How to: Analysis in the phase diagram

Disclaimer: This note adjusts an earlier note on the Ramsey model from a continuous time model into a discrete time model. In general this means that the figures might include notation such as $\dot{c} = 0$, which in continous time implies steady state for consumption. In the discrete time notation, this should read $c_{t+1} = c_t$. Furthermore, the trajectories in the phase-diagrams describing how the economy evolves over time should not be continous functions, but 'jumps' as you might have seen in lecture slides.

The main purpose is to give a structured approach to analysis in the phase-diagram mainly focusing on the technical part, thus leaving out quite a lot of intuition along the way. It is not meant as an instruction on how to answer an exam question, but more as an elaborated explanation of how you could start working with analysis in the phase diagram. E.g. should you encounter the shock described in section (2.3.1), a sufficient answer (on the technical part) is to only use figure 10 and equation (2.5) and (2.6).

1 Introduction

In the standard solution approach, in the standard Ramsey setup, we are at this point at step 4:

1. **Setting the stage of the problem (STS):**

   i) A representative household, solving:

$$\max U = \sum_{t=0}^{\infty} [\beta(1+n)]^t \frac{c_t^{1-\sigma}}{1-\sigma}$$
$$\text{s.t. } a_{t+1}(1+n) = a_t R_t + w_t - c_t, \qquad a_t = k_t + b_t$$

   along with some boundary conditions. (NPGC, TVC, initial k_0 etc.)

   ii) A representative firm, which maximizes profits (under perfect competition and assuming CRS production function):

$$\Pi(K_t, L_t) = F(K_t, L_t) - w_t L_t - (r_t + \delta)K_t$$

2. **Microfoundations:**

   i) Households: Setting up a Lagrangian and maximizing wrt. c_t, a_{t+1} we obtain:

$$c_t^{-\sigma} = \lambda_t \tag{1.1}$$
$$\lambda_t = \beta\lambda_{t+1}R_{t+1} \tag{1.2}$$
$$\frac{c_{t+1}}{c_t} = [\beta R_{t+1}]^{\frac{1}{\sigma}}, \tag{1.3}$$

   which along with the TVC condition and the budget describes the optimal household behavior. ii) Firms: The first order conditions couple factor prices to the capital intensity (k_t)

$$w_t = f(k_t) - f'(k_t)k_t \tag{1.4}$$
$$r_t = f'(k_t) - \delta \tag{1.5}$$

3. **Equilibrium results and the phase diagram:**

   Following the procedure in 'How to: The Phase Diagram', we combine all available information on c_t, k_t and t in one diagram[1].

4. **Analysis.**

[1] I have not yet adjusted the 'How to: The Phase Diagram' note from continous to discrete time. However, the short introduction note on dynamic models includes a somewhat thorough explanation of this part.

2 How to: A classic analysis in the phase diagram.

In the following we will discuss how to trace short, medium and long-run effects of a given shock in the Ramsey model. In general we consider 3 cases of shocks in the classic Ramsey model: Unanticipated and permanent shocks, anticipated and permanent shocks, and anticipated and temporary shocks.

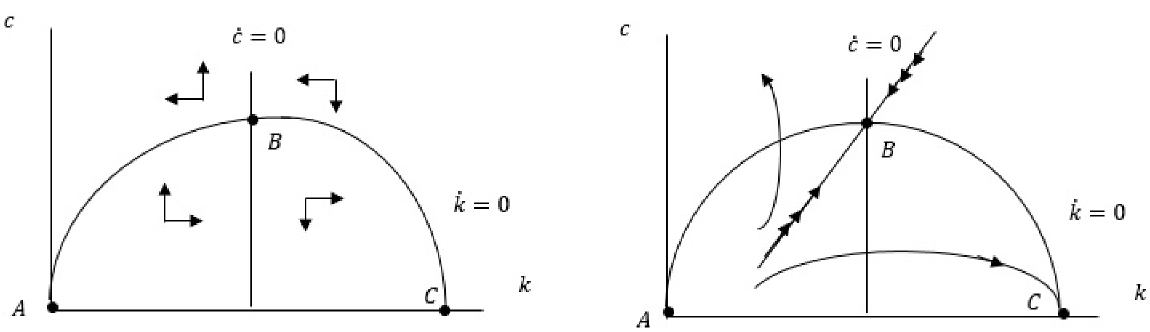
In the classic Ramsey model we construct the phase diagram mainly using the steady state functions:

$$c_{t+1} = c_t \qquad \Rightarrow \qquad f'(k^*) = \frac{1-\beta(1-\delta)}{\beta} \tag{2.1}$$

$$k_{t+1} = k_t \qquad \Rightarrow \qquad c^{ss} = f(k^{ss}) - (n+\delta)k^{ss}, \tag{2.2}$$

where we are summing up the dynamics in the standard phase diagram below:

Figure 1: The Phase diagram



Furthermore, in the current model we can trace the choice of consumption from recursively substitution of the Euler equations and budget constraints:

$$c_{t_0} = \frac{a_{t_0} + W_{t_0}}{\sum_{t=t_0}^{\infty} [\beta(1+n)]^t \prod_{i=t_0}^{t} [\beta R_i]^{\frac{1-\sigma}{\sigma}}} \tag{2.3}$$

where W_{t_0} denotes the present value of all future labor income. Later, we can use this to trace what happens to initial consumption in the case of a shock where we have both substitution and income effects.

2.1 The unanticipated and permanent shock

To illustrate the method, let's consider the case where δ increases unanticipated at time t_0:

1. First, we look at how the **phase diagram** and how the steady state functions change:

   For the steady state function of $c_{t+1} = c_t$ a permanent increase in δ implies a decrease in k^*. To see this mathematically, differentiate wrt. δ on both sides of (2.1) and apply the chain rule:

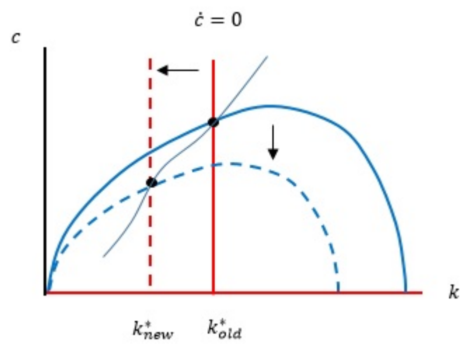
$$\frac{\partial f'(k^*)}{\partial k^*}\frac{\partial k^*}{\partial \delta} = 1 \qquad \Rightarrow \frac{\partial k^*}{\partial \delta} = \frac{1}{f''(k^*)} < 0 \tag{2.4}$$

   For the steady state function of $k_{t+1} = k_t$ we see that the function pushes downwards:

$$c^{ss} = f(k^{ss}) - (n+\delta)k^{ss} \qquad \Rightarrow \qquad \frac{\partial c^{ss}}{\partial \delta} = -k^{ss}$$

   with the difference k_t. Thus we can sum up the phase diagram in figure 2[2]:

   Figure 2: Change in steady state functions



2. **The short-run effects:**
   In general note that the saddle-path is constructed **as the optimal combination of c_t and k_t** in a permanent system. Thus with this shock we must adjust consumption directly to the new saddle-path. In our case we have opposing substitution and income effects though, which means that we cannot be certain whether to increase or decrease consumption initially:

   - Substitution effect (From Euler equation): When δ increases, the interest rate decreases and thus the relative price on current consumption decreases $\Rightarrow c_{t_0} \uparrow$.
   - Income effect (from steady state of $k_{t+1} = k_t$): When δ increases, the interest rate decreases and thus savings generate less income in the future. In essence, we become poorer $\Rightarrow c_{t_0} \downarrow$.

   The aggregate effect essentially depends on the risk-aversion parameter, σ[3]:

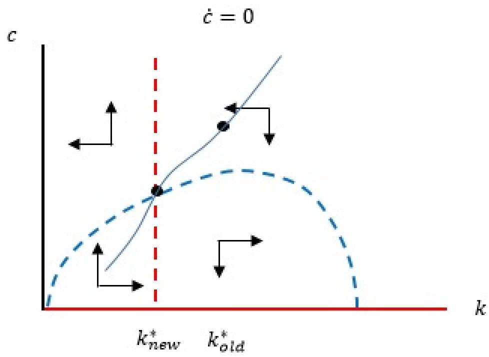
   - If $\sigma > 1 \Rightarrow$ income effect dominates $\Rightarrow c_{t_0} \downarrow$.
     The household is relatively risk averse, thus values a smooth consumption path. In essence this implies that the substitution effect is relatively small: The household is not willing to change consumption path when relative prices change.
   - If $\sigma = 1$ (log-preferences)$\Rightarrow$ income effect and substitution effects perfectly offsets each other $\Rightarrow \Delta c_{t_0} = 0$.
   - If $\sigma < 1 \Rightarrow$ substitution effects dominates $\Rightarrow c_{t_0} \uparrow$.

[2]Note that the trajectory we follow is actually not continuous as drawn in figure 2 as time is discrete not continuous.
[3]To see how exactly, see equation (2.3)

3. **Medium and Long-run effects:**

To tell the story of the medium and long-run effects, let's look at the phase diagram in figure 3 after the shock has occured:

Figure 3: Medium and long-run effects



After δ has increased capital and consumption will decrease over time: consumption due to the relatively low interest rate and capital mechanically due to the larger depreciation rate.

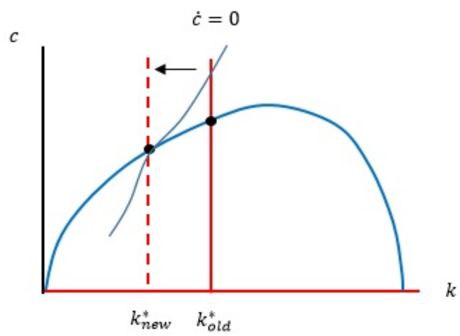
As capital and consumption decreases, the marginal product of capital increases such that changes in consumption becomes smaller. Thus we converge towards steady state. Note that the trajectory in figure 3 is continuous, however, this is not actually the case as we have discrete time.

2.2 The anticipated and permanent shock

In essence, we are going to repeat the same 3 steps, but in a different way. Furthermore, to illustrate the difference from an unanticipated shock, this time we consider an decrease in β instead of an increase in δ. We assume that t_0 is the time where households become aware that β will change, and t_1 is the time of impact (where β actually changes)[4].

1. **The Phase diagram:**
   A decrease in β implies the changes in the phase diagram summarized in figure 4:

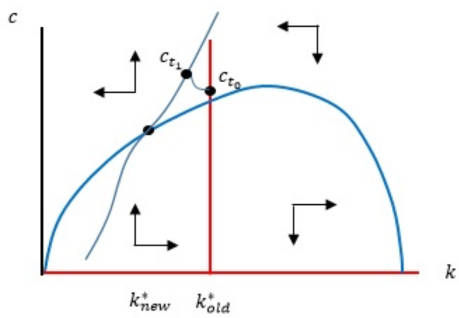
   Figure 4: The Phase Diagram

   If the shock was unanticipated, we would unambiguously jump directly up to the new saddle-path.

2. **The short-run effect: Between t_0 (time of announcement) and t_1 (time of shock)**
   When we derive the short run effect on consumption and capital, we use the fact that at time t_1 when the shock hits the economy, we must be on the new saddle-path. The optimal way to do this, is to increase consumption partially at time t_0 and follow the dynamics of the phase diagram between t_0 and t_1 as noted in figure 5:

   Figure 5: Optimal response to an anticipated shock

   In the time interval between the announcement and acutal shock, $[t_0, t_1[$, consumption will increase and capital decrease. The initial jump in consumption is constructed such that the household end up on the saddlepath at exactly time t_1, where the dynamics change.

3. **Medium and long-run effects:**
   Equivalent to the analysis in section 2.1: We follow the saddle-path towards new steady state.

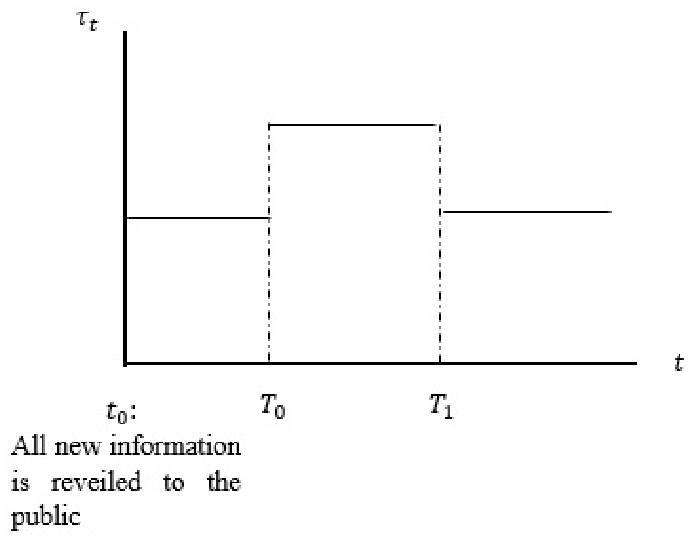
[4]Admittedly this is a weird shock to 'anticipate' as it is solely determined by the household. However, this note focuses primarily on the technical part and a shock implying an unambiguous effect on initial consumption is easier to track.

2.3 The anticipated and temporary shock

To illustrate how to analyze an anticipated and temporary shock, let's consider two different scenarios of anticipated and temporary shocks:

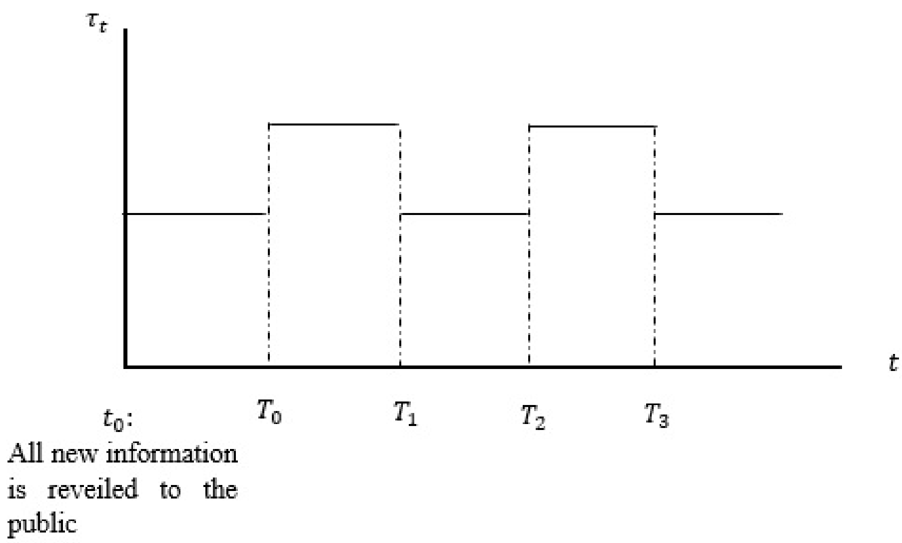
1. A left-wing government announces at time t_0 that they will introduce an increase in labor-taxes time T_0. However, households anticipate (correctly) that a right-wing government at time T_1 will eliminate the tax-hike and restore taxes to the original level. That is:

Figure 6: Tax rate in scenario 1



2. The scenario 1 repeats itself cyclically, thus we have no permanent long-run structure in the economy:

Figure 7: Tax rate in scenario 2



In both scenarios, we assume that the government uses labor-income taxes to purchase public goods, not affecting household utility.

2.3.1 An anticipated and temporary shock, scenario 1:

When there is a permanent long-run structure (at some point in time we expect the phase-diagram to be constant over time) we can approach the problem as follows:

1. The initial point of our analysis is to describe **the phase-diagram** in the low and high tax rate scenario:

   In the case of a tax on labor income, the relevant equations describing the household solution is given by[5]:

$$\frac{c_{t+1}}{c_t} = [\beta R_{t+1}]^{\frac{1}{\sigma}} \quad \Rightarrow \quad \frac{c_{t+1}}{c_t} = [\beta(1+f'(k_{t+1})-\delta)]^{\frac{1}{\sigma}}$$

$$a_{t+1}(1+n) = a_t R_t + w_t(1-\tau_t) - c_t \quad \Rightarrow \quad k_{t+1} = \frac{1}{1+n}\left(f(k_t) - (n+\delta)k_t - c_t - \tau_t(f(k_t) - f'(k_t)k_t)\right)$$

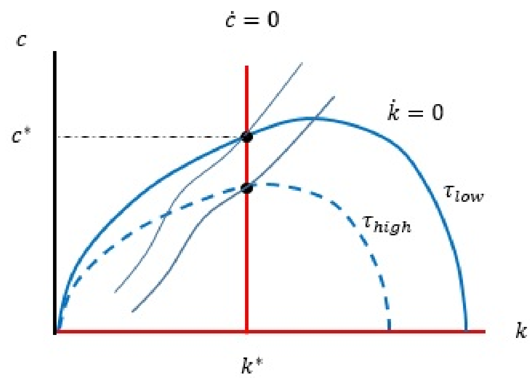
   Thus we construct the relevant steady state functions as

$$\frac{c_{t+1}}{c_t} = 1 \quad \Rightarrow \quad f'(k^*) = \frac{1-\beta(1-\delta)}{\beta} \tag{2.5}$$

$$k_{t+1} = k_t \quad \Rightarrow \quad c^{ss} = f(k^{ss}) - (n+\delta)k^{ss} - \tau_t(f(k^{ss}) - f'(k^{ss})k^{ss}). \tag{2.6}$$

   From (2.5) and (2.6) we see that the k^* is unaffected by the labor-income-tax, while the steady state function for k shiftes downwards from an increase in τ_t. Thus we get a scenario ressembling the phase diagram in figure 8:

   Figure 8: Phase diagram in scenario 1



2. **Inferring the level of c_{t_0}, c_{T_0} and c_{T_1}:**

   Firstly, in scenario 1, we know that we can track the direction of the initial jump in consumption from equation 2.3. That is, we know that $c_{t_0} < c^*$, due to the fact that between T_0 and T_1, the household is impacted by a negative income shock.

   Secondly, in scenario 1, we know that at time T_1, we must end up on the saddle-path for an economy with a permanent τ_{low}, that is, the upper saddle-path drawn in figure 8[6].

   Thus, we need to adjust c_{t_0} downwards initially, such that if I follow the standard dynamics of the phase diagram, we end up on the saddle-path at time T_1.

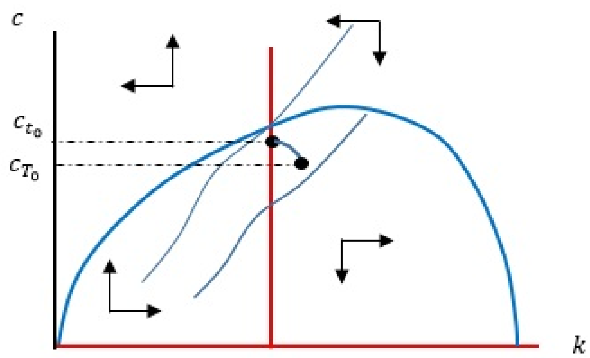
[5] Along with boundary conditions not relevant for this part.

[6] Notice that if the tax was a lump-sum tax, the $\dot{k} = 0$ curve would shift downwards with the same difference for all k, and thus not go through origo.

Let's split the trajectory up into two parts:

Between t_0 **and** T_0: We jump down to c_{t_0} and thus we accumulate capital. As capital increases, the interest rate decreases and we further lower consumption over time untill the first shock actually hits at T_0. The scenario is illustrated in figure 9:

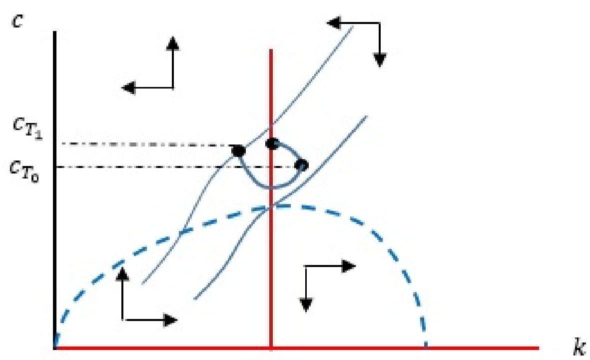
Figure 9: Trajectory between announcement, t_0, and time of first shock, T_0



Trajectory between T_0 **and** T_1: It turns out that it is important that the initial decrease in consumption, is sufficiently small, such that at time T_0 we have not yet passed the second saddle-path.

If we have not yet crossed the second saddle-path, the dynamics between T_0 and T_1 are as follows: Now that the labor-tax has increased, our after-tax income is not sufficiently high to maintain the same capital stock, thus capital decreases over time. Initially, the interest rate is still relatively small and consumption will decrease further. However, as capital now decreases, the interest rate is increasing and thus at some point consumption increases as well:

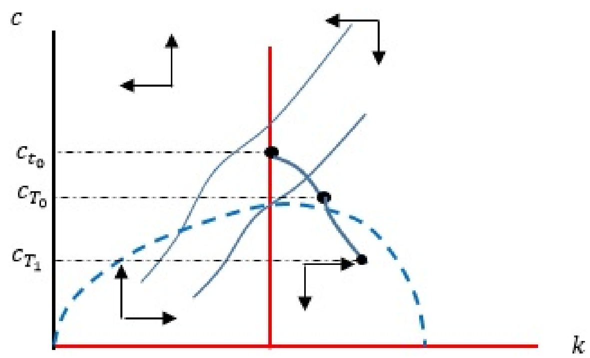
Figure 10: Trajectory between T_0 and T_1



When the tax rate is lowered again at time T_1, we will now simply converge via the saddle-path towards the original steady state level.

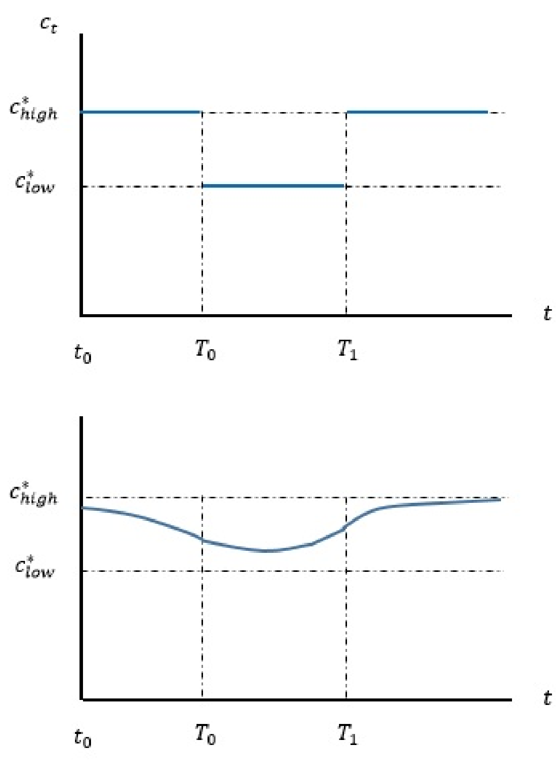
If our initial adjustment in consumption had led us to cross the second saddle-path, we would essentially diverge towards the steady state of zero consumption as depicted in figure 11:

Figure 11: Divergence with a suboptimally large adjustment of c_{t_0}



The general intuition of the adjustment process, is once again one of consumption smoothing: Even though we in the time interval $[t_0, T_0[$ could have enjoyed a larger consumption, we smooth out the tax burden over time. Note that one other trivial solution that would imply that we ended in the same steady state, is not to smooth the tax burden, but to make two jumps in consumption: We could at time T_0 have jumped from one steady state directly into the lower steady state, and then at time T_1 jumped directly back the original steady state once the taxes were lowered again:

Figure 12: Smoothening the tax-burden



As illustrated in figure 12, we choose the smoothening of consumption / tax burden, due to our assumption of risk-aversion.

2.3.2 An anticipated and temporary shock, scenario 2