Q&A session

1 The Ramsey model

- **What is the TVC condition?** The transversality condition is a boundary condition on a mathematical problem:

  1. Why do we need a TVC condition (nice to know): Think of a classic dynamic problem such as optimal savings behavior. Assume that we live from $t = 0$ to $t = T$. In any point in time where $t < T$ we derive a first order condition that states how much we should consume at point t, compared to the near future, $t + 1$. This first order condition describes our optimal behavior.

     However, at the boundary of our problem when $t = T$ we can no longer use the first order condition comparing consumption at point T to the near future $T + 1$, as we per construction do not care about $T + 1$. The TVC condition states the requirement for optimality on the boundary where the usual first order conditions are not sufficient.

  2. The TVC condition in the Ramsey model (need to know): The TVC condition is in scenarios a so-called complementary slackness condition of

     $$\lim_{T\rightarrow\infty}(q_T a_{T+1}) = 0, \qquad q_T \equiv \prod_{t=0}^{T}\frac{1+n}{R_t}. \tag{1.1}$$

     The TVC condition states that the present value of wealth must be zero in the limit. To see why this is a requirement we follow the two arguments:

     1) As a technical requirement, we will not allow $\lim_{T\rightarrow}(q_T a_{T+1}) < 0$, as this would imply that we could simply finance consumption in the long run using debt. (The No Ponzi-Game Condition)

     2) As an condition for optimal behavior we will not allow for $\lim_{T\rightarrow\infty}(q_T a_{T+1}) > 0$, as this would imply that we could increase consumption today (at least marginally) without lowering future consumption levels. Put differently; when the economy 'ends' so to speak, we are still saving wealth for future consumption.

- **PS1 12 and PS2 part 2b:)** *Showing that the NPGC must hold with equality (basically showing the TVC condition is optimal)*

  In my opinion not at the core of the curriculum. To show it we need the Ramsey setup:

  $$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u(c_t), \qquad \text{s.t. } c_t = a_t R_t + w_t - a_{t+1} \qquad \text{and } \lim_{T\rightarrow\infty}(q_T a_{T+1}) \geq 0. \tag{1.2}$$

  From this we see that we can rewrite the problem as

  $$\max_{t+1\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u(a_t R_t + w_t - a_{t+1}), \tag{1.3}$$

and that the problem implies the standard Euler equation

$$u'_t = \beta R_{t+1} u'_{t+1}, \qquad u' \equiv \frac{\partial u}{\partial c}. \tag{1.4}$$

Now assume that we have two different solutions to this problem: Denote $\hat{a}$ the solution that also obeys the TVC condition, and an alternative solution $\bar{a}$, which only obeys the NPGC. Both solutions will have the same initial wealth $a_0 = \bar{a}_0 = \hat{a}_0$, but over time one of the plans will drive the present value of wealth to zero, while the other will just have non-negative wealth. We will now show that the plan $\hat{a}$ is the preferred plan and thus that we need the TVC condition to choose the optimal plan from the following steps:

– **Perform a first order Taylor expansion around** $\hat{u}$ and denote the approximate value $\tilde{u}$:[1]

$$\tilde{u}_t = \hat{u}_t + R_t \hat{u}'_t (a_t - \hat{a}_t) - \hat{u}'_t (a_{t+1} - \hat{a}_{t+1}), \tag{1.5}$$

and define the approximation error simply as $\tilde{u}_t - u_t$.

– **Derive inequality from approximation:**

For a concave function u the minimum of an approximation error is given in the point of approximation, here $\hat{u}$. We also know that around the point of approximation, $\hat{u}$, the approximation error is zero. This implies that

$$\underbrace{\tilde{u}_t - u_t}_{\text{approx. error}} = \hat{u}_t + R_t \hat{u}'_t (a_t - \hat{a}_t) - \hat{u}'_t (a_{t+1} - \hat{a}_{t+1}) - u_t \geq 0,$$

or rewritten slightly that

$$\overbrace{R_t \hat{u}'_t (a_t - \hat{a}_t) - \hat{u}'_t (a_{t+1} - \hat{a}_{t+1})}^{=\text{LHS}_t} > \overbrace{u_t - \hat{u}_t}^{\text{RHS}_t}, \qquad \text{for } (a_t, a_{t+1}) \neq (\hat{a}_t, \hat{a}_{t+1}). \tag{1.6}$$

– **Now consider the two alternative solutions,** $\hat{a}$ **and** $\bar{a}$**, and sum the approximation errors:**

$$\sum_{t=0}^{T} \beta^t \hat{u}'_t (R_t (\bar{a}_t - \hat{a}_t) - (\bar{a}_{t+1} - \hat{a}_{t+1})) > \sum_{t=0}^{T} \beta^t (\bar{u}_t - \hat{u}_t), \tag{1.7}$$

where the inequality holds due to condition (1.6). Next we use the Euler equation in (1.7). To see this write out the sum in (1.7) (here the parts with $\hat{a}$):

$$\begin{aligned}
\sum_{t=0}^{T} \beta^t \hat{u}'_t (-R_t \hat{a}_t + \hat{a}_{t+1}) = &- \hat{u}'_0 R_0 \hat{a}_0 \\
&+ \hat{u}'_0 \hat{a}_1 - \beta \hat{u}'_1 R_1 \hat{a}_1 \\
&+ \beta \hat{u}'_1 \hat{a}_2 - \beta^2 \hat{u}'_2 R_2 \hat{a}_2 \\
&\vdots \\
&+ \beta^{T-1} \hat{u}'_{T-1} \hat{a}_T - \beta^T \hat{u}'_T R_T \hat{a}_T \\
&+ \beta^T \hat{u}'_T \hat{a}_{T+1}.
\end{aligned}$$

Note that using the Euler equation implies that every part cancels out except for the first and last line as

$$\hat{u}'_0 \hat{a}_1 - \beta \hat{u}'_1 R_1 \hat{a}_1 = \hat{a}_1 \underbrace{(\hat{u}'_0 - \beta R_1 \hat{u}'_1)}_{=0 \text{ by Euler}}.$$

[1] Recall that for a function f with two variables around (x_0, y_0) this is given by:

$$f(x, y) \approx f(x_0, y_0) + \frac{\partial f(x, y)}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f(x, y)}{\partial y}(x_0, y_0)(y - y_0).$$

Furthermore, as the level a_0 is the same for both solutions, the first line will cancel out in (1.7) as well. The implication for (1.7) is thus that

$$\beta^T \hat{u}'_T(\hat{a}_{T+1} - \bar{a}_{T+1}) > \sum_{t=0}^{T} \beta^t(\bar{u}_t - \hat{u}_t). \tag{1.8}$$

– **Lastly, use the TVC and NPGC condition:**

Using the definition of the prices q_T we can substitute for

$$\beta^T \hat{u}'_T = q_T \hat{u}'_0, \tag{1.9}$$

and thus rewriting inequality (1.8) as

$$\hat{u}'_0 \left(\underbrace{q_T \hat{a}_{T+1}}_{=0 \text{ by TVC}} - \underbrace{q_T \bar{a}_{t+1}}_{>0 \text{ by NPGC}} \right) > \sum_{t=0}^{\infty} \beta^t(\bar{u}_t - \hat{u}_t). \tag{1.10}$$

The left hand side is negative, thus $\hat{u} > \bar{u}$ showing that the TVC condition is indeed optimal.

An alternative approach to showing the TVC:

Let us more flexibly assume the condition that

$$\lim_{T\to\infty} (q_T a_{T+1}) = k \geq 0, \tag{1.11}$$

and let k be a variable under household control. We will now proceed to find the intertemporal budget by iterative substitution:

$$\begin{aligned} c_0 + (1+n)a_1 &= a_0 R_0 + w_0 \\ c_0 + \frac{1+n}{R_1}c_1 + \frac{(1+n)^2}{R_1}a_2 &= w_0 + \frac{1+n}{R_1}w_1 + a_0 R_0 \\ &\vdots \\ \sum_{t=0}^{\infty} c_t q_t &= \sum_{t=0}^{\infty} w_t q_t + a_0 R_0 - \lim_{T\to\infty}\left[q_T a_{T+1}\right], \end{aligned}$$

where the prices q_t are defined as usual. Now using the flexible TVC condition in (1.11) we rewrite the problem as

$$\sum_{t=0}^{\infty} c_t q_t = a_0 R_0 + W_0 - k. \tag{1.12}$$

Intuitively we are done here: We should choose the lowest possible k, as this increases the present value of consumption.

To finish the argument we could proceed as follows: Note that implicitly from (1.12) we have that a change in k implies that consumption will be changed as

$$\sum_{t=0}^{\infty} q_t \frac{\partial c_t}{\partial k} = -1. \tag{1.13}$$

Furthermore, the from the Euler equation we know that equation (14) must hold regardless of k:

$$\frac{\partial u_t}{\partial c_t} = \beta R_{t+1} \frac{\partial u_{t+1}}{\partial c_{t+1}}. \tag{1.14}$$

Differentiating on both sides wrt k we then get

$$u''_t \frac{\partial c_t}{\partial k} = \beta R_{t+1} u''_{t+1} \frac{\partial c_{t+1}}{\partial k} \qquad \Rightarrow \qquad \text{sign}\left(\frac{\partial c_t}{\partial k}\right) = \text{sign}\left(\frac{\partial c_{t+1}}{\partial k}\right) \tag{1.15}$$

Equation (1.15) states that if k changes the optimal thing we can do is to change c_t and c_{t+1} in the same direction. From (1.13) and (1.15) we then can now say that increasing k would mean lower consumption at all points in time. Thus the TVC setting k as small as possible is clearly optimal.

- **Substitution and income effects, initial level of consumption after a shock:**
  1. If the steady state function for $c_{t+1} = c_t$ is affected by shock $\Rightarrow$ substitution effect.
  2. If the steady state function for $k_{t+1} = k_t$ is affected by the shock $\Rightarrow$ income effect.
  3. In a scenario where the depreciation rate increases, $\delta \uparrow$, the substitution effect would be a lower interest rate $\Rightarrow$ increase current consumption. The income effect would be to decrease current consumption. What happens to consumption initially then? If we have CRRA preferences, that is

  $$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta} \tag{1.16}$$

  the substitution effect will dominate if $\theta < 1 \Rightarrow$ consumption initally increases. The income effect will dominate if $\theta > 1 \Rightarrow$ consumption initially decrease. Remember that log-preferences, $u(c_t) = \ln(c_t)$, corresponds to $\theta = 1$.

- **Deriving and setting up the phase-diagram in the Ramsey model**

  Assume the relatively simple setup with capital taxation:

$$\begin{aligned}
(1+n)a_{t+1} &= a_t(1 + r_t(1-\tau)) + w_t - c_t + T_t && (1.17)\\
\frac{c_{t+1}}{c_t} &= \left[\beta(1 + r_{t+1}(1-\tau))\right]^{\frac{1}{\theta}} && (1.18)\\
w_t &= f(k_t) - f'(k_t)k_t && (1.19)\\
r_t &= f'(k_t) && (1.20)\\
T_t &= a_t r_t \tau. && (1.21)
\end{aligned}$$

  Constructing the phase diagram involves:

  – Derive steady state function for consumption:

    Use (1.20) in (1.18) and impose steady state requirement that $c_{t+1}/c_t = 1$:

    $$\beta(1 + f'(k)(1-\tau)) = 1 \qquad \Rightarrow \qquad f'(k^*) = \frac{1-\beta}{\beta(1-\tau)}. \tag{1.22}$$

    When capital is large r_{t+1} is relatively small thus consumption decreases over time and vice versa.

  – Derive steady state function for capital:

    Use (1.19)-(1.21) in (1.17) as well as $a_t = k_t$. Using (1.21) in (1.17) note that the tax parts cancel out:

    $$\begin{aligned}
    (1+n)k_{t+1} &= k_t(1 + f'(k_t)) + f(k_t) - f'(k_t)k_t - c_t \\
    &= k_t + f(k_t) - c_t. && (1.23)
    \end{aligned}$$

In steady state this implies

$$c^{ss} = f(k^{ss}) - nk^{ss}. \tag{1.24}$$

When consumption is larger than (1.24) we are not investing enough to maintain k^{ss} and thus capital decreases over time.

2 The OLG model

- **Dynamic inefficiency:** Start by need to know, end with nice to know (math).

  1) In an OLG setup we can have the possibility of dynamic inefficiency. Generally, this means that we have overaccumulated capital to an extent, where a population transfer system would be more efficient to use than capital markets. Savings through capital market yields return $1 + r_{t+1}$. Savings through transfers yields return $1 + n$.

  2) What does this have to do with $k^* > k^G$? Well in general, the aggregate ressource constraint in the economy is given by (assuming the standard setup):

  $$c_t + k_{t+1}(1+n) = f(k_t) + k_t \tag{2.1}$$

  Assuming that we are in a steady state implies that

  $$c^{ss} = f(k^{ss}) - nk^{ss} \tag{2.2}$$

  The golden rule of capital, is the level of k^{ss} that maximizes c^{ss}. We find this by the first order condition

  $$\frac{\partial c^{ss}}{\partial k^{ss}} = f'(k^{ss}) - n = 0 \qquad \Rightarrow \qquad f'(k^G) = n. \tag{2.3}$$

  We argued intuitively above that we have dynamic inefficiency whenever $r < n$. Lastly, note that $f'(k) = r$ implying that for us to have dynamic inefficiency we must have $k^{ss} > k^G$.

  3) Mathematically showing inefficiency when $r^{ss} < n$: The easiest way to show that something is inefficient is to show that a social planner could reallocate ressources and make all households better off than before.

  Consider the social planner strategy: In period t_0 change the capital level from k^* to k^G. In all following periods, let capital stay at the level k^G. At time t_0 the social planner changes consumption with:

  $$\Delta c_{t_0} = \underbrace{f(k^*) - nk^G + (k^* - k^G)}_{\text{Consumption with SP change}} - \underbrace{(f(k^*) - nk^*)}_{\text{Steady state consumption without SP change}} = (k^* - k^G)(1+n)$$

  If $k^* > k^G$ the social planner will lower the investment level and use it on consumption instead. Thus consumption will be larger in this case. For all future periods the change in consumption will be:

  $$\Delta c_{t_0+i} = \underbrace{f(k^G) - nk^G}_{=c^G} - \underbrace{(f(k^*) - nk^*)}_{=c^*} \geq 0 \tag{2.4}$$

  By construction, the golden rule level of capital will imply the largest possible steady state consumption. Thus consumption will increase in all future periods as well.

Questions for PS5 part 2

The setup in PS5 part 2 is the OLG model with productive externalities and an elimination of a pay-as-you go pension system. The approach we usually use in OLG is:

- Solve firm problem:

$$r = \alpha A \tag{2.5}$$
$$w_t = (1-\alpha)Ak_t \tag{2.6}$$

- Solve household problem (here with PAYG system):

$$c_{1t} = \frac{1+\rho}{2+\rho}W_t, \qquad W_t = w_t - d\frac{r}{1+r} \tag{2.7}$$
$$c_{2t+1} = \frac{1+r}{2+\rho}W_t \tag{2.8}$$
$$s_t = \frac{w_t}{2+\rho} - \frac{d}{2+\rho}\left(1+\frac{1+\rho}{1+r}\right). \tag{2.9}$$

- Look at how aggregate economy evolves over time (derive transition diagram / capital accumulation scheme):

  In this simple OLG economy we have $k_{t+1} = s_t$, thus:

$$k_{t+1} = \frac{(1-\alpha)Ak_t}{2+\rho} - \frac{d}{2+\rho}\left(1+\frac{1+\rho}{1+\alpha A}\right). \tag{2.10}$$

  We note that in a (k_t, k_{t+1}) diagram this is a linear function. In steady state we have the requirement that $k_{t+1} = k_t$ implying that

$$k^*\left(1-\frac{(1-\alpha)A}{2+\rho}\right) = -\frac{d}{2+\rho}\left(1+\frac{1+\rho}{1+\alpha A}\right) \qquad \Rightarrow \qquad k^* = d\frac{2+\rho+\alpha A}{(1-\alpha)A-(2+\rho)}. \tag{2.11}$$

  However, there is an unstable steady state: If we start with $k_0 < k^*$ capital will go to zero over time, and with $k_0 > k^*$ our model will converge towards a constant growth rate of consumption. Thus the expression in (2.11) is not that meaningful.

In part c) through e) we look at an elimination of the pension system. The main points are:

- For **political support** of the reform we need to look at welfare levels for two generations:
  - **Generation young at time t_0:** They pay a share of pension benefits, γd but does not receive any benefits.
    If $\gamma = 0$ they support the reform as our economy is dynamically efficient, $r > n$, and eliminating the PAYG system implies that they are allowed to save using capital markets with a return r instead of PAYG with return n.
    If $\gamma = 1$ they do not support reform: They pay entire pension d, but do not receive any benefits.
  - **Generation old at time t_0:** No change.

- For **capital accumulation / savings:**

  When $\gamma = 0$ we have opposing effects: The elimination of the pension benefit clearly increases private savings. However, the government has to issue debt now to cover the last pension payment, thus crowding out investment in capital as we have:

$$k_{t_0+1} + \gamma d = s_{t_0} \qquad \Rightarrow \qquad k_{t_0+1} = s_{t_0} - \gamma d. \tag{2.12}$$

It turns out that in this case, the crowding out effect dominates and $k \downarrow$.

When $\gamma = 1$ we also have opposing effects: The elimination of pension system still implies larger private savings. However, with $\gamma = 1$ young households are poorer than before implying lower private savings. It turns out that in this case, the first effect dominates and $k \uparrow$.

3 Nominal rigidities

- **The use of** $y = m - p$**:** Is a simplification of the demand side of an economy. It simply states that output is equal to the real money balance, $Y = M/P$. Called the quantity theory of money.
- In some of the nominal rigitidies models we have encountered, the flex-price/long-run equilibrium is given by $p = m$ and $y = 0$. This is simply a **normalization** assumed when deriving optimal prices $p_i = \phi m + (1 - \phi)p$. No intuitive explanation.

4 Menu costs

We've seen menu costs in two setups: The simple duopoly setup in exercises and the Blanchard-Kiyotaki (BK) model in lectures (lecture 10). In the BK model we have:

- n firms producing n different goods that are imperfect substitutes.
- n households supplying specific labor.
- Monopolistic competition between firms.
- Money provides a liquidity service for households.

When there is no nominal frictions money is neutral. With menu costs this is no longer the case. Then if aggregate money increases we can interpret it as an increase in nominal demand: For a given price level of consumption goods, we are now richer. If the firms' menu costs are larger than the benefits of adjusting prices prices will in fact be unchanged $\Rightarrow$ output increases.

5 Monetary policy

- The intuition behind commitment policy outcome: We have assumed that social loss is given by deviations in inflation or output:

$$L_t = \frac{1}{2}\left(\pi_t^2 + \lambda(x_t - \bar{x})^2\right)$$

In the standard model, the commitment policy outcome is given by

$$\pi_t^C = 0 \tag{5.1}$$
$$x_t^C = \theta_t \tag{5.2}$$

First of all, if we could simply pick x_t **and** π_t we would choose $\pi_t = 0$ and $x_t = \bar{x}$. However, it turns out it is not possible to simply pick $\bar{x}$. The best we can do is to pick $x_t^C = \theta_t$. We can explain this in two ways. Firstly, the strictly intuitive approach:

1. Even if we used inflation to push output constantly towards the level $\bar{x}$, this would easily be anticipated by private agents and thus we would only end up with an inflation bias and no increase in output. Thus we choose to stabilize x_t^C around a mean of zero instead of $\bar{x}$.

2. Ideally, we would prefer to stabilize some of the θ_t shock to output using inflation. However, as private agents can observe the θ_t shock, they would easily anticipate our change in inflation. Thus once again we simply choose to not try and stabilize θ_t shocks to output, as this would only end up having an effect on inflation.

Secondly, we can see this using the first order conditions of optimal commitment policy. If our monetary policy is given by:

$$\pi_t = \eta_0 + \eta_1 \theta_t + \eta_2 \epsilon_t, \tag{5.3}$$

and we have the usual timing, we have the first order conditions:

$$\frac{\partial E[L_t]}{\partial \eta_0} = \eta_0 = 0 \tag{5.4}$$

$$\frac{\partial E[L_t]}{\partial \eta_1} = \eta_1 \sigma_\theta^2 = 0 \tag{5.5}$$

$$\frac{\partial E[L_t]}{\partial \eta_2} = \eta_2 \sigma_\epsilon^2 + \lambda(\eta_2 - 1)\sigma_\epsilon^2 = 0 \tag{5.6}$$

The first order condition in (5.4) states that if we change the constant inflation level, η_0 from zero, this will simply increase our expected loss by η_0 units. The reason for this is that a constant level of inflation is easily anticipated by private agents, thus we can never use η_0 to stabilize employment; $x_t = \theta_t + \pi_t - \pi_t^e - \epsilon_t$.

The first order condition in (5.5) states that if we change how inflation should react to θ_t shocks, η_1, the expected loss will increase by $\eta_1 \sigma_\theta^2$ units. The reason for this is once again that as we have assumed that private agents observe θ_t before forming expectations, they will easily anticipate any change in inflation aimed at stabilizing the θ_t shock. Thus changing η_1 from zero would only increase loss from inflation fluctuations, but not stabilize the output shock.

The first order condition in (5.6) states that if we change how inflation should react to ϵ_t shocks, η_2, we will have both a marginal cost and a marginal benefit:

1. Marginal cost: When η_2 increases, we have an increase in expected loss of $\eta_2 \sigma_\epsilon^2$ units. This is because we could choose $\eta_2 = 0$ in which case we would always simply have zero inflation, but instead we let inflation change with ϵ_t, which induces a loss.
2. Marginal benefit: As long as $\eta_2 < 1$, letting inflation change with ϵ_t shocks implies that the ϵ_t shocks to output are partially stabilized. This is the case because of the critical assumption that private agents do not observe ϵ_t shocks before forming expectations.

A good answer at the exam need not go this much into detail, but simply state that we let $\eta_0 = \eta_1 = 0$, as these cannot be used to stabilize output, as they can be perfectly anticipated by private agents. However, $\eta_2 \neq 0$, as we can use surprise inflation to stabilize ϵ_t shocks to output.

- **Optimal Central banker:** See note on Monetary Policy, pages 6-7.
- **Inflation bias under discretionary policy and the credibility problem:** See note on Monetary Policy, pages 4-5.

6 Other

- **Decentralized vs. centralized equilibrium:** Decentralized equilibrium is the one obtained without social planner. A hybrid form of decentralized/centralized equilibrium is the classic market equilibrium, but with state intervention through some sort of taxation (I would say).

- *When capital is predetermined, does this count for wealth as well?* Well, it depends on the timing of the model, but in the Ramsey-type setups we have encountered, yes.
- **CRRA** stands for Constant Relative Risk Aversion. You are allowed to use the term: Inverse of Intertemporal Elasticity of Substitution if you want. Remember however, that the greek letters σ and θ for instance, does not always hold the same meaning: It depends on the setting. In this setting, θ is the CRRA and σ the IES:

  $$u(c_t) = \frac{c_t^{1-\theta}}{1-\theta} = \frac{c_t^{1-1/\sigma}}{1-1/\sigma}$$

  But I could just as well call it the opposite:

  $$u(c_t) = \frac{c_t^{1-1/\theta}}{1-1/\theta} = \frac{c_t^{1-\sigma}}{1-\sigma}$$

- **The use of a social planner:** In general, you can distinguish a social planner setup and a representative household setup by the difference in budgets.[2] For example, see PS2 part 1 for a social planner approach to the Ramsey model and PS3 for the representative household setup.

  The classic use of a social planner is to derive the best possible outcome and then later see if government intervention can secure this outcome. Besides setting up the social planner problem directly, we often also use the first welfare theorem to state in which scenarios the social planner solution coincides with household solution.

7 A note on the Lucas model

This part is mainly an explanation of the technical part of the Lucas model. The question is regarding slidedeck 8: We're considering the Lucas model where the solution is given by:

$$y_i = \frac{1}{\gamma - 1}(p_i - p). \tag{7.1}$$

$$y_i = m - p + z_i - \eta(p_i - p) \tag{7.2}$$

Where (7.1) describes the supply of good i and (7.2) describes demand. The main driver of the Lucas model is that the individual producer cannot distinguish between local and aggregate shocks. If a change in price is due to a local shock $\Rightarrow$ optimal to increase prices and output. If it's an aggregate shock $\Rightarrow$ only increase price. When there is uncertainty, we get a 'monetary transmission mechanism' out of the blue, even without assuming nominal rigidities.

Assume that both aggregate shocks occur (with changes in m) and local shocks occur (with changes in z_i). Furthermore, assume for simplicity that both shocks are IID:

$$m \sim \mathcal{N}\Big(E[m], \quad V_m\Big) \tag{7.3}$$

$$z_i \sim \mathcal{N}\Big(0, \quad V_z\Big) \tag{7.4}$$

where V_m and V_z are variances. Next, use that producers observe p_i, but do not observe p. Thus for the individual producer to follow his supply rule of (7.1), we need to estimate the relative price $r_i = p_i - p$:[3]

$$E[r_i|p_i] = \frac{V_r}{V_r + V_p}(p_i - E[p]) \tag{7.5}$$

[2] If households are not representative, the welfare-function is different as well however.

[3] This comes from a linear least square projection $\min_\kappa[\hat{u}_t - u_t]^2$, where $\hat{u}_t = \kappa(u_t + u_t^i)$, under the assumption that u_i and u are uncorrelated. See appendix A

where V_r is the variance of r_i and V_p is the variance of p. As is noted in slides, if we assume certainty equivalence, we can use a individual supply curve of the form:

$$y_i = \frac{1}{\gamma - 1}\frac{V_r}{V_r + V_p}(p_i - E[p]) = b(p_i - E[p]) \tag{7.6}$$

Aggregating over producers we obtain the aggreagte supply curve $y = b(p - E[p])$. Using the aggregate demand side as well, $y = m - p$, we have the equilibrium result:

$$p = \frac{1}{1+b}m + \frac{b}{1+b}E[p] \tag{7.7}$$

$$y = \frac{b}{1+b}m - \frac{b}{1+b}E[p] \tag{7.8}$$

Taking expectations of (7.7) we can furthermore see that $E[p] = E[m]$. This gives the equilibrium:

$$p = E[m] + \frac{1}{1+b}(m - E[m]) \tag{7.9}$$

$$y = \frac{b}{1+b}(m - E[m]) \tag{7.10}$$

Note that now we can rewrite the supply curve in (7.6) as:

$$\begin{aligned} y_i &= b(p_i - p) + b(p - E[p]) \\ &= b(p_i - p) + b\left(\overbrace{E[m] + \frac{1}{1+b}(m - E[m])}^{=p} - \overbrace{E[m]}^{E[p]}\right) \\ &= b(p_i - p) + \frac{b}{1+b}(m - E[m]) \end{aligned}$$

Likewise, we can rewrite the demand side:

$$y_i = \overbrace{\frac{b}{1+b}(m - E[m])}^{=y} + z_i - \eta(p_i - p)$$

Combining the new individual demand and supply equations we get

$$p_i - p = \frac{z_i}{\eta + b} \tag{7.11}$$

The last part of the Lucas model is characterizing the b parameter. We defined b in (7.6) as a function of variance of $p_i - p$ and the variance of p. We can now derive the variance expressions from (7.11) and (7.9):

$$Var(p_i - p) = \frac{V_z}{(\eta + b)^2} \tag{7.12}$$

$$Var(p) = \frac{V_m}{(1 + b)^2} \tag{7.13}$$

Plugging into our definition of b we get the result that

$$b = \frac{1}{\gamma - 1}\frac{V_r}{V_r + V_p} = \frac{1}{\gamma - 1}\frac{\frac{V_z}{(\eta+b)^2}}{\frac{V_z}{(\eta+b)^2} + \frac{V_m}{(1+b)^2}} = \frac{V_z}{V_z + V_m\frac{(\eta+b)^2}{(1+b)^2}}. \tag{7.14}$$

With the specification of the general price level and output in equations (7.9)-(7.10), we finally add an assumption of demand:

$$m_t = m_{t-1} + c + u_t \qquad \Rightarrow \qquad E[m_t] = m_{t-1} + c. \tag{7.15}$$

Using this in equation (7.9)-(7.10) we get

$$p_t = \overbrace{m_{t-1} + c}^{=E[m]} + \frac{1}{1+b} \overbrace{u_t}^{=m-E[m]} \tag{7.16}$$

$$y_t = \frac{b}{1+b} u_t. \tag{7.17}$$

Using this we can find the inflation rate

$$\begin{aligned}
\pi_t = p_t - p_{t-1} =& m_{t-1} + c + \frac{1}{1+b} u_t - \left(m_{t-2} + c + \frac{1}{1+b} u_{t-1} \right) \\
=& \overbrace{m_{t-2} + c + u_{t-1}}^{=m_{t-1}} + \frac{1}{1+b} u_t - m_{t-2} - \frac{1}{1+b} u_{t-1} \\
=& c + \frac{1}{1+b} u_t + \frac{b}{1+b} u_{t-1}.
\end{aligned} \tag{7.18}$$

Finally we note that by taking expectations to (7.18) and using (7.17) we see

$$E[\pi_t] = c + \frac{b}{1+b} u_{t-1} \tag{7.19}$$

$$\frac{1}{1+b} u_t = \frac{1}{b} \underbrace{\frac{b}{1+b} u_t}_{=y_t}. \tag{7.20}$$

Finally we thus end up with the Phillips curve

$$\pi_t = E[\pi_t] + \frac{1}{b} y_t. \tag{7.21}$$

Appendices

A Linear least squares projection

We want to find the expected level of p, by minimizing the squared residual:

$$\begin{aligned}\min_{\kappa} E[\hat{p}-p]^2 &= E\left[\overbrace{\kappa(p+p_i)}^{=\hat{p}}-p\right]^2 \\ &= E\left[\kappa^2(p+p_i)^2+p^2-2\kappa(p+p_i)p\right]\end{aligned}$$

Assuming that p and p_i are independent this becomes:

$$\min_{\kappa} \kappa^2 Var(p+p_i)+Var(p)-2\kappa Var(p)$$

From this our FOC yields:

$$\kappa = \frac{Var(p)}{Var(p)+Var(p_i)}.$$

The intuition is that if we in general observe large shocks in p compared to p_i, we largely expect fluctuations in $p_i - p$ to stem from aggregate shocks, thus $\kappa \rightarrow 1$.