How to: OLG solution

1 Introduction

In the OLG, we will generally follow a cook-book approach, resembling that from the Ramsey model:

1. Describe the agents and the structure of the economy, as well as their behavioral objectives.

2. Derive the agents' behavioral rules.

3. Derive results for the aggregate economy: How is $k_{t+1}(k_t)$ evolving?

4. Analysis.

In general, each step should be revised dependent on the type of economy we are working with and what sort of analysis we are carrying out. In the following, we will focus on a simple OLG model, where we implement a pay-as-you go pension system.

2 How to: The OLG model

Initially, let's assume that the economy is described (almost) as in PS9:

Consider an economy where individuals live for two periods. Utility for young individuals born in period t is

$$U_t = \ln(c_{1,t}) + \frac{1}{1+\rho}\ln(c_{2,t+1}), \qquad \rho > -1 \tag{2.1}$$

where $c_{1,t}$ is consumption when young, and $c_{2,t+1}$ consumption when old. Young agents work a unit of time (i.e. their labor income is equal to the wage they receive). Old agents do not work and must provide consumption through saving (in capital and/or government debt, if this exists). Production for firm i that hires labor and capital is given by:

$$Y_t^i = A(K_t^i)^{\alpha}(L_t^i)^{1-\alpha}, \tag{2.2}$$

where $A > 0$, K is aggregate capital stock in the economy, K^i and L^i are the amounts of capital and labor hired by the firm. Markets for factors are competetive resulting in factors being rewarded their marginal products.

Suppose that the government runs a pay-as-you-go social security system in which the young contribute an amount d that is received by the old.

1. Agents, structure of the economy and behavioral objectives

The first part of the solution approach is to make sure that we get an overview of the economy. Thus, the first part is solely for your own benefit, making sure that you understand what is going on in the problem text. A brief answer could focus on agents:

- Households faces the optimization problem of:

$$\max U_t = \ln(c_{1,t}) + \frac{1}{1+\rho}\ln(c_{2,t+1}) \tag{2.3}$$

$$\text{s.t. } c_{1,t} = w_t - s_t - d \tag{2.4}$$

$$\text{s.t. } c_{2,t+1} = (1 + r_{t+1})s_t + d(1+n), \tag{2.5}$$

  applying the assumptions of the problem text above. Note that the pay-as-you-go pension system has a 'return' equivalent to the population growth, n: If you pay d as young, to the current old generation, then, when you are old, there will be $(1+n)$ young households contributing the benefit d to you.

- Firms face the optimization problem:

$$\max \Pi^i(K_t^i, L_t^i) = A(K_t^i)^{\alpha}(L_t^i)^{1-\alpha} - w_t L_t^i - r_t K_t^i. \tag{2.6}$$

  We could note that as all, say Q, firms are assumed to be identical, we have that

$$K_t = \sum_{i=1}^{Q} K_t^i = QK_t^i, \text{ and} \qquad L_t = \sum_{i=1}^{Q} L_t^i = QL_t^i \qquad \Rightarrow k_t = \frac{K_t}{L_t} = \frac{QK_t^i}{QL_t^i} = k_t^i, \tag{2.7}$$

  that is, the individual firm's k_t^i is equal to the aggregate k_t.

- The government: Only objective, is to run the pay-as-you-go pension system. If applied properly in the household setup, we need no further restrictions on the government.

2. Derive the agents' behavioral rules.

The solution follows from:

- Household solution: Here by substitution

$$
\begin{aligned}
U_t &= \ln(w_t - s_t - d) + \frac{1}{1+\rho}\ln((1+r_{t+1})s_t + d(1+n)) \\
\frac{\partial U_t}{\partial s_t} &= \frac{-1}{w_t - s_t - d} + \frac{1}{1+\rho}\frac{1+r_{t+1}}{(1+r_{t+1})s_t + d(1+n)} = 0 \\
\Rightarrow s_t(2+\rho) &= w_t - d - d\frac{1+\rho}{1+r_{t+1}}(1+n) \\
\Rightarrow s_t &= \frac{w_t}{2+\rho} - \frac{d}{2+\rho}\left(1 + \frac{(1+\rho)(1+n)}{1+r_{t+1}}\right) \qquad (2.8)
\end{aligned}
$$

From (2.9) we see that savings are lowered through two channels: 1) As the pension system poses a tax when young, the household lowers savings to increase consumption as young. 2) As the pension system increases income as old for a given amount of savings, the household lowers savings to increase consumption as young as well.

From (2.8) we can characterize the consumption levels as well, by plugging in s_t in the budget constraints:

$$
\begin{aligned}
c_{1,t} &= w_t - s_t - d &&= w_t\frac{1+\rho}{2+\rho} - d\left(1 - \frac{1}{2+\rho}\left(1 + \frac{(1+\rho)(1+n)}{1+r_{t+1}}\right)\right) \qquad (2.9) \\
c_{2,t+1} &= (1+r_{t+1})s_t + (1+n)d &&= w_t\frac{1+r_{t+1}}{2+\rho} + d\left(1 + n - \frac{1+r_{t+1}}{2+\rho}\left(1 + \frac{(1+\rho)(1+n)}{1+r_{t+1}}\right)\right) \qquad (2.10)
\end{aligned}
$$

By some tedious algebraic manipulation, the consumption levels can be expressed as

$$
\begin{aligned}
c_{1,t} &= w_t\frac{1+\rho}{2+\rho} - d\left(1 - \frac{1+r_{t+1} + (1+\rho)(1+n)}{(2+\rho)(1+r_{t+1})}\right) \\
c_{1,t} &= w_t\frac{1+\rho}{2+\rho} + d\frac{(1+\rho)(n - r_{t+1})}{(2+\rho)(1+r_{t+1})} \qquad (2.11)
\end{aligned}
$$

$$
\begin{aligned}
c_{2,t+1} &= w_t\frac{1+r_{t+1}}{2+\rho} + d\left(1 + n - \frac{1+r_{t+1} + (1+\rho)(1+n)}{2+\rho}\right) \\
c_{2,t+1} &= w_t\frac{1+r_{t+1}}{2+\rho} + d\frac{n - r_{t+1}}{2+\rho} \qquad (2.12)
\end{aligned}
$$

The derivation of consumption levels can be the source of mistakes algebraically, but can be essential to the analysis later on. In part 3 we derive a shortcut that may be usefull to apply in some settings.

- Firm solution: Follows the simple one from the Ramsey model

$$
\begin{aligned}
\frac{\partial \Pi_t^i}{\partial K_t^i} &= \alpha A (K_t^i)^{\alpha-1}(L_t^i)^{1-\alpha} - r_t = 0 && \Rightarrow && r_t = \alpha A k_t^{\alpha} \qquad (2.13) \\
\frac{\partial \Pi_t^i}{\partial L_t^i} &= (1-\alpha) A (K_t^i)^{\alpha}(L_t^i)^{-\alpha} - w_t = 0 && \Rightarrow && w_t = (1-\alpha) A k_t^{\alpha} \qquad (2.14)
\end{aligned}
$$

- Government solution: No restriction is required in this setup.

3. Derive results for the aggregate economy

In general, when describing the aggregate dynamics, we will go through the steps:

- State the capital accumulation scheme of the economy (often in the form of (2.17)).
- Impose relevant household and firm solutions to obtain $k_{t+1}(k_t)$.
- Draw the transition diagram, plotting k_{t+1} against k_t, and explain convergence towards a potential steady state of capital.

We start by deriving the per capita capital accumulation scheme in general. Start from the capital accounting identity:

$$K_{t+1} = K_t(1-\delta) + S_{1,t} + S_{2,t+1}, \tag{2.15}$$

where $S_{1,t}$ denotes the aggregate savings of the young generation at time t, and $S_{2,t+1}$ denotes the aggregate savings of the old generation at time $t+1$. Note that the old generation owns the entire existing capital stock. As the old generation has no incentive to save (they do not live in the future), we impose the condition that

$$S_{2,t+1} = -K_t(1-\delta). \tag{2.16}$$

Thus we have that

$$\begin{aligned} K_{t+1} =& S_{1,t} \\ \Rightarrow k_{t+1} =& \frac{s_t}{1+n}. \end{aligned} \tag{2.17}$$

In general you can skip the first part and just jump to the identity in (2.17). Next, we turn to imposing results from the household and firm solution:

$$\begin{aligned} k_{t+1} =& \frac{s_t}{1+n} \\ k_{t+1} =& \frac{1}{1+n}\left(\frac{w_t}{2+\rho} - \frac{d}{2+\rho}\frac{1+r_{t+1}+(1+\rho)(1+n)}{1+r_{t+1}}\right) \\ k_{t+1} =& \frac{1}{1+n}\left(\frac{(1-\alpha)Ak_t^{\alpha}}{2+\rho} - \frac{d}{2+\rho}\left(\frac{1+\alpha Ak_{t+1}^{\alpha-1}+(1+\rho)(1+n)}{1+\alpha Ak_{t+1}^{\alpha-1}}\right)\right). \end{aligned} \tag{2.18}$$

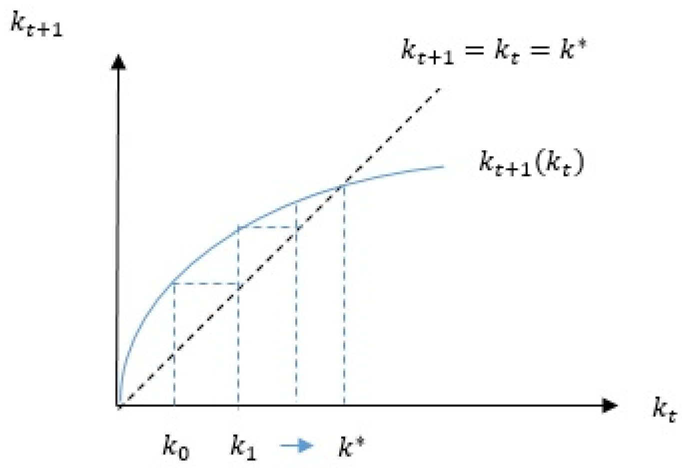
Ideally, we would prefer to have k_{t+1} on the left hand side (LHS) only as a function of constants and k_t on the right hand side (RHS). However, with the pay-as-you-go system, we have no analytic solution to (2.18).

Just to illustrate what we would ideally do, assume for know that we have no pension system, that is, assume that $d = 0$. In this case we have that

$$k_{t+1} = \frac{(1-\alpha)Ak_t^{\alpha}}{(1+n)(2+\rho)}. \tag{2.19}$$

With this capital accumulation scheme, we can trace how capital evolves over time in a transitional diagram as in figure 1.

Figure 1: Capital accumulation scheme in standard OLG model



4. Analysis

How to deal with a shock to the economy, largely depends on the sort of shock as well as the type of OLG model we are looking at. In a relatively simple model, where we have a well-defined $k_{t+1}(k_t)$ function as in (2.19), and where the shock is that a parameter changes, we will typically apply the following steps:

- Look at changes in the $k_{t+1}(k_t)$, and how the aggregate economy evolves over time.
- Explain how capital evolves in the short run, using the households' incentives to save and factor prices (short run effects).
- Explain how capital converges towards the new steady state (long-run effects).
- If relevant (depends on exercise), compute consumption levels before and after shock to compare welfare levels.

Other traditional types of shocks and methods can be:

- Change the model setup: Implement a tax, pension system etc..
  Solution: We have to go through the 3 first steps, before we can go to the analysis above.
- In an economy without an analytical solution to the capital accumulation scheme (as in (2.18)): Assume convergence towards a steady state, and look at k^*, which can be solved from (2.18). Then we can still conclude on long run effects of a change.

3 EXTRA I: The indirect utility function and a shortcut to welfare analysis

A general source of algebraic mistakes in the OLG model, is when we derive consumption levels, for instance in a setup with pay-as-you go pension system. Sometimes, we can circumvent calculating consumption levels (expenditure side) and instead just focus on the income side. Consider the household problem from earlier:

$$\max U_t = \ln(c_{1,t}) + \frac{1}{1+\rho}\ln(c_{2,t+1}) \tag{3.1}$$

$$\text{s.t. } c_{1,t} = w_t - s_t - d \tag{3.2}$$

$$\text{s.t. } c_{2,t+1} = (1+r_{t+1})s_t + (1+n)d \tag{3.3}$$

Instead of solving the problem by substitution, combine the two budget constraints to

$$c_{1,t} + \frac{c_{2,t+1}}{1+r_{t+1}} = \underbrace{w_t - s_t - d}_{\equiv c_{1,t}} + \underbrace{s_t + \frac{1+n}{1+r_{t+1}}d}_{\equiv c_{2,t+1}/(1+r_{t+1})} = w_t + d\frac{n-r_{t+1}}{1+r_{t+1}} \equiv W_t, \tag{3.4}$$

where W_t now denotes total present value income. In this case, we solve by Lagrange:

$$\mathcal{L}_t = \ln(c_{1,t}) + \frac{1}{1+\rho}\ln(c_{2,t+1}) + \lambda_t\left(W_t - c_{1,t} - \frac{c_{2,t+1}}{1+r_{t+1}}\right) \tag{3.5}$$

which yields the first order conditions:

$$\frac{\partial \mathcal{L}_t}{\partial c_{1,t}} = \frac{1}{c_{1,t}} - \lambda_t = 0 \quad \Rightarrow \quad c_{1,t} = \lambda_t^{-1} \tag{3.6}$$

$$\frac{\partial \mathcal{L}_t}{\partial c_{2,t+1}} = \frac{1}{(1+\rho)c_{2,t+1}} - \frac{\lambda_t}{1+r_{t+1}} = 0 \quad \Rightarrow \quad c_{2,t+1} = \lambda_t^{-1}\frac{1+r_{t+1}}{1+\rho}. \tag{3.7}$$

Combining (3.6), (3.7) and (3.4) we can get the solution that

$$c_{1,t}^* = \frac{1+\rho}{2+\rho}W_t \tag{3.8}$$

$$c_{2,t+1}^* = \frac{1+r_{t+1}}{2+\rho}W_t. \tag{3.9}$$

Define the *indirect utility function/value function*, V_t, as the utility obtained by consuming the optimal $c_{1,t}^*$ and $c_{2,t+1}^*$:

$$\begin{aligned} V_t(W_t) &\equiv \ln(c_{1,t}^*) + \frac{1}{1+\rho}\ln(c_{2,t+1}^*) \\ &= \ln\left(\frac{1+\rho}{2+\rho}W_t\right) + \frac{1}{1+\rho}\ln\left(\frac{1+r_{t+1}}{2+\rho}W_t\right) \end{aligned} \tag{3.10}$$

All we need now, is to note that

$$\frac{\partial V_t}{\partial W_t} > 0. \tag{3.11}$$

The conclusion is: If a given shock to the economy (such as implementing a pay-as-you go pension system), increases W_t, then it will in this case improve the welfare of households. Instead of calculating levels of $c_{1,t}$ and $c_{2,t+1}$ and comparing to before and after the pension system, I can just look at my expression for W_t in (3.4) and conclude that if $n > r_{t+1}$, then the pay-as-you go pension system is welfare improving.

4 EXTRA II: From aggregate capital to the per capita capital scheme