POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

10 Structural Dynamics: Introduction to Random Vibrations

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Material

Masarati Chapter 6
Preumont Chapter 8

Additional material

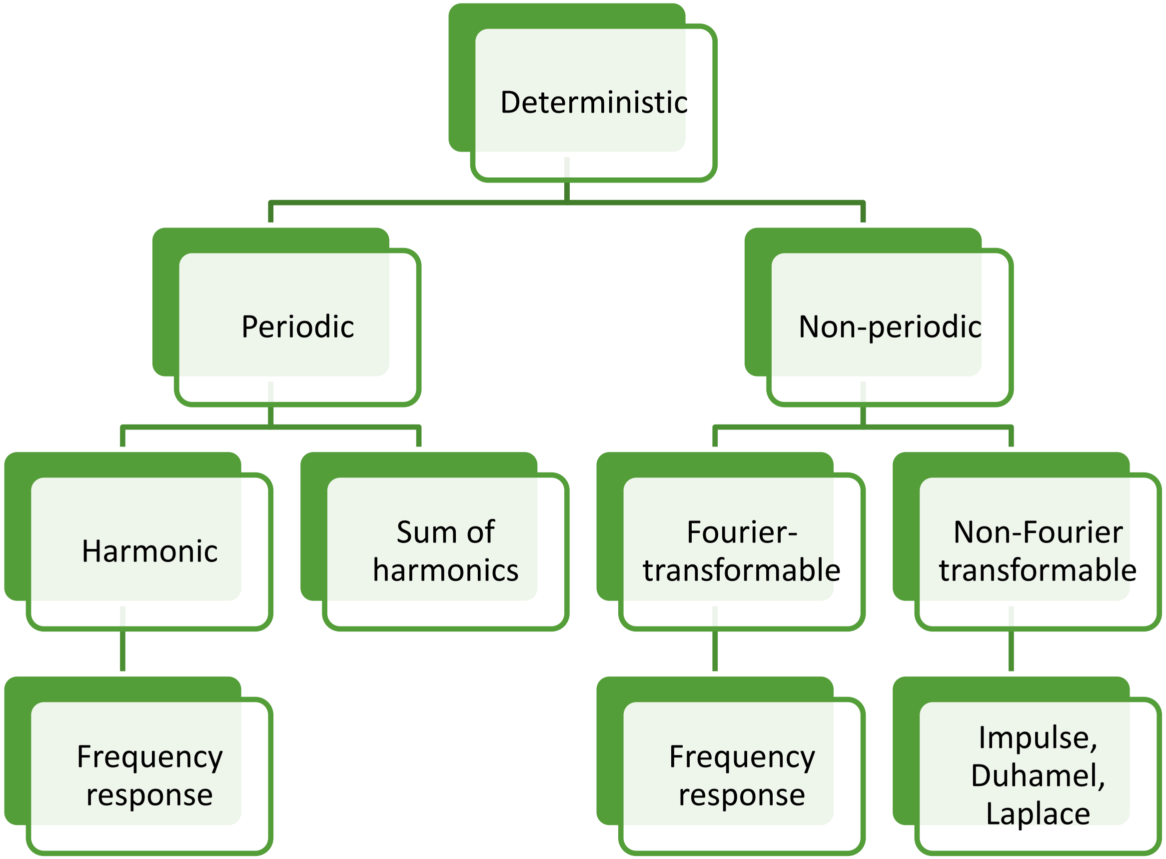
Cheli Diana Chapter 7 (accessible to Polimi students through
https://link.springer.com/book/10.1007%2F978-3-319-18200-1

Motivation: deterministic excitation

Up to now we considered only <u>*deterministic excitations*</u> to the considered structures, i.e., excitation for whom at an assigned position in space and at a precise instant in time the value of the excitation signal is perfectly known.

Definition:

A process is deterministic in so far as it represents a mathematical description whose input parameters has all been predetermined and remains unchanged.



Motivation: non-deterministic excitation

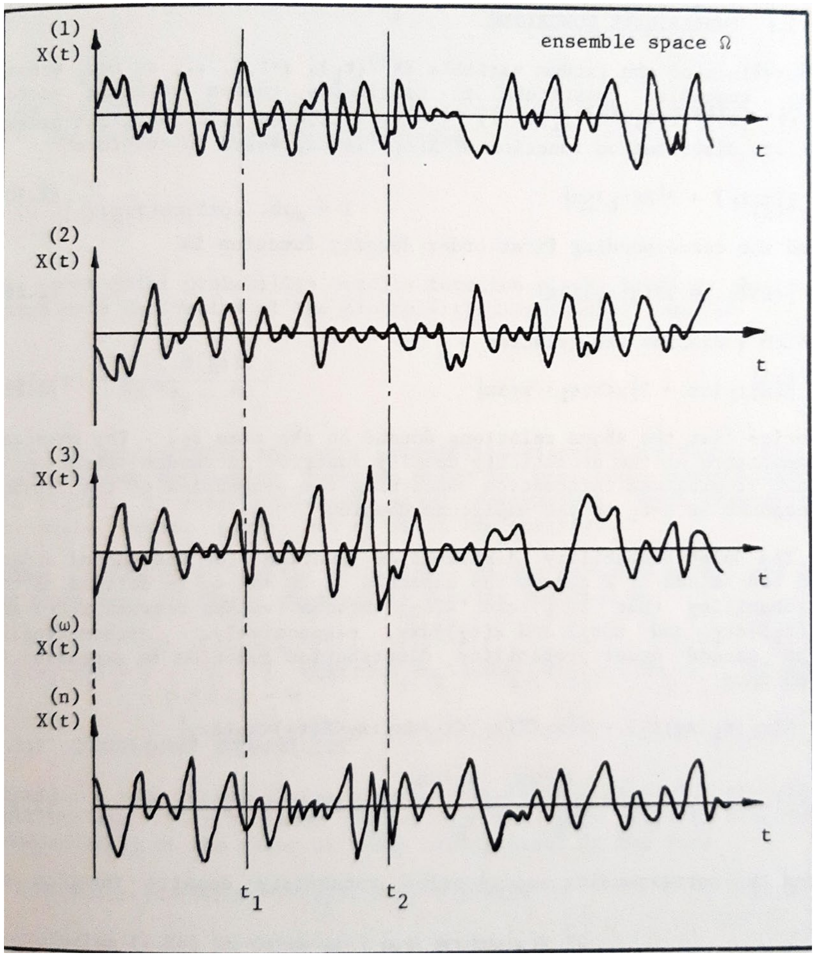
There are cases where the *excitation signal* in terms of value assumed in space and time *is random* and *can only be defined through a probability of occurrence.*

In this case the excitation is called *non-deterministic*, and it defined though *a stochastic process.*

The physical characteristics of a stochastic process are described by its *statistical properties*.

An attempt to predict its value at a certain position and time can only be performed in a statistical sense. An observed set of records cannot precisely be repeated, but *it will follow a certain pattern that may only be mathematically represented by statistics.*

Motivation: Random process



An observation of a stochastic process is called a *realization,* i.e., a sample signal none of which can, with certainty, be repeated even if the conditions are seemingly the same, and so it must be considered random.

The collection of all possible realizations is called *ensemble* or *Stochastic (random) process*.

Motivation: Random process

If the excitation random process is the cause of another process, this will also be a stochastic process. A stochastic input will provide a stochastic output.

So, in the case of a deterministic model of structural dynamics subject to a random input the output must be computed using statistical approach.

Case where the input should be considered stochastic

- **Response of an aircraft to atmospheric turbulence**
- **Response of a structure due to noise and vibrations of an engine**
- **Response of vehicles to uneven routes**
- **Response of inner flows (turbomachinery etc.) to turbulent motion of fluids**
- Response of offshore structures to wave forces
- Response of structures to earthquakes

The output is typically:

- Displacement or accelerations (performance, comfort, etc..)
- Stress (maximum stress reached, fatigue accumulations ecc.)

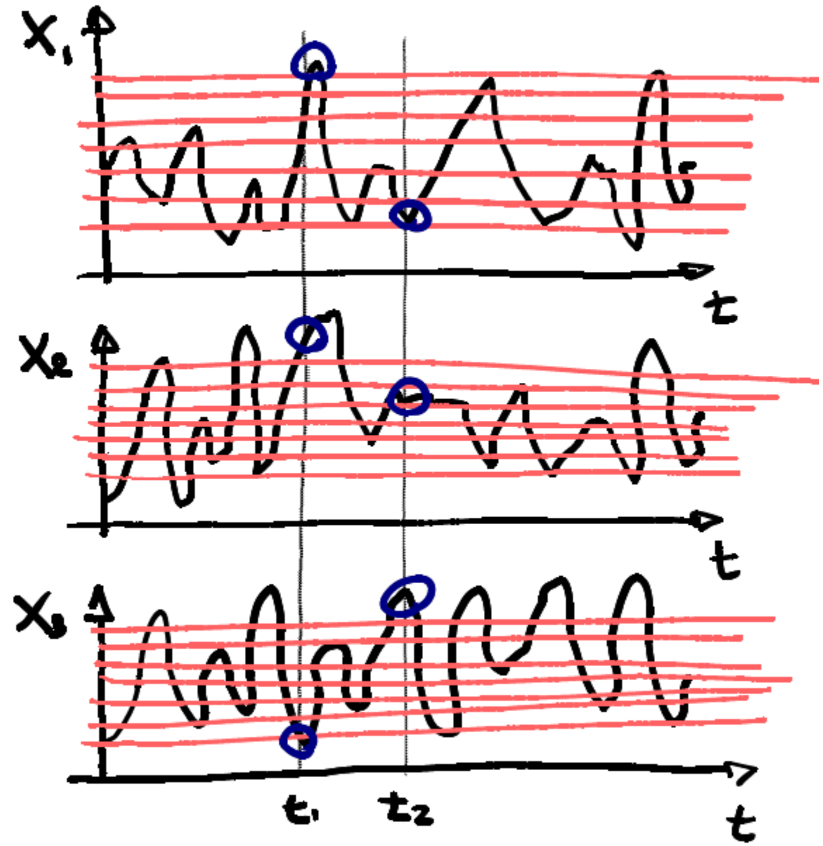
Probability distribution

X is a random variable i.e., the outcome of a random experiment (1,2,3…)

x is particular value assumed by that random variable

For a continuous variable we divide the range existence into several levels a compute the number of time $X(t_j)$ assumes a value within that range

In this way we produce a **histogram**



$$x_{i-1} < X(t_j) \leq x_i \;\rightarrow X(t_j) \,\text{is the event}\, L_i$$

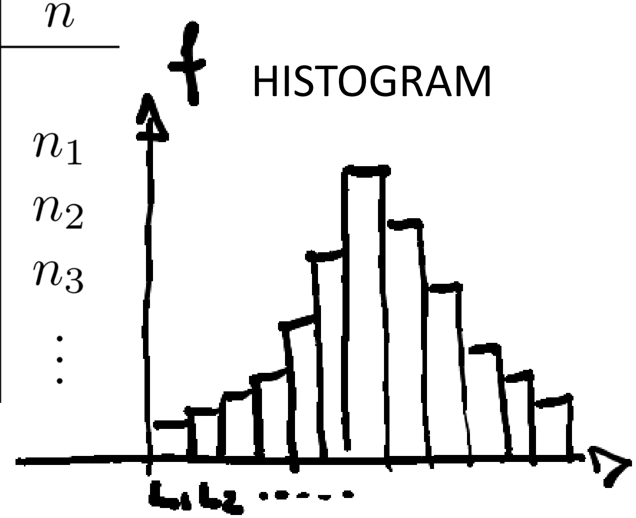
Probability distribution

Consider a finite set of events L_i. n_i is the number of times the event L_i is repeated. The *relative frequency* of the event L_i is

$$f_i = f_i(L_i) = \frac{n_i}{\sum_i n_i} = \frac{n_i}{N}$$

where N is the number of trials.

$X(t_j)$	n
L_1	n_1
L_2	n_2
L_3	n_3
$\vdots$	$\vdots$



If we repete the exact same experiment many times, it is possible to observe that the corresponding relative frequencies practically coincide when the number of trails is large.

It is so possible to define the *PROBABILITY* of the event L_i as

$$\mathcal{P}_i = \mathcal{P}(L_i) = \lim_{N \to \infty} \frac{n_i}{N}$$

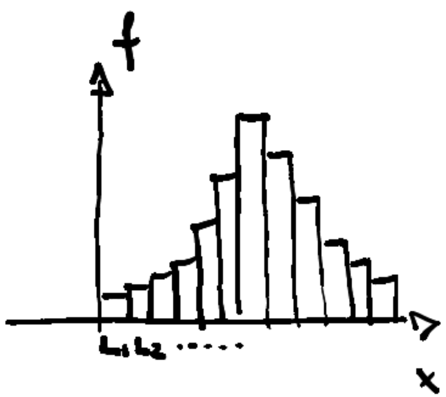
This is called the BERNOULLI Law od Large Numbers.

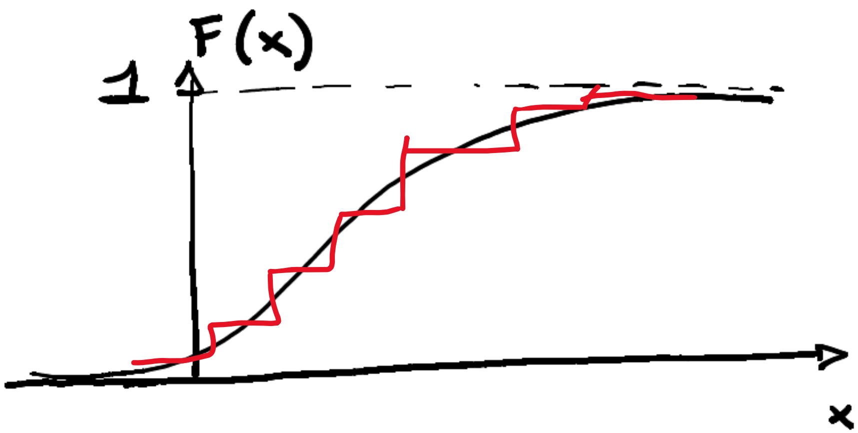
Probability distribution

This probability definition satisfies the three Probability Axioms defined by Kolmogorov

1. $\mathcal{P}(L_i) \geq 0 \, \forall X_i$
2. If we call Ω the entire sample space $\mathcal{P}(\Omega) = 1$
3. If $L_1, L_2, L_3, \ldots$ are mutually exclusive events then $\mathcal{P}(L_1 + L_2 + L_3 + \ldots) = \mathcal{P}(L_1) + \mathcal{P}(L_2) + \mathcal{P}(L_3) + \ldots$

Cumulative Probability Function (CPF)





$$F(x) = \mathcal{P}(X \leq x)$$

$$\begin{cases} F(-\infty) = \mathcal{P}(X \leq -\infty) = 0 \\ F(\infty) = \mathcal{P}(X \leq \infty) = 1 \end{cases}$$

if $x_1 < x_2 \ \rightarrow F(x_1) < F(x_2)$

$$F(x_1 < X < x_2) = F(x_2) - F(x_1)$$

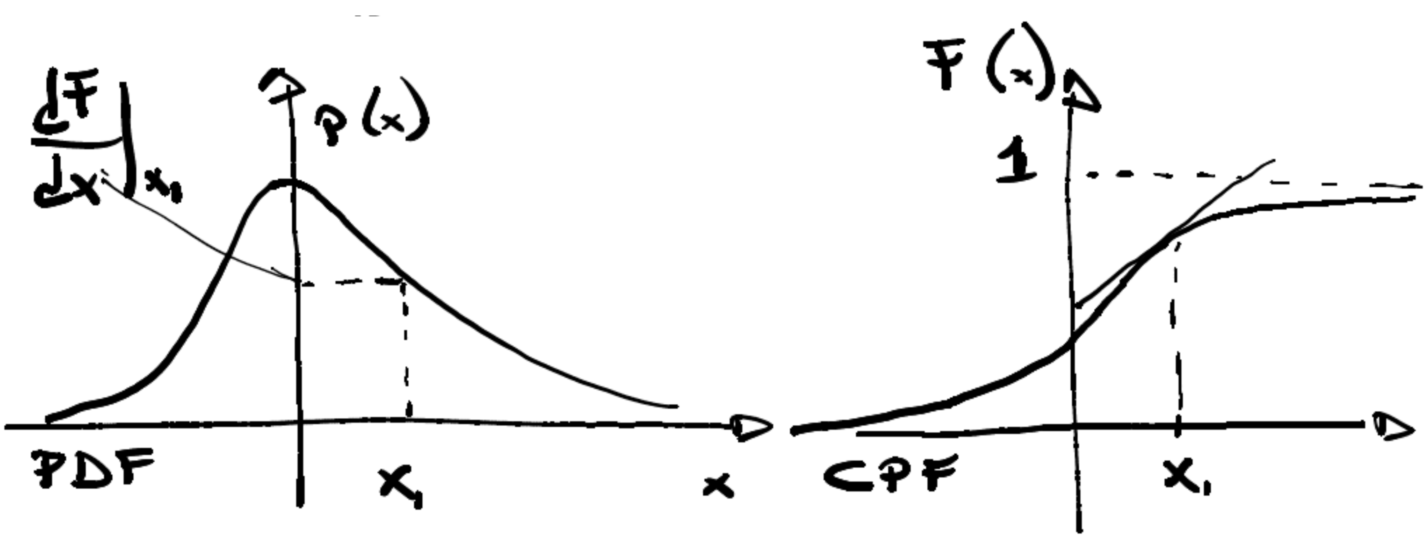
The F function is monotone

Probability Density Function (PDF)

It is the natural extension of the relative frequency to a continuous process

$$p(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} = \frac{\mathrm{d}F(x)}{\mathrm{d}x}$$

$$F(x_1) = \int_{-\infty}^{x_1} p(x)\mathrm{d}x \qquad \int_{-\infty}^{+\infty} p(x)\mathrm{d}x = 1$$



Expected values

The expected values is equal to the *ensemble or statistical average,* also known as *mean*

$$m = \frac{\sum_i x_i f_i}{\sum_i f_i}$$

$$E[X] = \int_{-\infty}^{+\infty} x p(x) \mathrm{d}x$$

It reminds the expression for the position of the center of mass and in fact this expression is also called *First Statistical Moment*

Expected value and statistical quantities

The expected value of a random number X is the arithmetic mean, i.e.,

$$E[X] = \int_{-\infty}^{+\infty} x p(x) \mathrm{d}x$$

The concept could be generalized to any function g(x)

$$E[g(X)] = \int_{-\infty}^{+\infty} g(x) p(x) \mathrm{d}x$$

<u>Mean</u>

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x p(x) \mathrm{d}x$$

<u>Mean Square Value</u>

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 p(x) \mathrm{d}x$$

<u>Root Mean Square</u>

$$RMS = \sqrt{E[X^2]}$$

Expected value and statistical quantities

Variance

$$
\begin{aligned}
\sigma_X^2 &= E[(X-\mu_X)^2] \\
\sigma_X^2 &= \int_{-\infty}^{+\infty} (x-\mu_X)^2 p(x)\mathrm{d}x \\
\sigma_X^2 &= \int_{-\infty}^{+\infty} x^2 p(x)\mathrm{d}x - 2\mu_X \int_{-\infty}^{+\infty} x p(x)\mathrm{d}x + \mu_X^2 \int_{-\infty}^{+\infty} p(x)\mathrm{d}x \\
\sigma_X^2 &= E[X^2] - \mu_X^2
\end{aligned}
$$

Standard Deviation

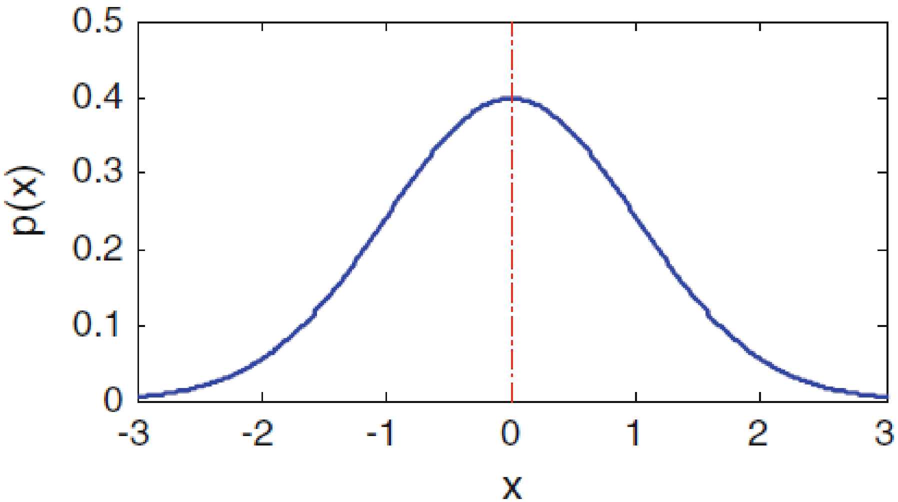
$$\sigma_X = \sqrt{\sigma_X^2}$$

Gaussian process

Central Limit Theorem: when the effects of independent random variables are summed together the resulting stochastic distribution tends to be *Gaussian* (or *Normal*). Many physical process are characterized by the combinatory effect of several independent variables.

The Normal distribution is fully characterized by its it mean and variance

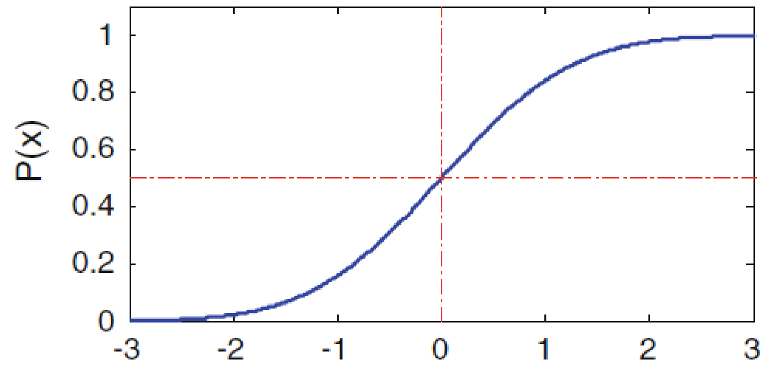
$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$



Gaussian process

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}\left(\frac{\xi-\mu}{\sigma}\right)^2}\,\mathrm{d}\xi$$

$$F(x) = \frac{1}{2} + \mathrm{erf}\left(\frac{\xi-\mu}{\sigma}\right)$$



where $\mathrm{erf}(x)$ is the error function, defined as

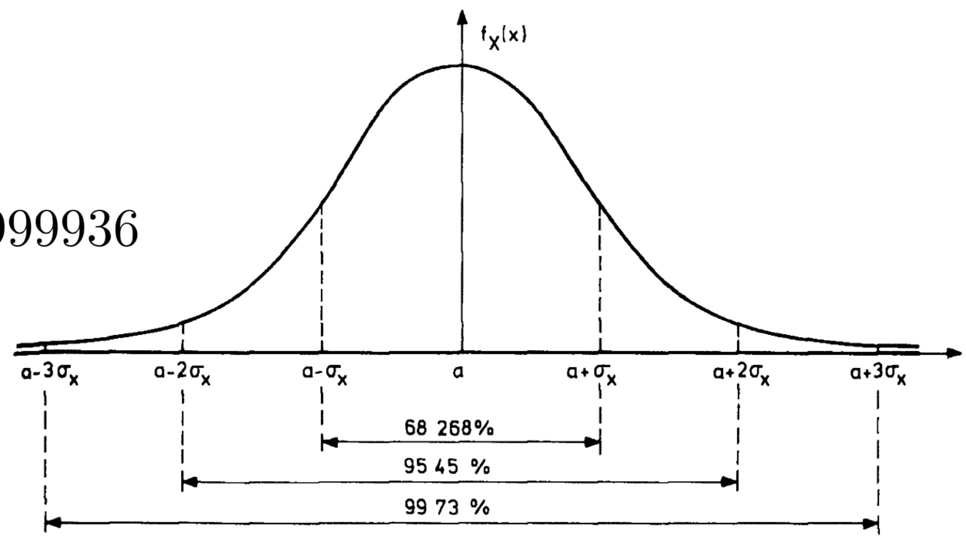
$$\mathrm{erf} = \frac{1}{\sqrt{2\pi}}\int_{0}^{x} \mathrm{e}^{-\frac{\xi^2}{2}}\,\mathrm{d}\xi$$

$$N(\mu, \sigma^2)$$

$$E[X] = \mu,\ E[(X-\mu)^2] = \sigma^2$$

$$\mathcal{P}(|X-\mu_x| \leq k\sigma_X) = 2\mathrm{erf}(k)$$

$$\mathcal{P}(|X-\mu_x| \leq 4\sigma_X) = 2\mathrm{erf}(4) = 0.999936$$



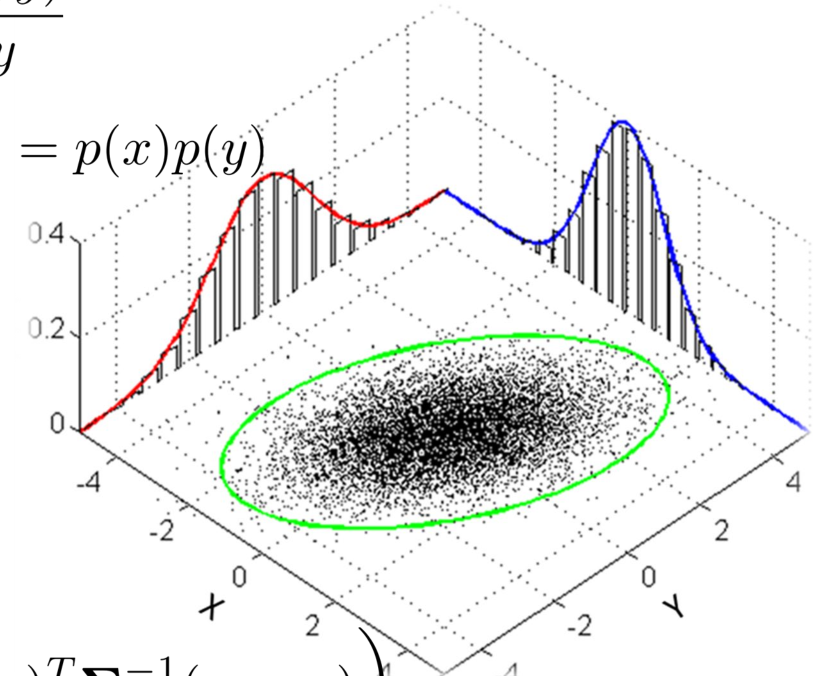
Gaussian process: joint (multivariate) distributions

X, Y two random processes, joint probability density

$$p(x,y) = \frac{\mathrm{d}^2\mathcal{P}(X \le x, Y \le y)}{\mathrm{d}x\,\mathrm{d}y} = \frac{\mathrm{d}^2 F(x,y)}{\mathrm{d}x\,\mathrm{d}y}$$

The two processes are independend iff $p(x,y) = p(x)p(y)$

$$F(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} p(\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$



$$p(x_1, x_2, \ldots, x_k) = \frac{1}{\sqrt{|\boldsymbol{\Sigma}|(2\pi)^k}} \exp\left(-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)$$

$$\Sigma_{ij} = E[(x_i - \mu_i)(x_j - \mu_j)]$$

If $\boldsymbol{\Sigma}$ is the matrix of VARIANCE (AUTO and CROSS). If it is diagonal the joint probability density is the union of k independent probability densities

Process function of time

For a specified time t the random signal $X(t)$ is a radom variable with a Cumulative probability function that depends on t and it is written as

$$F_X(x,t) = \mathcal{P}(X(t) \leq x)$$

The corresponding probability density function is

$$p(x,t) = \frac{\partial \mathcal{P}(X(t) \leq x)}{\partial x}$$

If we now take the value assumed by the random signal at two instants in time $X(t_1)$ and $X(t_2)$ then the signal is charaterized by the knowledge of the CPF

$$F_X(x_1, x_2; t_1, t_2) = \mathcal{P}(X(t_1) \leq x_1, X(t_2) \leq x_2)$$

Process function of time

The corresponding probability density function is

$$p(x_1, x_2; t_1, t_2) = \frac{\partial^2 F_X(x_1, x_2; t_1, t_2)}{\partial x_1 \partial x_2}$$

The mean and variance for fixed t is defined as

$$\begin{aligned} \mu_X(t) &= E[X(t)] = \int_{-\infty}^{+\infty} x p(x; t) \mathrm{d}x \\ \sigma^2_{XX}(t) &= E[(X(t) - \mu_X(t))^2] \end{aligned}$$

Autocorrelation and cross-correlation

It is defined as the joint moment of the random variable at two instant in time t_1 and t_2

$$
\begin{aligned}
R_{XX}(t_1.t_2) &= E[X(t_1)X(t_2)] \\
R_{XX}(t_1.t_2) &= \int_{-\infty}^{+\infty} x_1 x_2 p(x_1, x_2; t_1, t_2)\,\mathrm{d}x_1\,\mathrm{d}x_2
\end{aligned}
$$

$$R_{XX}(t_1, t_1) = R_{XX}(t_1) = E[X^2(t_1)]$$

It allows us to understand how statistics of the time history of a variable is related to itself shifted in time.

$$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$$

It is possible to do the same using two different random variables X and Y in case it is called cross-correlation.

Stationary process

Consider the process $X(t)$. The probability density function of $X(t_1), X(t_2), \ldots, X(t_n)$ is $p(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$.

The process is stationary if taken the signals $X(t_1+\varepsilon), X(t_2+\varepsilon), \ldots, X(t_n+\varepsilon)$ the probability density coincide with the previous one for any n and any ε.

$$p(x;t) = p(x;t+\varepsilon)\,\forall\varepsilon \;\rightarrow\; p(x;t) = p(x)$$

The probability density is independent of a shift of the time origin. Is is easy to see that the mean does not depend on time any more

$$\mu_X(t) = \int_{+\infty}^{-\infty} xp(x;t)\mathrm{d}x = \int_{+\infty}^{-\infty} xp(x)\mathrm{d}x = \mu_X$$

The same could be demostrated for the variance.

Stationary process

Taking two instants in time

$$p(x_1, x_2; t_1, t_2) = p(x_1, x_2; t_1 + \varepsilon, t_2 + \varepsilon)$$

If we take $\varepsilon = -t_1$ and we call $\tau = t_2 - t_1$ the time interval

$$p(x_1, x_2; t_1, t_2) = p(x_1, x_2; \tau)$$

Now the correlation becomes only funcion of the time shift τ

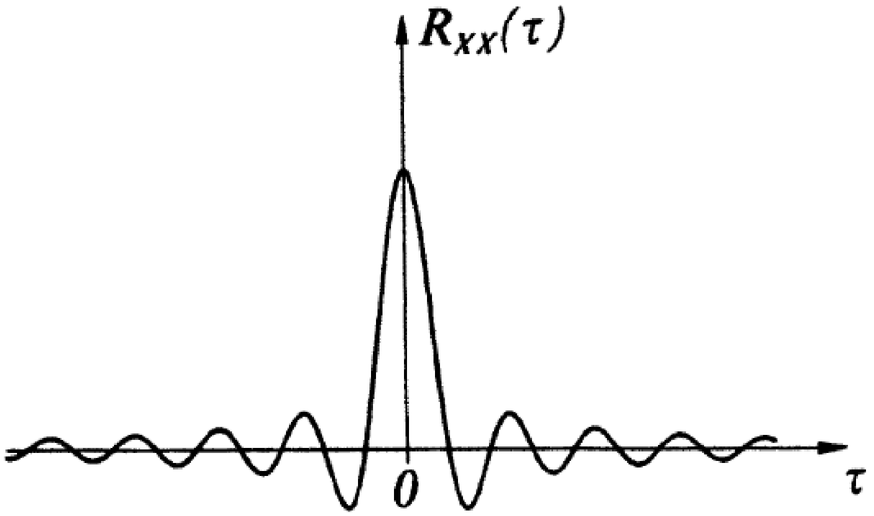
$$R_{XX}(t_1, t_2) = R_{XX}(\tau) = E[X(t)X(t+\tau)]$$

In case of two stationary signals also the cross-correlation depends only on the interval τ

$$R_{XY}(t_1, t_2) = R_{XX}(\tau) = E[X(t)Y(t+\tau)]$$

Properties of correlation

$$R_{XX}(\tau) = R_{XX}(-\tau)$$
$$R_{XX}(0) \geq R_{XX}(\tau)\,\forall\tau$$
$$\sqrt{R_{XX}(0) + R_{YY}(0)} \geq R_{XY}(\tau)\,\forall\tau$$

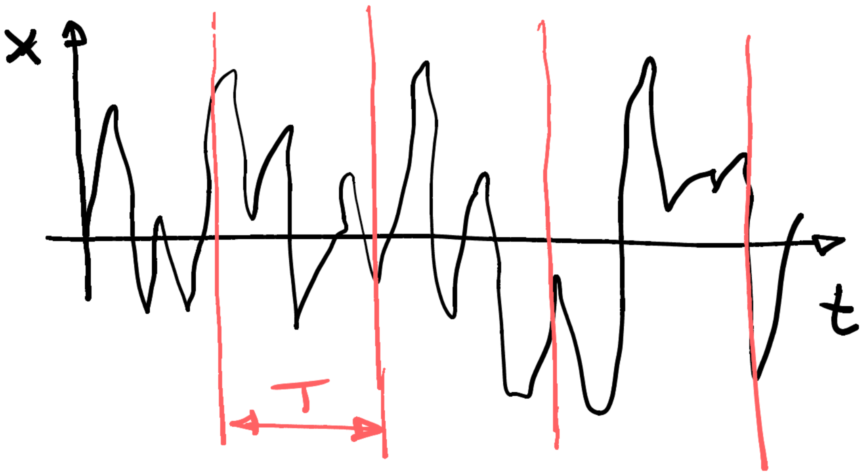


The autocorrelation is a function that has its maximum at τ = 0 and then decays toward infinity

Ergodic process

The determination of the statistics from an ensemble poses same practical problems related to the repetition of experiments. It is often difficult to have sufficient repetitions to obtain accurate estimation of probability densities. However, if the process is stationary, it seems reasonable to take one realization and replace ensemble averages with time averages. A stochastic process for which this exchange is possible is called *ergodic.*

It is possible to start by taking the time history dividing it in several portions of size T and use them as realizations



Ergodic process

Changing T it is possible to generate a sequence of random variables and the statistics could be computed taking the limit of this sequence

$$E[X] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} x(t) \mathrm{d}t$$

$$E[\ldots] = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{+T} \ldots \mathrm{d}t = \fint \ldots \mathrm{d}t$$

Mean integral

$$R_X(\tau) = E[X(t+\tau)X(t)] = \fint x(t+\tau)x(t) \mathrm{d}t$$

Auto-Covariance and Cross-covariance

$$
\begin{aligned}
k_{XX}(\tau) &= E[(X(t+\tau)-\mu_X)(X(t)-\mu_X)] \\
k_{XX}(\tau) &= \int (x(t+\tau)-\mu_X)(x(t)-\mu_X)\,\mathrm{d}t \\
k_{XX}(\tau) &= \int x(t+\tau)x(t)\,\mathrm{d}t + \\
&- \mu_X \int x(t+\tau)\,\mathrm{d}t - \mu_X \int x(t)\,\mathrm{d}t + \mu_X^2
\end{aligned}
$$

The autocovariance, is the equivalent of the autocorrelation taking off the mean by the signal.

The autocovariance at τ = 0 is equal to the variance

$$k_{XX}(\tau) = R_{XX}(\tau) - \mu_X^2$$

$$k_{XX}(0) = R_{XX}(0) - \mu_X^2 = \sigma_X^2$$

Cross-covariance and coherence

$$k_{XY}(\tau) = E[(X(t+\tau)-\mu_X)(Y(t+\tau)-\mu_Y)]$$
$$k_{XY}(\tau) = R_{XY}(\tau)-\mu_X\mu_Y$$

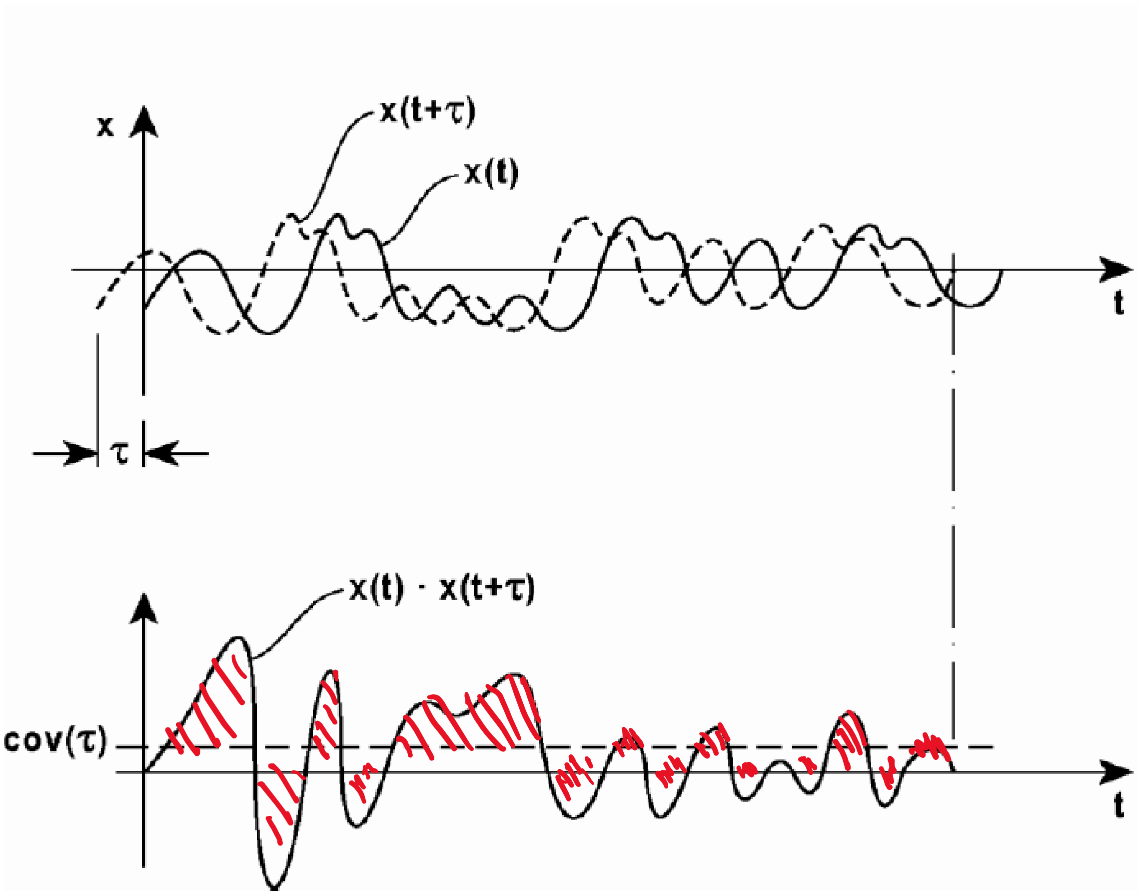
Of course, in a similar way it is possible to define a cross-covariance between two random signals X and Y.

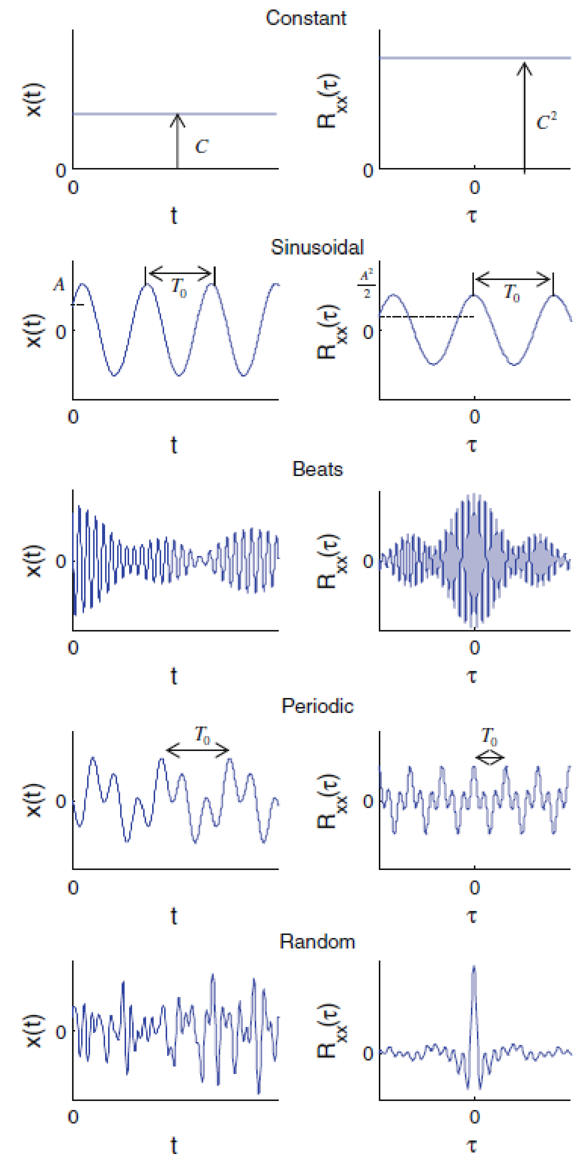
$$\text{COHERENCE}\ \rho_{XY} = \frac{k_{XY}(0)}{\sigma_x\sigma_Y}$$

$$-1 \leq \rho_{XY} \leq 1$$

$$X \text{ and } Y \text{ are } \underline{\text{Uncorrelated}} \text{ iff } \rho_{XY} = 0$$

Auto-covariance





Autocovariance as a mean to de-noise a signal

If $Y(t)$ is a noisy random signal so that $y(t) = x(t) + n(t)$, where $N(t)$ is a noise

$$
\begin{aligned}
R_{YY}(\tau) &= \int (X(t) + n(t))\,(x(t+\tau) + n(t+\tau))\;\mathrm{d}t \\
R_{YY}(\tau) &= R_{XX}(\tau) + 2R_{NX}(\tau) + R_{NN}(\tau)
\end{aligned}
$$

$R_{NN}(\tau) \to 0$ rapidly
$R_{NX}(\tau) \to 0$ because N and X are uncorrelated. So

$$\Rightarrow\ R_{YY} \approx R_{XX}(\tau)$$

Power Spectral Density (PSD)

If we are interested in processing the stochastic response in frequency domain, it is now possible to do the Fourier transform of an ergodic signal.

So, the frequency content of an ergodic signal may be found looking at the Fourier transform of the autocovariance:

$$\Phi_{XX}(\omega) = \mathcal{F}[k_{XX}(\tau)] = \int_{-\infty}^{+\infty} k_{XX}(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau$$

Since it is related to the square of a signal ($E[X^2]$) it is considered a measure of energy content

Power Spectral Density (PSD)

$$\Phi_{XX}\left[\frac{(\text{unit of } x)^2\text{s}}{\text{rad}}\right]$$

$\Phi_{XX}(\omega) \in \mathbb{R}$ if $k_{XX}(\tau)$ is EVEN

$\Phi_{XX} \geq 0$

$\Phi_{XX}(-\omega) = \Phi_{XX}(\omega)$

$$k_{XX}(\tau) = \mathcal{F}^{-1}[\Phi_{XX}(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{XX}(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

This expression represents an easy way to obtain the variance through an integral instead of the necessity to go through the computation of the autocovariance

The PSD is the variance spectral density

If $\tau = 0$

$$k_{XX}(0) = \sigma_X^2 = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Phi_{XX}(\omega)\mathrm{d}\omega$$

$$\sigma_X^2 = \frac{1}{\pi}\int_{0}^{+\infty} \Phi_{XX}(\omega)\mathrm{d}\omega$$

Power Spectral Density (PSD)

Narrow band stochastic process

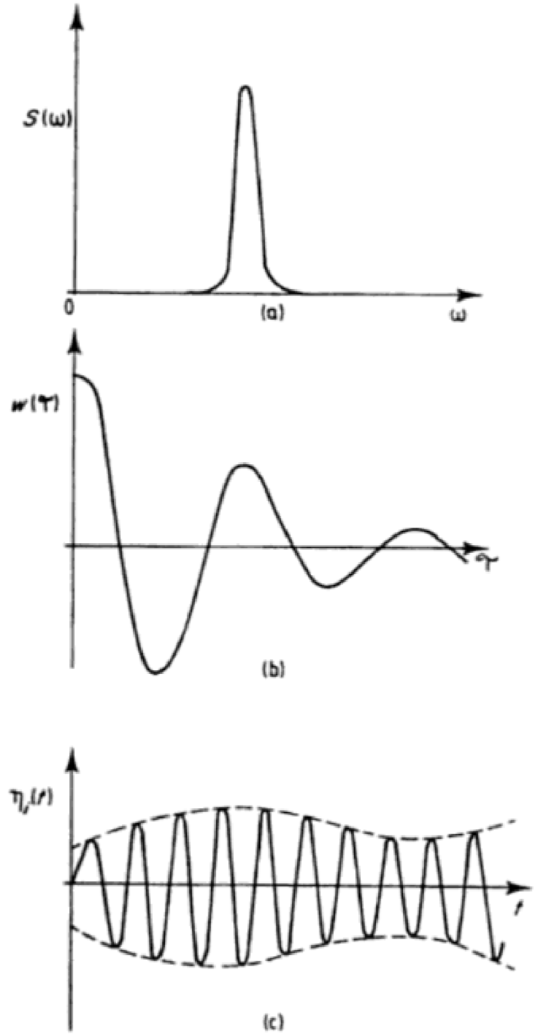
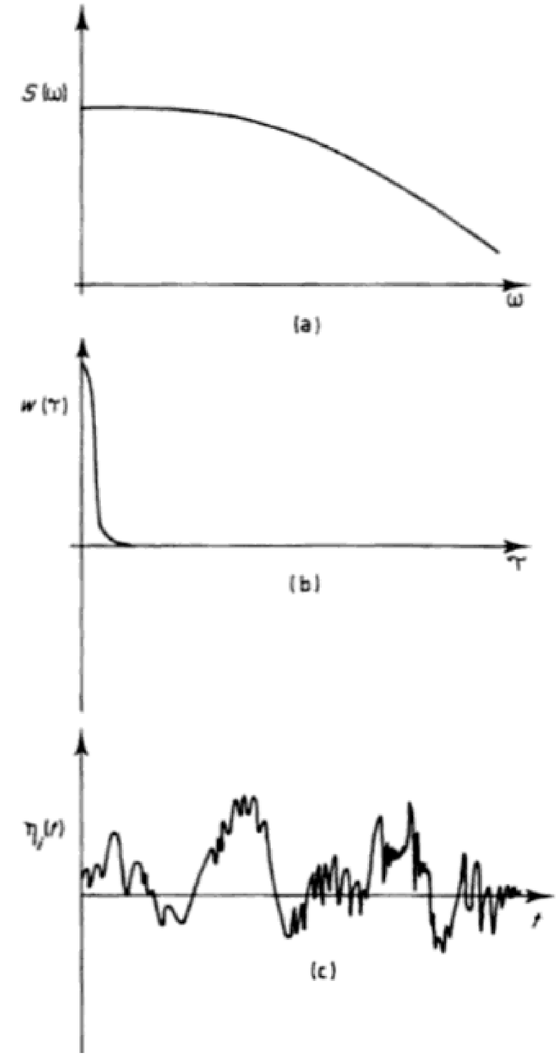


Figure 3.2

Wide band stochastic process



White noise

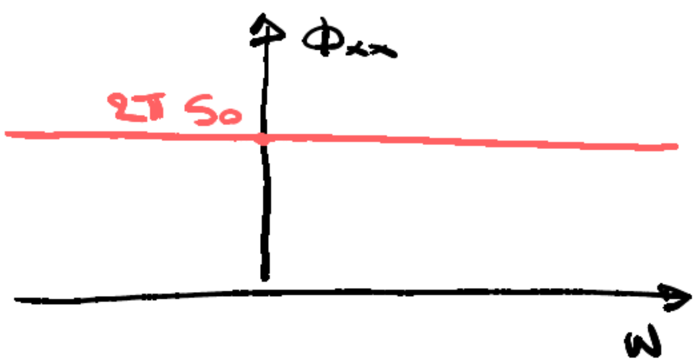
White noise is a process with uniform PSD. The autocovariance is the Dirac delta

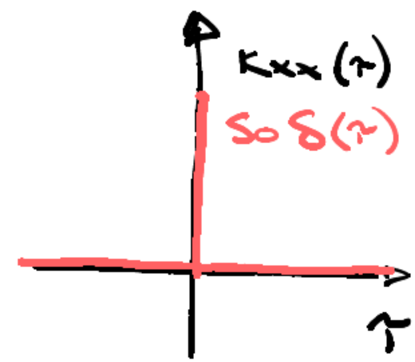
$$\Phi_{XX}(\omega) = 2\pi S_0 \qquad -\infty < \omega < +\infty$$

$$\Rightarrow k_{XX}(\tau) = \frac{1}{2\pi} 2\pi S_0 \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{j}\omega\tau} \mathrm{d}\omega$$

$$k_{XX}(\tau) = S_0 \delta(\tau)$$

This process is not physically realizable because there is an infinite area (power) under the spectrum





It is a convenient approximation of some processes.

Band-limited white noise

Called also ideal low-pass process.

The higher is ω_0 the more the autocovariance approximate a Dirac delta.

$$\Phi_{XX}(\omega) = 2\pi S_0 \quad |\omega| < \omega_0$$

$$\Rightarrow k_{XX}(\tau) = \frac{2\pi S_0}{2\pi} \int_{-\omega_0}^{+\omega_0} e^{j\omega\tau} d\omega$$

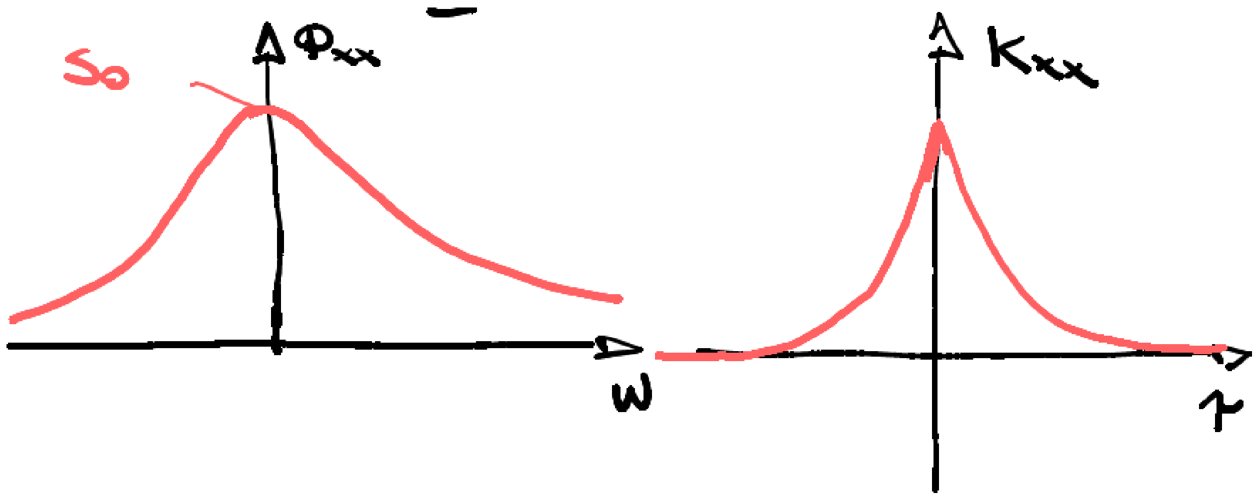
$$k_{XX}(\tau) = 2S_0 \frac{\sin \omega_0 \tau}{\tau}$$

Process with exponential correlation

$$\Phi_{XX}(\omega) = S_0 \frac{\beta^2}{\beta^2 + \omega^2}$$

$$\Rightarrow k_{XX}(\tau) = \frac{S_0}{2\pi} \int_{-\infty}^{+\infty} \frac{\beta^2}{\beta^2 + \omega^2} \mathrm{e}^{\mathrm{j}\omega\tau} \mathrm{d}\omega$$

$$k_{XX}(\tau) = \frac{S_0 \beta}{2} \mathrm{e}^{-\beta|\tau|}$$



Autocovariance

Let's evaluate the square of the difference between the stochastic process x and the same process shifted by t

$$\Delta X(t) = X(t) - \mu_x$$

$$\begin{aligned} E[(\Delta X(t) - \Delta X(t+\tau))^2] &= \int \Delta x^2(t)\mathrm{d}t - 2\int \Delta x(t)\Delta x(t+\tau)\mathrm{d}t + \\ &\quad \int \Delta x^2(t+\tau)\mathrm{d}t \\ E[(\Delta X(t) - \Delta X(t+\tau))^2] &= 2\left(\sigma^2_{XX} - k_{XX}(\tau)\right) \end{aligned}$$

This expected value will be null if the two time histories are coincident. The larger is the value larger is the difference.

Autocovariance

More rapidly the autocovariance decays with τ the less the shifted time histories are similar.

So more rapidly the autocovariance decays and more casual or chaotic is the signal.

The autocovariance allows to characterize exactly how a stochastic process behaves, in a sort of deterministic way through a function.