1. Demonstrate that the Shape Lemma is plausible (ask somebody what this lemma is if necessary) using the example in the accompanying `.m2` file.

2. Using the same example consider the multiplication operators on $A = R/I$, the quotient algebra: for $f \in R$ define $m_f : A \to A$ by $m_f([g]) = [fg]$.

   (a) Fix a monomial order (e.g., use the M2 default order), build a basis B of A, and find the matrix of the operator $m_{(y^2-x)}$ relative this basis.

   Hints:

   - Define the quotient algebra `A` and try `B = basis A`.
   - `coefficients(f,Monomials=>B)` is one way to get the coefficients of `f` corresponding to monomials in `B`.
   - See what happens when if `f` above is replaced with a matrix, for example, `B`.

   (b) Eigenvalues of m_f have a meaning, what is it? Compute these for some f to experiment.

3. The map

$$(x_0 : x_1 : x_2) \mapsto (x_0^2 + x_1^2 : x_1^2 + x_2^2)$$

   defines a polynomial (rational) map

$$\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1.$$

   (a) Is ϕ a morphism on $\mathbb{P}^2$?

   (b) Show that the restriction of ϕ to the projective variety $V = \mathbb{V}(x_0 - x_1) \subset \mathbb{P}^2$ is a morphism $V \to \mathbb{P}^1$.

   (A map $\phi = (\phi_0 : \cdots : \phi_m)$ defines a morphism on V if the ideal $\mathbb{I}(V) + \langle \phi_0, \ldots, \phi_m \rangle$ is irrelevant.)

4. Let $V \subset \mathbb{P}^n$ be a projective variety. What is the expected dimension of the secant variety $\mathrm{Sec}(V)$ when $\dim V = m$? Prove by computation that the dimension "drops" for the secant variety of the Veronese $\nu_2(\mathbb{P}^2) \subset \mathbb{P}^5$. .

5. Consider 4 points on the circle of radius r forming a quadrilateral with the sides a, b, c, d. Try to find a formula for (the square of) the area S of this quadrilateral in terms of a, b, c, dr.

6. Given $A \in \mathbb{C}^{n\times n}$ find the eigenvalues and eigenvectors of A using homotopy continuation.

   Consider the case $n = 2$. Let
$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad v = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

   There are three unknowns in the eigenproblem: the eigenvalue λ and the coordinates of the vector v. We seek solutions of the system
$$(Av - \lambda v) = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 - \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 - \lambda x_2 \end{pmatrix}.$$

   Augment the system with one linear equation:
$$F = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 - \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 - \lambda x_2 \\ b_1x_1 + b_2x_2 + 1 \end{pmatrix}.$$

   Assume that the eigenspaces of A are one-dimensional (this is the case, in particular, when the eigenvalues are not repeated). If $b_1, b_2 \in \mathbb{C}$ are generic, the last equation in the system picks out one nonzero vector in each eigenspace. We conclude that F is a 0-dimensional system generically, since a randomly picked matrix has distinct eigenvalues.

   (a) Solve the system F for a random A using the total-degree homotopy.

   (b) Alternatively, take a start system that arises from the eigenproblem that we know a solution to: take a diagonal matrix D with distinct entries on the diagonal,
$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}, \quad d_1 \neq d_2$$
   and `track` from the known start system to the target system F.

   (c) Solve F using `MonodromySolver` package function `monodromySolve`.

   (d) Write code that does *all of the above* for an arbitrary n.

7. (Composed by Mike Stillman. May be understood better after the tutorial by Dan Grayson.)

   Work through and understand the `eg-pappus.m2` which provides a computer proof of Pappus' theorem. There are a number of ways to actually construct an ideal, some are simpler than others. Contest: who can find the simplest?