LEZIONE 1 04/11/2024

Exam will be written : answer two exercise

theory and implementation 26 maximum

A project is needed → groups of 4 people

Discuss indipendently of the written exam. Every one discuss

Project on something not done at lesson.

Research and figure out what to do. We can suggest topic. We get halfway during the course.

NUMERICAL MODELS FOR DIFFERENTIAL PROBLEMS (3rd edition SPRINGER NATURE)

We don't know the physical solution.

We have a physical problem. We know how to model it mathematically

PHYSICAL PROBLEM u_{ph} → MATH PROBLEM u_m $\xrightarrow{\text{numerical approximation}}$ NUMERICAL PROBLEM u_n

↓

numerical algorithm

↓ Programming language

COMPUTER u_c

global error

$$\|u_{ph} - u_c\| \le \underbrace{\|u_{ph} - u_m\|}_{\text{Modelling error}} + \underbrace{\|u_m - u_n\|}_{\text{numerical error}} + \underbrace{\|u_n - u_c\|}_{\text{computer error}}$$

Modelling error → Accurate to model the problem

computer error based on ROUNDOFF

We start from the math problem and how to solve it.

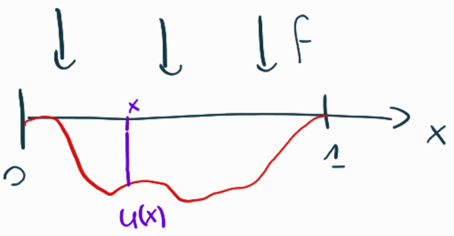
We don't have interest in computer error.

Example of math problem

The function are un because stand for unknown

1D

Elastic string



The equation describing this problem is

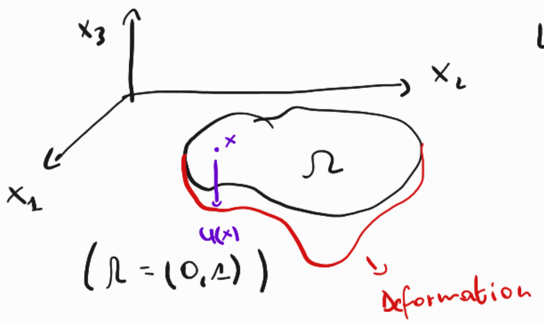
$$\begin{cases} -u''(x) = f(x) & 0 < x < 1 \quad \text{PDE (partial differential equation)} \\ u(0) = u(1) = 0 & \text{BC (boundary condition)} \end{cases}$$

This is the easiest possible

2D

The same problem became

Ω = Domain



Laplacian

$$-\Delta u = \begin{cases} -\dfrac{\partial^2 u}{\partial x_1^2} - \dfrac{\partial^2 u}{\partial x_2^2} = f & x \in \Omega \quad \text{(PDE)} \\ u(x) = 0 & x \in \partial\Omega \quad \text{(BC's)} \end{cases}$$

In 3D: $\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^{2,3} \\ u = 0 & \text{on } \partial\Omega \end{cases}$

These are $\boxed{\text{ELLIPTIC EQUATIONS}}$

EX 2

Information

$$\begin{cases} \text{(PDE)} \quad \dfrac{\partial u}{\partial t} - \Delta \boxed{u} = \boxed{f} & x \in \boxed{\Omega}, \quad 0 < t < \boxed{T} \\ \text{(BC)} \quad u = \boxed{0} & x \in \partial\Omega, \quad 0 < t < T \\ \text{(IC)} \quad u = \boxed{u_0} & x \in \Omega, \quad t = 0 \end{cases}$$

Initial condition

Time is only one dimension

$u(x,t;\text{data})$ Is the solution

For the temperature in a room we need 3 information

These are $\boxed{\text{PARABOLIC EQUATION}}$

Waves propagate in time.

$$\begin{cases} \text{PDE} & \dfrac{\partial^2 u}{\partial t^2} - \Delta u = f \\ \text{BC} & u = 0 \qquad x \in \partial\Omega,\ t > 0 \\ \text{IC} & \begin{cases} u = u_0 & x \in \Omega,\ t = 0 \\ \dfrac{\partial u}{\partial t} = x_0 & x \in \Omega,\ t = 0 \end{cases} \end{cases}$$

$\boxed{\text{HYPERBOLIC EQUATIONS}}$

NAVIER STOKES EQUATION for dynamic and fluids

ELLIPTIC EQS

$$(P) \begin{cases} Lu = -\underbrace{(\mu u')'}_{\text{diffusion}} + \underbrace{b u'}_{\text{transport/correction}} + \underbrace{\sigma u}_{\text{reaction}} = f \quad (\text{PDE}) \qquad \Omega = (0,1) \quad 0 < x < 1 \\ \left.\begin{array}{l} u(0) = \varphi \\ \mu u'(1) = \psi \end{array}\right\} \text{BC} \end{cases}$$

$u = u(x)$ solution ≡ concentration of pollution at point x

We have a river channel

I throw a polluted into the channel.

I threw oil in a river. The oil will be transported, the drop will be delayed. Diffusion due to molecular caratteristic. Transportation because the water is moving (b it's the velocity). The reaction is where the pollution go

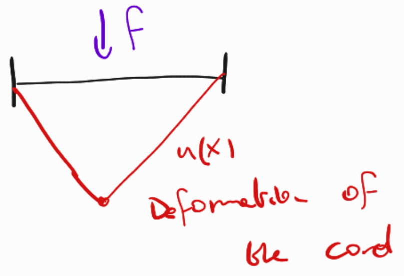
$\left.\begin{array}{l} \mu > 0 \\ b \\ \sigma \geq 0 \end{array}\right\}$ prom. data $\qquad \varphi$ concentration of pollution

f
φ, ψ, Ω

These problem can be used in other case

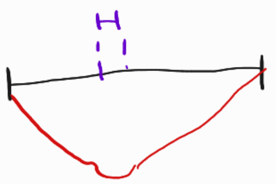
To resolve this problem I need to transform that math problem in another.

I want to transform because is this th most simple representation?



it's a cord

This is C^0 continuous
$\notin C^1$



$\notin C^2$
$\in C^1$

$-\mu u'' = f$

If f is not smooth enough it's not C^2

Transform problem in a problem without a second derivate.

I don't want second derivate

We do integration by part

Transforming (P) into (PW) → weak formulation

I take v (arbitrary function)

$Lu = f \longrightarrow Luv = fv \quad \forall v \in V$

$\Rightarrow \int_\Omega Luv = \int_\Omega vf$ now I integrate by part and get (PW)

So

$-(\mu u')'v + bu'v + \sigma uv = fv \quad \forall v \in V$

$\int_\Omega -(\mu u')'v + \int_\Omega bu'v + \int_\Omega \sigma uv = \int_\Omega fv$

$$\int_\Omega (\mu u')' v' - \underbrace{[\mu u' v]_0^1}_{\boxed{-\mu u'(1)} v(1) + \mu u'(0) \boxed{v(0)}} + \int_\Omega b u' v + \int_\Omega \sigma u v = \int_\Omega f v$$

$-\mu u'(1) = -\psi$, $\quad v(0) = 0$

Now I use the boundary condition

Boundary terms are horrible, we should avoid it. (The devil make it)

I choose to put $v(0)$ as 0.

IT'S MY CHOICE

All what is known goes to the right.

The unknown to the left

$$\int_\Omega (\mu u')' v' + \int_\Omega b u' v + \int_\Omega \sigma u v = \int_\Omega f v + \psi v(1) \qquad \forall v \in V$$

$$V = \{ v : (0,1) \to \mathbb{R}, \ \ldots\ldots, \ v(0) = 0 \}$$

This is my TRANSFORM PROBLEM

TEST FUNCTIONS $\quad v \in C^1 \Rightarrow$ OK

Sufficient condition

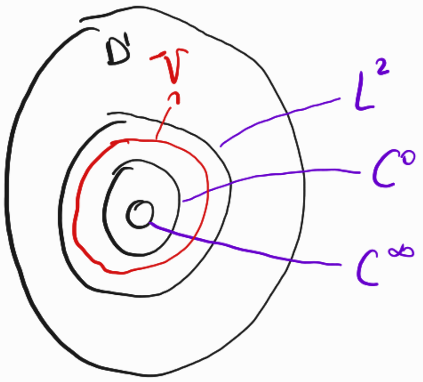
Advantage : no second derivate

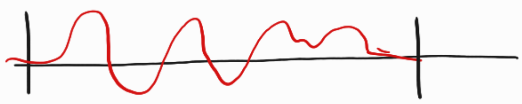
Disadvantage: I have infinite equations

DISTRIBUTION OF DERIVATIVES

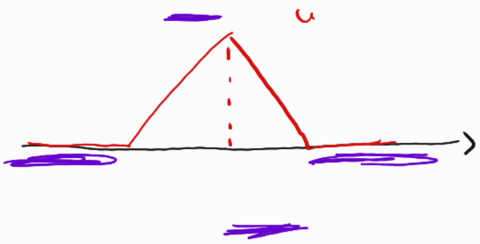
$$Du = g \quad \text{Iff.} \quad \int_{\mathbb{R}} g \varphi = - \int_{\mathbb{R}} u \varphi' \qquad \forall \varphi \in C^\infty(\mathbb{R})$$

φ goes to zero with all is derivaties as $|x| \to \infty$





Compact support



$g = Du$

This is the distribution derivative

So

$$v \in V = \{ v : (0,1) \to \mathbb{R},\ v \in L^2(0,1),\ Dv \in L^2(0,1),\ v(0) = 0 \} =$$

$$= \{ v \in H^1(0,1) : v(0) = 0 \}$$

$$L^2(\Omega) = \{ v : \Omega \to \mathbb{R},\ \int_\Omega v^2(x)\,dx < +\infty \}$$

Lebesgue

$$\boxed{H^1(\Omega) = \{ v : \Omega \to \mathbb{R},\ v \in L^2(\Omega),\ D_{x_i} v \in L^2(\Omega),\ i = 1, \dots, d \}}$$

SOBOLEV SPACES

So

$$\boxed{(PW) : ?\ u \in V : \int_\Omega \mu u'v' + \int_\Omega b u'v + \int_\Omega \sigma u v = F(v) \quad \forall v \in V}$$

I want to find u st

$$F(v) = \int_\Omega f v + \psi v(1)$$

I know how to approximate it.

From a linear PDE I get always a linear system

From a non linear PDE I get a non linear system

Use simple notation when possible

$$a(u,v) = \int_\Omega \mu u'v' + \int_\Omega b u'v + \int_\Omega \sigma u v$$

$a : V \times V \to \mathbb{R}$ bilinear

So the problem become

$(PW) : ?\ u \in V : a(u,v) = F(v) \quad \forall v \in V$

$F : V \to \mathbb{R}$ linear functional

Every elliptic problem can be written in this form

We need to transform in a finite dimensional problem.

Use V_h so $\dim V_h = N_h < \infty$

$$(PW_h): \ ?\, u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$\{\varphi_J\}_{J=1}^{N_h}$ basis of V_h

unknowns

$$u_h(x) = \sum_{J=1}^{N_h} \boxed{u_J}\, \varphi_J(x)$$

So $u_h = \varphi_i \qquad \forall i = 1, \ldots, N_h$

Linear $\longrightarrow$ $a\left(\sum_J u_J \varphi_J, \varphi_h\right) = F(\varphi_i) \quad \forall i = 1, \ldots, N_h$

$$\Rightarrow \sum_J u_J \underbrace{a(\varphi_J, \varphi_h)}_{A_{iJ}} = \underbrace{F(\varphi_i)}_{F_i} \qquad \forall i = 1, \ldots, N_h$$

This is a matrix $\Rightarrow$ $\boxed{A\vec{u} = \vec{F}}$ Linear Algebraic System

LEZIONE 2 05/11/2024

$$(P)\begin{cases} Lu = -(\mu u')' + bu' + \sigma u = f & x \in \Omega = (0,1) \\ u = \varphi & x = 0 \\ \mu u' = \psi & x = 1 \end{cases}$$

} example of boundary condition

$$\boxed{\text{Look for } u \in V : a(u,v) = F(v) \quad \forall v \in V}$$

In this case $\varphi = 0$

IMPORTANT

If $\varphi \neq 0$ we have a different type of problem

$V = \{ v \in H^1(0,1) : v(0) = 0 \}$

$a : V \times V \to \mathbb{R}, \; a(u,v) = \int_0^1 \mu u'v' + bu'v + \sigma uv$

$F : V \to \mathbb{R}, \; F(v) = \int_0^1 f v(0) + \psi v(1)$

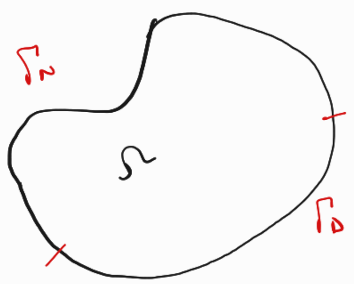
So let's consider

$\Omega \subset \mathbb{R}^d, \; d = 2,3$ divergence

(PDE) $Lu = -\mathrm{div}(\mu \nabla u) + \vec{b} \cdot \nabla u + \sigma u$ in Ω

$$(BC)\begin{cases} u = \varphi & \text{on } \Gamma_D \subset \partial\Omega \quad (D \text{ condition}) \\ \mu \nabla u \cdot \vec{n} = \psi & \text{on } \Gamma_N \subset \partial\Omega \quad (N \text{ condition}) \end{cases}$$

Ω is a set of $\mathbb{R}^2$ or $\mathbb{R}^3$ for example I want to solve a problem in this domain



$\partial\Omega$ = boundary of Ω

$\partial\Omega = \Gamma_D \cup \Gamma_N$

$\Gamma_D \cap \Gamma_N = \emptyset$

I chop $\partial\Omega$ in two part with no intersection

Γ_D = Dirichlet boundary $\to$ describe $u \to u$ = something

Γ_N = Neuman boundary $\to$ describe $\nabla u \to \nabla u$ = something

TERMINOLOGY

If $\Gamma_N = \emptyset \Rightarrow$ Dirichlet problem

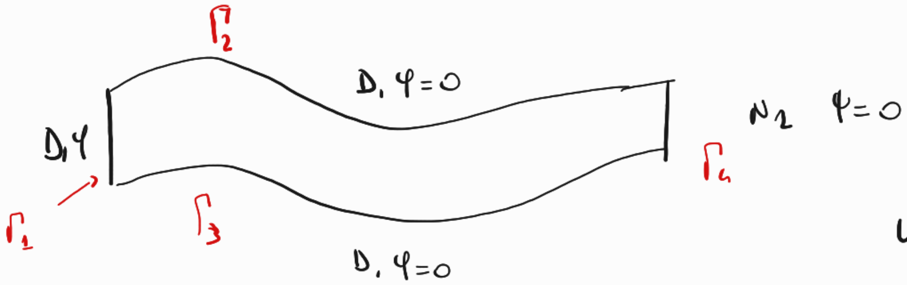
If $\Gamma_D = \emptyset \Rightarrow$ Neumann problem

If $\Gamma_D \neq \emptyset, \Gamma_N \neq \emptyset \Rightarrow$ mixed D-N problem

If $\varphi, \psi \neq 0 \Rightarrow$ non-homogeneous problem

If $\varphi = \psi = 0 \Rightarrow$ homogeneous problem

So with the river problem, the example



So $\Gamma_D = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$\Gamma_N = \Gamma_4$

We have mixed problem in reality

DEFINITIONS

$$v' \cdot v = \frac{d}{dx} v^2$$

$V : \Omega \to \mathbb{R}$

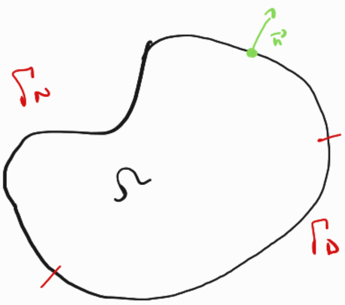
$$\nabla v = \begin{bmatrix} \partial v / \partial x_1 \\ \vdots \\ \partial v / \partial x_d \end{bmatrix} \quad \boxed{\text{gradient}}$$

$$\int \frac{\partial v}{\partial x} +$$

$$\nabla v \cdot v = \frac{\nabla v}{2}$$

$\vec{w} : \Omega \to \mathbb{R}^d$, $\quad \operatorname{div} \vec{w} = \sum_{i=1}^{d} \frac{\partial w_i}{\partial x_i}$ $\quad \boxed{\text{divergence}}$

$v : \Omega \to \mathbb{R}$, $\quad \Delta v = \sum_{i=1}^{d} \frac{\partial^2 v}{\partial x_i^2}$ $\quad \boxed{\text{Laplacian}} = \operatorname{div}(\nabla v)$



$\vec{n}$ normal to Γ_D Γ_N

$|\vec{n}| = 1$

So

$$\nabla u \cdot \vec{n} = \frac{\partial u}{\partial x_1} n_1 + \dots\dots + \frac{\partial u}{\partial x_d} n_d = \frac{\partial u}{\partial n}$$

normal derivative of u with respect to Γ_N

$$\mu \nabla u \cdot \vec{n} = \mu \frac{\partial u}{\partial n} \quad \boxed{\text{NORMAL FLUX}}$$

How to go from (P) to (PW)

$$Lu=f \rightarrow Lu\,v = fv \rightarrow \int_\Omega Luv = \int fv \rightarrow \text{integration by part}$$

We get

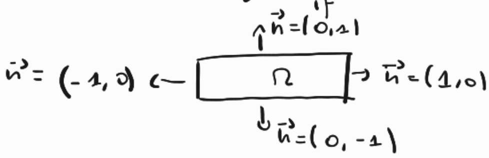
$$-\underbrace{\int_\Omega \mathrm{div}(\mu\nabla u)v}_{(*)} + \int_\Omega \vec{b}\cdot\nabla u\, v + \int_\Omega \sigma uv = \int_\Omega f v$$

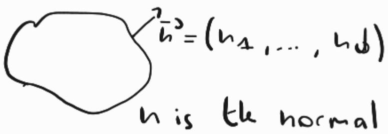
$$(*) = \int_\Omega \mu\nabla u\cdot\nabla v - \int_{\partial\Omega}\mu\nabla u\cdot\vec{n}\, v$$

This is the GREEN FORMULA in $\mathbb{R}^d$ $(d>1)$

$$\int \frac{\partial g}{\partial x_i} h = -\int_\Omega g\frac{\partial h}{\partial x_i} + \int_{\partial\Omega} g n_i h$$

$\vec{n}$ is uscente so if





n is the normal vector

Proof of (*)

$$-\int_\Omega \mathrm{div}(\underbrace{\mu\nabla u}_{\vec{w}})\, v \overset{\text{def of div}}{=} -\sum_i \int_\Omega \frac{\partial w_i}{\partial x_i} v \overset{G \text{ (Green)}}{=} \sum_i \int_\Omega w_i \frac{\partial v}{\partial x_i} - \sum_i \int_{\partial\Omega} w_i v n_i \overset{\text{I rewrite } w_i}{=}$$

$$= \sum_i \int \mu \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i} - \sum_i \int_{\partial\Omega} \mu \frac{\partial u}{\partial x_i} n_i v =$$

$$= \int_\Omega \mu\nabla u\cdot\nabla v - \int_{\partial\Omega}\mu\frac{\partial u}{\partial n} v$$

This is what we obtain

So returning to the original formula we can add that

$$\int_\Omega \mu\nabla u\cdot\nabla v - \int_{\partial\Omega}\mu\nabla u\cdot\vec{n}\, v = -\int_{\Gamma_D}\mu\nabla u\cdot n\, \underbrace{v}_{\text{I put } v=0} - \int \overbrace{\boxed{\mu\nabla u\cdot n}}^{\psi} v$$

And so watching the boundary conditions.

So to summarize we obtain

$$\underbrace{\int_\Omega \mu \nabla u \cdot \nabla v + \int_\Omega \vec{b} \cdot \nabla u \, v + \int_\Omega \sigma u v}_{a(u,v)} = \int_\Omega f v + \int_{\Gamma_N} \psi v = F(v)$$

$F: V \to \mathbb{R}$

$a: V \times V \to \mathbb{R}$

$$\Rightarrow \boxed{\text{Find } u \in V: \ a(u,v) = F(v) \quad \forall v \in V} \qquad (PW)$$

$$V = \{ v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D \}$$

$$H^1(\Omega) = \left\{ v: \Omega \to \mathbb{R}, \ v \in L^2(\Omega), \ \frac{\partial v}{\partial x_i} \in L^2(\Omega), \ i = 1, \dots, d \right\}$$

REMARK (on problem's data)

$$\underbrace{\int_\Omega \mu \nabla u \cdot \nabla v}_{\text{diffusion term}} + \underbrace{\int_\Omega \vec{b} \cdot \nabla u \, v}_{\text{transport term}} + \underbrace{\int_\Omega \sigma u v}_{\text{reaction term}}$$

It's a function

$\mu = \mu(x) \geq \mu_0 > 0$ diffusion coefficient

$\varphi = \varphi(x)$ Dirichlet data

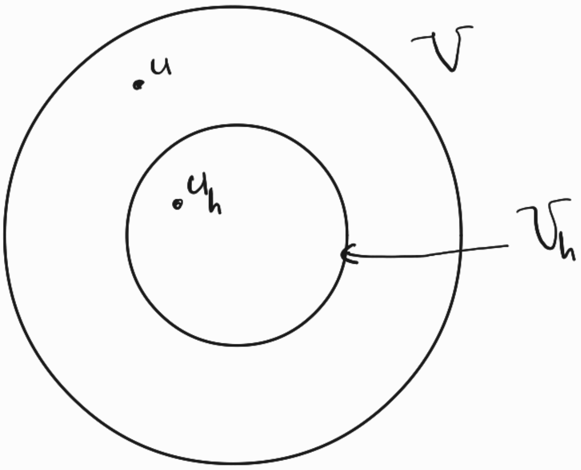
$\vec{b} = \begin{bmatrix} b_1(x) \\ \vdots \\ b_d(x) \end{bmatrix}$ transport (convection) term

$\psi = \psi(x)$ Neumann data (flux)

$\sigma = \sigma(x) \geq 0$ reaction term

$f = f(x)$ source term

NUMERICAL APPROXIMATION



$V_h \subset V$, vector space,

$\dim V_h = N_h$

$$\boxed{\text{Find } u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h} \quad (P_h)$$

Galerkin

Let $\{\varphi_i\}_{i=1}^{N_h}$ be a basis of V_h, then:

$$u_h(x) = \sum_{J=1}^{N_h} u_J \, \varphi_J(x)$$

u_J are unknown → if I find that I resolve the problem

Take $v_h = \varphi_i$ in (P_h)

As see yesterday $\Rightarrow$ $\sum_J u_J \underbrace{a(\varphi_J, \varphi_i)}_{A_{iJ}} = \underbrace{F(\varphi_i)}_{F_i} \qquad i = 1, \dots, d$

So $\boxed{\text{Find } u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h} \Rightarrow \boxed{A\vec{u} = \vec{F}}$

The problem is to solve an algebraic system when I don't know $\vec{u}$.

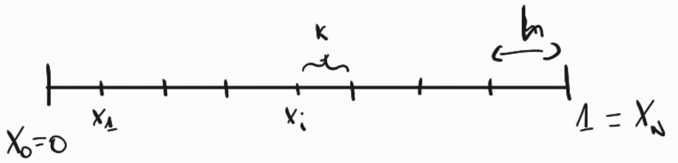
$A\vec{u} = \vec{F}$ $\quad N_h \times N_h$ Linear System

The choich of v_h is crucial to find the correct solution.
We need for accuracy and computational costs.

We use the $\boxed{\text{G - GALERKIN METHOD}}$

A special family of G-Methods: The FINITE ELEMENT METHOD (FEM)
(It's the only discussed in this course because is the most general and in average is the best)

1D $\Omega = (0,1)$



h = length of each intervals
K = every element

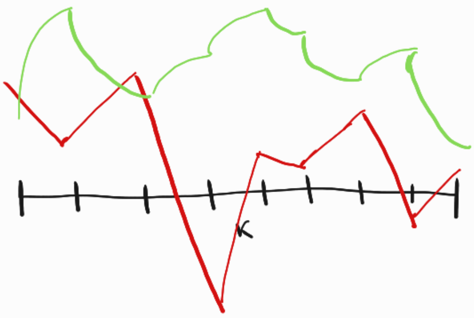
$V_h = V \cap X_h$ da aggiungere

$\{x_i\}$ i nodes

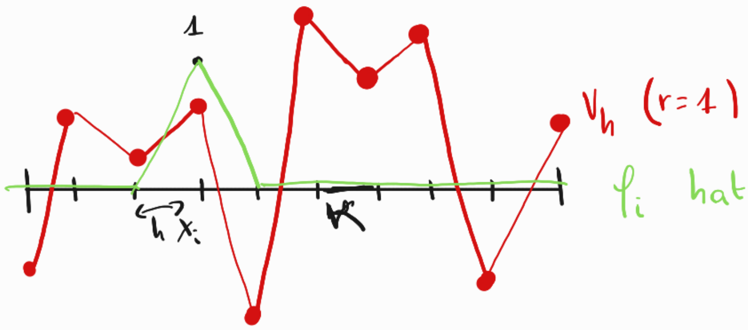
$\mathcal{T}_h = \bigcup K$ "triangulation" of Ω

$$X_h = \left\{ v_h : (0,1) \to \mathbb{R},\ v_h \in C^0([0,1]),\ v_h|_K \in \mathbb{P}^r(K)\ \forall K \in \mathcal{T}_h \right\},\ r \geq 1$$

FINITE ELEMENT SPACE



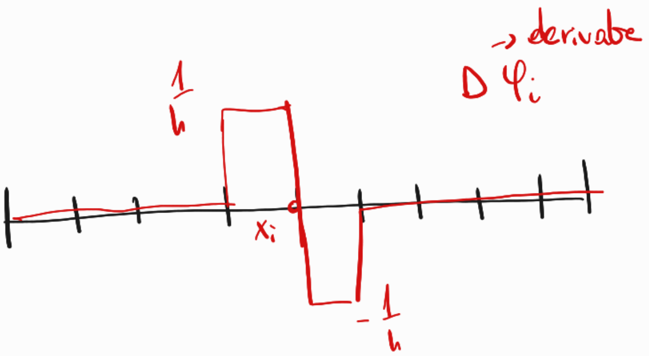
$r=1$ line and continuos
$r=2$ parabol and continuous
$r=3$ ....



This is a vector space. How I define a basis?

I want to find

$$\varphi_i \in X_h : \varphi_i(x_j) = \delta_{ij} = \begin{cases} 1 & j=i \\ \emptyset & \text{otherwise} \end{cases} \quad i=0,\dots,N$$

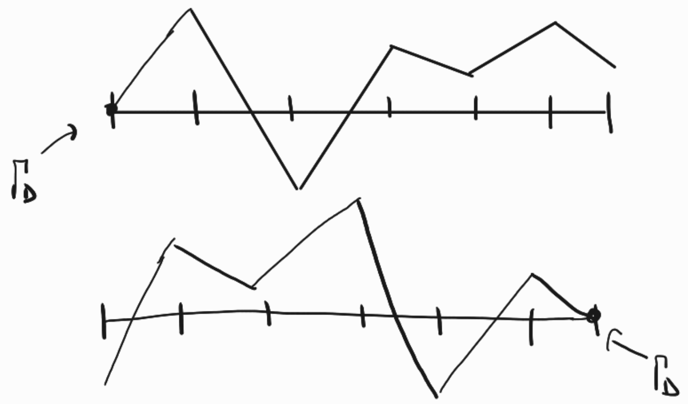


$\forall v_h \in V_h, \quad v_h(x) = \sum_{J=0}^{N} \boxed{v_J}\, \varphi_J(x)_i$ $\quad$ $v_h(x_i)$

$x = x_i \qquad v_h(x_i) = \sum_J v_J \underbrace{\varphi_J(x_i)}_{\delta_{iJ}} = v_i$

The coefficient are the known values

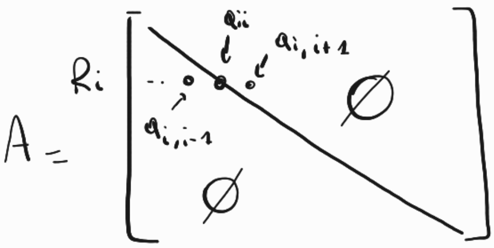
$$V_h = \{ v_h \in X_h : v_h = 0 \ \text{at}\ \Gamma_D \}$$



BASIS FUNCTION POSSIBILITY

$i = 0, \dots, N$	$\Gamma_N = \partial\Omega$	$(N_h = N+1)$
$i = 1, \dots, N$	$\Gamma_D = \{0\}$	$(N_h = N)$
$i = 0, \dots, N-1$	$\Gamma_D = \{1\}$	$(N_h = N)$
$i = 1, \dots, N-1$	$\Gamma_D = \{0, 1\}$	$(N_h = N - 1)$

Our matrix as only 3 element different from 0.



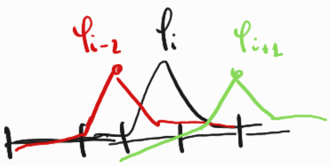
All the other elements are 0.
It's a tridiagonal matrix.

Go in one dimensional case

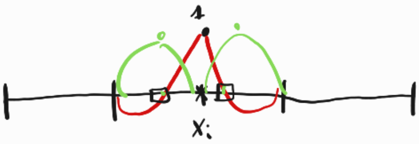
$$a_{iJ} = \int_0^1 \mu \varphi_J' \varphi_i' + \int_0^1 b\, \varphi_J' \varphi_i + \int_0^1 \sigma\, \varphi_J \varphi_i =$$

$$= \sum_{K=0}^{N-1} \left[\int_K \mu \varphi_J' \varphi_i' + \int_K b\, \varphi_J' \varphi_i + \int_K \sigma\, \varphi_J \varphi_i \right]$$

Pick i. φ_i lead on K_i, K_{i+1} so interact only with this two. You only see the previous and after φ_i.



With $r=2$ it's more subtle



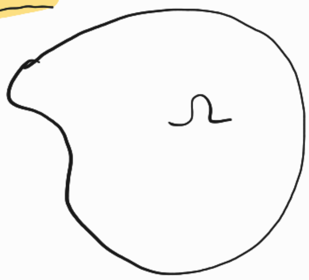
It's 0 or 2 points. We need more points

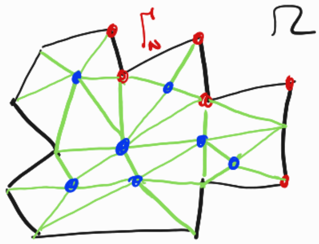
We have the middle point

I need $N+1+N$ but I throw away as before as the boundary conditions. (D or N)

We have 5 entries, a PENTADIAGONAL MATRIX (still sparse)

2D





Ω polynomial domain

$$\mathcal{T}_h = \{K\}$$

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$$

We make triangular to create triangul.

$$h = \max \underbrace{|K|}$$

$$\text{diam}(K) = \max_{x_i, x_J \in K} |x_i - x_J|$$

If the triangular are the same h is constant

$$X_h = \{ v_h : \Omega \to \mathbb{R},\ v_h \in C^0(\bar{\Omega}),\ v_h|_K \in \mathbb{P}^r(K)\ \forall K \in \mathcal{T}_h \} ; \quad r \geq 1$$

$r=1$ $\quad v_h(x)\big|_K = a + bx_1 + cx_2$

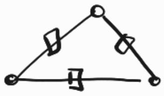
You only need the 3 vertices

The interaction is between the nodes collegate.

This is not going to be a six diagonal matrix.

It depends on the values. It's a few number of entries but we don't know where this value are. The matrix is sparse.

$r=2$ $\quad U_h(x) = a + bx_1 + cx_2 + dx_1x_2 + ex_1^2 + fx_2^2$



We need 6 points

$\|u - u_h\| \simeq C h^r$

h = geometrical function

r = polynomial function

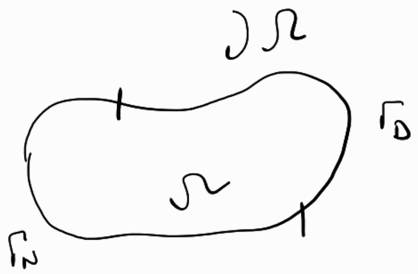
The matrix is still sparse but with more entries non null.

If you reduce h the better is the method

LEZIONE 3 11 / 11 / 2024 (Youtube)

Problem

1) (P) $\begin{cases} Lu = f & x \in \Omega \\ BC \begin{cases} D & \Gamma_D \\ N & \Gamma_N \end{cases} & x \in \partial\Omega \end{cases}$



WEAK FORMULATION

2) (PW) $u \in V : a(u,v) = F(v) \quad \forall v \in V$?

V space of functions

$a : V \times V \to \mathbb{R}$ "form"

$F : V \to \mathbb{R}$ "functional"

GALERKIN

3) $V_h \subset V$, $\dim(V_h) = N_h > 0$

(P_h) $u_h \in V_h : a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h$? $\hookrightarrow$ GALERKIN APPROXIMATION (G)

LINEAR

4) $\{\varphi_i\}_{i=1}^{N_h}$ basis functions for V_h

$(P_h) \iff A\vec{u} = \vec{f}$ [linear system]

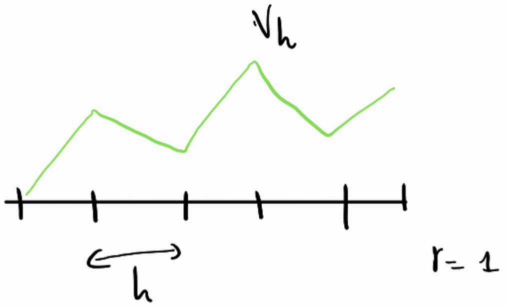
where $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N_h} \end{bmatrix}$, $u_h(x) = \sum_J u_J \varphi_J(x)$

$A = (a_{iJ}) = a(\varphi_J, \varphi_i)$, $\vec{F} = (F_i) = F(\varphi_i)$

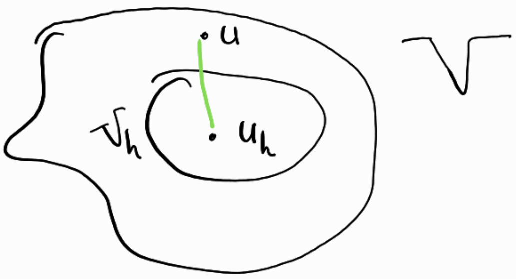
1 example of (G)

FEM

1D



ANALYSIS :



1) Is (Pw) well posed?

2) Is (P_h) well posed

3) How does u_h converges to u? $\iff$ error estimate $\simeq ch^r$

DEF

A well posed problem is one for u wich:

- A solution exist
- It is unique
- It continuosly depends on data

Consider (Pw) $\boxed{\text{Assumptions}}$

V is a Hilbert space (Linear space, banach, with a norm induced by a scalar product)

(The elements of this space are functions)

Banach = all the cauchy sequence converges

$$\{u_n\}_{n=0}^{\infty} \qquad \|u_\infty - u_n\| \longrightarrow 0$$

The norm is $\|\cdot\| : V \to \mathbb{R}^+$: Properties

$\|v\| \geq 0 \quad \forall v \in V$

$\|v\| = 0 \implies v = 0$

$\|\lambda v\| = \|\lambda\| \|v\| \quad \forall \lambda \in \mathbb{R}, \forall v \in V$

$\|v+w\| \leq \|v\| + \|w\|$ Triangular inequity

Scalar product: $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$

Properties

$(v,w) = (w,v) \quad \forall v,w \in V$ (simmetry)

$(\lambda v, w) = \lambda (v,w)$

$(v, \mu w) = \mu (v,w)$ $\quad \forall \lambda, \mu \in \mathbb{R}$ (homosensity)

$(v+w, z) = (v,z) + (w,z)$ (Linearity)

So $\quad \|v\| = \sqrt{(v,v)}$

<u>examples</u>

1) $V = L^2(\Omega) = \left\{ v : \Omega \to \mathbb{R} \, , \, \int_\Omega |v|^2 dx < +\infty \right\} \qquad \Omega \in \mathbb{R}^d \; (d \geq 1)$

$\hookrightarrow \begin{cases} (v,w)_{L^2(\Omega)} = \int_\Omega vw \, dx \\ \|v\| = \sqrt{\int_\Omega v^2 dx} \end{cases}$

2) $V = H^1(\Omega) = \left\{ v : \Omega \to \mathbb{R}, \, v \in L^2(\Omega), \, \nabla v \in (L^2(\Omega))^d \right\}$

$$(v,w)_{H^1(\Omega)} = (v,w)_{L^2(\Omega)} + (\nabla v, \nabla w)_{L^2(\Omega)} = \int_\Omega vw \, dx + \int_\Omega \nabla v \nabla w \, dx =$$

$$= \int_\Omega vw \, dx + \int_\Omega \sum \frac{\partial v}{\partial x_j} \frac{\partial w}{\partial x_j} dx$$

$$\|v\|_{H^1(\Omega)} = \sqrt{(v,v)_{H^1(\Omega)}} = \sqrt{(v,v)_{L^2(\Omega)} + (\nabla v, \nabla v)_{L^2(\Omega)}} =$$

$$= \sqrt{\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}}$$

3) $V = H^1_\Gamma(\Omega) = \{ v : \Omega \to \mathbb{R} ,\ v \in H^1(\Omega) ,\ v|_\Gamma = 0 \}$

$\hookrightarrow \Gamma = \partial\Omega \quad , \quad H^1_\Gamma(\Omega) \equiv H^1_0(\Omega)$

$\hookrightarrow$ 1st choice: $\begin{cases} (v,w)_{H^1_\Gamma(\Omega)} = (v,w)_{H^1(\omega)} \\ \|v\|_{H^1_\Gamma(\Omega)} = \|v\|_{H^1(\Omega)} \end{cases}$

2nd choice: $\begin{cases} (v,w)_{H^1_\Gamma(\Omega)} = (\nabla v, \nabla w)_{L^2(\Omega)} \\ \|v\|_{H^1_\Gamma(\Omega)} = \sqrt{(\nabla v, \nabla v)_{L^2(\Omega)}} = \sqrt{\int_\Omega |\nabla v|^2 \, dx} = \sqrt{\|\nabla v\|^2_{L^2_\Omega}} \end{cases}$

when D problem has Γ not empty

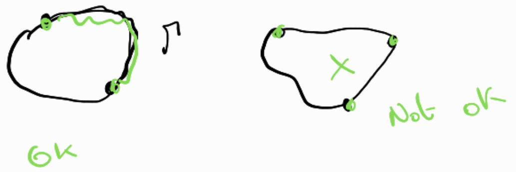
(This is better)

if Γ not empty means $(\Gamma) > 0$

1D

We need at least 1 Dirichlet condition

2D



DEF

Assume that I have 2 norms for the space V

$\|\cdot\|_1$ and $\|\cdot\|_2$

These 2 norms are said to be <u>EQUIVALENT</u>

$$\boxed{\text{if } \exists\ C_1, C_2 > 0 : \quad C_1 \|v\|_1 \leq \|v\|_2 \leq C_2 \|v\|_1 \quad \forall v \in V}$$

Lest's use the norm of 1st and 2nd choice

1st $\quad \|v\|_{H^1(\Omega)}$

2nd $\quad |v|_{H^1(\Omega)}$ SEMINORM $\rightarrow$ The 2nd property of the norm is not fullfiled

<u>PROPERTY</u>

If $\text{meas}(\Gamma) > 0$, then $\|\cdot\|_{H^1(\Omega)}$ and $|\cdot|_{H^1_\Omega}$ are EQUIVALENT norms

<u>Prof</u>

$$|v|_{H^1(\Omega)} = \sqrt{\|\nabla v\|^2_{L^2(\Omega)}} \leq \|v\|_{H^1(\Omega)} = \sqrt{\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}} \quad *$$

<u>Property</u> (Poincarre inequity)

$$\boxed{\exists\ C_p > 0 : \ \|v\|_{L^2(\Omega)} < C_p \|\nabla v\|_{L^2(\Omega)} \quad \forall v \in H^1_\Gamma(\Omega)}$$

So because of Poincarre

$$* \leq \sqrt{C_p \|\nabla v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}} = \underbrace{\sqrt{(1 + C_p^2)}}_{C_2} \overbrace{\|\nabla u\|_{L^2(\Omega)}}^{|v|_{H^1(\Omega)}}$$

So

$$\underbrace{1}_{C_1} \cdot |v|_{H^1(\Omega)} \le \|v\|_{H^1(\Omega)} \le \underbrace{C_2}_{\sqrt{(1+C_p^2)}} |v|_{H^1(\Omega)}$$

So it's verified

I can use $|v|_{H^1_\Gamma(\Omega)}$ as the norm of the space

•] I verified V is an HILBERT SPACE : $L^2(\Omega)$, $H^1(\Omega)$, $H^1_\Gamma(\Omega)$ are Hilbert space

$\|\cdot\|$ norm of V

•] $a : V \times V \to \mathbb{R}$

•) bilinear
$$a(v+w, z) = a(v,z) + a(w,z)$$
$$a(v, w+z) = a(v,w) + a(v,z)$$
$$a(\lambda v, w) = \lambda a(v,w) = a(v, \lambda w)$$

•) continuous: $\exists M > 0$ s.t. $|a(v,w)| \le M \cdot \|v\| \|w\| \quad \forall v,w \in V$

•) coercivity (positivity) : $\exists \alpha > 0$ s.t. $a(v,v) \ge \alpha \|v\|^2 \quad \forall v \in V$

•] $F : V \to \mathbb{R}$

•) linear $\quad F(v+w) = F(v) + F(w)$

$F(\lambda v) = \lambda F(v) \qquad \forall v \in V$

•) continuous (bounded) : $\exists K > 0$ s.t. $|F(v)| \le K \|v\| \quad \forall v \in V$

$\Rightarrow F \in V'$ dual space of V

Def

$$\|F\|_* := \sup_{\substack{v \in V \\ v \neq 0}} \frac{F(v)}{\|v\|}$$

NORM of F in the space called V'

(linear and continuos functional)

We can take $K = \|F\|_*$

$$\left(\frac{F(v)}{\|v\|} \leq \|F\|_* \right)$$

SUMMARY

(i) V: Hilbert space

(ii) $a: V \times V \to \mathbb{R}$ bilinear, continuous, coercive

(iii) $F: V \to \mathbb{R}$ linear, continuous

THEOREM (LAX-MILGRAM) (LM)

Consider (PW) Assure that properties (i), (ii), (iii) are satisfied then there exists a unique solution u to (PW)

Moreover: $\|u\| \leq \frac{1}{\alpha} \|F\|_*$ ⊙ well-posed !!

data

Proof $\odot$

Take (P_W) and set $v = u$

$a(u,u) = F(u)$ from that I can say that

$$\alpha \|u\|^2 \underset{\substack{\downarrow \\ \text{coercivity}}}{\leq} a(u,u) = F(u) \underset{\substack{\downarrow \\ F \in V'}}{\leq} \|F\|_* \|u\|$$

$$\Rightarrow \alpha(u) \leq \|F\|_*$$

I can apply the same theorem LM to (P_h)

The properties are satisfied because V_h is an underspace

$\Rightarrow$ $\exists!\, u_h$ solution to (P_h)

Moreover, $\odot$ because $\|u\|_h \leq \underbrace{\frac{1}{\alpha} \|F\|_*}_{\text{Independent of } h}$

$\Downarrow$

This means STABILITY

1D ADR PROBLEM

vector diffusion reaction problem

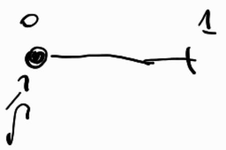
PDE $Lu = -(\mu u')' + bu' + \sigma u = f \qquad 0 < x < 1$

BC $\quad u(0) = 0$
$\quad \mu u'(1) = \psi$

I assume that $\mu = \mu(x) \geq \mu_0 > 0$
$b = b(x)$
$\sigma = \sigma(x) > 0$
$f \in L^2(0,1)\ ,\ \psi \in \mathbb{R}$

(PW) Find $u \in V \in H^1_\Gamma(0,1)$: $a(u,v) = F(v) \quad \forall\, v \in V$

$$V = H^1_\Gamma(0,1) = \{ v : \Omega \to \mathbb{R}\ ,\ v \in H^1(0,1),\ v(0) = 0 \}$$



$$\boxed{\|v\| = \|v'\|_{L^2(0,1)} \quad \text{norm of } V}$$ (Γ is not empty)

This is my selection of norm

$$a(u,v) = \int_0^1 \mu u'v' + bu'v + \sigma uv$$

$$F(v) = \int_0^1 fv + \psi\, v(1)$$

$a(\cdot)$ is bilinear

1) **Continuity**: $\boxed{\exists\ M > 0 \quad |a(u,v)| \leq M \|u\| \|v\|} \quad \forall u, v \in V$

I want to prove that

$$|a(u,v)| \leq \left|\int_0^1 \mu u'v'\right| + \left|\int_0^1 bu'v\right| + \left|\int_0^1 \sigma uv\right| \leq (*)$$

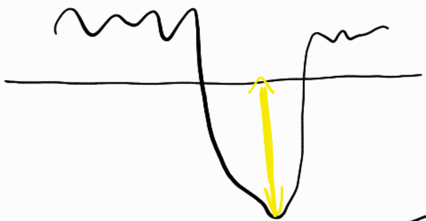
Property $\quad (v,w)_{L^2} \le \|v\|_{L^2}\|w\|_{L^2} \quad \forall v,w \in L^2$ CAUCHY-SCHWARZ INEQUALITY

I use this prop. to every term so

$$(*) \le \|\mu\|_\infty \int_0^1 |u'v'| + \|b\|_{L^\infty} \int_0^1 |u'v| + \|\sigma\|_{L^\infty} \int_0^1 |uv| \underset{CS}{\le}$$

Cauchy-Schwarz

$$\|v\|_\infty = \sup_{0<x<1} |v(x)|$$



$$\underset{CS}{\le} \|\mu\|_\infty \underbrace{\|u'\|_{L^2}}\,\underbrace{\|v'\|_{L^2}} + \|b\|_\infty \underbrace{\|u'\|_{L^2}}\,\underbrace{\|v\|_{L^2}}_{\le C_p\|v'\|_{L^2}} + \|\sigma\| \, \underbrace{\|u\|_{L^\infty}}_{\le C_p\|u'\|_{L^2}} \, \underbrace{\|v\|_{L^\infty}}_{\le C_p\|v'\|_{L^2}} \le$$

This is correct for us as we define this norm

The problem is the other and we bounded using the previous theorem

So we obtain $\le \underbrace{\|u'\|_{L^2}}_{\|u\|}\cdot\underbrace{\|v'\|_{L^2}}_{\|v\|} \underbrace{\left(\|\mu\|_{L^\infty} + C_p\|b\|_\infty + C_p^2 \|\sigma\|_\infty \right)}_{\text{This is a number}}$

$\|u\| \cdot \|v\|$

I prove what I want to prove

$$\boxed{\begin{array}{l} 1 < \mu_0 \le \mu \le \mu_1 < \infty \\ |b(x)| \le b_1 < \infty \\ |\theta(x)| \le \theta_1 < \infty \end{array}}$$ New assumptions

We need the

- **coercivity**: $\exists\ \alpha > 0 : \ a(v,v) \ge \alpha \|v\|^2 \quad \forall v \in V$

$$a(v,v) = \underbrace{\int_0^1 \mu (v')^2}_{\ge \mu_0 \int_0^1 |v'|^2} + \underbrace{\int_0^1 b v' v}_{} + \underbrace{\int_0^1 \theta v^2}_{\ge 0} \ge$$

(it's enough, θ non negative)

$$v' v = \frac{d}{dx}\left(\frac{1}{2} v^2\right)$$

by part

$$\int_0^1 b \frac{d}{dx}\left(\frac{1}{2} v^2\right) = -\int_0^1 b' \frac{1}{2} v^2 + \left[\frac{1}{2} b v^2\right]_0^1 =$$

$$= -\int_0^1 \left(\frac{1}{2} b'\right) v^2 + \frac{1}{2} b(1) v^2(1) - \frac{1}{2} \underbrace{\cancel{b(0) v^2(0)}}_{0}$$

So we obtain

$$a(v,v) \ge \mu_0 \|v\|^2 + \int_0^1 \left(\theta - \frac{b'}{2}\right) v^2 \, dx + \frac{1}{2} b(1) v^2(1)$$

We make new assumptions

I need that

$$\boxed{\begin{array}{l} \bullet\ \theta - \dfrac{b'}{2} \geq 0 \\[2ex] \bullet\ b(1) \geq 0 \end{array}}$$

for $\alpha = \mu_0 \implies$ coercivity.

So far we have

$$\left.\begin{array}{l} \bullet)\ 1 < \mu_0 \leq \mu \leq \mu_1 < \infty \\ \bullet)\ |b(x)| \leq b_1 < \infty \\ \bullet)\ |\theta(x)| \leq \theta_1 < \infty \\ \bullet)\ \theta - \dfrac{b'}{2} \geq 0 \\ \bullet)\ b(1) \geq 0 \end{array}\right\} \overset{\text{assumption}}{\implies} \begin{cases} a(\cdot)\ \text{bilinear} \\ \text{Continuity} \\ \text{Coercivity} \end{cases}$$

•) <u>We have to prove that F is linear and bounded</u>

F is linear (obvious)

$$F(v) = \int_0^1 f v + \psi\, v(1)$$

| bounded using the absolute value

$$|F(v)| = \Big| \int_0^1 f v \Big| + |\psi\, v(1)|$$

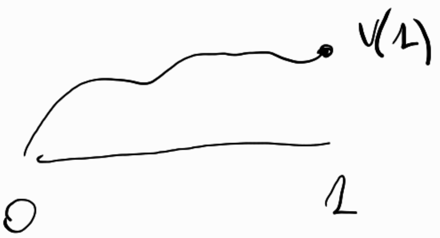
$$\le_{CS} \|f\|_{L^2} \|v\|_{L^2} \underset{\text{Poincarre}}{\le} \|f\|_{L^2} C_P \|v'\|_{L^2} =$$

$$= \left(C_P \|f\|_{L^2} \right) \|v\|$$

$$|\psi\, v(1)| = |\psi|\, |v(1)|$$

| want to bound $v(1)$ with the norm of v

$$v(1) = v(0) + \int_0^1 v'(x)\,dx = \int_0^1 v'(x)\,dx$$



$$|v(1)| = \Big| \int_0^1 1 \cdot v'(x)\,dx \Big| = |(1 \cdot v')| \le_{CS} \sqrt{\|1\|^2_{L^2} \cdot \|v'\|^2_{L^2}} =$$

$$= \|v'\|_{L^2} = \|v\|$$

So at the end

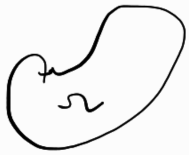
$$F(v) \leq \left(C_p \|f\|_{L^2} + |\Psi| \right) \|v\| = K \|v\|$$

Apply the LM theorem to conclude that $\exists! \, u$ to (PW) (ADR in 1D)

$$\|u\| \leq \frac{1}{\alpha} \|F\|_* = \frac{1}{\mu_0} \left(C_p \|f\|_{L^2} + |\Psi| \right)$$

LEZIONE 4 (19/11/2024)

$$(P) \begin{cases} Lu = f & \text{in } \Omega \\ + BC & \text{on } u \; \partial\Omega \end{cases}$$



$\Omega \subset \mathbb{R}^d \quad d = 1, 2, 3, \dots$

$\Downarrow$

$(PW) \; ?\, u \in V : \; a(u,v) = F(v) \quad \forall\, v \in V$

Galerkin problem

$(P_h) \; ?\, u_h \in V_h : \; a(u_h, v_h) = F(v_h) \quad \forall\, v_h \in V_h \qquad ((G))$

$\Updownarrow$

$A\vec{u} = \vec{F}$ linear algebraic system

ANALYSIS

① Lax-Milgram Lemma $\Rightarrow \exists!\, u$ of (PW), $\exists!\, u_h \in P_h$,

$$\|u_h\| \leq \frac{1}{\alpha} \|F\|_{V'}$$

$\alpha =$ coercivity constant of $a(\cdot,\cdot)$

② We applied the LM lemma to:

$$ADR: \begin{cases} Lu := -(\mu u')' + bu' + \sigma u = f & 0 < x < 1 \\ + BCs & x = 0, 1 \end{cases}$$

With the same arguments, we can prove $\exists!$ of ADR in $d = 2, 3, \dots$

$$\begin{cases} Lu = -\operatorname{div}(\mu \nabla u) + \boxed{\vec{b} \cdot \nabla u} + \sigma u = f & \text{in } \Omega \subset \mathbb{R}^d \\ + B.C.s & \text{on } \partial\Omega \end{cases}$$

$\operatorname{div}(\vec{b}u)$

Prove by ourself

Remark

$$\mathrm{div}(\vec{b}\,u) = \vec{b}\cdot\nabla u + u\,\mathrm{div}\,\vec{b}$$

Let's try to prove the case with

$d = 2, 3, \dots$

$$\int_\Omega \mu \nabla u\cdot\nabla v - \int_{\partial\Omega}\mu \nabla u\cdot\vec{n}\,v = -\int_{\Gamma_D}\mu\nabla u\cdot n\,v - \int \overbrace{\boxed{\mu\nabla u\cdot n}}^{\psi}\,v \quad +(b\cdot n)u$$

I put $v=0$

$$\int_\Omega \mu\nabla u\cdot\nabla v + \int_\Omega \vec{b}\cdot\nabla u\,v + \int_\Omega \sigma u v = \int_\Omega f v + \int_{\Gamma_N}\psi v = F(v)$$

$$\boxed{\mathrm{div}(\vec{b}\,u) = -b\,u\,\mathrm{div}\,v + \int b u\cdot n\cdot v}$$

AMBIGUO

Now we work on (P_h) so

$$(P_h) \ \textcircled{2} \Rightarrow \exists !\, u_h \in V_h \, , \ \|u_h\|_V \leq \frac{\|F\|_{V'}}{\alpha} \qquad \text{STABLE}$$

We also need <u>CONVERGENCE</u>

$$\lim_{h \to 0} u_h \equiv u \iff \lim_{h \to 0} \|u - u_h\|_V = 0$$

THEOREM (EQUIVALENCE THM)

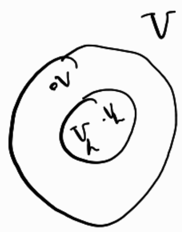
If (P_h) is <u>consistent</u> with (P) then STABILITY $\iff$ CONVERGENCE

$$P_h(u) \xrightarrow[h \to 0]{} 0$$

$$a(u, v_h) - F(v_h) \xrightarrow[h \to 0]{} 0 \qquad \text{This is consistency}$$

Have we got consistency?

In our case:



$$a(u, v_h) - F(v_h) = 0$$

$\Rightarrow$ Consistency Because V_h is into V

So it's very STRONG

How do we prove CONVERGENCE?

(A) Take $v = v_h$ in (PW)

$$a(u, v_h) = F(v_h) \quad \forall v_h \in V_h$$

$$(P_h) \quad a(u_h, v_h) = F(v_h) \quad \text{"} \quad \text{"}$$

subtract term by term

$$\Rightarrow \quad a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0$$

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$ GALERKIN ORTHOGONALITY (GO)

(B) Continuing by proof

$$\alpha \|u - u_h\|_V^2 \leq a(u - u_h, u - u_h) =$$ | sum and subtract w_h

Coercivity

$$= a(u - u_h, u - w_h) + a(u - u_h, \underbrace{w_h - u_h}_{\in V_h})\ (*) \quad \forall w_h \in V_h$$

↳ It's an element of V_h

So thanks to (GO) this go to 0

$$(*) \quad \leq M \|u - u_h\|_V \|u - w_h\|_V$$

continuity of $a(\cdot, \cdot)$

So

$$\alpha \|u-u_h\|_V^{\cancel{2}} \leq M \cancel{\|u-u_h\|_V} \|u-w_h\|_V$$

(C) $\|u-u_h\|_V \leq \frac{1}{\alpha} \|u-w_h\|_V \quad \forall\, w_h \in V_h$

CÉA LEMMA

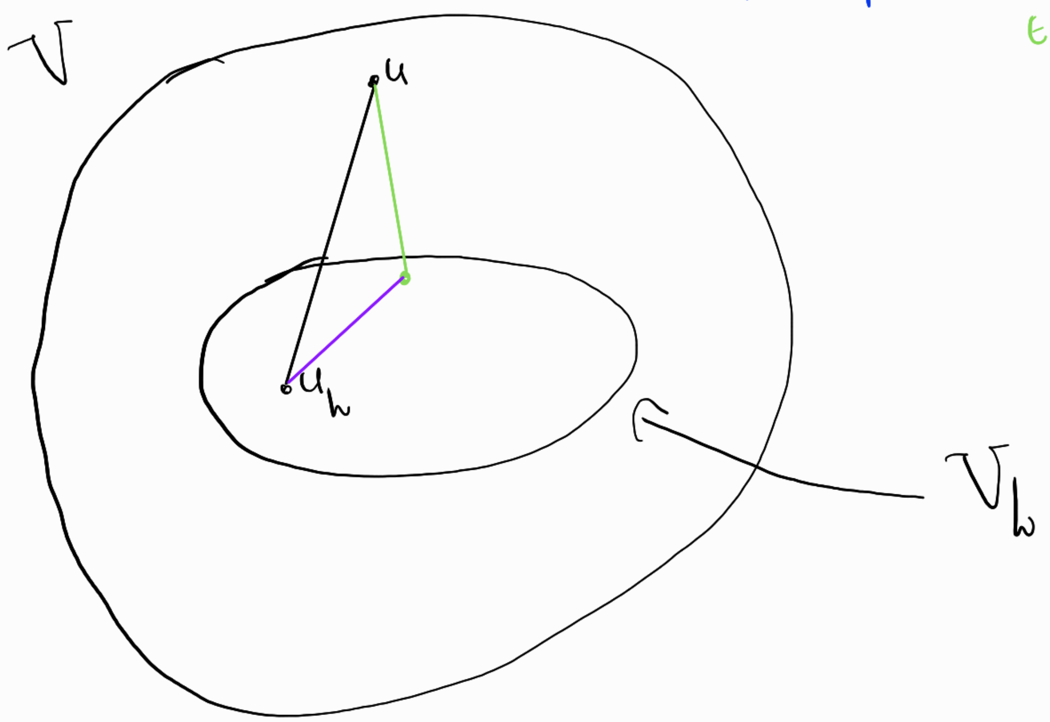
ERROR (real number)

constant, depends on P but NOT on h

(D) $\underbrace{\|u-u_h\|_V}_{} \leq \boxed{\frac{M}{\alpha}} \inf_{v_h \in V_h} \underbrace{\|u-v_h\|_V}_{}$

Best possible approximation ever

ERROR = minimal possible distance



COROLLARY

If the best approximation error $\xrightarrow[h\to 0]{} 0$ then

my $\|u-u_h\| \xrightarrow[h\to 0]{} 0 \rightarrow$ CONVERGENCE !

If I chose V_h in a good way the method converge

PROPERTY

If V_h is a FE (finite element) space, then BAE $\xrightarrow[h \to 0]{} 0$

best approximation error

Indeed $\inf_{v_h \in V_h} \| u - v_h \|_V \leq c h^r$

Then $\| u - u_h \|_V \underset{(D)}{\leq} \frac{M}{\alpha} c h^r \xrightarrow[h \searrow 0]{} 0$ CONVERGENCE

r = order of convergence

Remember r = polynomial degree on every element $K \in \mathcal{T}_h$

Gaet lik orthogonalism

If we think that A is symmetric we get a scalar product and we obtain that $u - u_h$ is orthogonal to v_h

We chose V_h in a suitable way that all above is good.

This arguments are good only for ELLIPTIC PROBLEM

Remarks on nature of $A\vec{u} = \vec{F}$

① If $a(\cdot,\cdot)$ is symmetric $\Rightarrow$ A is symm

Proof

$$A = (a_{ij}) = \big(a(\varphi_j, \varphi_i)\big)$$

$$a_{ij} = a(\varphi_j, \varphi_i) \underset{\substack{\downarrow \\ a \text{ symm}}}{=} a(\varphi_i, \varphi_j) = a_{ji} \qquad \forall i,j$$

✓

② If $a(\cdot)$ is coercive $\Rightarrow$ A is SPD

Proof

$$v_h \in V_h \rightarrow v_h(x) = \sum_{j=1}^{N_h} v_j \varphi_{ij}(x) \qquad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_{N_h} \end{bmatrix} \in \mathbb{R}^{N_h}$$

Every function is fully caracterized by the components of the linear combination

A is PD iff (if and only if)

$$\forall \vec{v} \in \mathbb{R}^{N_h}, \quad \vec{v}^T A \vec{v} \geq 0; \quad = 0 \quad \text{iff} \quad \vec{v} = 0$$

$$\vec{v}^T A \vec{v} = \sum_{i,j=1}^{N_h} a_{ij} v_i v_j = \sum_{ij} a(\varphi_j, \varphi_i) v_j v_i = \sum_{ij} a(v_j \varphi_j, v_i \varphi_i) =$$

$$v_h \in V_h : v_h(x) = \sum_j v_j \varphi_j$$

$$= a\Big(\sum_j v_j \varphi_j, \sum_i v_i \varphi_i\Big) = a(v_h, v_h) \overset{\substack{\text{coercivity} \\ \downarrow}}{\geq} \alpha \|v_h\|_V^2$$

$$\Rightarrow v^T A \vec{v} \geq \alpha \|v_h\|_V^2 \begin{cases} > 0 \\ = 0 \iff v_h = 0 \iff \vec{v} = \vec{0} \end{cases} \Rightarrow A \text{ PD}$$

PD: positive definite

$\Rightarrow$ symm. if a is symm

The EIGENVALUES are real and positive if A is SPD

A PD $\Rightarrow Re(\lambda) > 0$

A SPD $\Rightarrow \lambda > 0$

<u>example</u>

A SPD

$$\left(\text{Gradient method}\right) \quad \|\vec{u} - \vec{u}^{(k)}\|_A \leq \boxed{\left(\frac{K(A)-1}{K(A)+1}\right)^k} \|\vec{u} - \vec{u}^{(0)}\|_A$$

$$c = \frac{\sqrt{K(A)} - 1}{\sqrt{K(A)} + 1}$$

$$\left(\text{CG method}\right) \quad \|\vec{u} - \vec{u}^{(k)}\|_A \leq \boxed{\frac{2c^k}{1+c^{2k}}} \|\vec{u} - \vec{u}^{(0)}\|$$

$$\simeq \left(\frac{\sqrt{K(A)} - 1}{K(A) + 1}\right)^k \|\vec{u} - \vec{u}^{(0)}\|$$

$K(A) = \frac{\lambda_{max}}{\lambda_{min}} > 1$ by definition

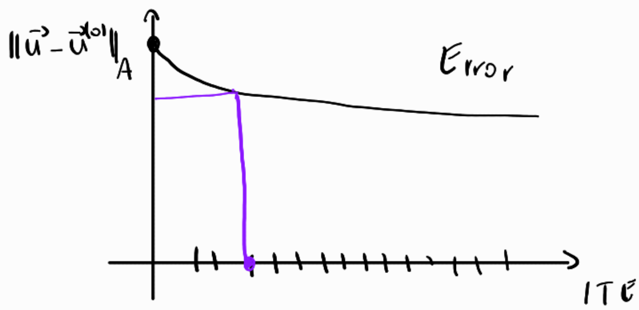
$K(A)$ is usually up to 10^4.

$$\boxed{K(A) \simeq C h^{-2}}$$

We use preconditioner to reduce the $K(A)$

<u>example</u> (Gradient Method)

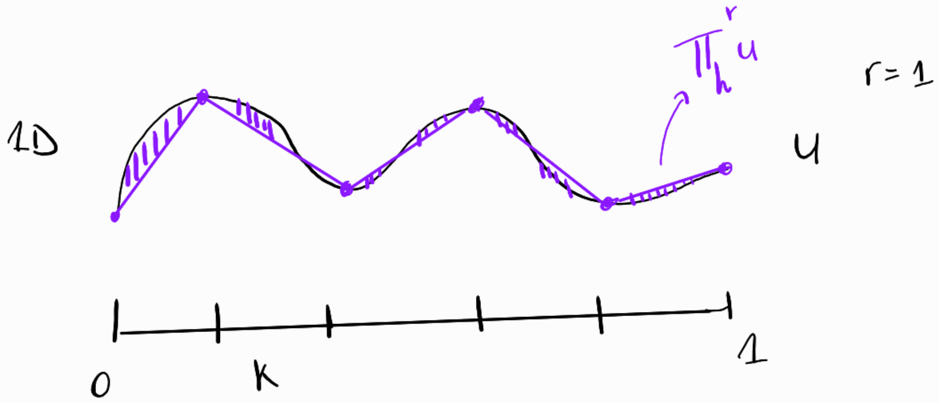
To reduce the initial error by a factor 10



$$\left(\frac{K(A)-1}{K(A)+1}\right)^{k^*} < \frac{1}{10}$$

$$\Leftrightarrow k^* > \frac{\log 10}{\log\left(\frac{K(A)-1}{K(A)+1}\right)} \quad , \quad k^* > \frac{\log(10)}{\log\left(\frac{\sqrt{K(A)}-1}{\sqrt{K(A)}+1}\right)} \quad \text{(with CG)}$$

BAE: $\inf_{v_h \in V_h} \|u - v_h\|_V \leq \|u - \bar{v}_h\| \quad \forall \bar{v}_h \in V_h$



We construct $\Pi_h^r u \in V_h$ interpolation

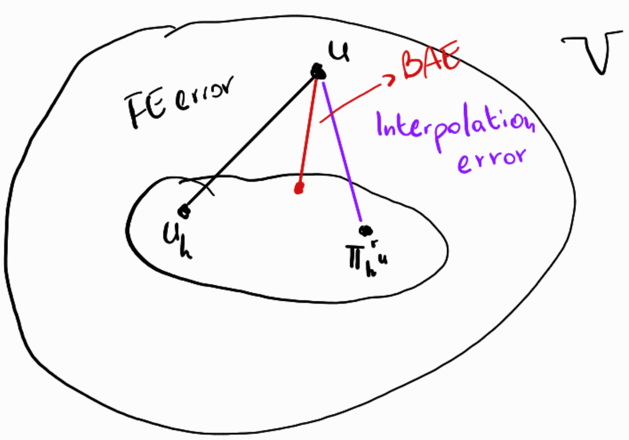
$$\Pi_h^r u(x_j) = u(x_j) \quad \forall \text{ node } x_j \in \mathcal{T}_h$$

$$\Pi_h^r u(x) = \sum_{j=1}^{N_h} u(x_j)\, \varphi_i(x)$$

If u is known I can create this function and we obtain

ERROR CHAIN

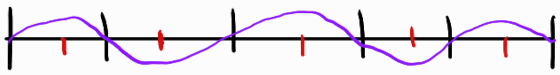
$$\boxed{\underbrace{\|u - u_h\|_V}_{\text{FE error}} \leq \frac{M}{\alpha} \underbrace{\inf_{v_h \in V_h} \|u - v_h\|_V}_{\text{BAE}} \leq \frac{M}{\alpha} \|u - \bar{v}_h\| \leq \frac{M}{\alpha} \underbrace{\|u - \Pi_h^r u\|_V}_{\text{Interpolation error}}}$$



So we can compute

$$= \frac{M}{\alpha}\left(\|u-u_h\|_{L^2}^2 + \|\nabla u - \nabla u_h\|_{L^2}^2 \right)^{\frac{1}{2}} = \frac{M}{\alpha}\left(\int_0^1 (u-u_h)^2 + \int_0^1 (u'-u_h') \right)^{\frac{1}{2}}$$

Using Rolle we use that the error in two consecutive node is 0.



Computin the derivative there are other points that are 0.

Also in more dimension the result is the same.

$$\Rightarrow \boxed{\|u-u_h\|_V \leq Ch^r \quad (*) \qquad C = c\,|u|_{H^{r+1}(\Omega)} \simeq c\,\|D^{r+1}u\|_{L^2}}$$

u has to be good enough.

What happen using only

$$\boxed{\|u-u_h\|_{L^2} \leq Ch^{r+1} \quad (**)}$$ we get a better approximation

FE CONVERGENCE TABLE

Assume $u \in H^{p+1}(\Omega)$ (regularity of u; the largest p, the more regular u)

- best for ⊛
- best for ⊛⊛

	$p=0$	$p=1$	$p=2$	$p=3$	$p \geq 3$
$r=1$		$O(h)$ h^2	$O(h)$ h^2	$O(h)$ h^2	$O(h)$ h^2
$r=2$		$O(h)$ h^2	$O(h^2)$ h^3	$O(h^2)$ h^3	$O(h^2)$ h^3
$r=3$		$O(h)$ h^2	$O(h^2)$ h^3	$O(h^3)$ h^4	$O(h^3)$ h^4

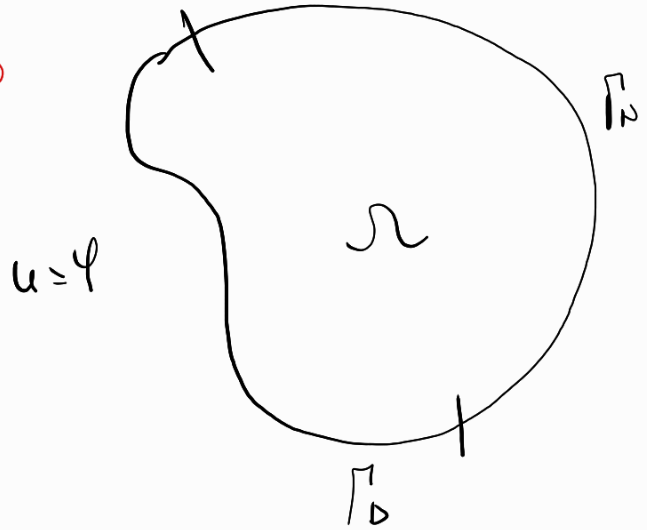
If $r > p$ we don't gain nothing. If $r < p$ we are not esploiting the regularity.

The regularity depends on the data.

All these problems are Dirichlet without boundary conditions.

$$\begin{cases} Lu = -\mathrm{div}(\mu u) + \vec{b}\cdot\nabla u + \sigma u = f & \text{in } \Omega \\ u = \boxed{\varphi} & \text{on } \Gamma_D \\ \mu \dfrac{\partial u}{\partial n} = \psi & \text{on } \Gamma_N \end{cases}$$

$\varphi \neq 0$



$\Rightarrow \quad \cdot v \quad \int_\Omega$ Gree formula, we integrate

$$-\int_{\Gamma_N} \mu \overbrace{\frac{\partial u}{\partial n} v}^{\psi} \qquad +$$

$$\underbrace{\int_\Omega \mu \nabla u \nabla v + \int_\Omega \vec{b}\, \nabla u\, v + \int_\Omega \sigma u v}_{a(u,v)} - \int_{\partial\Omega} \mu \frac{\partial u}{\partial n} v = \underbrace{\int_\Omega f v}_{F(v)} -$$

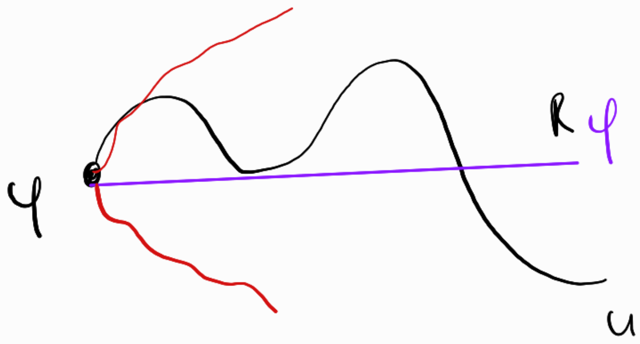
$v=0$ on Γ_D

So

(as usual)

(PW) Find u: $a(\overset{\mathring{u}+R\varphi}{u},v) = F(v) \quad \forall v \in V$

$$\forall v \in V = H^1_{\Gamma_D}(\Omega) = \{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \}$$



φ = constant function

There are many possible function

$\mathring{u} = u - R\varphi$

$\mathring{u}|_{\Gamma_D} = 0$ — on Γ_D

$$a(u,v) = a(\mathring{u},v) + a(R\varphi,v) \Rightarrow a(\mathring{u},v) = \overbrace{F(v) - a(R\varphi,v)}^{\tilde{F}(v)} \quad \forall v \in V$$

In 1D

In 2D it's difficult and we have extantion

LEZIONE 5 20/11/2024

PARABOLIC PROBLEMS

bilinear, continuous, coersive

Ch 1 Elliptic eqs $\begin{cases} Lu = f & \text{in } \Omega \\ +BC & \text{on } \partial\Omega \end{cases} \to (PW)\ \exists!\, u \in V : \underline{a(u,v) = F(v)}$

$\forall\, v \in V$

$\Omega \subset \mathbb{R}^d$

$u = u(x)$ independent of t.

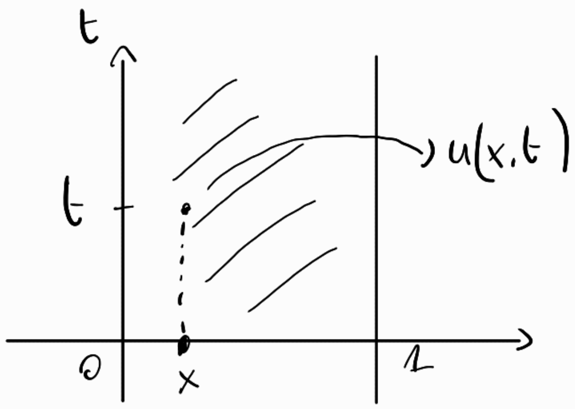
PARABOLIC PROBLEMS/EQS

$$\begin{cases} \dfrac{\partial u}{\partial t} + Lu = f & x \in \Omega,\ t > 0 \\ +BC \text{ on } \partial\Omega, & \forall t > 0 \\ + \text{I.C.} & \text{in } \Omega,\ t = 0 \end{cases} \qquad u = u(x,t)$$

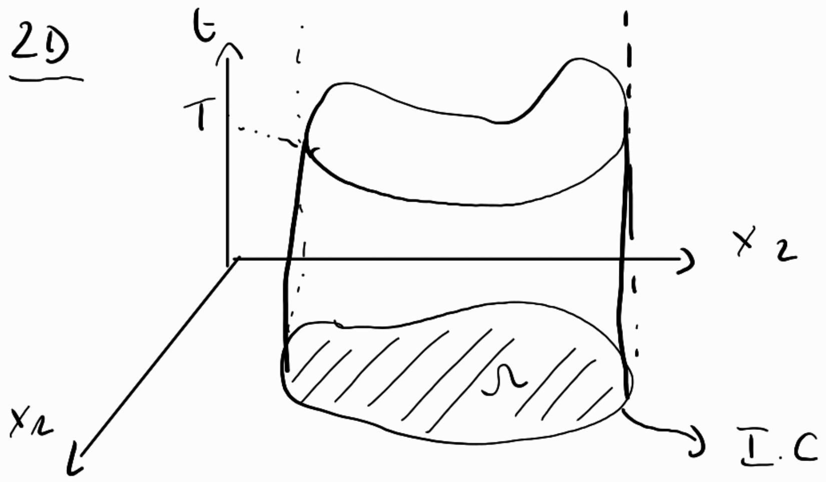
$t > 0 \to 0 < t < T$

I.C. → Initial condition

1D



2D



$Q = \Omega \times (0, +\infty)$ Cylinder

$Q_T = \Omega \times (0, T)$ "

$\Sigma_T = \partial\Omega \times (0, T)$ We impose boundary conditions

The boundary is all the surface of the cylinder

ANALYSIS (PP) → Parabolic problem

Going from (PP) to (PPW)

$\forall t > 0$, $*\, v(x) \longrightarrow \int_\Omega$ → Green Formula

multiply

• Let us do it on a specific example:

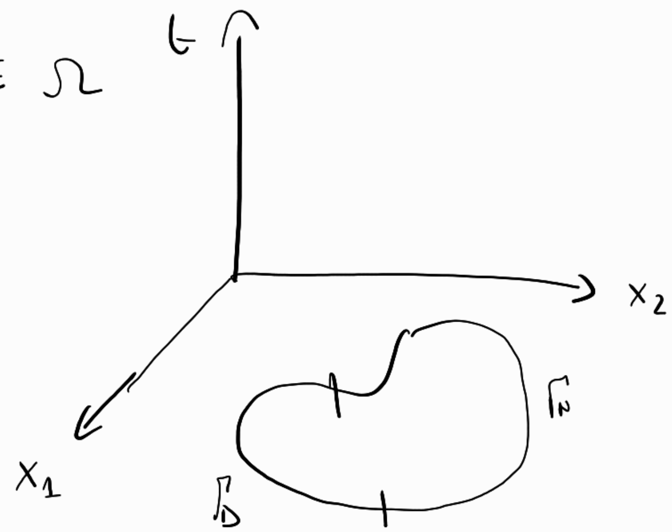
$$Lu = -(\operatorname{div}\, \mu \nabla u) + \vec{b} \cdot \nabla u + \sigma u$$

(PDE) $\dfrac{\partial u}{\partial t} + Lu = f \qquad x \in \Omega,\ 0 < t < T$

(D) $u(x,t) = \varphi(x,t) \qquad x \in \Gamma_D,\ 0 < t < T$

(N) $\mu \dfrac{\partial u}{\partial n}(x,t) = \psi(x,t) \qquad x \in \Gamma_N \quad 0 < t < T$

(IC) $u(x, t=0) = u_0(x) \qquad x \in \Omega$



So

$*\, v(x)$:

$$\int_\Omega \frac{\partial u}{\partial t} v \, dx + \int_\Omega \mu \nabla u \cdot \nabla v \, dx + \int_\Omega \vec{b} \cdot \nabla u \, v \, dx + \int_\Omega \sigma u v \, dx \underbrace{- \int_{\partial\Omega} \mu \frac{\partial u}{\partial n} v \, d\gamma}_{\cancel{- \int_{\Gamma_D} \mu \frac{\partial u}{\partial n} \underset{=0}{v}} \, - \int_{\Gamma_N} \mu \frac{\partial u}{\partial n} v} = \int_\Omega f v \, dx$$

$$\mu \frac{\partial u}{\partial n} \to \varphi$$

So we obtain

$$\underbrace{\int_\Omega \frac{\partial u}{\partial t} v \, dx + \int_\Omega \mu \nabla u \cdot \nabla v \, dx + \int_\Omega \vec{b} \cdot \nabla u \, v \, dx + \int_\Omega \sigma u v \, dx}_{a(u,v)} = \underbrace{\int_\Omega f v \, dx + \int_{\Gamma_N} \varphi v \, d\gamma}_{F(v)}$$

$$\forall t > 0 \quad \int_\Omega \frac{\partial u}{\partial t} v \, dx + a(u,v) = F(v) \qquad \forall v \in V$$

$$V = \left\{ v : \Omega \to \mathbb{R}, \ v \in H^1(\Omega), \ v|_{\Gamma_D} = 0 \right\} = H^1_{\Gamma_D}(\Omega)$$

Test function $v(x)$ (Depends only on x)

I need to add the I.C.

$$u(x, t=0) = u_0(x) \qquad x \in \Omega$$

2 CASES

1) $\varphi \equiv 0 \quad \forall t > 0 \quad u \in V$ (easy case)

I'm happy

$$\begin{cases} \forall t > 0 \quad \int_\Omega \frac{\partial u}{\partial t} v \, dx + a(u,v) = F(v) \qquad \forall v \in V \\ \text{I.C.} \quad u(x, t=0) = u_0(x) \qquad x \in \Omega \end{cases}$$

2) $\varphi \neq 0 \Rightarrow$ we use the lifting operator (done in the previous lecture)

In this case

$$\begin{cases} \mathring{u}(x,t) = u(x,t) - R\varphi(x,t) \\ \forall t>0 \,, \quad \mathring{u}=0 \quad x\in\Gamma_D \end{cases}$$

Introducing this function we obtain

$$\forall t>0 \int_\Omega \frac{\partial}{\partial t}(\mathring{u}+R\varphi)\, v\,dx + a(\mathring{u}+R\varphi, v) = F(v) \quad \forall v\in V$$

$$\Rightarrow \forall t>0 \int_\Omega \frac{\partial \mathring{u}}{\partial t} v\,dx + a(\mathring{u}, v) = \underbrace{F(v) - \int_\Omega \frac{\partial R\varphi}{\partial t} v - a(R\varphi, v)}_{\tilde{F}(v)} \quad \forall v \in V$$

I know this element

For the initial condition

$$\mathring{u}(x,t=0) = \overbrace{u(x,t=0)}^{u_0(x)} - R\varphi(x,t=0) \quad x\in\Omega$$

So the problem began

$$\begin{cases} \forall t>0, \ ? \ \mathring{u}\in V: \displaystyle\int_\Omega \boxed{\frac{\partial \mathring{u}}{\partial t}}\, v\,dx + a(\mathring{u},v) = \tilde{F}(v) \\ \text{I.C.} \quad \mathring{u}(x,t=0) = u_0(x) - R\varphi(x,t=0) \quad x\in\Omega \end{cases}$$

$$\begin{cases} =0 & \text{if } \varphi = \varphi(x) \\ \neq 0 & \text{if } \varphi = \varphi(x,t) \end{cases}$$

$\Rightarrow$ In all cases I will end up with a problem that may be written as:

$$\begin{cases} \forall t: 0 < t < T, \quad \text{Find } u \in V: \displaystyle\int_\Omega \partial_t \bar{u} v \, dx + a(\bar{u}, v) = F(v) \quad \forall v \in V \\ \bar{u}(x, t=0) = \bar{u}_0(x) \quad x \in \Omega \\ \text{with } \begin{cases} \bar{u} = u & \varphi = 0 \\ \bar{u} = u - R\varphi & \varphi \neq 0 \end{cases} \end{cases}$$

Stands for $\frac{\partial u}{\partial t}$

From now on I will use u instead of $\bar{u}$

Theorem

If $a(\cdot,\cdot)$ is bilinear, continuous and coercive

$\Rightarrow \exists!$ solution u to (PPW)

Moreover the same conclusion holds if $a()$ is weakly-coercive

Coercivity: $\exists \alpha > 0 \quad a(v,v) \geq \alpha \|v\|^2_V$

Weak coercivity: $\exists \lambda \geq 0, \exists \alpha > 0 : \quad a(v,v) + \lambda \|v\|^2_{L^2(\Omega)} \geq \alpha \|v\|^2_V$

We add the red to the coercivity definition example

$\partial\Omega = \Gamma_N$ Neumann problem

$\sigma = 0$, $\vec{b} = 0$, $\mu = 1$

$\Rightarrow a(u,v) = \int_\Omega \mu \nabla u \nabla v$ $V = \{ v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \} = H^1(\Omega)$

$\|v\|_V = \|v\|_{H^1(\Omega)} = \left(\|v\|^2_{L^2} + \|\nabla v\|^2_{L^2} \right)^{1/2}$ I don't have coercivity in this case

$\Rightarrow a(v,v) = \int_\Omega \mu |\nabla v|^2 = \|\nabla v\|^2_{L^2} \ngeq \|v\|^2_V$

"If I get help" with the weak definition

However: $\underbrace{a(v,v) + \|v\|^2_{L^2}}_{} \geq \|\nabla v\|^2_{L^2} + 1\, \|v\|^2_{L^2} = \|v\|^2_V$

$\Rightarrow a(\cdot,\cdot)$ is weak coercive (with $\alpha = 1$)

Coercivity $\underset{\not\Leftarrow}{\Rightarrow}$ weak coercivity $(\lambda = 0)$

Why weak coercivity is enough?

Assure $a(\cdot,\cdot)$ weakly coercive but NOT Coercive.

Consider the following change of variable

$\boxed{u = e^{\lambda t} w}$ (PPW) : $\int_\Omega \partial_t (e^{\lambda t} w) v \, dx + a(e^{\lambda t} w, v) = F(v)$

— function of time $\lambda \int_\Omega e^{\lambda t} w v \, dx + \int_\Omega e^{\lambda t} \partial_t w v \, dx + a(e^{\lambda t} w, v) = F(v)$

So

$$e^{\lambda t}\left[\int_\Omega \lambda w v + \int_\Omega \partial_t w v + a(w,v)\right] = F(v)$$

$$\Rightarrow \int_\Omega \partial_t w v + \underbrace{a(w,v) + \lambda \int_\Omega w v}_{\bar{a}(w,v)} = \underbrace{e^{-\lambda t} F(v)}_{\bar{F}(v)}$$

$$\Rightarrow \int_\Omega \partial_t w v + \bar{a}(w,v) = \bar{F}(v)$$ $\rightarrow$ This problem is coercive

Note : $\bar{a}(v,v) = a(v,v) + \lambda \|v\|^2_{L^2}$ COERCIVE

We have the existence and uniquness of w. w is correleted to u so it good

(This is all theory, in reality we resolve the problem with u)

$\boxed{\text{APPROXIMATION}}$ (First on x, then on t)

APPROXIMATION ON X

Bound as the elliptic case : $V_h \subset V$, $\dim V_h = N_h$

then approximate (PPW) by

$$\begin{cases} \forall t > 0 \; ? \; v_h \in V_h : \int_\Omega \partial_t u_h v_h \, dx + a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h \\ u_h(x, t=0) = u_{0h} \in V_h \end{cases}$$

Let $\{\varphi_J\}_{J=1}$ be a basis of V_h

Unknown

$$\Rightarrow \forall t>0, \quad u_h(x,t) = \sum_{J=1}^{N_h} \boxed{u_J(t)} \varphi_J(x)$$

Plug u_h into (P_h) and take $v_h = \varphi_i$

$$\Rightarrow \begin{cases} 0<t<T & \int_\Omega \partial_t \left(\sum_J u_J(t) \varphi_J \right) \varphi_i \, dx + a\left(\sum_J u_J(t) \varphi_J, \varphi_i \right) = F(\varphi_i) \quad 1 \leq i \leq N_h \\ & u_h(x,0) = u_{0_h}(x) \qquad x \in \Omega \end{cases}$$

$$\Rightarrow 0<t<T \qquad \sum_J \mathring{u}_J(t) \underbrace{\int_\Omega \varphi_J \varphi_i \, dx}_{m_{iJ}} + \sum_J u_J(t) \underbrace{a(\varphi_J, \varphi_i)}_{a_{iJ}} = \underbrace{F(\varphi_i)}_{F_i} \qquad 1 \leq i \leq N_h$$

$M = m_{iJ}$ MASS MATRIX

$A = a_{iJ}$ STIFFNESS MATRIX

$\vec{F} = (F_i)$ R.h.s. vector

This is my problem

$$\Rightarrow \begin{cases} M \mathring{\vec{u}}(t) + A \vec{u}(t) = \vec{F}(t) & 0<t<T \\ \vec{u}_0 = u_0(x_J) & 1 \leq J \leq N_h \end{cases}$$

$$\vec{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_{N_h}(t) \end{bmatrix}$$

This is called CAUCHY-PROBLEM

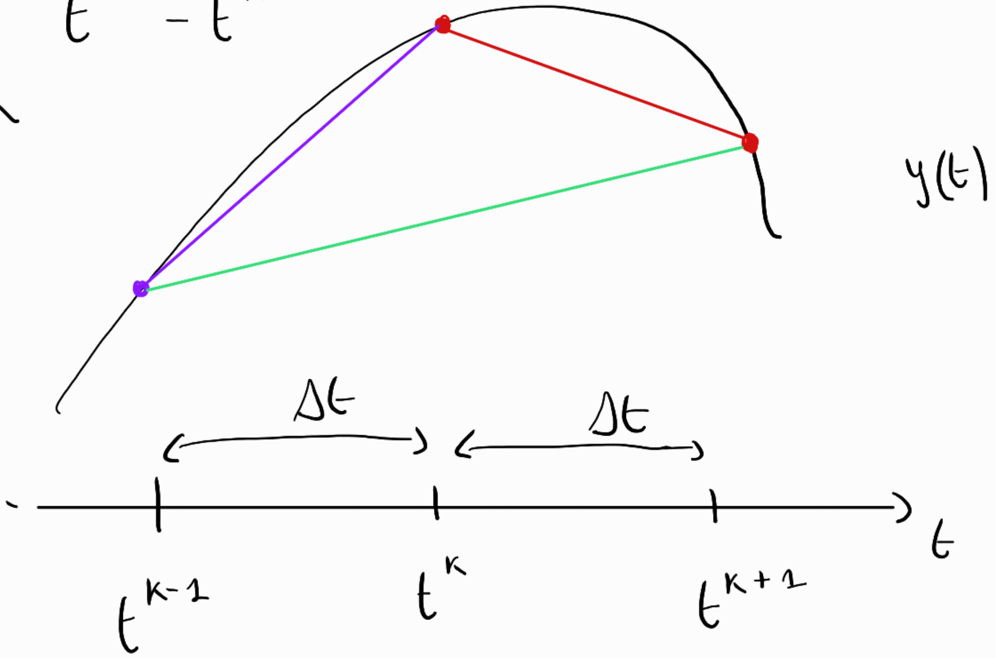
APPROXIMATION IN TIME

Forward Euler

$$\frac{dy}{dt}(t) = \lim_{\delta \to 0} \frac{y(t+\delta)-y(t)}{\delta}$$

$$\frac{dy}{dt}(t^K) \simeq \begin{cases} 1) \ \dfrac{y(t^{K+1}) - y(t^K)}{t^{K+1} - t^K} & (FE) \\[2ex] 2) \ \dfrac{y(t^K) - y(t^{K-1})}{t^{K+1} - t^K} & (BE) \\[2ex] 3) \ \dfrac{y(t^{K+1}) - y(t^{K-1})}{t^{K+1} - t^{K-1}} & (MP) \end{cases}$$

Backword Euler

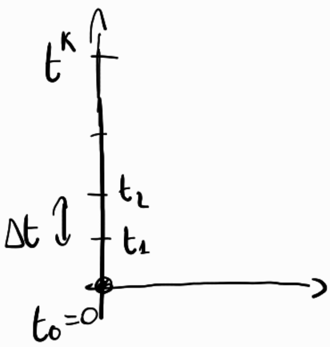


Errors are

$O(\Delta t)$, $O(\Delta t)$, $O(\Delta t^2)$

So

$\vec{u}^K = \vec{u}(t^K)$ $\quad t^K = t^0 + K\Delta t \quad \longrightarrow$



1) $M \dfrac{\vec{u}^{K+1} - \vec{u}^K}{\Delta t} + A\vec{u}^K = \vec{F}^K$ (FE)

2) $M \dfrac{\vec{u}^K - \vec{u}^{K-1}}{\Delta t} + A\vec{u}^K = \vec{F}^K$ $\quad \forall K$ (BE)

3) $M \dfrac{\vec{u}^{K+1} - \vec{u}^{K-1}}{2\Delta t} + A\vec{u}^K = \vec{F}^K$ (MP)

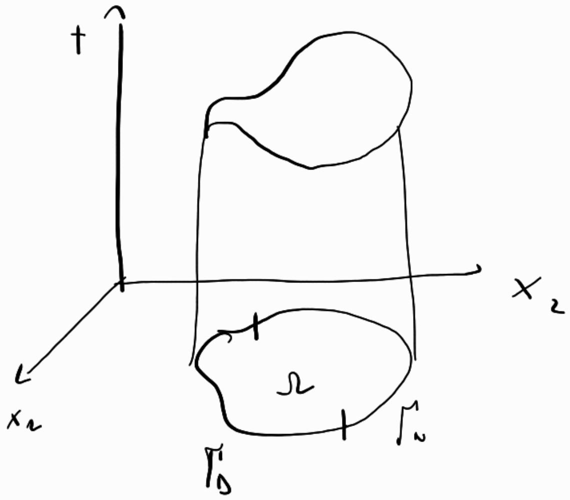
$\vec{u}^{K+1}$ UNKNOWN

Algebraic linear system

LEZIONE 6 26/11/2024

Parabolic Problem

(P) $\partial_t u + Lu = f \quad x \in \Omega \quad 0 < t < T$



$$BC \begin{cases} u = \varphi & \text{on } \Gamma_D \times (0,T) \\ \mu \dfrac{\partial u}{\partial n} = \psi & \text{on } \Gamma_N \times (0,T) \end{cases}$$

$u = u_0 \qquad x \in \Omega, \quad t = 0$

$Lu = -\mathrm{div}(\mu \nabla u) + \vec{b} \cdot \nabla u + \sigma u$

f, φ, ψ, u_0 are given function

(PW)

$\forall\, 0 < t < T$ find $u \in V$:

$$\int_\Omega \partial_t u v \, dx + a(u,v) = F(v) - \int_\Omega \partial_t R\varphi \, v \, dx + \int_{\Gamma_N} \psi v \, d\gamma + a(R\varphi, v) \quad \forall v \in V$$

$V = \{ v : \Omega \to \mathbb{R},\ v \in H^1(\Omega),\ v|_{\Gamma_D} = 0 \}$

$R\varphi \in H^1(\Omega) : R\varphi|_{\Gamma_D} = \varphi$

$Fv = \int_\Omega f v \, dx$

$a(u,v) = \int_\Omega \mu \nabla v \nabla u + \int_\Omega \vec{b} \cdot \nabla u \, v + \int_\Omega \sigma u v$

If data are "regular enough" and $a: V \times V \to \mathbb{R}$ is continuous and WEAKLY COERCIVE then $\exists!$ solution of (PW)

Semi-discrete (SD) in approximation in (space only)

$V_h \subset V$, $\dim V_h = N_h$

$$(SD) \quad \begin{cases} \forall t \in (0,T), \text{ find } u_h(t): \int_\Omega \partial_t u_h v_h \, dx + a(u_h, v_h) = \tilde{F}(v_h) \quad \forall v_h \in V_h \\ u_h(t=0) = u_{0_h} \in V_h \end{cases}$$

$\Updownarrow$

$\forall t \in (0,T)$

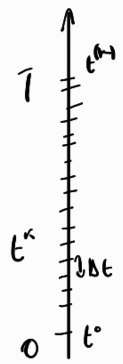
$$M\dot{\vec{u}}(t) + A\vec{u}(t) = \tilde{F}(t) \qquad M = (m_{ij}) = \left(\int_\Omega \varphi_j \varphi_i\right)$$

$$\vec{u}(t=0) = \vec{u}_{0,h} \qquad A = (a_{ij}) = \left(a(\varphi_i, \varphi_j)\right)$$

$$\tilde{\tilde{F}}_i = \tilde{F}(\varphi_i)$$

Full discretization FD

(FD) $\vec{u}^{(k)} = \vec{u}(t^k) \qquad t^k = k\Delta t$



$$(BE) \quad M \frac{\vec{u}^{(k+1)} - \vec{u}^{(k)}}{\Delta t} + A\vec{u}^{(k+1)} = \vec{\tilde{F}}^{(k+1)}$$

$$(FE) \quad M \frac{\vec{u}^{(k+1)} - u^{(k)}}{\Delta t} + A\vec{u}^{(k)} = \tilde{\tilde{\vec{F}}}^{(k)}$$

$$(MP) \quad M \frac{\vec{u}^{(k+1)} - \vec{u}^{(k-1)}}{2\Delta t} + A\vec{u}^{(k)} = \tilde{\tilde{\vec{F}}}^{(k)}$$

Linear algebraic systems

$\vartheta \in \mathbb{R}$, $0 \le \vartheta \le 1$

$\boxed{\text{COMPUTATIONAL FORM}}$

$$(\vartheta) \begin{cases} M \dfrac{\vec{u}^{(k+1)} - \vec{u}^{(k)}}{\Delta t} + \vartheta A\vec{u}^{(k+1)} + (1-\vartheta)Au^{(k)} - \vartheta \vec{\tilde{F}}^{(k+1)} + (1-\vartheta)\vec{\tilde{F}}^{(k)} & 0 \le k \le K-1 \\ (\vartheta) \Rightarrow \underbrace{\left(\dfrac{1}{\Delta t} M + \vartheta A\right)}_{B} \vec{u}^{(k+1)} = \underbrace{\vartheta \vec{\tilde{F}}^{(k+1)} + (1-\vartheta)\vec{\tilde{F}}^{(k)} + \dfrac{1}{\Delta t} M\vec{u}^{(k)} + (\vartheta - 1) A u^{(k)}}_{\vec{C}^{(k)}} \\ \vec{u}^{(0)} = \vec{u}_{0,h} \end{cases}$$

ϑ method :

Error $\|u_{ex}(t^k) - u_h(t^k)\| = O(\Delta t)$

Special cases

$$\begin{cases} \vartheta = 0 \Rightarrow (FE) \\ \vartheta = 1 \Rightarrow (BE) \end{cases} \Bigg\} \text{ 1st order method}$$

$\vartheta = \frac{1}{2} \Rightarrow (CN)$ ↳ Crank Nikolson

$$M \frac{\vec{u}^{(k+1)} - \vec{u}^{(k)}}{\Delta t} + \frac{1}{2} A\vec{u}^{(k+1)} + \frac{1}{2} A\vec{u}^{(k)} = \frac{1}{2}\vec{\tilde{F}}^{(k+1)} + \frac{1}{2}\vec{\tilde{F}}^{(k)}$$

Second order method

$$\|u(t^k) - u_h(t^k)\| = O(\Delta t^2)$$

ERROR ESTIMATE FOR THE ϑ-METHOD

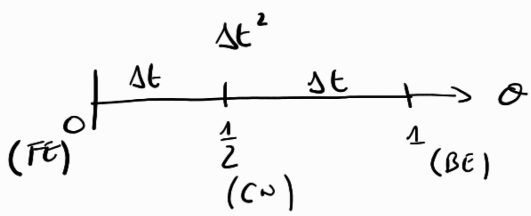
$\boxed{\text{CONVERGENCE}}$

$$\|u(t^{(k)}) - u_h(t^{(k)})\|_{L^2(\Omega)} \le C\left(\underline{h^r} + \underline{\Delta t^{p(\vartheta)}}\right)$$

Exact → SD SD → F.D

$$p(\vartheta) = \begin{cases} 2 & (CN) \\ 1 & \text{otherwise } (\forall \vartheta \ne \frac{1}{2}) \end{cases}$$

Sintetic view of our results



STABILITY

Is a necessary condition for convergence

Stability of (SD)

$\forall t \in (0,t)$ take $v = u_h(t)$

$$\Longrightarrow \underbrace{\int_\Omega \partial_t u_h(t)\,dx}_{\frac{1}{2}\partial_t [u_h(t)]^2} + \underbrace{a(u_h(t), u_h(t))}_{\downarrow \text{ coercivity } \alpha \|u_h(t)\|_V^2} = \underbrace{\tilde{F}(u_h, t)}_{\hookrightarrow \text{ if } \varphi = \psi = 0}$$

$$\frac{1}{2}\partial_t \underbrace{\int_\Omega (u_h(t))^2}_{\|u_h(t)\|^2_{L^2(\Omega)}}$$

$$\int_\Omega f(t)\, u_h(t) \le CS \;\|f(t)\|_{L^2(\Omega)} \|u_h(t_h)\|_{L^2(\Omega)}$$

$$\Longrightarrow \boxed{\partial_t \|u_h(t)\|^2_{L^2(\Omega)} + 2\alpha \|u_h(t)\|_V^2 \le 2\|f(t)\|_{L^2(\Omega)} \|u_h(t)\|_{L^2(\Omega)}}$$

Simplest case : $f = 0$

$$\int_0^t \Rightarrow \|u_h(t)\|^2_{L^2(\Omega)} - \|u_h(0)\|^2_{L^2(\Omega)} + 2\alpha \int_0^t \|u_h(s)\|_V^2\, ds = 0$$

I integrate the red formula

$$\Rightarrow \boxed{\|u_h(t)\|^2_{L^2(\Omega)} + 2\alpha \int_0^t \|u_h(s)\|^2_V \, ds \leq \|u_{0_h}\|^2_{L^2(\Omega)}} \qquad 0 < t < T$$

The norm is boundary by the norm of the initial guess.

In particular $\Rightarrow \|u_h(t)\|_{L^2(\Omega)} < \|u_{0_h}\|_{L^2(\Omega)}$ (It's very good)

•) if $f \neq 0$

$$\|u_h(t)\|^2_{L^2(\Omega)} - \|u_h(0)\|^2_{L^2(\Omega)} + 2\alpha \int_0^t \|u_h(s)\|^2_V \, ds \leq 2\|f(t)\|_{L^2(\Omega)} \|u_h(t)\|_{L^2(\Omega)}$$

$$\Rightarrow \|u_h(t)\|^2_{L^2(\Omega)} + 2\alpha \int_0^t \|u_h(s)\|^2_V \, ds \leq \|u_{0_h}\|^2_{L^2(\Omega)} + 2\int_0^t \|f(s)\|_{L^2(\Omega)} \|u_h(s)\|_{L^2(\Omega)}$$

I have the solution in the right part also, we need the YOUNG INEQUALITY

$$\forall a,b \in \mathbb{R}, \ \forall \varepsilon > 0, \ ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \rightsquigarrow \left(\sqrt{\varepsilon}\, a - \frac{1}{2\sqrt{\varepsilon}} b\right)^2 \geq 0$$

$$\Rightarrow \|u_h(t)\|^2_{L^2(\Omega)} + 2\alpha \int_0^t \|u_h(s)\|^2_V \, ds \leq \|u_{0_h}\|^2_{L^2(\Omega)} + 2\left(\varepsilon \int_0^t \|f(s)\|^2_{L^2(\Omega)} \, ds + \frac{1}{4\varepsilon} \int_0^t \|u_h(s)\|^2_{L^2(\Omega)} ds\right.$$

$$\Rightarrow \text{LHS} \leq \|u_{0h}\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^t \|f(s)\|^2_{L^2(\Omega)}\, ds + \frac{1}{2\varepsilon} \underbrace{\int_0^t \|u_h(s)\|^2_{L^2(\Omega)}\, ds}_{\leq \int_0^t \|u_h(s)\|^2_V\, ds}$$

Take ε s.t. $\frac{1}{2\varepsilon} < 2\alpha$ (for instance $2\varepsilon > \frac{1}{2\alpha}$, $\varepsilon > \frac{1}{4\alpha}$; $\varepsilon = \frac{1}{2\alpha}$)

So I can move it to the left

$$\Rightarrow \|u_h(t)\|^2_{L^2(\Omega)} + \overbrace{\left(2\alpha - \frac{1}{2\varepsilon}\right)}^{\bar{\alpha} > 0} \int_0^t \|u_h(s)\|^2_V\, ds \leq \|u_{0h}\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^t \|f(s)\|^2_{L^2(\Omega)}\, ds$$

The conclusion is that

$$\Rightarrow \boxed{\|u_h(t)\|^2_{L^2} + \bar{\alpha} \int_0^t \|u_h(s)\|^2_V\, ds \leq \|u_{0h}\|^2_{L^2(\Omega)} + 2\varepsilon \int_0^t \|f(s)\|^2_{L^2(\Omega)}\, ds} \Rightarrow \text{STABILITY}$$

Regularity of data

$$\|f\|^2_{L^2(\Omega)\times(0,T)}$$

$u_0 \in L^2(\Omega)$, $u_{0_h} \in L^2(\Omega)$, $f \in L^2(x,t) : \left| \int_0^t \int_\Omega f^2(x,s)\,dx\,ds \right| < \infty \quad \forall\, 0<t<T$

Stability of θ-method

ANALYSIS FORMULATION

$$(\theta) \Leftrightarrow \begin{cases} \forall\, k=0,\dots,K-1 \quad \text{find } u_h^{(k+1)} \in V_h \text{ s.t.} \\ \displaystyle\int_\Omega \frac{u_h^{(k+1)} - u_h^{(k)}}{\Delta t} \cdot v_h\,dx + \theta\, a(u_h^{(k+1)}, v_h) + (1-\theta)\, a(u_h^{(k)}, v_h) = \theta \int_\Omega f^{(k+1)} v_h\,dx + (1-\theta)\int_\Omega f^{(k)} v_h\,dx \\ \qquad \forall v_h \in V_h \\ u_h^{(0)} = u_{0_h} \end{cases}$$

$(\theta = 1) \Leftrightarrow (BE)$ take $v_h = u_h^{(k+1)}$

$$\Rightarrow \int_\Omega \frac{\overbrace{u_h^{(k+1)}}^{a} - \overbrace{u_h^{(k)}}^{b}}{\Delta t}\, \overbrace{u_h^{(k+1)}}^{a}\,dx + \underbrace{a(u_h^{(k+1)}, u_h^{(k+1)})}_{\geq \alpha \|u_h^{(k+1)}\|^2_V} = \int_\Omega f^{(k+1)} u_h^{(k+1)}\,dx$$

$(*)$

$\to$ coercivity

ALGEBRAIC INEQUALITY

$\forall\, a,b \in \mathbb{R} \quad (a-b)\cdot a \geq \frac{1}{2}a^2 - \frac{1}{2}b^2$

dim

$a(a-b) = a^2 - ab \qquad a^2 - ab \geq \frac{1}{2}a^2 - \frac{1}{2}b^2$

I know that $\quad ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2 \quad$ so $\quad -ab \geq -\frac{1}{2}a^2 - \frac{1}{2}b^2$

$$\Downarrow$$

$$(f-g, f) \geq \frac{1}{2}\|f\|^2_{L^2} - \frac{1}{2}\|g\|^2_{L^2}$$

$\square$

So (*) became

$$\geq \frac{1}{2}\|u_h^{(k+1)}\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_h^{(k)}\|^2_{L^2(\Omega)}$$

$$\Rightarrow \|u_h^{(k+1)}\|^2_{L^2(\Omega)} + 2\alpha\Delta t\,\|u_h^{(k+1)}\|^2_V \leq \|u_h^{(k)}\|^2_{L^2(\Omega)} + 2\Delta t\int_\Omega f^{(k+1)} u_h^{(k+1)}\,dx$$

So we can sum up for every K

$$\sum_{K=0}^{m} \Rightarrow \boxed{\sum_{K=0}^{m}\left(\|u_h^{(k+1)}\|^2_{L^2(\Omega)} - \|u_h^{(k)}\|^2_{L^2(\Omega)}\right)} + 2\alpha\Delta t\sum_{K=0}^{m}\|u_h^{(k+1)}\|^2_V \leq 2\Delta t\sum_{K=0}^{m}\int_\Omega f^{(k+1)} u_h^{(k+1)}$$

$m = K-1$

Telescopic sum

$$\|u_h^{(m+1)}\|^2_{L^2(\Omega)} - \|u_h^{(0)}\|^2_{L^2(\Omega)}$$

$$\Rightarrow \|u_h^{(m+1)}\|^2_{L^2(\Omega)} + 2\alpha\Delta t\sum_{K=0}^{m}\|u_h^{(k+1)}\|^2_V \leq \|u_{0_h}\|^2_{L^2(\Omega)} + 2\Delta t\sum_{K=0}^{m}\int_\Omega f^{(k+1)} u_h^{(k+1)}$$

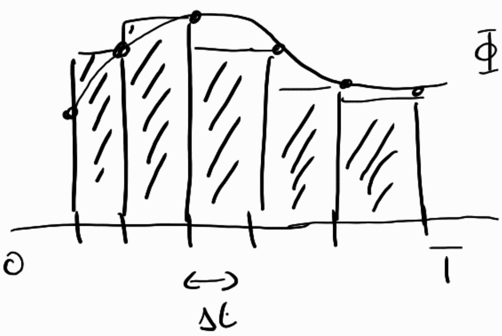
SIMPLEST CASE

$\bullet)\ f = 0$

$$\simeq \int_0^{t_{n+1}} \|u_h(s)\|^2_V\,ds$$

$$\Phi : (0,T) \to \mathbb{R}$$

$$\Rightarrow \|u_h^{(m+1)}\|^2_{L^2(\Omega)} + 2\alpha\Delta t\sum_{K=0}^{m}\|u_h^{(k+1)}\|^2_V \leq \|u_{0_h}\|^2_{L^2(\Omega)}$$



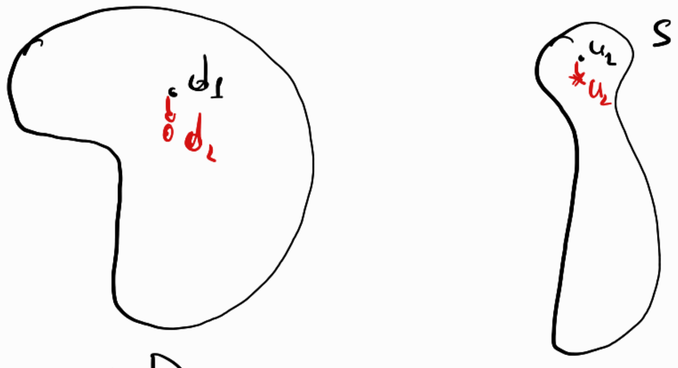
$$\Rightarrow \|u_h^{(m+1)}\|_{L^2(\Omega)} < \|u_{0_h}\|_{L^2(\Omega)} \qquad 0 \leq m \leq K-1$$

· if $f \neq 0$

$\boxed{} \leq CS \leq \checkmark$ Exercise, as before

Stability : $\begin{cases} P(u_1, \dots, d_1) = 0 \\ P(u_2, \dots, d_2) = 0 \end{cases}$ P = problem

If $d_1 - d_2$ is small $\Rightarrow$ $u_1 - u_2$ is small



$\to$ Spazio u_1, u_2

Spazio f_1, f_2 $\subset D$

$\exists \; c > 0 : \|u_1 - u_2\|_S \leq C \|d_1 - d_2\|_D \quad \forall \, d_1, d_2$

<u>Boundedness</u> $P(u, d) = 0$

$\exists \; c > 0 : \|u\|_S \leq c \|d\|_D \quad \forall \, d \in D$

If P is a linear problem then boundedness $\Rightarrow$ Stability

<u>Ex.</u> P: $\boxed{Lu = f}$

$\begin{cases} Lu_1 = f_1 \\ Lu_2 = f_2 \end{cases}$ sub member by member $\Rightarrow$ $Lu_1 - Lu_2 = f_1 - f_2$

$\overset{\text{Linear}}{\Rightarrow}$ $L(\underbrace{u_1 - u_2}_{w}) = \underbrace{f_1 - f_2}_{g}$

If I have bundedness $\Rightarrow$ $\|w\|_S \leq C \|g\|_D$

$\Leftrightarrow$ $\|u_1 - u_2\|_S \leq C \|f_1 - f_2\|_D$ and we have STABILITY

STABILITY

For $\vartheta = 1 \Rightarrow$ The (BE) method is (unconditionally) STABLE
$$\Updownarrow$$
$$\forall \Delta t > 0$$

For $0 \le \theta \le 1$: Assume $a(\cdot,\cdot)$ symmetric $\left(a(u,v) = a(v,u) \quad \forall u,v \in V \right)$

We know that $A\vec{w} = \lambda \vec{w}$, so

Eigenvalues of $a(\cdot,\cdot)$ $\quad \boxed{a(w,v) = \lambda \underbrace{(w,v)}_{\int_\Omega w v \, dx} \quad \forall v \in V}$ $\quad \lambda \in \mathbb{R}$ because a symmetric

$\int_\Omega w v\, dx \to$ scalar product

(λ, w) : eigenvalue/eigenfunction of $a(\cdot,\cdot)$

$\lambda \in \mathbb{R}^+$ because a is coercive, I have real and positive eigenvalues.

DISCRETE EIGENVALUES OF $a(\cdot,\cdot)$

$$a(w_h, v_h) = \lambda_h (w_h, v_h) \qquad \forall v_h \in V_h \qquad \lambda_h > 0$$

$$0 < \lambda_1 \le \lambda_2 \le \dots \le \lambda_{N_h} \xrightarrow[N_h \uparrow +\infty]{} +\infty$$

(λ_h, w_h): discrete eigenvalue/eigenfunctions

$\lambda_h^1 \; \dots \; \lambda_h^{N_h}$

$w_h^1 \; \dots \dots \; w_h^{N_h}$

ORTHO-NORMAL

$$(w_h^i - w_h^J) = \begin{cases} 0 & i \ne J \\ 1 & i = J \end{cases}$$

$\Rightarrow \{w_h^i\}_{i=1}^{N_h}$ basis for V_h

$$\Rightarrow V_h \ni u_h^{(k)}(x) = \sum_{J=1}^{N_h} u_J w_h^J(x)$$

θ - method

$f = 0$

$$\int_\Omega \frac{u_h^{(k+1)} - u_h^{(k)}}{\Delta t} v_h \, dx + \theta \, a(u_h^{(k+1)}, v_h) + (1-\theta) \, a(u_h^{(k)}, v_h) = 0$$

Take $v_h = w_h^i$

$$\Rightarrow \sum_{J=1}^{N_h} \frac{1}{\Delta t} \int \left(u_J^{(k+1)} - u_J^{(k)}\right) w_h^J w_h^i \, dx + \sum_{J=1}^{N_h} a\left(\left[\theta u_J^{(k+1)} + (1-\theta) u_J^{(k)}\right] w_h^J, w_h^i\right) = 0$$

Now I exploit first element of the sum

$$\frac{1}{\Delta t} \sum_J \left(u_J^{(k+1)} - u_J^{(k)}\right) \int_\Omega w_h^J w_h^i \, dx \longrightarrow \delta_{iJ} \quad \text{for orthonormality}$$

The second element become

$$\sum_J \left[\theta u_J^{(k+1)} + (1-\theta) u_J^{(k)}\right] a(w_h^J, w_h^i) \longrightarrow \lambda_h^i (w_h^J, w_h^i) = \lambda_h^i \delta_{iJ}$$

for orthonormality

$$\Rightarrow \frac{u_i^{(k+1)} - u_i^{(k)}}{\Delta t} + \lambda_h^i \left[\theta u_i^{(k+1)} + (1-\theta) u_i^{(k)}\right] = 0$$

$$\left[1 + \Delta t \, \lambda_h^i \theta\right] u_i^{(k+1)} = \left[1 - \Delta t \, \lambda_h^i (1-\theta)\right] u_i^{(k)}$$

$$u_i^{(k+1)} = \left(\frac{1 - \Delta t (1-\theta) \lambda_h^i}{1 + \Delta t \, \theta \lambda_h^i}\right) u_i^{(k)} \longrightarrow \text{Constant that depends on } i$$

$$\delta(i, \Delta t)$$

LEZIONE 7 27/11/2024

$$u_h^{(k)}(x) = \sum_{j=1}^{N_h} u_j^{(k)} \omega_h^j(x)$$

$$u_j^{(k+1)} = \left[\frac{1-(1-\theta)\lambda_h^j \Delta t}{1+\theta \lambda_h^j \Delta t} \right] u_j^{(k)} \quad (*)$$

This is a constant $\delta_{j,\theta,\Delta t}$

Assume $|\delta_{j,\theta,\Delta t}| < 1 \quad \forall j \quad (**)$

Then:

$$\|u_h^{(k+1)}\|^2_{L^2(\Omega)} = \left(u_h^{(k+1)}, u_h^{(k+1)}\right)_{L^2(\Omega)} = \left(\sum_j u_j^{(k+1)} \omega_h^j, \sum_i u_i^{(k+1)} \omega_h^i\right)_{L^2(\Omega)} =$$

$$= \sum_i \sum_j u_j^{(k+1)} u_i^{(k+1)} \underbrace{\left(\omega_h^j, \omega_h^i\right)_{L^2(\Omega)}}_{\delta_{i,j}} = \sum_{i=1}^{N_h} \left(u_i^{(k+1)}\right)^2 =$$

$$\stackrel{(*)}{=} \sum_i |\delta_{i,\theta,\Delta t}|^2 \left(u_i^{(k)}\right)^2 \stackrel{(**)}{<} \sum_i \left(u_i^{(k)}\right)^2 = \|u_h^{(k)}\|^2_{L^2(\Omega)}$$

Conclusion

If $(**)$ holds, then $\|u_h^{(k+1)}\|_{L^2(\Omega)} < \|u_h^{(k)}\|_{L^2(\Omega)} < \dots < \|u_{0_h}\|_{L^2(\Omega)}$

STABILITY

So $(**)$ began

$$-1 < \delta_{j,\theta,\Delta t} < 1$$

$$\Leftrightarrow \quad -1-\theta \Delta t \lambda_h^j < 1-(1-\theta)\lambda_h^j \Delta t < 1+\theta \Delta t \lambda_h^j \qquad \forall j$$

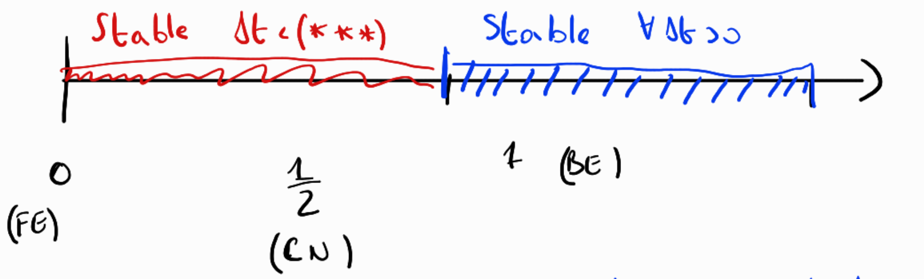
$$\Rightarrow \quad -2 - \theta \Delta t \lambda_h^j < (\theta - 1)\lambda_h^j \Delta t < \theta \Delta t \lambda_h^j \qquad \forall j$$

always true

$$-\frac{2}{\lambda_h^j \Delta t} - \theta < \theta - 1$$

$$\frac{-2}{\lambda_h^j \Delta t} < 2\theta - 1 \Rightarrow \begin{cases} \frac{1}{2} \le \theta \le 1 & \text{TRUE } \forall \Delta t \\ 0 \le \theta < \frac{1}{2} & \frac{2}{\lambda_h^j \Delta t} > 1 - 2\theta \iff \boxed{\Delta t < \frac{2}{\lambda_h^j (1-2\theta)} \quad \forall j = 1, \dots, N_h} \end{cases}$$

(***)



Conditionally stable

Unconditionally stable

If $\Delta t < \frac{2}{\lambda_h^{N_h}(1-2\theta)}$ it's good the (***) because

$$\boxed{\lambda_h^{N_h} \simeq C h^{-2}}$$

$\lambda_h^{N_h}$ it's the biggest eigenvalue

$$\Downarrow$$

$$\boxed{\Delta t \le C h^2}$$

Don't confuse STABILITY with convergence

STABILITY: numerical solution bounded by data

convergence: error goes to 0

$\theta = 1$ (BE) $\quad \displaystyle\int_\Omega \frac{u_h^{k+1} - u_h^k}{\Delta t} v_h + a(u_h^{k+1}, v_h) = 0$

<u>Consistency</u>

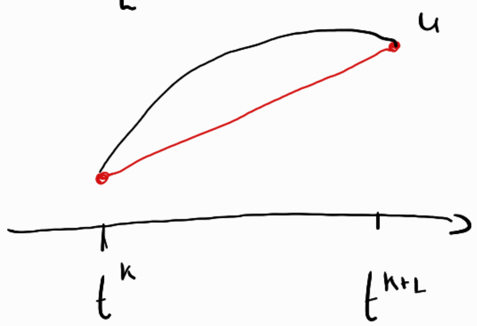
u = exact solution $\Rightarrow$ $$\int_\Omega \frac{u(t^{k+1}) - u(t^k)}{\Delta t} V_h + a(u(t^{k+1}), v_h) = O(\Delta t) \xrightarrow[h,\Delta t \to 0]{} 0$$

empirically the exact solution is

$$\int_\Omega \partial_t u v + a(u,v) = 0 \qquad \forall\, v \in V$$

$$\odot \int_\Omega \partial_t u v_h + a(u, v_h) = 0 \qquad \forall v_h \in V_h$$

red become $\quad - \displaystyle\int_\Omega \partial_t u(t^{k+1}) v_h$ $\quad$ because of $\odot$



$u(t^{k+1}) = u(t^k) - \Delta t \cdot u'(t^{k+1}) + O(\Delta t^2)$ $\quad$ Taylor expansion

$\downarrow$

goes in the rhs

So $\quad$ blue become that

$$\Rightarrow \cancel{\int_\Omega \partial_t u(t^{k+1}) v_h} + O(\Delta t) - \cancel{\int_\Omega \partial_t u(t^{k+1}) v_h} \longrightarrow 0$$

NAVIER-STOKES EQUATIONS (time dependent)

Used in fluid flow. Two differn study. If is compressed or incompressed.

We study INCOMPRESSIBLE (VISCOUS).

$\vec{u}$ = vector of velocity of the fluid
p = pressure

} unknowns

=0 STEADY NAVIER-STOKES EQ

non linear

Domain when the fluid is put

$$(PDE)\ (*)\ \begin{cases} \underbrace{\boxed{\partial_t \vec{u}}}_{=0} - \nu \Delta \vec{u} + \boxed{(\vec{u} \cdot \nabla)\vec{u}} + \nabla p = \vec{f} & x \in \Omega,\ 0 < t < T \\ \operatorname{div} \vec{u} = 0 \end{cases}$$

momentum eq → conservation of the linear momentum

continuity eq → conservation of mass

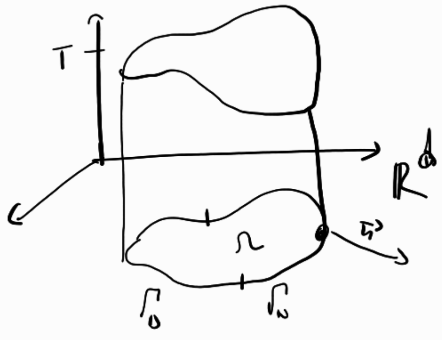
$\Downarrow$

$$\begin{cases} \partial_t u_i - \nu \Delta u_i + \sum_j \left(u_j \frac{\partial}{\partial x_j}\right) u_i + \frac{\partial p}{\partial x_i} = f_i & i = 1, \dots, d\ (=2,3) \\ \sum_j \frac{\partial u_j}{\partial x_j} = 0 \end{cases}$$

We have 4 equations

Data: $\nu > 0$, $\vec{f}, \vec{\varphi}, \vec{\psi}, \vec{u}_0$

$$(BC)\ (**)\ \begin{cases} (D) & \vec{u} = \vec{\varphi} \quad \text{on } \Gamma_D \times (0,T) \\ (N) & \nu \frac{\partial \vec{u}}{\partial n} - p\vec{n} = \vec{\psi} \quad \text{on } \Gamma_N \times (0,T) \end{cases}$$



$$(IC)\quad \vec{u} = \vec{u}_0 \quad \text{in } \Omega,\ t = 0$$

It's (*) + (**) that determine the problem

(*) IT'S NOT LINEAR

RK ADR

$\vec{b}\,\nabla u$ (transport, here $\vec{b} = \vec{u}$)

$|\vec{u}|$ = caracteristic velocity

$1 \ll Re = \dfrac{|\vec{u}|L}{\nu}$ Reynolds number

(It's this problem well posed in 3D? We DON'T KNOW if there is a unique solution)

We can consider a simpler problem with none of nonelinear part

STOKES PROBLEM (Time dependent)

$$\begin{cases} \boxed{\partial_t \vec{u}} - \nu \Delta \vec{u} + \nabla p = \vec{f} \\ \operatorname{div} \vec{u} = 0 \\ + BC's \\ + IC \end{cases}$$

$\partial_t \vec{u} = 0$ Steady STOKES EQ

$\longrightarrow$ We change this equation

It's a linear problem

If $\vec{u}$ is small we can erase the non linear term.

In some case we don't want the change of time (they will be irrilevant, don't change in time) and we can drop the time derivative

When $\partial_t u = 0$ STEADY (NAVIER)-STOKES EQ

In STOKES problem we obtain

STEADY STOKES PROBLEM

$$\begin{cases} \alpha \vec{u} - \nu \Delta \vec{u} + \nabla p = \vec{f} \\ \operatorname{div} \vec{u} = 0 \\ + BC's \;\; * \end{cases} \qquad \alpha \geq 0$$

NS time-dep. $\xrightarrow{\partial_t \vec{u} = \vec{0}}$ NS steady

$(\vec{u}\cdot\nabla)\vec{u} \simeq 0$ (Re small) $\downarrow$ $\qquad\qquad$ $\downarrow (u\cdot\nabla)\vec{u} \simeq 0$

Stokes time-dep $\xrightarrow[\text{or } 2) \text{ Using BE to go from } t^k \text{ to } t^{k+1}]{1)\ \partial_t \vec{u} = \vec{0}}$ $\boxed{\text{Stokes steady}}$ Very usefull to study

We can go from S time-dep to S steady using BE method

$$\begin{cases} \dfrac{\overbrace{\vec{u}(t^{k+1})}^{\vec{w}} - \vec{u}(t^k)}{\Delta t} - \nu\Delta\vec{u}(t^{k+1}) + \nabla \overbrace{p(t^{k+1})}^{\pi} = \vec{f}(t^{k+1}) \\ \text{div}\, \vec{u}(t^{k+1}) = 0 \end{cases}$$

I can move these term to rhs

So I obtain

$$\begin{cases} \underbrace{\dfrac{1}{\Delta t}}_{\alpha}\vec{w} - \nu\Delta\vec{w} + \nabla\pi = \vec{f}(t^{k+1}) + \dfrac{1}{\Delta t}\vec{u}(t^k) \equiv \tilde{\vec{f}} \\ \text{div}\, \vec{w} = 0 \end{cases} \Rightarrow \begin{cases} \alpha\vec{w} - \nu\vec{w} + \nabla\pi = \tilde{\vec{f}} \\ \text{div}\, \vec{w} = 0 \end{cases}$$

STEADY STOKES PROBLEM (time t^{k+1})

Very often for different reason I endup in the STEADY STOKES PROBLEM.

If this problem don't have a solution I don't know how to solve the general problem

Assume that $\partial\Omega = \Gamma_D$ $(\Gamma_N = \emptyset)$ (all bounderies is a Dirichlet baundary)

The pressure will not be unique. Every traslation of the pressure is still a solution.

In the fully D problem p is not unique.

If p is a solution $\Rightarrow$ $p+c$ is a solution $\forall c \in \mathbb{R}$

$\Gamma_N \neq \emptyset$ in this cas p can be unique

LEZIONE 8 04/12/2024

NS EQS

$$\partial_t \vec{u} - \mu \Delta \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \nabla \phi = \vec{f} \qquad \vec{x} \in \Omega, \quad 0 < t < T \quad * \vec{v}$$

$$\text{div}\, \vec{u} = 0 \qquad \text{"} \qquad \text{"} \qquad * q$$

$$(N) \quad \mu \frac{\partial \vec{u}}{\partial n} - p\vec{n} = \vec{\varphi} \qquad \vec{x} \in \Gamma_N \times (0,T)$$

$$(D) \quad \vec{u} = \vec{g} \qquad \vec{x} \in \Gamma_D \times (0,T)$$

$$\vec{u} = \vec{u}_0 \qquad x \in \Omega, t = 0$$

$\Rightarrow$ Weak formulation

$$\int_\Omega \partial_t \vec{u}\, \vec{v}\, d\vec{x} + \int_\Omega \mu \nabla \vec{u} \cdot \nabla \vec{v}\, d\vec{x} + \int_\Omega (\vec{u} \cdot \nabla)\vec{u}\, \vec{v}\, dx - \int_\Omega p\, \text{div}\, \vec{v}\, dx$$

$$\boxed{- \int_{\partial\Omega} \mu \frac{\partial \vec{u}}{\partial n} \vec{v}\, d\gamma + \int_{\partial\Omega} p \vec{n} \vec{v}\, d\gamma} = \int \vec{f}\, \vec{v}\, d\vec{x} \qquad \forall \vec{v} \in V$$

$$\int_\Omega \text{div}\, \vec{u}\, q\, d\vec{x} = 0 \qquad \forall q \in Q \qquad \left(\vec{v}\big|_{\Gamma_D} = 0 ! \right)$$

$$-\int_{\Gamma_N} \underbrace{\left(\mu \frac{\partial \vec{u}}{\partial n} - p\vec{n}\right)}_{\vec{\psi}} \vec{v}\, d\gamma \longrightarrow \text{This move to rhs}$$

My weak equation become

(NS-W)

$\forall t \in (0,T)$

$$\begin{cases} \int_\Omega \partial_t \vec{u}\,\vec{v} + \int_\Omega \mu \nabla\vec{u} \cdot \nabla\vec{v} + \int_\Omega (\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v} - \int_\Omega p \,\mathrm{div}\, \vec{v} = \int_\Omega \vec{f}\cdot\vec{v} + \int_{\Gamma_N} \vec{\psi}\,\vec{v} \qquad \forall \vec{v} \in V \\ \int \mathrm{div}\,\vec{u}\, q = 0 \qquad \forall q \in Q \\ \vec{u}\big|_{t=0} = \vec{u}_0 \end{cases}$$

The gradient of $\nabla\vec{u}\cdot\nabla\vec{v} = \sum_{i,j} \frac{\partial u_i}{\partial x_j} \frac{\partial v_i}{\partial x_j}$

How to chose V, Q?

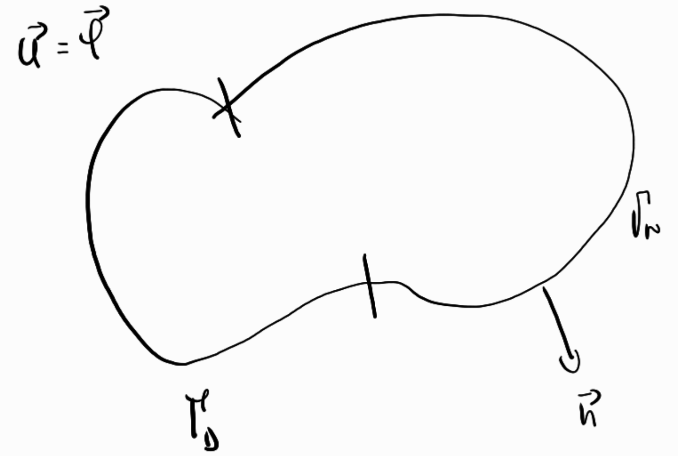
$$V = \left\{ \vec{v} \in [H^1(\Omega)]^2 ,\ \vec{v}\big|_{\Gamma_D} = \vec{0} \right\}$$

$Q = L^2_0(\Omega)$ if $\Gamma_D = \partial\Omega$, $L^2_0(\Omega) = \left\{ q \in L^2(\Omega) \, , \int_\Omega q = 0 \right\}$

$\downarrow$

(Space of the pressure)

Otherwise $Q = L^2(\Omega)$



$\mu \frac{\partial \vec{u}}{\partial n} - p\vec{n} = \vec{\psi}$

1st case in homogeneous (D) condition : $\vec{\varphi} = \vec{0}$

$? \ \vec{u}(t) \in V$, $p(t) \in Q$:

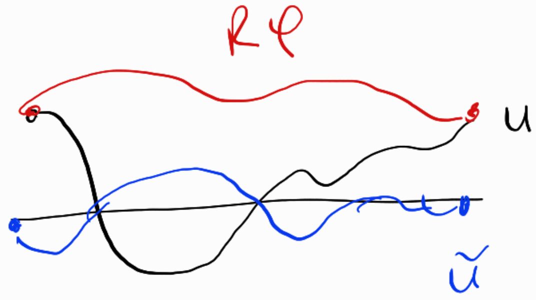
(NS-W)

$\forall t \in (0,T)$

$$\begin{cases} \int_\Omega \partial_t \vec{u}\, \vec{v} + \int_\Omega \mu \nabla\vec{u} \cdot \nabla\vec{v} + \int_\Omega (\vec{u}\cdot\nabla)\vec{u}\cdot\vec{v} - \int_\Omega p \, \mathrm{div}\, \vec{v} = \int_\Omega \vec{f}\cdot\vec{v} + \int_{\Gamma_N} \vec{\psi}\, \vec{v} \quad \forall \vec{v} \in V \\ \int_\Omega \mathrm{div}\, \vec{u}\, q = 0 \quad \forall q \in Q \\ \vec{u}\big|_{t=0} = \vec{u}_0 \end{cases}$$

2nd case : non homogeneous (D) condition $\vec{\varphi} \neq \vec{0}$

$$\vec{u} = \vec{\tilde{u}} + R\vec{\varphi}$$



$$\vec{\tilde{u}} = \bar{u} - R\vec{\varphi} \in \mathcal{V}\left(\vec{\tilde{u}}\Big|_{\Gamma_D} = \vec{0}\right)$$

$\Rightarrow$ Non homogeneous (D) condition

$$\forall t \in (0,T), \ ? \ \tilde{u} \in \mathcal{V}, \ p \in Q :$$

$$\int_\Omega \partial_t \tilde{u}\, v + \int_\Omega \mu \nabla\tilde{u} \cdot \nabla v + \overbrace{\int_\Omega \left(\vec{\tilde{u}} + R\vec{\varphi}\right) \cdot \nabla(\tilde{u} + R\varphi)\, v}^{(*)} - \int_\Omega p \operatorname{div} \vec{v} =$$

$$= \int_\Omega \vec{f}\, \vec{v} + \int_{\Gamma_N} \vec{\psi} \cdot \vec{v} - \int_\Omega \partial_t R\varphi\, v - \int_\Omega \mu \nabla R\varphi \nabla v \qquad \forall \vec{v} \in \mathcal{V}$$

$$\int_\Omega \operatorname{div} \vec{\tilde{u}}\, q = -\int_\Omega \operatorname{div} R\varphi \cdot q \qquad \forall q \in Q$$

$$(*) = \int_\Omega \vec{\tilde{u}}\, \nabla \tilde{u}\, v + \int_\Omega \tilde{u} \nabla R\varphi\, v + \int_\Omega R\varphi \nabla \tilde{u}\, v + \underbrace{\int_\Omega R\varphi \nabla R\varphi\, v}_{\text{Rhs}}$$

$F \quad \vec{\varphi} = \vec{0}$

$\forall\, t \in (0,T)$? $\vec{u} \in V$, $p \in Q$:

$$(NS\text{-}w)\begin{cases} \int_\Omega \partial_t \vec{u}\,\vec{v} + \int_\Omega \mu \nabla\vec{u}\nabla\vec{v} + \int_\Omega (\vec{u}\nabla)\vec{u}\,v - \int p \,\mathrm{div}\,\vec{v} = \vec{F}(\vec{v}) & \forall\, \vec{v} \in V \\ \int_\Omega \mathrm{div}\,\vec{u}\, q = 0 & \forall\, q \in Q \\ \vec{u}\big|_{t=0} = \vec{u}_0 \end{cases}$$

$$V = \left\{ v \in \left[H^1(\Omega)\right]^{\overset{\to \text{dimension}}{d}} : \vec{v}\big|_{\Gamma_D} = \vec{0} \right\}, \quad Q = \begin{cases} L^2(\Omega) & \text{if } \operatorname{meas}(\Gamma_N) = 0 \\ L^2_0(\Omega) & \text{if } \Gamma_D = \partial\Omega \end{cases}$$

Stokes Eqs

? $\vec{u} \in V$, $p \in Q$:

$$\begin{cases} \underbrace{\int_\Omega \sigma \vec{u}\,\vec{v} + \int_\Omega \mu \nabla\vec{u}\cdot\nabla\vec{v}}_{a(\vec{u},\vec{v})} \underbrace{- \int p \,\mathrm{div}\,\vec{v}}_{b(\vec{v},p)} = \int_\Omega \vec{f}\,\vec{v} + \int_{\Gamma_N} \vec{\varphi}\,\vec{v}\, d\gamma & \forall\, \vec{v} \in V \\ \boxed{\int_\Omega \mathrm{div}\,\vec{u}\, q} = 0 & \forall\, q \in Q \end{cases}$$

I can remove that in some case

$-b(\vec{u},q) \to$ I can change notation because it's equal to 0 $\quad +b(\vec{u},q)$

So (STOKES-w) weak formulation

Find $\vec{u} \in V$, $p \in Q$:

$$\begin{cases} a(\vec{u},\vec{v}) + b(\vec{v},p) = F(\vec{v}) & \forall\, \vec{v} \in V \\ b(\vec{u},q) = 0 & \forall\, q \in Q \end{cases}$$

Two unknowns and 2 equation

$\vec{u}$ is a velocity in L^1

p is in L^2

$$v = u \Rightarrow \begin{cases} a(u,u) + b(u,p) = F(u) \\ b(u,p) = 0 \end{cases} \quad q = p \rightarrow$$

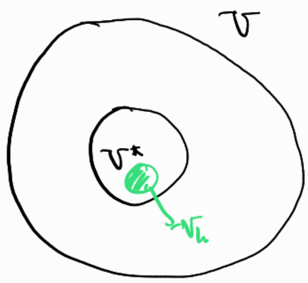
coercivity
$\leadsto$

$$\alpha \|u\|_V^{\cancel{2}} \le a(u,u) = F(u) \le \|F\|_* \cancel{\|u\|_V}$$

$$\|u\|_V \le \frac{1}{\alpha} \|F\|_*$$

$$(\text{Stokes-w}) \Rightarrow u \in V^* = \{ v \in V : b(\vec{v}, q) = 0 \quad \forall q \in Q \}$$

"$\operatorname{div} \vec{v} = 0$"



$$\Rightarrow \vec{u} \in V^* : a(u,v) + \underbrace{\boxed{b(v,q)}}_{=0} = F(v) \quad \forall v \in V^*$$

$$\Rightarrow \boxed{\vec{u} \in V^* : a(\vec{u}, \vec{v}) = F(\vec{v}) \quad \forall v \in V^*}$$ THIS IS MY PROBLEM

(I should find subspace and this construction are not easy) (V_h)

$V_h \subset V \qquad \dim(V_h) = N_h < \infty$

$Q_h \subset Q \qquad \dim(Q_h) = M_h < \infty$

$$(\text{Stokes-w})_h : \begin{cases} \text{Find } \vec{u}_h \in V_h, \; p_h \in Q_h : \\ a(u_h, v_h) + b(v_h, p_h) = F(v_h) \quad \forall v_h \in V_h \\ b(u_h, q_h) = 0 \quad \forall q_h \in Q_h \end{cases}$$

We should obtain a Linear system

We need to find basis functions.

$\{\vec{\varphi}_j\}$ basis functions for $V_h \qquad \vec{u}_h(\vec{x}) = \sum_j u_j \vec{\varphi}_j(\vec{x})$

$\{\psi_k\}$ " " " $Q_h \qquad p_h(\vec{x}) = \sum_k p_k \psi_k(\vec{x})$

Pick: $\vec{v}_h = \vec{\varphi}_i$, $q_h = \psi_m$

$$\Rightarrow \begin{cases} a\left(\sum_j \vec{u}_j \vec{\varphi}_j, \vec{\varphi}_i\right) + b\left(\vec{\varphi}_i, \sum_k p_k \psi_k\right) = F(\vec{\varphi}_i) & \forall \vec{\varphi}_i \\ b\left(\sum_j \vec{u}_j \vec{\varphi}_j, \psi_m\right) = 0 & \forall \psi_m \end{cases}$$

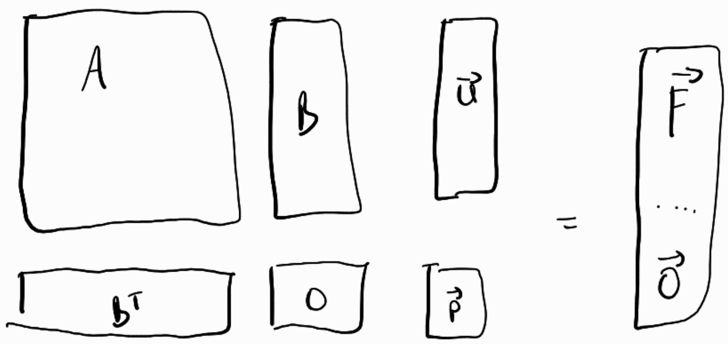
$$\Rightarrow \begin{cases} \sum_j \vec{u}_j \underbrace{\boxed{a(\vec{\varphi}_j, \vec{\varphi}_i)}}_{a_{ij}} + \sum_k p_k \underbrace{\boxed{b(\vec{\varphi}_i, \psi_k)}}_{b_{ik}} = \underbrace{\boxed{F(\vec{\varphi}_i)}}_{F_i} & \forall \vec{\varphi}_i \\ \sum_j \vec{u}_j \boxed{b(\vec{\varphi}_j, \psi_m)} = 0 \end{cases}$$

$b_{jm} = (b_{mj})^T$

B = rectangular matrix

$$\Rightarrow \begin{cases} A\vec{u} + B\vec{p} = \vec{F} \\ B^T \vec{u} = \vec{0} \end{cases}$$

$$\begin{array}{cc} & \begin{array}{cc} N_h & M_h \end{array} \\ \begin{array}{c} N_h \\ M_h \end{array} & \left[\begin{array}{c|c} A & B \\ \hline B^T & 0 \end{array}\right] \end{array} \begin{bmatrix} \vec{u} \\ \vec{p} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ \vec{0} \end{bmatrix}$$



Is this system non singular? NOT ALWAYS. We have to make assumption.

ANALYSIS OF $(\text{STOKES}-\omega)_h$

① Assume $a(\cdot,\cdot)$ continuous and coercive on $V_h \times V_h$

② Assume $F \in V'$ (dual of V)

LBB: Ladyzenskaija-Brezzi-Babiska
Inf-sup condition $\rightarrow$ Compatibility

$\Longleftrightarrow \exists \beta_h > 0 : \inf_{q_h \in Q_h} \sup_{v_h \in V_h} \dfrac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geq \beta_h$

③ Assume $\boxed{\exists\, \beta_h > 0 : \forall q_h \in Q_h, \quad \exists\, \vec{v}_h \in V_h : \quad b(\vec{v}_h, q_h) \geq \beta_h \|v_h\|_V \|q_h\|_Q}$

Then there exists unique solution of $(\text{STOKES}-\omega)_h$

LEZIONE 9 10/12/2024

$\boxed{\text{STOKES PROBLEM}}$

$$\begin{cases} \sigma\vec{u} - \mu\Delta\vec{u} + \nabla p = \vec{f} & \text{in } \Omega \subset \mathbb{R}^d \quad d = 2,3 \\ \operatorname{div}\vec{u} = 0 \end{cases}$$

$+ \text{BC} \ (\vec{u}|_{\partial\Omega} = 0)$

$$a(u,v) = \int_\Omega \sigma u v + \int_\Omega \mu \nabla u \nabla v$$

$?\ \vec{u} \in V,\ p \in Q:$

$$\begin{cases} a(\vec{u},\vec{v}) + b(\vec{v},p) = \vec{F}(\vec{v}) & \forall \vec{v} \in V \\ b(\vec{u},q) = 0 & \forall q \in Q \end{cases}$$

$$b(v,q) = -\int_\Omega q \operatorname{div} v$$

$$\vec{F}(v) = \int_\Omega f v + \int_{\Gamma_N} \varphi v$$

$$V = \{\vec{v} \in [H^1(\Omega)]^d,\ \vec{v}|_{\Gamma_D} = 0\}$$

$$Q = \begin{cases} L^2_0(\Omega) & \text{if } \partial\Omega = \Gamma_D \\ L^2(\Omega) & \text{otherwise} \end{cases}$$

Approximation

$?\ u_h \in V_h,\ p_h \in Q_h$

$$(S)_h \begin{cases} a(u_h,v_h) + b(v_h,p_h) = F(v_h) & \forall v_h \in V_h \\ b(u_h,q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

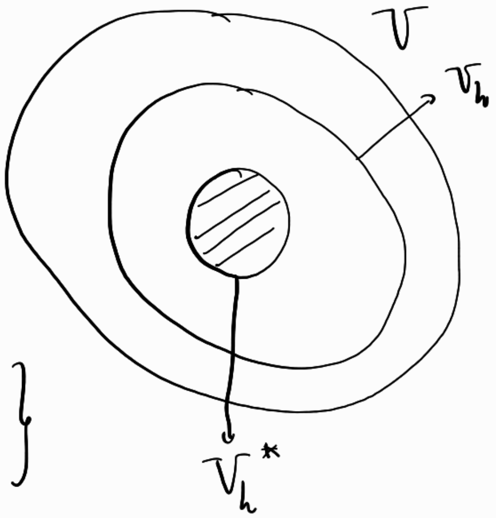
<u>Analysis</u>

1. $a: V \times V \to \mathbb{R}$ continuous and coercive
2. $b: V \times Q \to \mathbb{R}$ should satisfy the LBB condition

$$\exists\ \beta > 0: \ \forall q_h \in Q_h \ \exists\ v_h \in V_h : b(v_h,q_h) \geq \beta \|v_h\|_V \|q_h\|_Q$$

Remark

① Is sufficient not necessary



The necessary condition:

$$
①_h \begin{cases} \exists\ \alpha_h > 0 : \quad a(v_h, v_h) \geq \alpha_h \|v_h\|^2_{\mathcal{V}} \\ \forall\ v_h \in V_h^* = \left\{ v_h \in \mathcal{V}_h : b(v_h, q_h) = 0 \quad \forall q_h \in Q_h \right\} \end{cases}
$$

$\left(\Longleftrightarrow \text{"} \operatorname{div} v_h = 0 \text{"} \right)$

$① \Longrightarrow ①_h$

$$
② \Longleftrightarrow \exists\ \beta > 0 : \inf_{\substack{q_h \in Q_h \\ q_h \neq 0}} \sup_{v_h \in \mathcal{V}_h} \frac{b(v_h, q_h)}{\|v\|_{\mathcal{V}} \|q_h\|_Q} \geq \beta
$$

THEOREM

If ① + ② satisfied, then $\exists!$ solution of $(S)_h$ $\rightarrow$ stokes, and

BOUNDEDNESS $\Rightarrow$ STABILITY

$$
③ \|u_h\|_{\mathcal{V}} \leq C(\alpha_h, \beta_h) \|F\|_{\mathcal{V}}, \quad ④ \|p_h\|_Q \leq \bar{C}(\alpha_h, \beta_h) \|F\|_{\mathcal{V}}
$$

CONVERGENCE:

$$
⑤ \|u - u_h\|_{\mathcal{V}} \leq C_1 \inf_{v_h \in \mathcal{V}_h} \|u - u_h\|_{\mathcal{V}} + C_2 \inf_{q \in Q_h} \|p - q_h\|_Q
$$

$$
⑥ \|p - p_h\|_Q \leq \quad '' \qquad\qquad ''
$$

$(S_h) \Rightarrow$ linear algebraic system:

basis for Q_h : $\{\psi_m\}_{m=1}^{M}$ $\Longrightarrow$ $p_h(x) = \sum_m \boxed{p_m} \psi_m(x)$

It's not a vector

basis for V_h : $\{\vec{\varphi}_k\}_{k=1}^{N_h}$ $\Longrightarrow$ $\vec{u}_h(x) = \sum_k \boxed{u_k} \vec{\varphi}_k(x)$

$$\vec{\varphi}_1 = \begin{pmatrix} \varphi_1^1 \\ 0 \\ 0 \end{pmatrix} \qquad \vec{\varphi}_2 = \begin{pmatrix} 0 \\ \varphi_2^2 \\ 0 \end{pmatrix} \qquad \vec{\varphi}_3 = \begin{pmatrix} 0 \\ 0 \\ \varphi_3^1 \end{pmatrix}$$

$$(S_h) : \begin{cases} a\left(\sum_k u_k \vec{\varphi}_k, \vec{\varphi}_i\right) + b\left(\vec{\varphi}_i, \sum_m p_m \psi_m\right) = F(\vec{\varphi}_i) & \forall i = 1, \dots, N_h \\ b\left(\sum_k u_k \vec{\varphi}_k, \psi_i\right) = 0 & \forall J = 1, \dots, M_h \end{cases}$$

$A = (a_{im})$ $\qquad$ $(F_i) = \vec{F}$

$$\begin{cases} \sum_k u_k \boxed{a(\vec{\varphi}_k, \vec{\varphi}_i)} + \sum_m p_m \boxed{b(\vec{\varphi}_i, \psi_m)} = \boxed{\vec{F}(\varphi_i)} & \forall i \\ \sum_k u \boxed{b(\vec{\varphi}_k, \psi_J)} = 0 & \forall J = 1, \dots, M_h \end{cases}$$

(b_{im})

$(b_{kJ}) = B$

$$\Rightarrow \begin{cases} A\vec{U} + B^T \vec{P} = \vec{F} \\ B\vec{U} = 0 \end{cases}$$

We use a different notation last time but this is more correct & we use this one

$$\begin{array}{c c} & \begin{array}{cc} N_h & M_h \end{array} \\ \begin{array}{c} N_h \\ M_h \end{array} & \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \end{array} \begin{bmatrix} \vec{U} \\ \vec{P} \end{bmatrix} = \begin{bmatrix} \vec{F} \\ 0 \end{bmatrix}$$

A and 0 square $\qquad$ B and B^T rectangular

$$\vec{U} = \begin{bmatrix} u_1 \\ \vdots \\ u_{N_h} \end{bmatrix} \qquad \vec{p} = \begin{bmatrix} p_1 \\ \vdots \\ p_{M_h} \end{bmatrix}$$

$$S = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$$

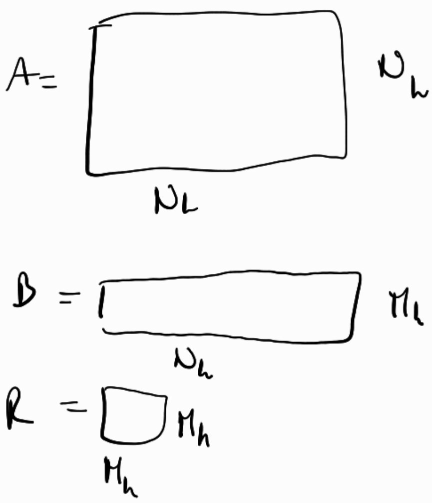
preview: S non singular iff ① + ② are satisfied

Proof: $\begin{cases} A\vec{U} + B^T\vec{p} = \vec{F} \\ B\vec{U} = \vec{0} \end{cases}$ ① $\Rightarrow A$ is non singular

$$\Rightarrow \begin{cases} \vec{U} = A^{-1}\left[\vec{F} - B^T\vec{p}\right] \\ \vec{0} = B\vec{U} = BA^{-1}F - BA^{-1}B^T\vec{p} \end{cases}$$

$$\Rightarrow \underbrace{(BA^{-1}B^T)}_{\mathbb{R}^{M_h\times M_h} \ni R}\underbrace{\vec{p}}_{\in \mathbb{R}^{M_h}} = BA^{-1}\vec{F} = \vec{c} \in \mathbb{R}^{M_h}$$

$$\Rightarrow \boxed{R\vec{p} = \vec{c}}$$



if R is non-sing $\Rightarrow \exists!$ solution

Non singularity of $R \iff$ Uniqueness of $\vec{p}$

$\iff$ $\boxed{\text{the only solution of } R\vec{p}^* = \vec{0} \text{ is } \vec{p}^* = 0}$

(*)

If we prove (*) I demonstrate the fucking theorem

Proof

Scalar product

$$R\vec{p}^{\,*} = \vec{0} \iff (BA^{-1}B^T)\,\vec{p}^{\,*} = \vec{0} \iff (BA^{-1}B^T p^*, q) = 0 \quad \forall \vec{q} \in \mathbb{R}^{M_h}$$

Move it here for the property of scalar product

$$\iff (A^{-1}B^T p^*, B^T \vec{q}) = 0$$

$$\vec{q} = \vec{p}^{\,*} \Rightarrow (\underbrace{A^{-1}B^T p^*}_{\vec{w}}, \underbrace{B^T p^*}_{\vec{w}}) = 0$$

Note: in the handwriting the first brace covers only $B^T p^*$, not $A^{-1}B^T p^*$.

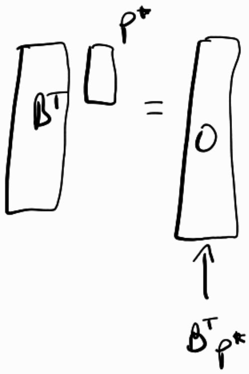
$$\Rightarrow (A^{-1}\vec{w}, \vec{w}) = 0$$

(Remember

$a(\cdot,\cdot)$ coercive $\Rightarrow$ A positive definite $\Rightarrow A^{-1}$ p.d.

$\Rightarrow (A^{-1}\vec{v}, \vec{v}) > 0, \quad (A^{-1}\vec{v}, \vec{v}) = 0 \Rightarrow \vec{v} = \vec{0}$)

$\Rightarrow \vec{w} = 0 \quad \Rightarrow B^T\vec{p}^{\,*} = \vec{0} \quad \overset{?}{\Rightarrow} \quad \vec{p}^{\,*} = \vec{0}\;?$



if and only if

It's true $\boxed{\text{iff } B^T \text{ has full rank } \in M_h}$

$\iff \boxed{\operatorname{Ker} B^T = 0} \underset{(**)}{\iff}$ LBB is verified

D

Proof of (**)

$(**) \iff \{$ if LBB violated $\Rightarrow \operatorname{Ker} B^T \neq \vec{0} \}$

$\Downarrow$

$$\forall \beta_h > 0 \;\; \exists \; p_h^* \in Q_h \quad \forall v_h \in V_h \quad \frac{b(v_h, p_h^*)}{\|v_h\|_V \|p_h^*\|_Q} < \beta_h$$

Take $-v_h$:

δ real number

$$\frac{b(-v_h, p_h^*)}{\|v_h\|_V \, \|p_h^*\|_Q} = \frac{-b(v_h, p_h^*)}{\|v_h\|_V \, \|p_h^*\|_Q} < \beta_h$$ → is arbitrary

$\Rightarrow \delta = 0$ since β_h is an arbitrary $\in \mathbb{R}^+$

$\exists p_h^* : \forall v_h \in V_h \quad b(v_h, p_h^*) = 0$

$p_h^* \in Q_h$: $\vec{p}^{\,*} = \begin{bmatrix} p_1 \\ \vdots \\ p_{N_h} \end{bmatrix}$

$\Downarrow$

$p_h^*(x) = \sum_J p_J \psi_J(x)$

$b(v_h, p_h^*) = 0 \quad \forall v_h \in V_h$

$(\vec{v}_h = \vec{\varphi}_i)$

$\Updownarrow$

$b(\vec{\varphi}_i, \sum_J p_J \psi_J) = 0 \quad \forall i$

$\sum_J p_J b(\vec{\varphi}_i, \psi_J) = 0 \iff \boxed{B^T \vec{p}^{\,*} = \vec{0}}$

Corollary

If LBB violated $\Rightarrow \boxed{\exists\, p_h^* \in Q_h : b(v_h, p_h^*) = 0 \quad \forall v_h \in V_h}$

$\iff \operatorname{Ker} B^T \neq 0$

$$\text{Ⓢ}_h : \begin{cases} a(u_h, v_h) + \underbrace{b(v_h, p_h)}_{\to + c\vec{p}_h^{\,*}} = F(v_h) & \forall v_h \in V_h \\ b(u_h, q_h) = 0 & \forall q_h \in Q_h \end{cases} \qquad \forall c \in \mathbb{R}$$

Non esiste unicità p_h^*

$b(v_h, p_h) + c\, \cancel{b(v_h, p_h^*)}$

↳ it's a ghost function

$\Rightarrow$ if $(\vec{u}_h, p_h)$ is a solution Ⓢ$_h$ $\to (\vec{u}, p_h + c\,p_h^*)$ is a solution too

All functions p_h^* that violate LBB are called

PARASITIC FUNCTIONS (parasitic modes)

Depending upon the way we choose V_h, Q_h

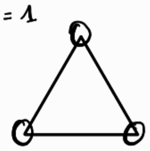
we will have LBB condition $\begin{cases} \text{VERIFIED} \rightarrow \exists! \, (u_h, p_h) \text{ of } (S)_h \\ \text{NON VERIFIED} \rightarrow \exists \text{ of parasitic pressure functions} \end{cases}$

Velocity space $V_h \subset [H^1(\Omega)]^d$; "morally", $\vec{v}_h \in [C^0(\Omega)]^d$

Pressure space $Q_h \subset L^2(\Omega)$, "morally", $q_h \in \begin{cases} C^0(\Omega) \\ \text{Discontinuous functions} \end{cases}$

<u>Let us consider FE spaces</u> $\begin{cases} \text{Triangles in 2D} \\ \text{Tetrahedrons in 3D} \end{cases}$

r=1 s=0



Velocity Pressure

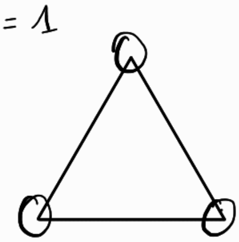
$$V_h = \{ \vec{v}_h \in [C^0(\Omega)]^d , \ \vec{v}_h|_K \in [\mathbb{P}^r]^d \ \forall K \in \mathcal{T}_h , \ \vec{v}_h|_{\Gamma_D} = 0 \}$$

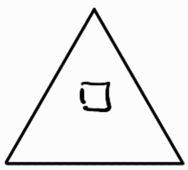
Continuous

$$Q_h^c = \{ q_h \in C^0(\Omega) : \underbrace{q_h|_K \in [\mathbb{P}^s]^d \ \forall K \in \mathcal{T}_h}_{} : \underbrace{\int_\Omega q_h = 0 \text{ if } \Gamma_D = \partial\Omega}_{} \}$$

$$Q_h^D = \{ q_h \in L^2(\Omega) : \quad \text{idem} \quad \text{idem} \}$$

r = 1 S=0





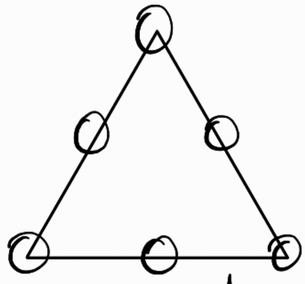
$(\mathbb{P}_1^c)^{d=2}$ $\quad$ $\mathbb{P}_0^{dis}$

NO LBB

○ $\{ v_h^1, v_h^2, v_h^3 \}$

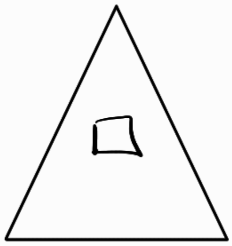
□ q_h

d= grade of liberty



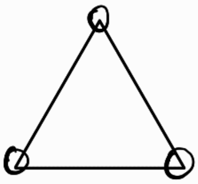
$(\mathbb{P}_2^c)^d$

$r = 2$



$\mathbb{P}_0^{dis} \to$ discontinuity

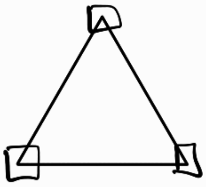
LBB



$(\mathbb{P}_1^c)^2$

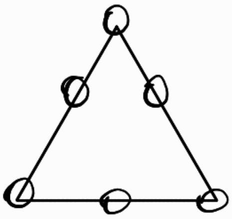
$r = 1$

NO LBB



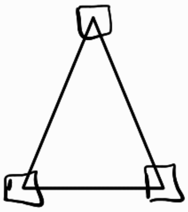
$\mathbb{P}_1^c$

$s = 1$



$(\mathbb{P}_2^c)^2$

LBB



$\mathbb{P}_1^c$

TAYLOR-HOOD ELEMENTS

$(\mathbb{P}_{k+1}^c)^2 \qquad \mathbb{P}_k^c$

$\forall k \geq 1$

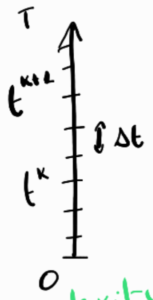
LBB

$$\|\vec{u} - \vec{u}_h\|_V + \|p - q_h\|_Q \leq C \boxed{h^{k+1}} \left(|\vec{u}|_{[H^{k+1}(\Omega)]^d} + |p|_{H^{k+1}(\Omega)} \right)$$

Type of norm defined in the previous lecture

$O(h^2)$

(NS) $\begin{cases} \partial_t u - \mu \Delta u + \nabla p + (u \cdot \nabla) u = f \\ \operatorname{div} u = 0 \end{cases}$



$$g(t) \to g^n \cong g(t^n)$$

complexity

Advance in time (NS)

[4] Fully implicit

unconditionally stable

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \mu \Delta u^{n+1} + \nabla p^{n+1} + (u^{n+1} \cdot \nabla) u^{n+1} = f^{n+1} \\ \operatorname{div} u^{n+1} = 0 \end{cases}$$

It's the problem to solve. It's always the same

[3] Linearly implicit

cond. stable

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \mu \Delta u^{n+1} + \nabla p^{n+1} + (u^n \cdot \nabla) u^n = f^{n+1} \\ \operatorname{div} u^{n+1} = 0 \end{cases}$$

$$\Rightarrow \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u^{n+1} \\ p^{n+1} \end{bmatrix} = F^{n+1}$$

$Y^n_{u^{n+1}}$

or $(u^n \cdot \nabla) u^{n+1}$

[2] Semi-implicit

cond. stable

$$\begin{cases} \dfrac{u^{n+1} - u^n}{\Delta t} - \mu \Delta u^{n+1} + \nabla p^{n+1} + \underbrace{(u^{n+1} \cdot \nabla) u^n}_{X^n \to u^{n+1}} = f^{n+1} \\ \operatorname{div} u^{n+1} = 0 \end{cases}$$

$$\begin{bmatrix} A + \begin{smallmatrix} X^n \\ Y^n \end{smallmatrix} & B^T \\ B & 0 \end{bmatrix}$$

In the end I obtain a NON LINEAR ALGEBRAIC SYSTEM in [4] and it's the worst.

The best → [3] and it's linear and INDEPENDENT FROM TIME

The [2] is linear but depend on the time

[1] Stable in time $\forall \Delta t > 0$

[2] $\nearrow X^n \quad \Delta t \le \dfrac{c}{\max [\hat{u}^n_h]}$

$\searrow Y^n \quad \Delta t \le \dfrac{c\, h}{\max [u^n_h]}$

[3] Like Y^n

NS + discretisation in time of order

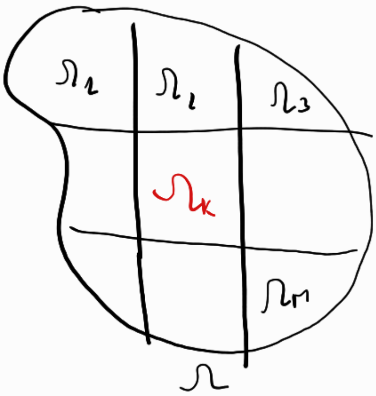
$$\|\vec{u}(t^n) - \vec{u}_h^n\|_V + \|p(t^n) - p_h^n\|_Q \leq C\left(\underbrace{\Delta t}_{\text{Time}} - \underbrace{h^{k+1}}_{\text{Space}}\right)$$

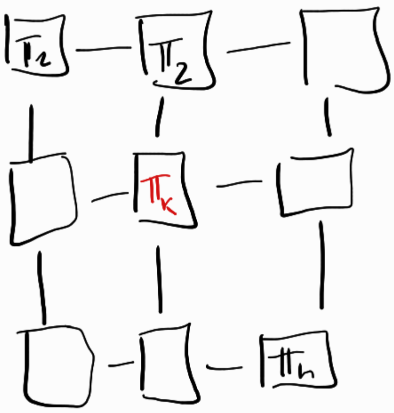
Taylor-Hood + Euler ($\boxed{1}$, $\boxed{2}$ or $\boxed{3}$)

LEZIONE 10 17/12/2024

DOMAIN DECOMPOSITION (DD)

$$(1)\begin{cases} Lu = f & \text{in } \Omega \\ + BC \end{cases}$$





$M \gg 1$ $\quad \{\Omega_k\}$ subdomains

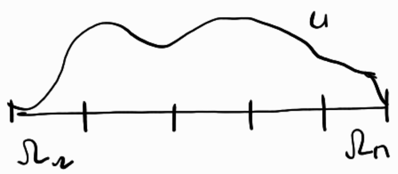
I want to resolve this problem for every subdomain

It's a global problem and we have to reduce it.

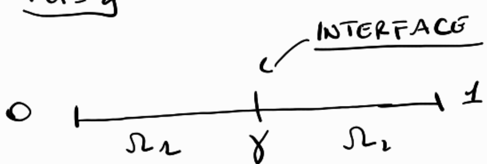
How to split the global problem (1) in Ω into a sequence of local problems in $\{\Omega_k\}$?

1D

$$(2)\begin{cases} Lu := -(\mu u')' + b u' + \sigma u = f & \text{in } \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$



$\boxed{u = ?}$



Remark

Problem (2) is equivalent to solve

$$u_i = u\Big|_{\Omega_i} \qquad (3) \begin{cases} Lu_1 = f & \text{in } \Omega_1 = (0,\gamma) \\ Lu_2 = f & \text{in } \Omega_2 = (\gamma, \gamma_2) \\ u_1 = u_2 \text{ on } \gamma, & \mu u_1' = \mu u_2' \text{ on } \gamma \end{cases}$$

INTERFACE CONDITION

also the normal derivative is continuous

I want that in the interface the solution don't brakes.

Now solve (3) has an iterative method:

DEF (DN METHOD)

Let $\lambda^{(1)}$ be give on γ, at every $k \geq 1$ we solve:

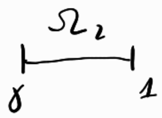
Dirichlet problem

$$(D) \begin{cases} Lu_1^{(k)} = f & \text{in } \Omega_1 \\ u_1^{(k)} = \lambda^{(k)} & \text{on } \gamma \end{cases}$$

Dirichlet-Neumann (DN) Method

Neumann

$$(N) \begin{cases} Lu_2^{(k)} = f & \text{in } \Omega_2 \\ \mu u_2^{(k)} = \mu^{(k)} u_1^{(k)} & \text{on } \Gamma \end{cases}$$



$$(\text{Relax}) \quad \lambda^{(k+1)} = \theta\, u_2^{(k)}(\gamma) + (1-\theta)\, \lambda^{(k)}$$

Example

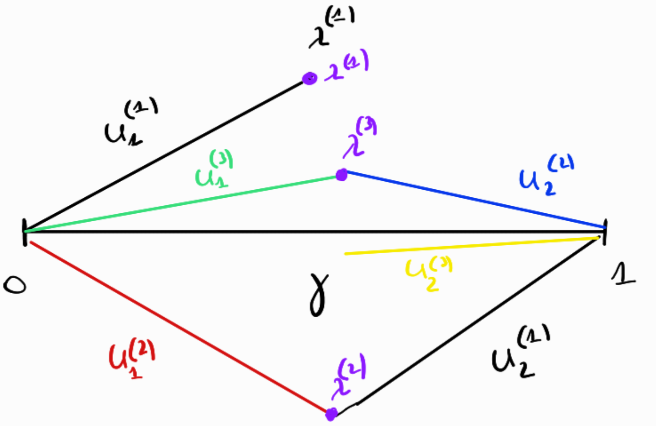
$$\begin{cases} Lu := -u'' = f & x \in \Omega = (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

Take $f = 0 \rightsquigarrow u(x) = 0$

$\theta = 1$ $\gamma > \frac{1}{2}$

$\lambda^{(1)} = 1 \cdot u_2^{(1)}(\gamma) + \emptyset$

$|\Omega_1| > |\Omega_2|$

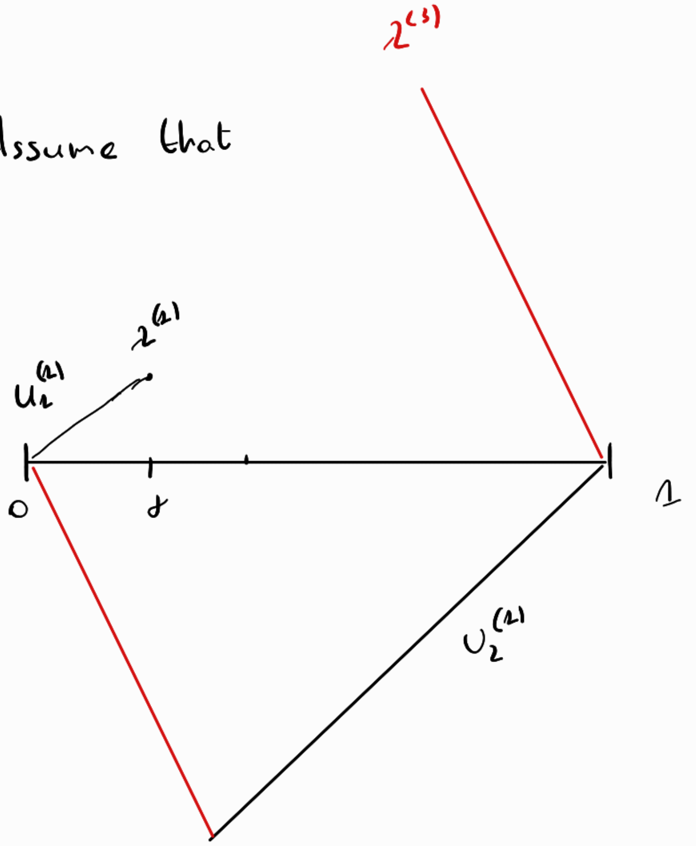


$\lambda^{(k)}$ as we want. We resolve the D problem. Then we resolve N.

$\Rightarrow |\lambda^{(k)}| \xrightarrow[k\uparrow\infty]{} 0$ $\Rightarrow u_1^{(k)} \xrightarrow[k\uparrow\infty]{} 0 = u_1$

$\Rightarrow u_2^{(k)} \xrightarrow[k\uparrow\infty]{} 0 = u_2$ CONVERGENCE !!

Assume that



At each step the line are parallel

$u_1^{(1)} // u_2^{(1)}$

$u_1^{(2)} // u_2^{(2)}$ and so on

In this way $|\lambda^{(k)}| \xrightarrow[k\uparrow\infty]{} \infty$

$\gamma < \frac{1}{2}$ DIVERGENCE !

It's θ that assure convergence

We chose θ such that:

$$0 = \lambda^{(1)} = \theta u_2^{(1)}(\gamma) + (1-\theta)\lambda^{(1)} =$$

$$= \theta(u_2^{(1)}(\gamma) - \lambda^{(1)}) + \lambda^{(1)}$$

$$\theta = \frac{-\lambda^{(1)}}{u_2^{(1)}(\gamma) - \lambda^{(1)}}$$

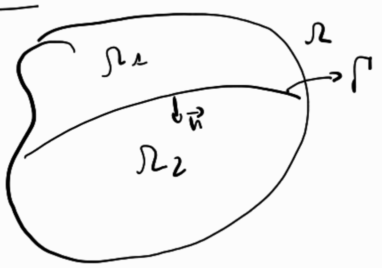
With θ properly the method converge properly

THEOREM

$\exists\ \theta_{max} > 0$: if $0 < \theta < \theta_{max}$

$\Rightarrow$ DN method convergence, $\forall\ \gamma \in (0,1)$

Case 2D, 3D



(Circle in 3D)

RK

Problem ① is equivalent to

$$Lu_1 = f \quad \text{in } \Omega_1$$

$$Lu_2 = f \quad \text{in } \Omega_2$$

$$u_1 = u_2 \text{ on } \Gamma \qquad \mu \frac{\partial u_1}{\partial n} = \mu \frac{\partial u_2}{\partial n} \text{ on } \Gamma$$

DN

Let $\lambda^{(1)}$ be given on Γ;

$\forall\ K \geq 1$

$$(D)\begin{cases} Lu_1^{(K)} = f & \text{in } \Omega_1 \\ u_1^{(K)} = \lambda^{(K)} & \text{on } \gamma \end{cases}$$

$$(N)\begin{cases} Lu_2^{(K)} = f & \text{in } \Omega_2 \\ \mu \dfrac{\partial u_2^{(K)}}{\partial n} = \mu \dfrac{\partial u_1^{(K)}}{\partial n} & \text{on } \Gamma \end{cases} \qquad \frac{\partial u_1^{(K-1)}}{\partial n}$$

$$(\text{Relax}) \quad \lambda^{(K+1)} = \theta\, u_2^{(K)}(\Gamma) + (1-\theta)\, \lambda^{(K)} \quad \text{on } \Gamma$$

(DN METHOD)

The result is the same as before. It's valid for all elliptic problem.

If we get $K-1$ we can solve the two indipendently.

This version is in parallel but is good as well as before.

SEQUENTIAL ≡ Multiplicative (DN)

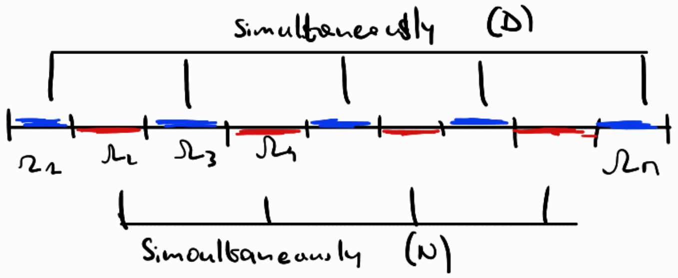
PARALLEL ≡ Additive (DN)

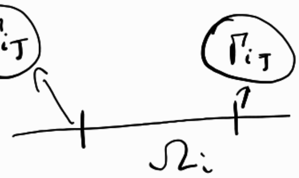
|M|>2 Many subdomains

1D

$\forall K \geq 1$

$$Lu_i^{(K)} = f \quad \text{in } \Omega_i \text{ (blue)}$$
$$Lu_J^{(K)} = f \quad \text{in } \Omega_J \text{ (red)}$$
$$u_i^{(K)} = \lambda^{(K)} \quad \text{on } \Gamma_{iJ}$$
$$\mu u_J^{(K)} = \mu u_i^{(K)} \quad \text{on } \Gamma_{iJ}$$

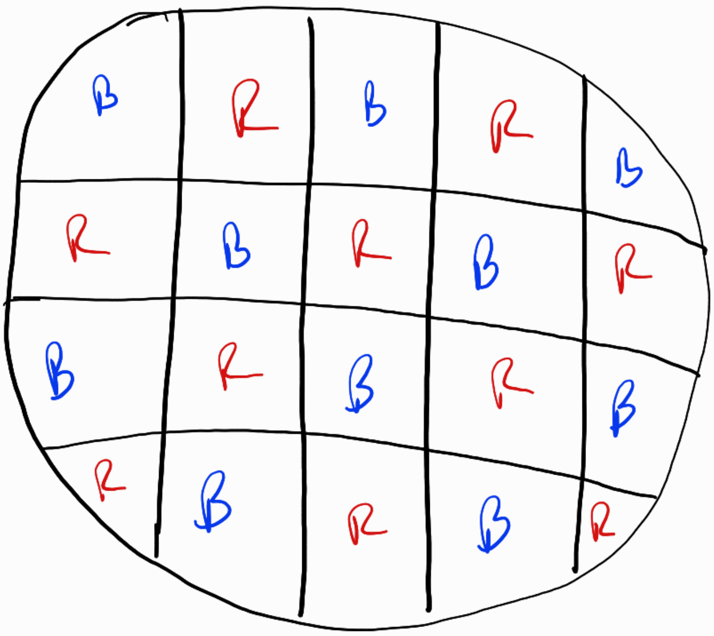




So I solve simoultaneously the blue and the red ones

2-3 D

skeleton Γ



The convergence result are the same.

The problem is that the partition is not always possible.

If the domain is different we can't gurantee the B and B are disjoint.

In 2D-3D we don't do ND decomposition, it's very difficult to setup.

We use a different decomposition

DEF

The (NN) (Neumann-Neumann) Method

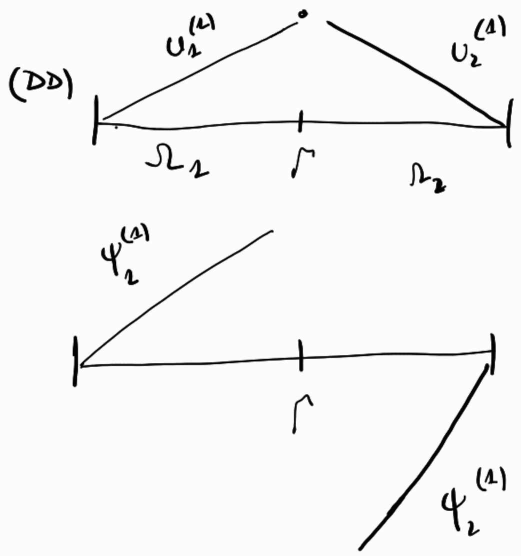
$M=2$, 1D

$\forall\ k \geq 1$

$$(DD)\begin{cases} Lu_i^{(k)} = f & \text{in } \Omega_i, \quad i=1,2 \\ u_i^{(k)} = \lambda^{(k)} & \text{on } \Gamma \end{cases}$$

$$(NN)\begin{cases} L\psi_i^{(k)} = 0 & \text{in } \Omega_i, \quad i=1,2 \\ \mu \dfrac{\partial \psi_i^{(k)}}{\partial n} = \mu \dfrac{\partial u_1^{(k)}}{\partial n} - \mu \dfrac{\partial u_2^{(k)}}{\partial n} & \text{on } \Gamma \end{cases}$$

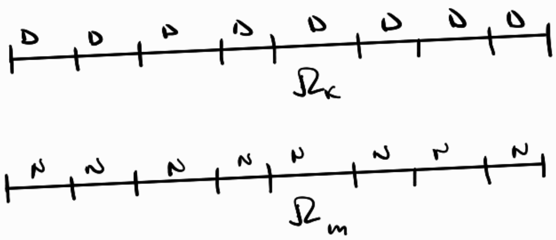
$$(Relax)\quad \lambda^{(k+1)} = \lambda^{(k)} - \theta\left(\sigma_1 \psi_1^{(k)} - \sigma_2 \psi_2^{(k)}\right)$$



σ_1 σ_2 are geometrical parameter.

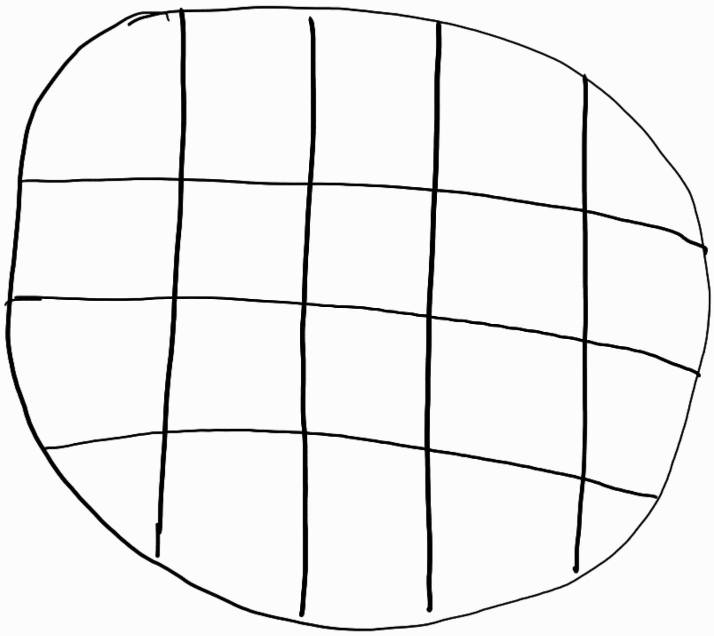
θ is as before to accellerate convergence

$M > 2$



I solve all D problem and then go on with N

3D



We solve all the D and then N and make correction.

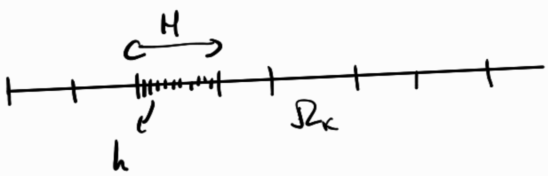
It's not as before

We do that in parallel to gain time also.

This is just on differential problem.

All of this is useless. We have FEM. We still need FEM.

These methods convergence but the convergence is slow. The convergence depends on 2 variable h and H. H is the size of the subdomain



H is for the subdomain
h is use in FEM

} These two are the velocity of convergation.

RK

Generally, convergence rate are both (DN) and (NN) will depend on h, H: $\rho = \rho(h, H)$

If you reduce H you increase number of processor and convergence will be slower.

Property

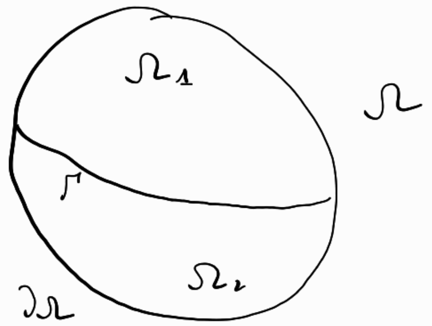
If ρ independent of h $\Rightarrow$ OPTIMALITY

If ρ indipendent of H $\rightarrow$ SCALABILITY

With 2 subdomains, if it's not the scalability we need to optimize the h.

We want to modife our methods to improve all OPTMALITY and SCALABILITY.

We need to work on the interface to improve our goals.



How to transform

$$(1) \begin{cases} Lu = f & \text{in } \Omega \\ + BCs & \text{on } \partial\Omega \end{cases}$$

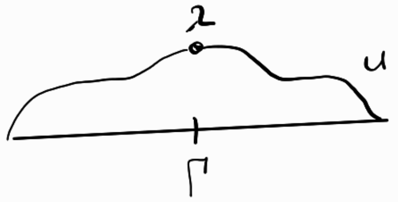
Into a problem $S\lambda = X$

Let $\lambda = u|_\Gamma$ (unknown!)

If I knew λ, then

(3) $\begin{cases} Lw_1 = f & \text{in } \Omega_1 \\ w_1 = \lambda & \text{on } \Gamma \end{cases}$ $w_1 = u|_{\Omega_1}$

$\begin{cases} Lw_2 = f & \text{in } \Omega_2 \\ w_2 = \lambda & \text{on } \Gamma \end{cases}$ $w_2 = u|_{\Omega_2}$



How to find λ?

We set (5) $\boxed{w_i = w_i^* + \mathring{u}_i}$ $i = 1, 2$

where w_i^* solve:

$\begin{cases} Lw_i^* = f & \text{in } \Omega_i \\ w_i^* = 0 & \text{on } \Gamma \end{cases} \Rightarrow w_i^* = T_i f$

Resolve that get us the solution of our problem.

Notation

$\begin{cases} L\mathring{u}_i = 0 & \text{in } \Omega_i \\ \mathring{u}_i = \lambda & \text{on } \Gamma \end{cases} \Rightarrow \mathring{u}_i \equiv H_i \lambda$

Remember

(1) $\Leftrightarrow$ (2) $\begin{cases} Lu_1 = f & \text{in } \Omega_1 \\ u_1 = u_2 & \text{on } \Gamma \\ Lu_2 = f & \text{in } \Omega_2 \\ \mu \frac{\partial u_2}{\partial n} = \mu \frac{\partial u_1}{\partial n} & \text{on } \Gamma \end{cases}$

We enforce another condition so in this way w_1 and w_2 will be the solution of the original problem (1)

We pretend that w_1 and w_2 satisfy. (4) $\mu \frac{\partial w_1}{\partial n} = \mu \frac{\partial w_2}{\partial n}$

If ④ is satisfied $\Longrightarrow$ $w_1 = u_1$, $w_2 = u_2$ $\longrightarrow$ $\boxed{\lambda = u|_\Gamma}$

$$④ \overset{}{\underset{⑤}{\Longleftrightarrow}} \mu\left(\frac{\partial w_1^*}{\partial n} + \frac{\partial \mathring{u}_1}{\partial n}\right) = \mu\left(\frac{\partial w_2^*}{\partial n} + \frac{\partial \mathring{u}_2}{\partial n}\right)$$

$$\Longleftrightarrow \mu\left(\frac{\partial \mathring{u}_1}{\partial n} - \frac{\partial \mathring{u}_2}{\partial n}\right) = \mu\left(\frac{\partial w_2^*}{\partial n} - \frac{\partial w_1^*}{\partial n}\right)$$

substitute

$$\Longleftrightarrow \mu \frac{\partial}{\partial n}\left(H_1\lambda - H_2\lambda\right) = \mu \frac{\partial}{\partial n}\left(T_2 f - T_1 f\right)$$

$$\boxed{\mu \frac{\partial}{\partial n}\left(H_1 - H_2\right)}\lambda = \boxed{\mu \frac{\partial}{\partial n}\left(T_2 - T_1\right)} f$$

$$S \cdot \lambda = \underbrace{X(f)}_{\text{given}}$$

INTERFACE PROBLEM

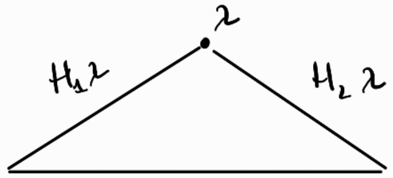
$\Longrightarrow$ $\boxed{S\lambda = X(f)}$ on Γ

$$\Updownarrow$$

$\boxed{\lambda = u|_\Gamma}$ $\longrightarrow$ Solving this problem is the same of the global problem

EX 1D $\quad Lu = -D^2 u$

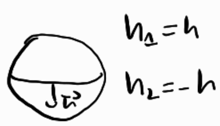
Action of S



$$S\lambda = (H_1\lambda)' - (H_2\lambda)'\Big|_\gamma$$

$$S\lambda = \underbrace{\mu\frac{\partial}{\partial n} H_1\lambda}_{S_1\lambda} \underset{+}{-} \underbrace{\mu\frac{\partial}{\partial n} H_2\lambda}_{S_2\lambda}$$

We need to considerate the -



$H_i\lambda$ = Harmonic Extension of λ in Ω_i

S = Steklov-Poincaré operator

$S_i \lambda = \mu \frac{\partial}{\partial n_i} H_i \lambda$

Theorem

Both (DN) and (NN) methods can be re-interpolated as ITERATING METHODS for the solution of the SP problem on Γ.

Precisely

One-step of (DN) corresponds to:

$$\boxed{S_2 \left(\lambda^{(k+1)} - \lambda^{(k)} \right) = \theta \left(X - S\lambda^{(k)} \right)}$$

One-step of (NN) corresponds to:

$$\boxed{\left(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1} \right) \left(\lambda^{(k+1)} - \lambda^{(k)} \right) = \theta \left(X - S\lambda^{(k)} \right)}$$

In compact form

$$\boxed{\underline{P} \left(\lambda^{(k+1)} - \lambda^{(k)} \right) = \theta \left(X - S\lambda^{(k)} \right) \qquad k \geq 0}$$

where:

$$\underline{P} = \begin{cases} S_2 & \text{for (DN)} \\ \left(\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1} \right)^{-1} & \text{for (NN)} \end{cases}$$

We can solve problem using these preconditioner

So we have seen

$A\vec{x} = \vec{b}$

Preconditioner

Richardson $\quad P \left(\vec{x}^{(k+1)} - x^{(k)} \right) = \theta \left(\vec{b} - A\vec{x}^{(k)} \right)$

Remark

I know to solve the problem in the INTERFACE. The matrix is small.

I never have constract S, I only know what it does.

I know what S does.

A good preconditioner should be A.

$$S = S_1 + S_2$$ the inverse is $S^{-1} = (S_1 + S_2)^{-1} =$

$\neq S_1^{-1} + S_2^{-1}$ it's a good approximation

So we use $S^{-1} = S_2$ in (DN)

$$\mathcal{S}^{-1} = \sigma_2 S_1^{-1} + \sigma_2 S_2^{-1}$$ that is a good approximation

I know that in Richardson if A, P are SPD, $\exists\ \theta \in (0, \theta_{max})$

$$\| \vec{x}^{(k+1)} - \vec{x} \|_A \simeq \rho^k \| \vec{x}^{(1)} - \vec{x} \|_A$$

$$\rho \sim \frac{\text{cond}(P^{-1}A) - 1}{\text{cond}(P^{-1}A) + 1}$$

$$\|\lambda^{(k+1)} - \lambda\|_S \simeq \rho^k \|\lambda^{(1)} - \lambda\|_S, \quad \rho \sim \frac{\text{cond}(P^{-1}S) - 1}{\text{cond}(P^{-1}S) + 1}$$

This is very formal but we can rewrite it as Matrices.

LEZIONE 11 (18/12/2024)

(DN) iterations

(NN) iterations

Preconditioned Richardson iterations

$$P(\lambda^{(k+1)} - \lambda^{(k)}) = \theta(\chi - S\lambda^{(k)})$$

$$P = \begin{cases} P_{DN} \\ P_{NN} \end{cases}$$

$$(P) \begin{cases} Lu = f \\ + BC \end{cases} \to (PW) \; ?\, u \in V : a(u,v) = F(v) \quad \forall v \in V$$

$$?\, \lambda \in \Lambda : \underbrace{s(\lambda,\mu)}_{{}_{\Lambda'}\langle S\lambda, \mu\rangle_{\Lambda}} = \langle \chi, \mu \rangle \quad \forall \mu \in \Lambda$$

$$\lambda = u|_\Gamma$$

$$S\lambda = \chi$$

$$(FE)\; ?\, u_h \in V_h : a(u_h, v_h) = F(v_h) \to (ALG)\; A\vec{u} = \vec{f} \xrightarrow{\vec{\lambda} = \vec{u}|_\Gamma} \Sigma_1 \vec{\lambda} = \vec{\chi}$$

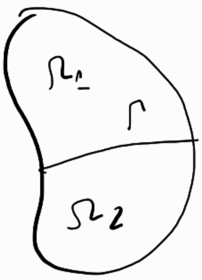
$$\lambda_h = u_h|_\Gamma$$

$$\Rightarrow P(\vec{\lambda}^{(k+1)} - \lambda^{(k)}) = \theta(\vec{\chi} - \Sigma_1 \vec{\lambda}^{(k)})$$

$$?\, \lambda_h \in \Lambda_h : s(\lambda_h, \mu_h) = \chi_h(\mu_h) \quad \forall \mu_h \in \Lambda_h$$

It's important to understand this scheme

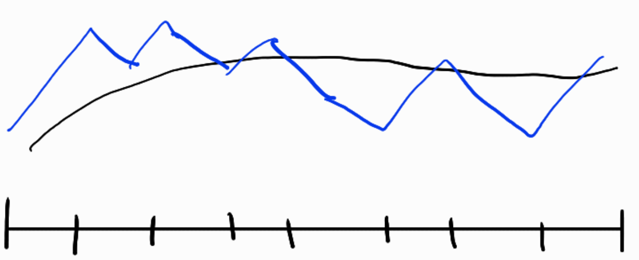
We reinterpretate (PW) using $S\lambda = \chi$



$\lambda = u|_\Gamma$ interface

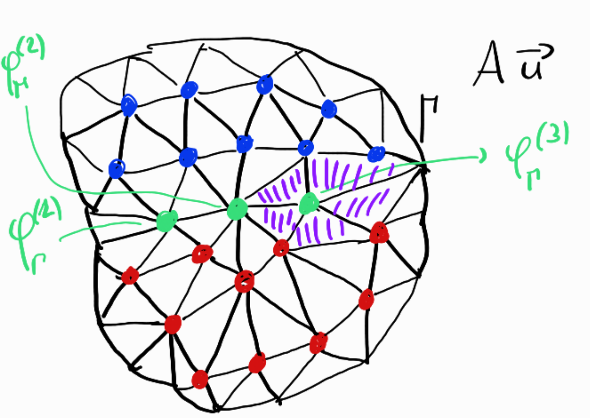
$V = H^1_{\Gamma_D}(\Omega)$

$$\Lambda = \{ \mu = v|_{\Gamma} \ , v \in V \}$$ it's the space of the traces.



$\mu_h \in \Lambda_h$

It's important to get $\sum_f \vec{\lambda} = \vec{x}$ to solve the problem. I need the action not the origin.



$$A \vec{u} = \vec{f}$$

Points of interest

internal Nodes in Ω_1

internal Nodes in Ω_2

//// Where $\varphi_\Gamma^{(3)}$ is not 0

Force the continous of the solution

$$\begin{bmatrix} \boxed{A_{11}} & \overset{0}{\boxed{A_{12}}} & A_{1\Gamma} \\ \underset{0}{\boxed{A_{21}}} & \boxed{A_{22}} & A_{2\Gamma} \\ A_{\Gamma 1} & A_{\Gamma 2} & \boxed{A_{\Gamma\Gamma}} \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_\Gamma \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_2 \\ \vec{f}_\Gamma \end{bmatrix}$$

$$\longrightarrow \begin{cases} \vec{u}_1 = A_{11}^{-1} (\vec{f}_1 - A_{12} \vec{u}_2 - A_{1\Gamma} \vec{u}_\Gamma) \\ \vec{u}_2 = A_{22}^{-1} (\vec{f}_2 - A_{21} \vec{u}_1 - A_{2\Gamma} \vec{u}_\Gamma) \\ \vec{u}_\Gamma = A_{\Gamma\Gamma}^{-1} (\vec{f}_\Gamma - A_{\Gamma 1} \vec{u}_1 - A_{\Gamma 2} \vec{u}_2) \end{cases}$$

So

$$\vec{u}_\Gamma = A_{\Gamma\Gamma}^{-1} \vec{f}_\Gamma - A_{\Gamma\Gamma}^{-1} A_{\Gamma 1} \vec{u}_1 - A_{\Gamma\Gamma}^{-1} A_{\Gamma 2} \vec{u}_2$$

The yellow Matrix are zeros

$A_{12} = a(\varphi_J^{(2)}, \varphi_i^{(1)})$ There is no intersection.

No communication between Ω_1 and Ω_2

$$\Rightarrow \begin{cases} \vec{u}_1 = A_{11}^{-1}(\vec{f}_1 - A_{1\Gamma}\vec{u}_\Gamma) \\ \vec{u}_2 = A_{22}^{-1}(\vec{f}_2 - A_{2\Gamma}\vec{u}_\Gamma) \\ \vec{u}_\Gamma = A_{\Gamma\Gamma}^{-1}\vec{f}_\Gamma - A_{\Gamma\Gamma}^{-1}A_{\Gamma 1}A_{11}^{-1}\vec{f}_1 + A_{\Gamma\Gamma}^{-1}A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma}\vec{u}_\Gamma - A_{\Gamma\Gamma}^{-1}A_{\Gamma 2}A_{22}^{-1}\vec{f}_2 \\ \quad + A_{\Gamma\Gamma}^{-1}A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}\vec{u}_\Gamma \end{cases}$$

$$\Rightarrow \underbrace{\left[A_{\Gamma\Gamma} - A_{\Gamma 1}A_{11}^{-1}A_{1\Gamma} - A_{\Gamma 2}A_{22}^{-1}A_{2\Gamma}\right]}_{\Sigma \in \mathbb{R}^{GREEN\times GREEN}} \underbrace{\vec{u}_\Gamma}_{\vec{\lambda}} = \overbrace{\vec{f}_\Gamma - A_{\Gamma 1}A_{11}^{-1}\vec{f}_1 - A_{\Gamma 2}A_{22}^{-1}\vec{f}_2}^{\vec{X} \in \mathbb{R}^{GREEN}}$$

GREEN = dimension of green (3 in this case)

$$\Rightarrow \boxed{\Sigma\vec{\lambda} = \vec{X}}$$

SCHUR - COMPLEMENT SYSTEM

Recall that

$S = S_1 + S_2$

$P_{DN} = S_2$ $\qquad P_{NN} = (\sigma_1 S_1^{-1} + \sigma_2 S_2^{-1})^{-1}$

if $\Sigma = \Sigma_1 + \Sigma_2$

$P_{DN} = \Sigma_2$ $\qquad P_{NN} = (\sigma_1 \Sigma_1^{-1} + \sigma_2 \Sigma_2^{-1})^{-1}$

The last step is finding σ_1 and σ_2 and then we are done.

$$\underbrace{\left[A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right]}_{\Sigma} \underbrace{\vec{u}_\Gamma}_{\vec{x}} = \underbrace{\vec{f}_\Gamma - A_{\Gamma 1} A_{11}^{-1} \vec{f}_1 - A_{\Gamma 2} A_{22}^{-1} \vec{f}_2}_{\vec{\chi}}$$

$$(A_{\Gamma\Gamma})_{pq} = a\left(\varphi_\Gamma^{(p)}, \varphi_\Gamma^{(q)}\right) = \int_{\cap\ \text{support}\ \left(\varphi_\Gamma^{(p)}, \varphi_\Gamma^{(q)}\right)} \nabla \varphi_\Gamma^{(p)} \nabla \varphi_\Gamma^{(q)} = \underbrace{\int_{\text{support in } \Omega_1} \cdots}_{(A_{\Gamma\Gamma})_{pq}^1} + \underbrace{\int_{\text{support in } \Omega_2}}_{(A_{\Gamma\Gamma})_{pq}^2}$$

$\uparrow Lu = -\Delta u$

So I can write

$A_{\Gamma\Gamma} = A_{\Gamma\Gamma}^{(1)} + A_{\Gamma\Gamma}^{(2)}$ because of the support

$$\Rightarrow \underbrace{\left(A_{\Gamma\Gamma}^{(1)} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}\right)}_{\Sigma_1} + \underbrace{\left(A_{\Gamma\Gamma}^{(2)} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}\right)}_{\Sigma_2} \equiv A_{\Gamma\Gamma} - A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma} - A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma}$$

$$\Sigma_1 + \Sigma_2 \equiv \Sigma$$

To be clear

$$\left[A_{1\Gamma}\right]_{pi} = a\left(\varphi_1^{(i)}, \varphi_\Gamma^{(p)}\right)$$

express interaction between INTERFACE and Ω_1

INTERFACE $\downarrow$ green point; Ω_1 $\downarrow$ blue point

Remark

If the start of the problem is symmetric it will be also the matrix A. Otherwise is no good.
If symmetrix I use whatever method I want.

If I want to improve I need to choose the good preconditioner.

$$\Sigma_1 \vec{\lambda} = \vec{X} \qquad \text{cond}\left(P^{-1}\Sigma\right) = F(h, H) \begin{cases} \text{if indip. of } h \to \text{OPTIMAL PREC.} \\ \text{if indip. of } H \to \text{SCALAR PREC.} \end{cases}$$

$\underline{P}$ = preconditioner

Analysis

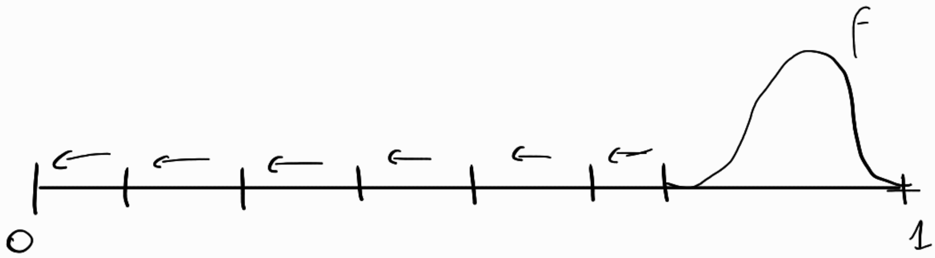
With 2 subdomains $\longrightarrow$ OPTIMALITY!

With $M \gg 1$ subdomains $\longrightarrow$ $\text{cond}\left(P_{NN}^{-1}\Sigma\right) \simeq C H^{-2}\left(1 + \log \frac{H}{h}\right)^2$

(It's not scalable and not optimal)

Every subdomains talk to each other. In a computer there is no global comunication.

$-u'' = f$



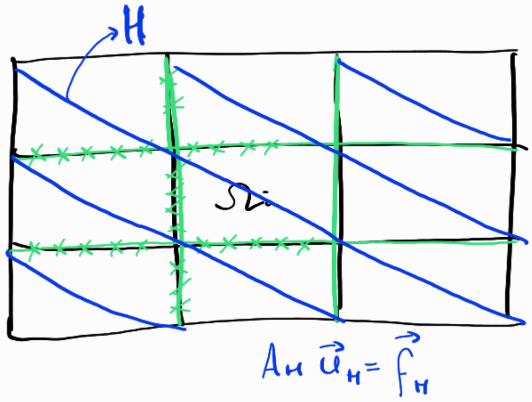
The spreading of information is slow. $H = \frac{1}{M}$

I need a course correction to make this faster.

$$\left(P_{NN}\right)^{-1} = \underset{i}{\overset{(\text{Sommatory})}{\text{Sum}}}\ R_{\Gamma_i}^T \underset{=}{D_i}\ \Sigma_i^{*\,(\text{Pseudoinverse})}\ \underset{=}{D_i}\ R_{\Gamma_i}$$

D_i = diagonal coefficient

R_{Γ_i} is essential



Skeleton

x are the values

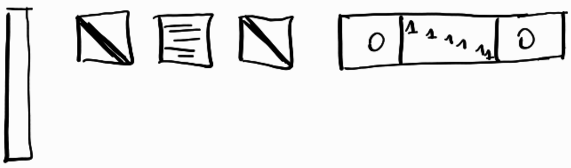
Σ_i is applicated only on Ω_i

R_{Γ_i} = RECTANGULAR RESTRICTION MATRIX

Restrict the values to the local knowns

propagate

$R^T_{\Gamma_i} D_i \Sigma_i^* D_i R_{\Gamma_i}$



This new system is small that we can use a direct method to solve it.

So

Previous preconditioner

$$\left(P_{NN}^{\text{Coarse}}\right)^{-1} = \Sigma_H^{-1} + \left(I - \Sigma_H^{-1}\,\Sigma\right)\boxed{\left(P_{NN}\right)^{-1}}\left(I - \Sigma\,\Sigma_H^{-1}\right)$$

$$\left(P_{NN}^{BNN}\right)^{-2}$$

Σ_H is the SCHUR COMPLEMENTATION

$$\text{Cond}\left(P_{NN}^{BNN^{-1}}\,\Sigma\right) \simeq C\left(1 + \log\frac{H}{h}\right)^2$$