GENERALIZED EULER SCHEME

Log-Euler Scheme:

Consider the lognormal process:

$dX(t) = \mu(t, X(t))\, X(t)\, dt + \sigma(t)\, X(t)\, dW(t)$

$(t,x) \longmapsto \mu(t,x)$, $t \longmapsto \sigma(t)$ are given deterministic function

(We consider: $\sigma(t, X(t)) = \sigma(t) \cdot X(t)$)

Euler Scheme:

$\tilde{X}(t_{i+1}) = \tilde{X}(t_i) + \mu(t_i, \tilde{X}(t_i))\, \tilde{X}(t_i)\, \Delta t_i + \sigma(t_i)\, \tilde{X}(t_i)\, \Delta W(t_i)$

$\tilde{X}(t_0) = X_0$ initial value (it is exact)

Discretization error: $X(t) - \tilde{X}(t)$ it could be very large!

- if $\mu = 0 \rightarrow \tilde{X}(t)$ is normal distributed while $X(t)$ is log-normal distributed
- $\tilde{X}$ can be negative while X cannot!

How improve it?

Let's consider $Y(t) = \log X(t)$; by Itô's lemma we get:

$$dY(t) = \frac{1}{X(t)}\, dX(t) - \frac{1}{2} \frac{1}{X^2(t)}\, d\langle X \rangle_t = \dots$$

$$= \left(\mu(t, X(t)) - \frac{\sigma^2}{2}(t) \right) dt + \sigma(t)\, dW(t)$$

Thus:

$$d \log X(t) = \left(\mu(t, X(t)) - \frac{\sigma^2}{2}(t) \right) dt + \sigma(t) \, dW(t)$$

We can discretize this process by using the Euler scheme:

$$\log \tilde{X}(t_{i+1}) = \log \tilde{X}(t_i) + \left(\mu(t_i, X(\tilde{t}_i)) - \frac{\sigma^2}{2}(t_i) \right) \Delta t_i + \sigma(t_i) \Delta W(t_i)$$

applying the exponential we get:

$$\tilde{X}(t_{i+1}) = \tilde{X}(t_i) \exp\left[\left(\mu(t_i, X(\tilde{t}_i)) - \frac{\sigma^2}{2}(t_i) \right) \Delta t_i + \sigma(t_i) \Delta W(t_i) \right]$$

$\rightarrow$ Better discretization

Note:

- We must apply exp( ) because we need $\tilde{X}(t_{i+1})$
- $\tilde{X}(t_{i+1})$ is non negative if $\tilde{X}(t_i)$ is non negative (i.e. if $X_0 > 0$ by induction)
- We lose the generality of the Euler scheme
  - ↳ OOP design problem (open-closed principle)
    - ↳ rewrite the abstract class in order to maintain the OCP.