The coordinate of the circle's center; the radius of this circle;
Objective:$\underset{R}{min}(\pi R^2)$
where R is the radius of the circle;
Constraints: $(X_i - C)^2 < R^2$
where C is the center's coordinate of the circle, X_i is the coordinate of the i_{th} point.

In [1]:

```
## generate the data points
using PyPlot, JuMP, Gurobi;

num = 50;
X = randn(2,num) .+ 4;

m1 = Model(Gurobi.Optimizer);
set_silent(m1);

C = @variable(m1, [1:2,1:1]); ## circle center
@variable(m1,R2); ## radius's square


for i = 1:num
   @constraint(m1, sum((X[:,i] - C).^2) <= R2);
end

@objective(m1, Min, pi * R2);
optimize!(m1);

```

```
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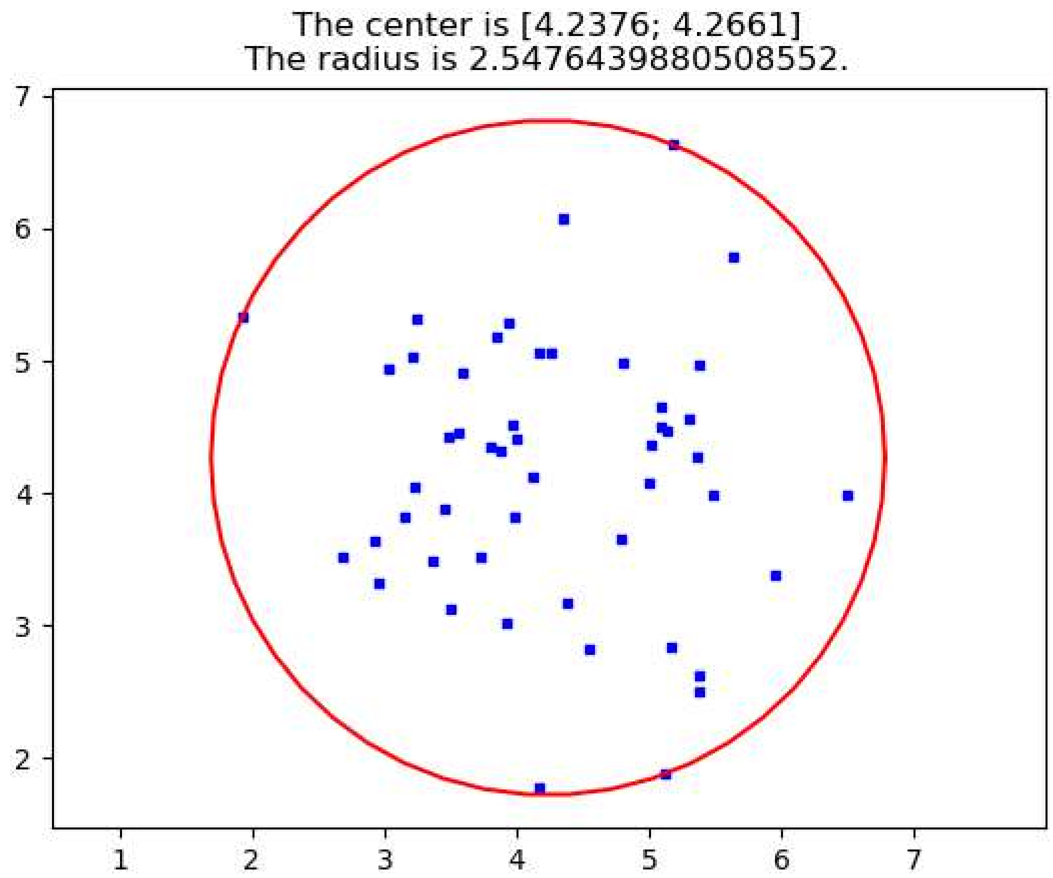
```

In [2]:

```
center = value.(C);
radius = (value.(R2))^0.5;

edge = 50;

figure();
scatter(X[1,:],X[2,:], s = 5, marker = "s", c = "blue");
t = (0:1:edge)./edge.*2*pi;
x = center[1] .+ cos.(t) .* radius;
y = center[2] .+ sin.(t) .* radius;
plot(x,y, c = "r");
title( "The center is $(round.(center,digits = 4))\nThe radius is $(radius).")
axis("equal");
```



2. The Huber loss.

a).

variables: slope, a, and intercept, b
objective: minimize $\sum_i (y_i - ax_i - b_i)^2$

In [3]:

```
x = 1:15;
y = [6.31 3.78 5.12 1.71 2.99 4.53 2.11 3.88 4.67 26 2.06 23 1.58 2.17 0.02];

X1 = [x';y];
X2 = zeros(2,13);

## generate the data without outliners
i1 = 1;
i2 = 1;
while i1 <= 15
  if (i1 != 10) && (i1 != 12)
    X2[:,i2] = X1[:,i1];
    i2 += 1;
  end
  i1 += 1;
end
```

In [4]:

```
using PyPlot, JuMP,  Gurobi;

m2a = Model(Gurobi.Optimizer);
set_silent(m2a);
#p1 = zeros(2,1); ## linear fit parameters (a and b)
#p2 = zeros(2,1);

ab = @variable(m2a, [1:2,1:1]);

@objective(m2a, Min, sum((X1[2,:] .- X1[1,:].*ab[1] .- ab[2] ).^2));
optimize!(m2a);
p1 = value.(ab);
error1 = JuMP.objective_value(m2a);
@objective(m2a, Min, sum((X2[2,:] .- X2[1,:].*ab[1] .- ab[2] ).^2));
optimize!(m2a);
p2 = value.(ab);
error2 = JuMP.objective_value(m2a);
```

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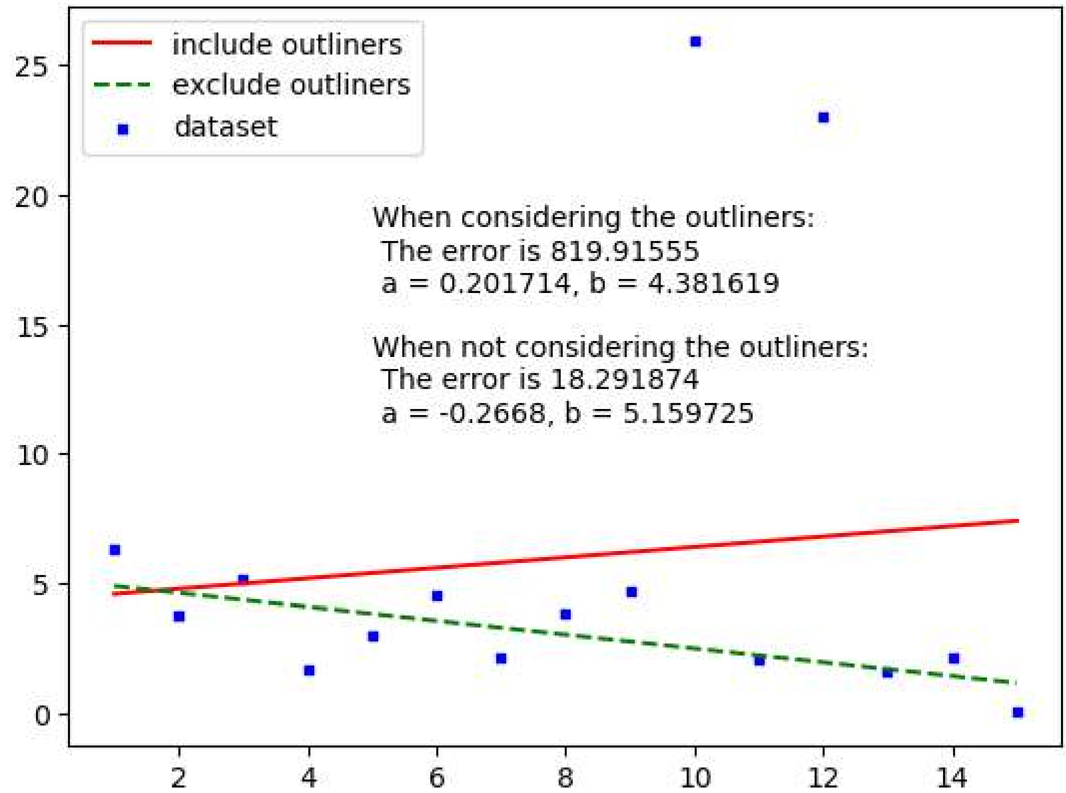
```

In [5]:

```
figure();
scatter(X1[1,:], X1[2,:], s = 5, c = "blue", marker = "s", label = "dataset");
plot(X1[1,:], p1[1].*X1[1,:] .+ p1[2], label = "include outliners", c = "r");
plot(X2[1,:], p2[1].*X2[1,:] .+ p2[2], label = "exclude outliners", c = "g", linestyle = "--");
text(5,15, "When considering the outliners:\n The error is $(round.(error1, digits = 6))\n a = $(
text(5,10, "When not considering the outliners:\n The error is $(round.(error2, digits = 6))\n a
legend();
```



Let's say fit_1 includes the outliners, while fit_2 doesn't. From the plots we can see fit_1 has a positive slope, but fit_2's slope is negative; in the other words, the a in fit_1 is positiv,but in fit_2 is negative;
The reason for this difference may be that the outliners are very far away from the expected locations. In l_2 fit, we minimize the difference in Euclid distance; but the outliners are far away, so their contribution to the objective is much greater than other points when squared. Thus we get a positive a value, even though the other data shows a decreasing trend.

b).

variables: slope, a, and intercept, b
objective: minimize $\sum_i |y_i - ax_i - b_i|$

In [6]:

```
m2b = Model(Gurobi.Optimizer);
set_silent(m2b);

## p3 as l1 fit

ab = @variable(m2b, [1:2,1:1]);
t = @variable(m2b, [1:length(X1[1,:])]);
@constraint(m2b, t .>= (X1[2,:] .- X1[1,:].*ab[1] .- ab[2] ) );
@constraint(m2b, t .>= -(X1[2,:] .- X1[1,:].*ab[1] .- ab[2] ) );
@objective(m2b, Min, sum(t));
optimize!(m2b);
p3 = value.(ab);
error3 = JuMP.objective_value(m2b);

```

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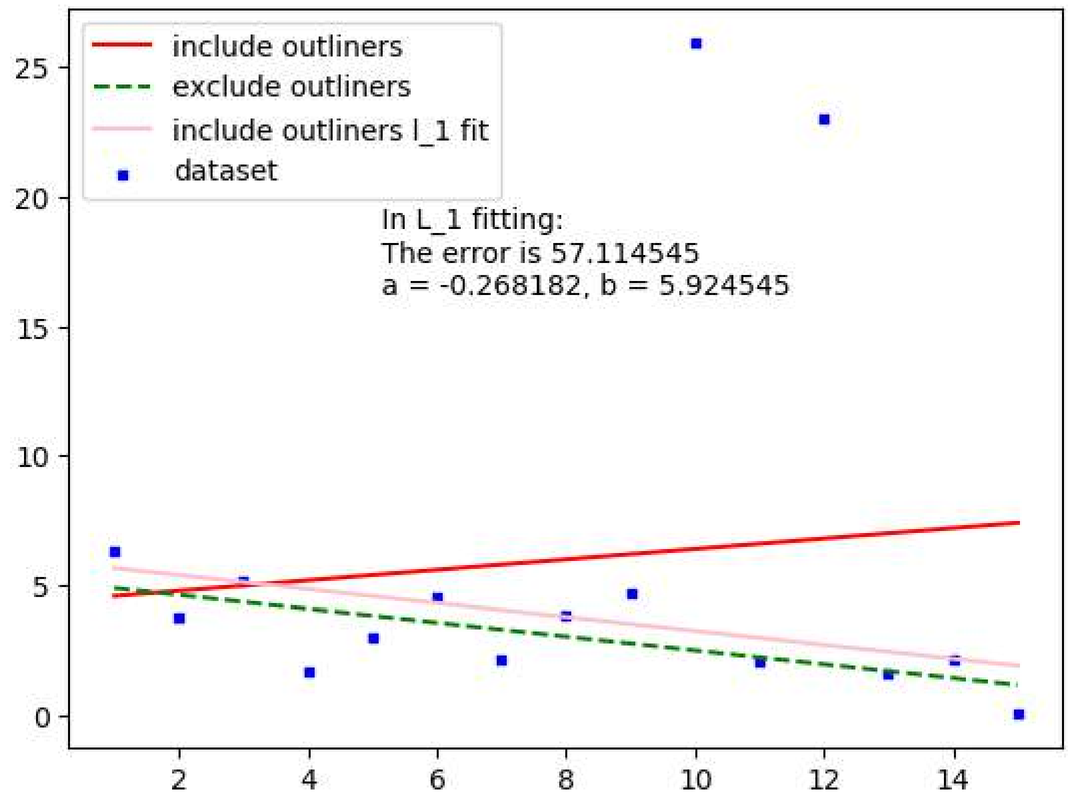
```

In [7]:

```
figure();
scatter(X1[1,:], X1[2,:], s = 5, c = "blue", marker = "s", label = "dataset");
plot(X1[1,:], p1[1].*X1[1,:] .+ p1[2], label = "include outliners", c = "r");
plot(X2[1,:], p2[1].*X2[1,:] .+ p2[2], label = "exclude outliners", c = "g", linestyle = "--");
plot(X1[1,:], p3[1].*X1[1,:] .+ p3[2], label = "include outliners l_1 fit", c = "pink");
text(5,15, " In L_1 fitting:\n The error is $(round.(error3, digits = 6))\n a = $(round.(p3[1],d:

legend();
```



From this figure we can see l_1 fit is better than l_2 fit when including the outliners, as the slope is almost the same as the l_2 fitting without outliners, and the only difference is the intercept, but not significant.

The reason may be that, in l_1 fit, the contribution from the outliners are still large, but not as significant as in l_2 fit, where their contributions are squared. So a few outliners will not affect the linear fitting greatly in l_1 approach.

c).

variables: slope, a, and intercept, b
objective: minimize $\sum_i \phi(y_i - ax_i - b)$

In [8]:

```
using PyPlot, JuMP, Gurobi;

function Huber_loss(M, x)
  m_Huber = Model(Gurobi.Optimizer);
  set_silent(m_Huber);
  @variable(m_Huber, w <= M);
  @variable(m_Huber, v >= 0);
  @constraint(m_Huber, w + v >= x);
  @constraint(m_Huber, w + v >= -x);
  @objective(m_Huber, Min, w^2 + 2*M*v);
  optimize!(m_Huber);
  return JuMP.objective_value(m_Huber);
end

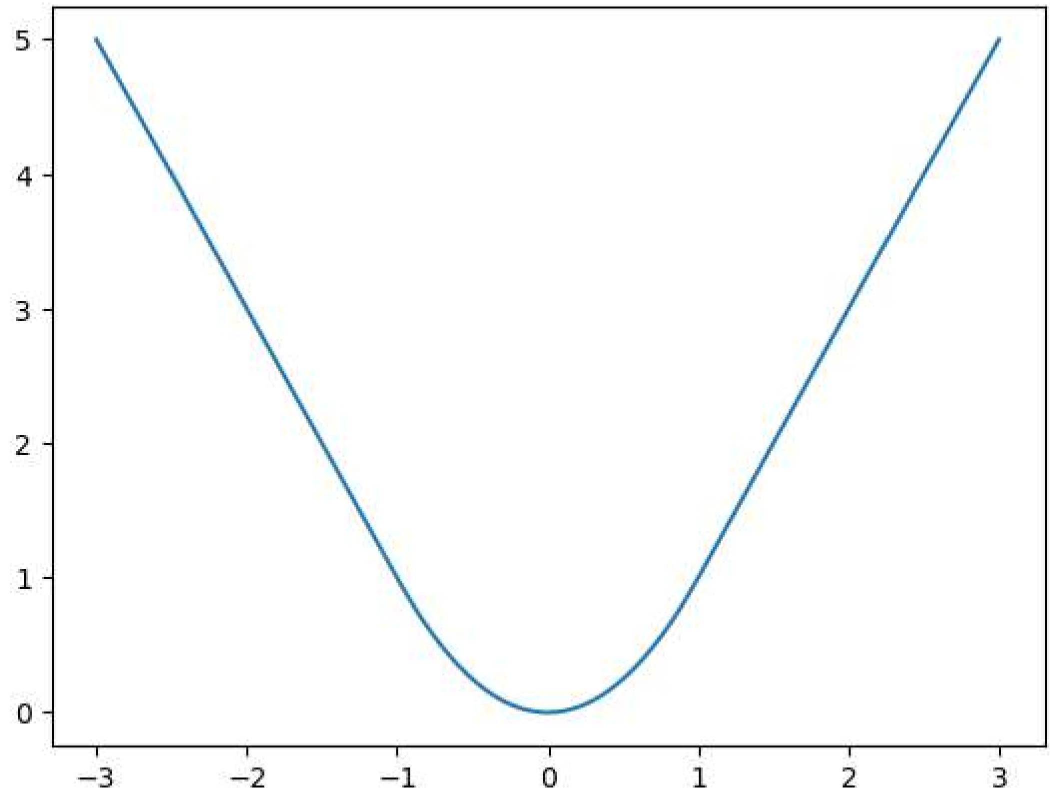
function Huber_loss_array(M, x)
  m_Huber = Model(Gurobi.Optimizer);
  set_silent(m_Huber);
  w = @variable(m_Huber,[1:length(x)]);
  v = @variable(m_Huber,[1:length(x)]);
  t = @variable(m_Huber,[1:length(x)]);
  @constraint(m_Huber, w .<= M);
  @constraint(m_Huber, v .>= 0);
  @constraint(m_Huber, w + v .>= x);
  @constraint(m_Huber, w + v .>= -x);
  @constraint(m_Huber, t.>= w.^2 .+ 2 .*M.*v);
  @objective(m_Huber, Min, sum(t));
  optimize!(m_Huber);
  return value.(t);
end

M = 1;

X_verify_m = 50;
X_verify = ((-X_verify_m:1:X_verify_m))./X_verify_m ;
X_verify = X_verify .*3;

Y_verify =  Huber_loss_array(M, X_verify);

#=
Y_verify = zeros(length(X_verify), 1);
for i = 1:length(Y_verify)
  Y_verify[i] = Huber_loss(M, X_verify[i]);
end
=#
figure();
plot(X_verify, Y_verify);
```

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```

Ans:
So we can see this $\phi(x)$ corresponds to the analytical expression in this question. We can see these two plots are exactly the same.

variables: w, v, a, b
constraints: $w_i \leq M, v_i \geq 0, w_i + v_i \geq |y_i - ax_i - b|$
objective: Minimize $\sum_i w_i^2 + 2Mv_i$

In [9]:

```

m2c = Model(Gurobi.Optimizer);
set_silent(m2c);
ab = @variable(m2c, [1:2,1:1]);
cost = @variable(m2c, [1:length(X1[1,:])]);
w = @variable(m2c, [1:length(X1[1,:])]);
v = @variable(m2c, [1:length(X1[1,:])]);

@constraint(m2c, (w .+ v) .>= (X1[2,:] .- X1[1,:].*ab[1] .- ab[2]));
@constraint(m2c, (w .+ v) .>= -(X1[2,:] .- X1[1,:].*ab[1] .- ab[2]));
@constraint(m2c, v .>= 0);
@constraint(m2c, w .<= M);
@constraint(m2c, cost .>= w.^2 + (2 * M) .* v);

@objective(m2c, Min, sum(cost));
optimize!(m2c);
p4 = value.(ab);
error4 = JuMP.objective_value(m2b);
```

```
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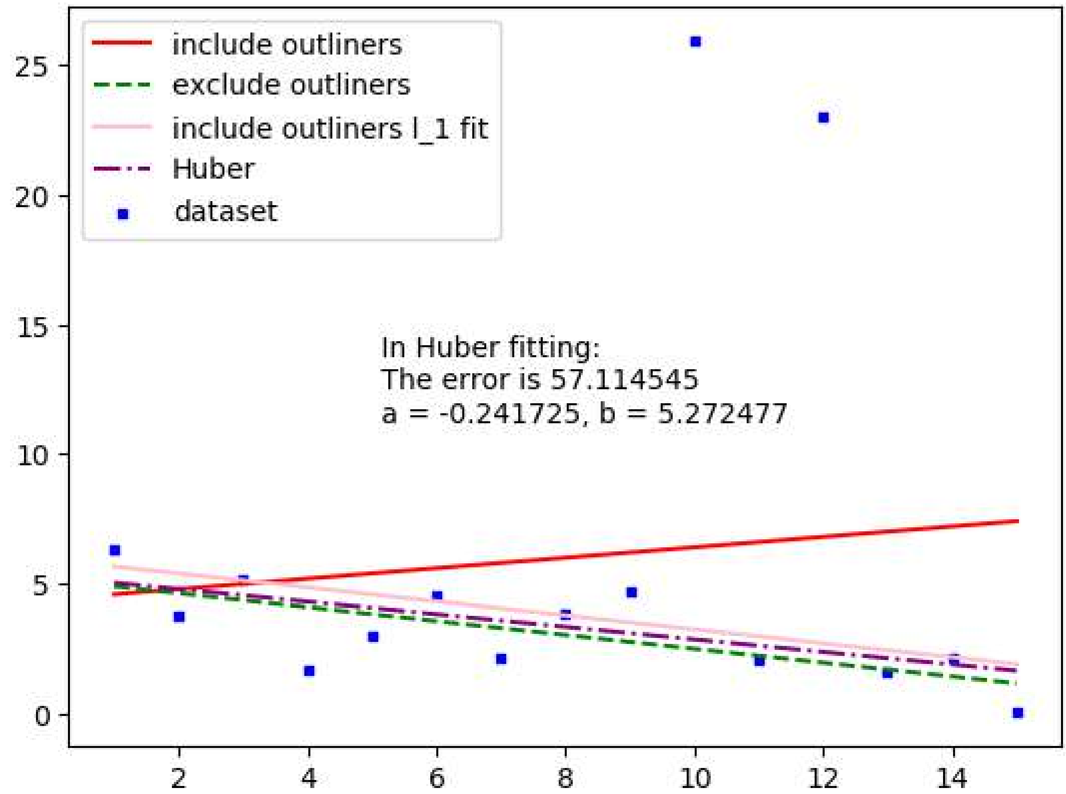
```

In [10]:

```julia
figure();
scatter(X1[1,:], X1[2,:], s = 5, c = "blue", marker = "s", label = "dataset");
plot(X1[1,:], p1[1].*X1[1,:] .+ p1[2], label = "include outliners", c = "r");
plot(X2[1,:], p2[1].*X2[1,:] .+ p2[2], label = "exclude outliners", c = "g", linestyle = "--");
plot(X1[1,:], p3[1].*X1[1,:] .+ p3[2], label = "include outliners l_1 fit", c = "pink");
plot(X1[1,:], p4[1].*X1[1,:] .+ p4[2], label = "Huber", c = "purple", linestyle = "-.");
text(5,10, " In Huber fitting:\n The error is $(round.(error4, digits = 6))\n a = $(round.(p4[1]

legend();
```



3.Heat pipe design.

a).

The velocity of the heat flow is $v = \alpha_4 T r^2$, so the amout of heat transferring in this tube is $\pi r^2 v = \pi \alpha_4 T r^4$
Convert this question into a geometric form, we thus have:

$$
\begin{aligned}
max \quad & \pi \alpha_4 T r^4 \\
s.t \quad & T_{min} \le T \le T_{max} \\
& r_{min} \le r \le r_{max} \\
& w_{min} \le w \le w_{max} \\
& w \le 0.1 r \\
& \alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 r w \le C_{max}
\end{aligned}
$$

Then we can convert it to a generic form;

$$
\begin{aligned}
-min \quad (-\pi \alpha_4 T r^4) \; s.t \quad & \frac{T_{min}}{T} \le 1 \quad \frac{T}{T_{max}} \le 1 \\
& \frac{r_{min}}{r} \le 1 \quad \frac{r}{r_{max}} \le 1
\end{aligned}
$$

$\frac{w_{min}}{w} \leq 1 \quad \frac{w}{w_{max}} \leq 1$
$\frac{10w}{r} \leq 1$
$\frac{\alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 rw}{C_{max}} \leq 1$

When apply logarithm on all the constriants, we have a convex optimization:

$min \quad -\log \pi\alpha_4 - \log T - 4\log r$
$s.t \quad \log T_{min} - \log T \leq 0 \quad \log T - \log T_{max} \leq 0$
$\log r_{min} - \log r \leq 0 \quad \log r - \log r_{max} \leq 0$
$\log w_{min} - \log w \leq 0 \quad \log w - \log w_{max} \leq 0$
$\log 10 + \log w - \log r \leq 0$
$\log \frac{\alpha_1 e^{\log T + \log r - \log w} + \alpha_2 e^{\log r} + \alpha_3 e^{\log r + \log w}}{C_{max}} \leq 0$

b).

Assume we have $x = \log T, y = \log r, z = \log w$
$\alpha_i = 1 \quad \forall i \in \{1,2,3,4\}$
$C_{max} = 500$
Hence we have:
$min \quad -\log \pi - x - 4y$
$s.t \quad \log 10 + z - y \leq 0$
$\log \frac{e^{x+y-z} + e^{y} + e^{y+z}}{500} \leq 0$

In [11]:

```
using JuMP, Ipopt,PyPlot;
m3 = Model(Ipopt.Optimizer);
set_silent(m3);
@variable(m3, x);
@variable(m3, y);
@variable(m3, z);

@constraint(m3, log(10) + z - y <= 0);
@NLconstraint(m3, log((exp(x+y-z) + exp(y) + exp(y+z))/500) <= 0);
@objective(m3, Min, -log(pi) - x - 4*y);
optimize!(m3);
```

```
Warning: Cutting back alpha due to evaluation error
The inequality constraints contain an invalid number
Warning: Cutting back alpha due to evaluation error
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Warning: Cutting back alpha due to evaluation error
The inequality constraints contain an invalid number
Warning: Cutting back alpha due to evaluation error
 11 -2.5671926e+03 6.71e+02 1.11e+04  -1.0 6.78e+08    -  1.27e-07 6.63e-07f 21
The inequality constraints contain an invalid number
Warning: Cutting back alpha due to evaluation error
The inequality constraints contain an invalid number
Warning: Cutting back alpha due to evaluation error
```

In [12]:

```
T = exp(value.(x));
r = exp(value.(y));
w = exp(value.(z));
Heat_max = pi * T * r^4;
print("The maximum heat flow in this pip is $(Heat_max), \nwhere T = $(T),r = $(r), w = $(w).\n");
print("The cost is $(T*r/w + r + r*w).")
```

```
The maximum heat flow in this pip is 3.0679616908019958e9,
where T = 0.24999999951949164,r = 250.0000024635135, w = 0.4999999993593595.
The cost is 500.0000046867728.
```

In [ ]: