The OLG model with money growth

We're considering the standard OLG model with money. Adding to this setup we assume that the money stock grows at a rate σ. This amount $M_t\sigma$ is printed by the government and distributed as lump-sum transfers to the old generation at each point in time. Thus we face the household setup:

$$\begin{aligned}
\max_{c_{1t},c_{2t+1}} \quad & u(c_{1t})+\beta u(c_{2t+1}) & (1)\\
\text{s.t} \quad & \frac{M_t^d}{P_t} = y_1 - c_{1t} & (2)\\
\text{and} \quad & \frac{M_t^d + \Delta M_t}{P_{t+1}} = c_{2t+1}, & (3)
\end{aligned}$$

where the households perceives ΔM_t to be lump-sum. The household solution to this problem can then be found by Lagrange

$$\mathcal{L} = u(c_{1t}) + \beta u(c_{2t+1}) + \lambda \left(y_1 + \Delta M_t \frac{P_{t+1}}{P_t} - c_{1t} - c_{2t+1}\frac{P_{t+1}}{P_t} \right) \tag{4}$$

with the first order conditions that

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_{1t}} &= u_1 - \lambda = 0 & (5)\\
\frac{\partial \mathcal{L}}{\partial c_{2t+1}} &= \beta u_2 - \lambda \frac{P_{t+1}}{P_t} = 0, \qquad u_i \equiv \frac{\partial u}{\partial c_i}. & (6)
\end{aligned}$$

The combination of the two yields a standard Euler equation describing that the marginal rate of substitution must be equal to relative prices, given by

$$\frac{u_1}{u_2} = \beta \frac{P_t}{P_{t+1}}. \tag{7}$$

Note that with $P_t/P_{t+1} = 1 + r_{t+1}$ this corresponds to the Euler equation in the standard OLG model with capital markets. This makes sense intuitively: If $P_t > P_{t+1}$, the price of consumption goods falls over time, implying that your 'savings' in terms of money becomes more valuable. Thus the relative price describes the 'return' on our savings from money.

Interpreting money, M_t^d/P_t, as savings with the relative price as the interest rate, we can describe the demand for money in the standard substitution / income effect perspective:

$$\frac{M_t^d}{P_t} = L\left(\frac{P_t}{P_{t+1}}, \frac{\Delta M_t}{P_{t+1}}\right), \qquad L_{\Delta M_t} < 0, \qquad L_{P_t/P_{t+1}} \lesseqgtr 0. \tag{8}$$

- Changing the relative price (analogously, the interest rate) has two effects:
  - Substitution effect: If P_t/P_{t+1} increases, consumption today becomes more expensive and we substitute towards consumption in the future $\Rightarrow M_t^d/P_t$ (savings) increases.
  - Income effect: If P_t/P_{t+1} increases, the return on money increases $\Rightarrow$ increase consumption both today and in the future (increase M_t^d/P_t).
- Changing the size of the lump-sum transfer to the old however, has an unambiguous effect: This increases income as old and keeps income as young fixed. In order to smooth out consumption over time this implies lower savings through a lower money demand.

As in the problem set we finally need to impose equilibrium in our markets.

Market clearing in the money market

By the assumption that the money supply grows at a constant rate σ the condition for market clearing is that

$$\underbrace{H_t = H_0(1+\sigma)^t}_{\text{Nominal supply of money}} = \underbrace{(1+n)^t N_0 M_t^d = N_t M_t^d}_{\text{Nominal demand for money}}.$$

Note that we here assume that N_t is the size of the population, which we assume grows at the rate n. Rewriting this market-clearing condition we have that

$$M_t^d = \frac{H_0}{N_0}\left(\frac{1+\sigma}{1+n}\right)^t. \tag{9}$$

Assuming a stationary equilibrium, where the real money demand is constant over time, we have that[1]

$$\frac{M_t^d}{P_t} = \frac{H_0}{N_0}\left(\frac{1+\sigma}{1+n}\right)^t \frac{1}{P_t} = \frac{H_0}{N_0}\left(\frac{1+\sigma}{1+n}\right)^{t+1} \frac{1}{P_{t+1}} = \frac{M_{t+1}^d}{P_{t+1}},$$

which implies that prices evolve according to

$$\frac{P_t}{P_{t+1}} = \frac{1+n}{1+\sigma}. \tag{10}$$

With the market clearing condition we can rewrite the money demand function in (8) as

$$\frac{M_t^d}{P_t} = L\left(\frac{1+n}{1+\sigma}, \frac{\sigma M_t}{P_{t+1}}\right), \qquad \Delta M_t = \sigma M_t,$$

where we can finally derive an expression for M_t/P_{t+1} as

$$\frac{M_t}{P_{t+1}} = \frac{M_t}{P_t}\frac{P_t}{P_{t+1}} = \frac{M_t}{P_t}\frac{1+n}{1+\sigma}, \tag{11}$$

which implies the final and general real money demand function

$$\frac{M_t^d}{P_t} = L\left(\frac{1+n}{1+\sigma}, \sigma\frac{M_t}{P_t}\frac{1+n}{1+\sigma}\right). \tag{12}$$

A final note is that we acutally still have a seperate equilibrium condition, that is our feasibility constraint. With population growth in the economy this is slightly different than the model in the problem set (part one), namely that

$$N_{1t}y_1 = N_{1t}c_{1t} + N_{2t}c_{2t}, \qquad N_{1t} = (1+n)N_{2t},$$

implying that

$$y_1 = (1+n)c_{1t} + c_{2t}. \tag{13}$$

As we will confirm in the last part, Walras law ensures that as long as prices clear the money market by following equation (10) the only other market, the goods market/feasibility constraint, will clear as well. Lastly, note that in a **stationary** equilibrium, which indeed will be the case in this model, the feasibility constraint can be written with $c_{2t} = c_{2t+1}$:

$$(1+n)y_1 = (1+n)c_{1t} + c_{2t+1}. \tag{14}$$

Thus (14) can be seen as an additional constraint on the household along with (2) − (3).

[1] Note that real money demand will in fact be constant as: 1) agents are identical with the same preferences over time, and 2) the endowment income y_1 is assumed to be constant over time.

Efficiency, feasibility constraint and money growth in equilibrium

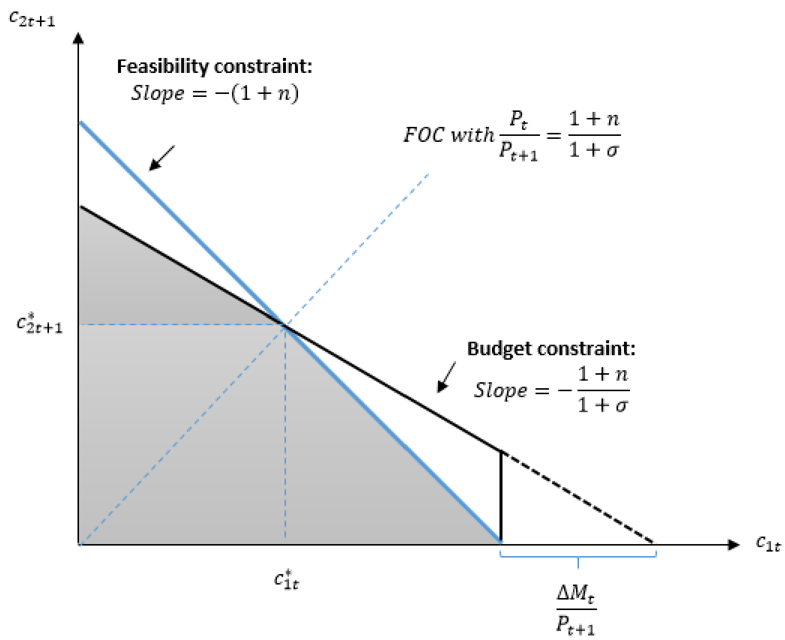
Illustrating the setup, note that in the case without money growth simply consists of the same derivation but with $\sigma = 0$. In the general case we have

$$c_{2t+1} = \frac{1+n}{1+\sigma}(y_1 - c_{1t}) + \frac{\Delta M_t}{P_{t+1}} \tag{15}$$

$$c_{2t+1} = (1+n)(y_1 - c_{1t}), \tag{16}$$

where (15) comes from combining household budgets and imposing equilibrium prices, and the second is simply the feasibility constraint. Illustrating the household decision consider figure 1:

Figure 1: Household decision with money growth



From the household setup the consumption-possibility frontier consists of the budget slope. However, adding the feasibility constraint as well, we note that the price mechanism will in fact ensure that the consumption possibility frontier will be the kinked line consisting of the budget constraint and the feasibility constraint. Put differently, the Consumption Possibility Subset (CPS) are given by the shaded area given by

$$CPS_{\sigma>0} = \left\{ (c_{1t},\ c_{2t+1}) \in \mathbb{R}^2_+ : c_{2t+1} + \frac{1+n}{1+\sigma} c_{1t} \leq \frac{1+n}{1+\sigma} y_1 + \frac{\Delta M_t}{P_{t+1}} \ \wedge\ c_{2t+1} + (1+n)c_{1t} \leq (1+n)y_1 \right\}. \tag{17}$$

It is clear that the case of $\sigma = 0$ would increase the CPS area to include the upper triangle in figure 1: In the case of $\sigma = 0$ the budget and feasibility constraint is the same, that is we have

$$CPS_{\sigma=0} = \left\{ (c_{1t},\ c_{2t+1}) \in \mathbb{R}^2_+ : c_{2t+1} + (1+n)c_{1t} \leq (1+n)y_1 \right\}. \tag{18}$$

Thus without further investigation we can conclude that the case with money growth cannot pareto-dominate the case with constant money stock, simply because households in the case of $\sigma = 0$ have all the same possibilities as with $\sigma > 0$ and actually more.