POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

06 Structural Dynamics: Introduction and Beam modeling

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References

Beam and structural modeling (any book is good)

Bauchau, Craig, Structural Analysis with application to aerospace structures Springer 2009 (Sec 1.2 1.4 2.1.1, 5, 7.1

BAH Chapter 3

Preumont “Twelve lectures on Structural dynamics” Chapter 3

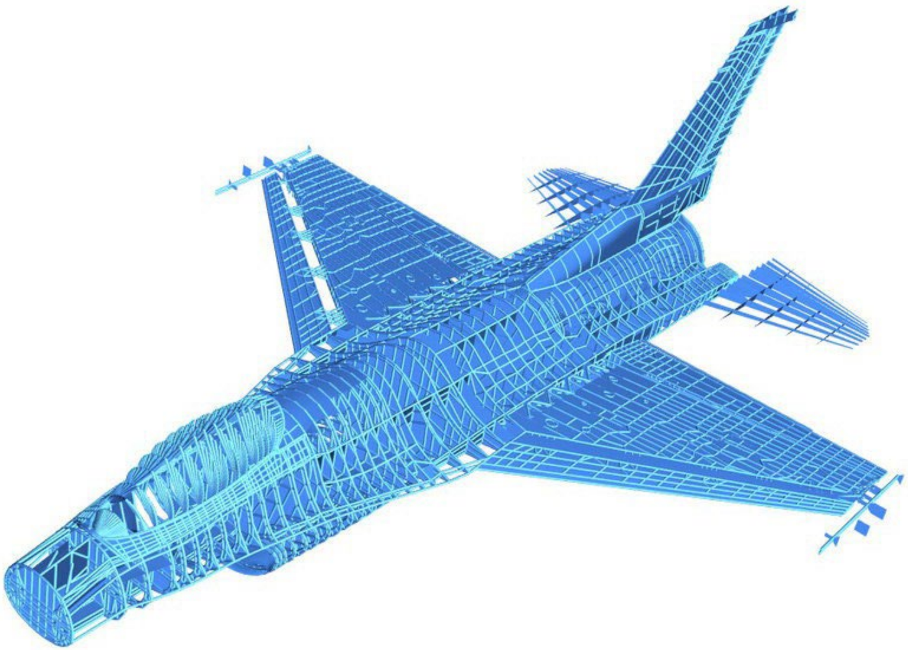
Masarati Chapter 4

Structural dynamics for aircraft structures

It is necessary to understand how the elastic structure of an aircraft behaves when subject to loads.

We are interested in the behavior of entire structure but in particular of the elements that may cause a significant change in aerodynamic forces.

A very detailed FEM (Finite Element Modeling) model may be useful for the determination of internal stresses but not appropriate for the determination of dynamic loads.



Different levels of structural modeling

Simpler model used in dynamic analysis to identify the correct magnitude of dynamic loads (so the correct global deformation and aeroelastic effects)
dofs (10 – 10k)

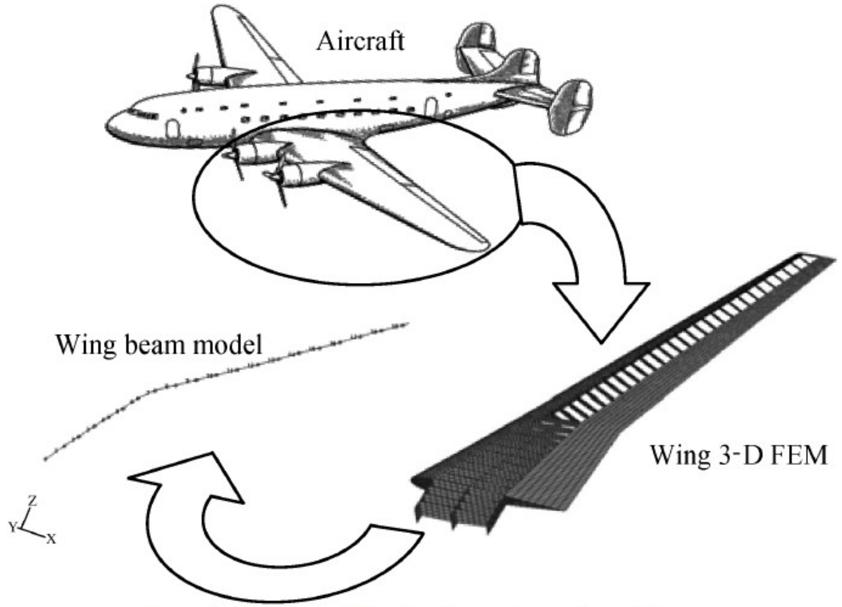
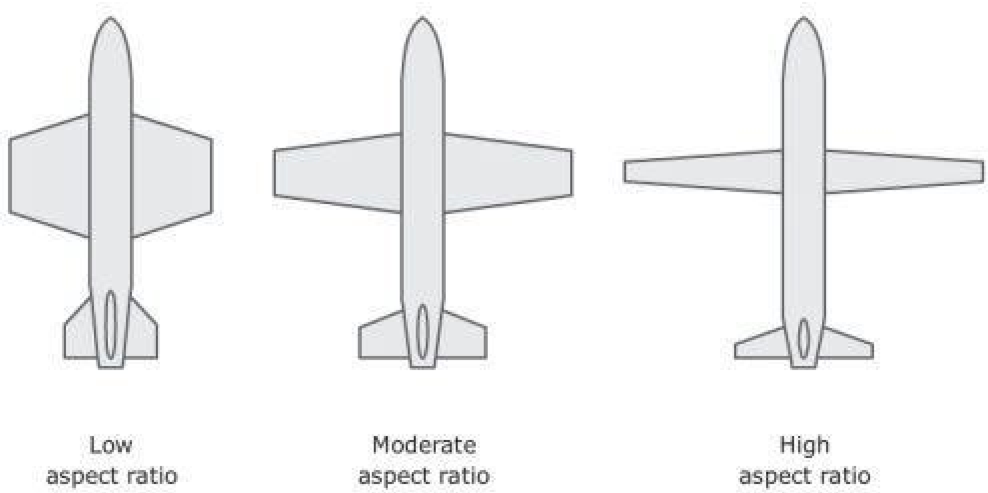


Fig. 1 Different levels of structural modeling.

Detailed Finite Element Model for stress analysis under static loads (or instantaneous dynamic loads)
dofs O(100k – 1M)

Types of reduced models



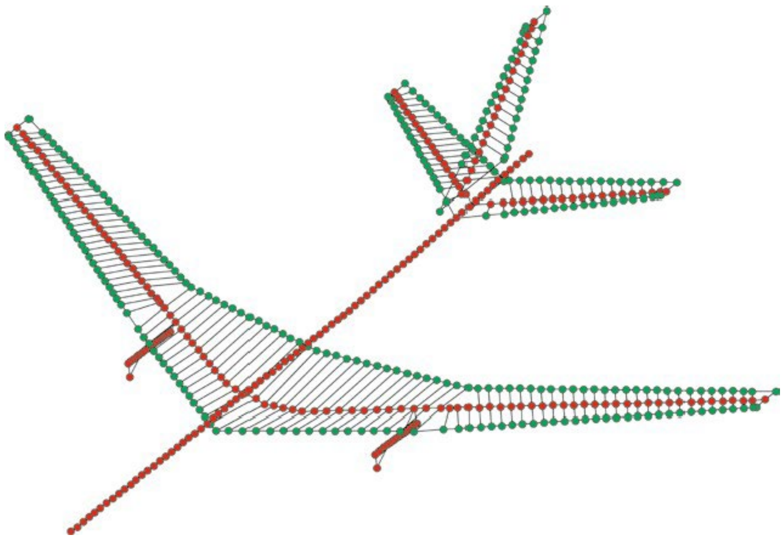
L1, L2, L3 characteristic Lengths along three orthogonal directions

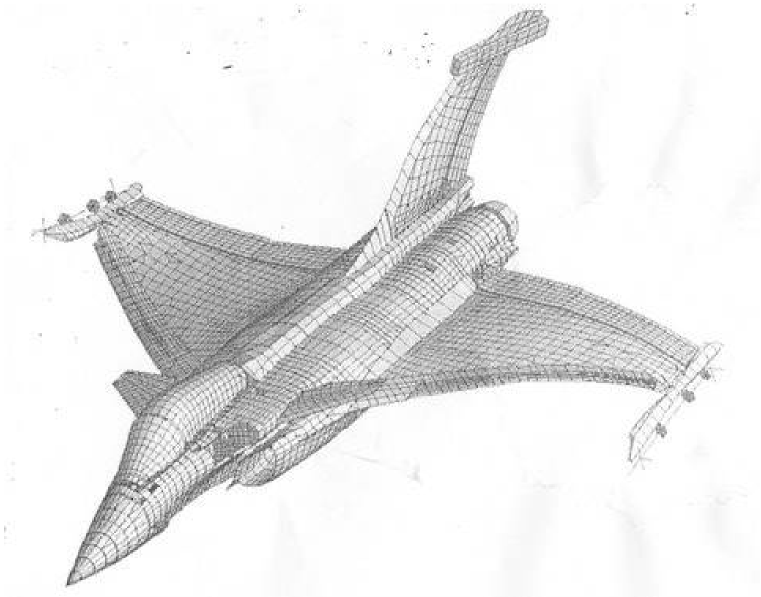
Aspect Ratio L1/L2 & L1/L3 $\gg 1$

Low aspect ratio structures (delta wings, wings of supersonic aircraft, etc.) often can be modelled using **PLATES**

High aspect ratio structures can be easily modelled as **SLENDER BEAMS** (high a.r. wings, fuselages, etc…)

Types of reduced models

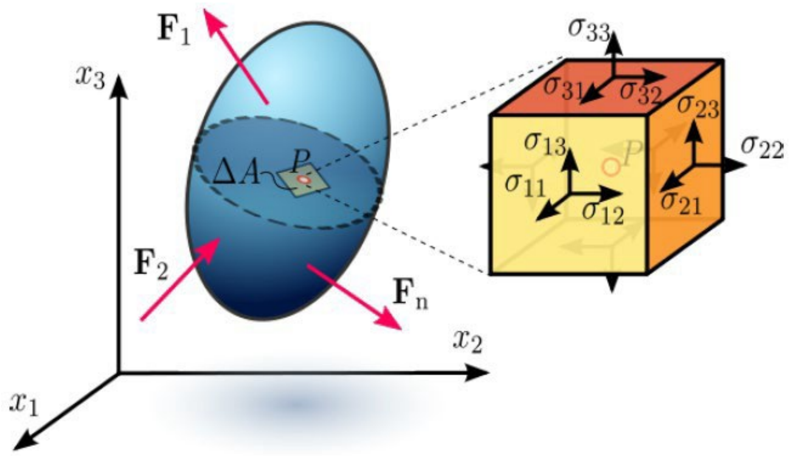




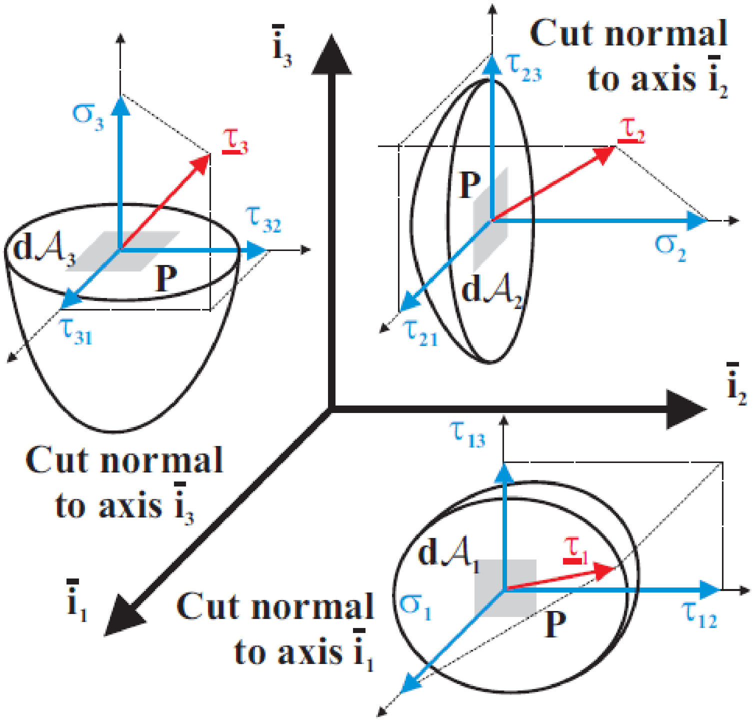
Continuous deformable structure: Hypothesis

Deformable structures are *arbitrarily shaped bodies* which are acted by a *distribution of surface forces* applied to the boundaries and *body forces* acting over their volume

- Small displacement
- Equilibrium equations could be written in the initial (undeformed) configuration



Stress: Cauchy stress tensor



$$\boldsymbol{\tau}_n = \lim_{\mathrm{d}\mathcal{A}_n \to 0} \frac{\mathbf{F}_n}{\mathrm{d}\mathcal{A}_n}$$

$$\begin{aligned}
\boldsymbol{\tau}_1 &= \sigma_1 \mathbf{i}_1 + \tau_{12}\mathbf{i}_2 + \tau_{13}\mathbf{i}_3 \\
\boldsymbol{\tau}_2 &= \tau_{21} \mathbf{i}_1 + \sigma_2\mathbf{i}_2 + \tau_{23}\mathbf{i}_3 \\
\boldsymbol{\tau}_3 &= \tau_{31} \mathbf{i}_1 + \tau_{32}\mathbf{i}_2 + \sigma_3\mathbf{i}_3
\end{aligned}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_1 & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_2 & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_3 \end{bmatrix}$$

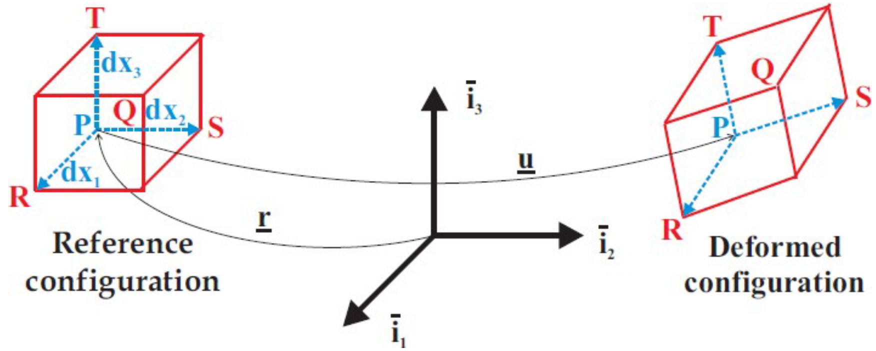
Second order tensor

$$\boldsymbol{\tau} = \boldsymbol{\sigma} \cdot \mathbf{n}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$$

The stress tensor is symmetric

Strain: Linear strain



$$\mathbf{u}(x_1(t), x_2(t), x_3(t)) = \mathbf{u}(\mathbf{x}(t)) = u_1(\mathbf{x}(t))\mathbf{i}_1 + u_2(\mathbf{x}(t))\mathbf{i}_2 + u_3(\mathbf{x}(t))\mathbf{i}_3$$

$$\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$$

if $||\mathbf{u}|| \ll 1$

$$\varepsilon = \frac{1}{2}\left(\boldsymbol{\nabla}_0\mathbf{u}^T + \boldsymbol{\nabla}_0\mathbf{u}\right)$$

$$\mathbf{E} = \frac{1}{2}\left(\boldsymbol{\nabla}_0\mathbf{u}^T + \boldsymbol{\nabla}_0\mathbf{u} + \boldsymbol{\nabla}_0\mathbf{u}^T\boldsymbol{\nabla}_0\mathbf{u}\right)$$

$\boldsymbol{\nabla}_0$ Gradient with respect to the reference system in the undeformed configuration

$$\boldsymbol{\nabla}_0 = \frac{\partial \mathbf{u}}{\partial \mathbf{x}}$$

Extensional and shear strains

Extensional strain

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_3 = \frac{\partial u_3}{\partial x_3}$$

Shear strain (or angular distortion)

$$\gamma_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}, \quad \gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}, \quad \gamma_{12} = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}.$$

$$\gamma_{ij} = 2\varepsilon_{ij}$$

Constitutive law: linear, isotropic, elastic material

Generalized Hook's law $\sigma_{ij} = C_{ijhk}\varepsilon_{hk}$

$$\begin{aligned} \varepsilon_1 &= \frac{1}{E}\left(\sigma_1 - \nu\left(\sigma_2 + \sigma_3\right)\right) \\ \varepsilon_2 &= \frac{1}{E}\left(\sigma_2 - \nu\left(\sigma_1 + \sigma_3\right)\right) \\ \varepsilon_3 &= \frac{1}{E}\left(\sigma_3 - \nu\left(\sigma_1 + \sigma_2\right)\right) \end{aligned}$$

$$\gamma_{23} = \tau_{23}/G, \quad \gamma_{13} = \tau_{13}/G, \quad \gamma_{12} = \tau_{12}/G.$$

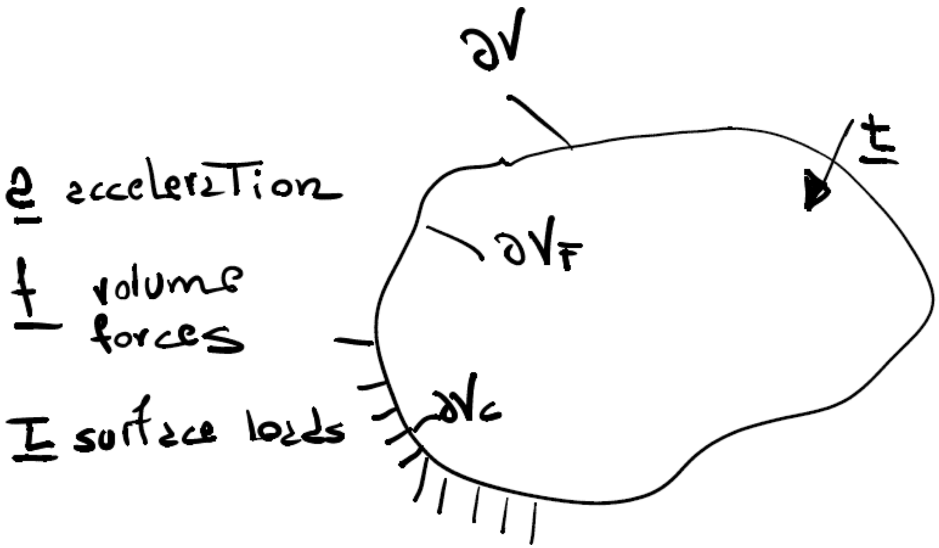
$$G = \frac{E}{2(1+\nu)}$$

$$\begin{aligned} \underline{\varepsilon} &= \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}\}^T \\ \underline{\sigma} &= \{\sigma_1, \sigma_2, \sigma_3, \tau_{23}, \tau_{13}, \tau_{12}\}^T \end{aligned}$$

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 & 0 & 0 \\ \nu & (1-\nu) & \nu & 0 & 0 & 0 \\ \nu & \nu & (1-\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\underline{\sigma} = \mathbf{C}\underline{\varepsilon}$$

Principle of Virtual Works



$$\delta W_e = \delta W_i$$

$$\begin{aligned} \delta W_e &= \int_V \delta \mathbf{u}^T(-\rho\mathbf{a}+\mathbf{f})\mathrm{d}v + \int_{\partial V_F} \delta\mathbf{u}^T\mathbf{t}\mathrm{d}a \\ \delta W_i &= \int_V \delta\boldsymbol{\varepsilon} : \boldsymbol{\sigma}\mathrm{d}v = \int_V \delta\underline{\boldsymbol{\varepsilon}}^T\,\underline{\boldsymbol{\sigma}}\mathrm{d}v \end{aligned}$$

$$\delta\boldsymbol{\varepsilon} = \boldsymbol{\nabla}_0\delta\mathbf{u}$$

Appendix: from weak to strong formulation

Using the divergence theorem

$$
\begin{aligned}
\int_V \delta \mathbf{u}^T \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \mathrm{d}v &= \int_V \boldsymbol{\nabla} \cdot \left(\delta \mathbf{u}^T \boldsymbol{\sigma}\right) \mathrm{d}v - \int_V \boldsymbol{\nabla} \delta \mathbf{u} : \boldsymbol{\sigma} \mathrm{d}v \\
\int_V \delta \boldsymbol{\varepsilon} : \boldsymbol{\sigma} \mathrm{d}v &= \int_V \boldsymbol{\nabla} \delta \mathbf{u} : \boldsymbol{\sigma} \mathrm{d}v = -\int_V \delta \mathbf{u}^T \boldsymbol{\nabla} \cdot \boldsymbol{\sigma} \mathrm{d}v + \int_{\partial V_F} \delta \mathbf{u}^T \boldsymbol{\sigma} \cdot \mathbf{n} \mathrm{d}a
\end{aligned}
$$

Given the arbitrariness of $\delta \mathbf{u}$ that must satisfy only the essential boundary conditions over V_C the following PDE results

$$
\begin{aligned}
&\boldsymbol{\nabla} \cdot \boldsymbol{\sigma} - \rho \mathbf{a} + \mathbf{f} = \mathbf{0} \quad \text{in } V \\
&\boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{t} \quad \text{on } \partial V_F \\
&\mathbf{u} = \mathbf{u}_c \quad \text{on } \partial V_C
\end{aligned}
$$

To these equations we have to addd the initial conditions

$$
\begin{aligned}
\mathbf{u}(0) &= \mathbf{u}_0 \\
\dot{\mathbf{u}}(0) &= \mathbf{v}_0
\end{aligned}
$$

Beams

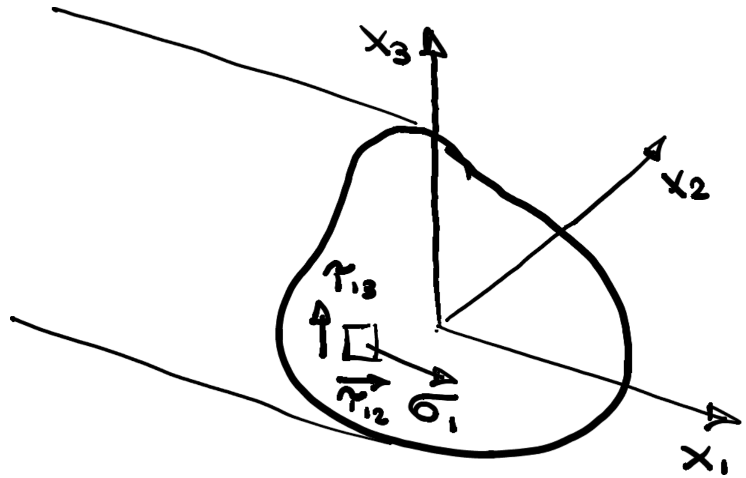
- A beam is a slender structural component, essentially mono-dimensional.
- Geometric and structural properties are assumed to change way more regularly spanwise than section-wise. This means that abrupt section changes, or abrupt bends are not allowed.
- To be representative, it is necessary to reduce the complex behavior of a 3D structure to these simple, and rather schematic, elements.
- From the forces point of view, the beam is represented by the resultant of stresses acting on its section.
- Form the kinematic point of view, the beam is described by the behavior of a reference line; generalized stain measures must be defined (associated with the change of shape of the reference line).

Beams

Approaches to develop a beam model

1. *Semi-inverse or induced*. Some ad-hoc hypothesis on the cross-section deformation are made which lead to the governing equations. Examples: De-Saint Venant theory, i.e. Euler-Bernoulli beam; Thimoshenko beam.
2. *Intrinsic theories:* Self-contained internally consistent characterization of the motion of a slender solid carried out by a finite set of strain parameters that depend on one independent space variable, which is typically the **arclength** coordinate along the beam. Various levels of detail can be incorporated by increasingly richer kinematic descriptions.

Stress resultants



Given the type of loading only the stresses on the section normal to the beam axis are different from zero

$$N = \int_{A_s} \sigma_1 \mathrm{d}a, \quad \text{AXIAL}$$

$$S_2 = S_y = \int_{A_s} \tau_{12} \mathrm{d}a, \quad S_3 = S_z = \int_{A_s} \tau_{13} \mathrm{d}a \quad \text{SHEAR}$$

$$M_2 = M_y = \int_{A_s} x_3 \sigma_1 \mathrm{d}a, \quad M_3 = M_z = -\int_{A_s} x_2 \sigma_1 \mathrm{d}a \quad \text{BENDING}$$

$$M_t = T = \int_{A_s} (x_2 \tau_{13} - x_3 \tau_{12}) \, \mathrm{d}a \quad \text{TORSION}$$

Euler-Bernoulli assumptions

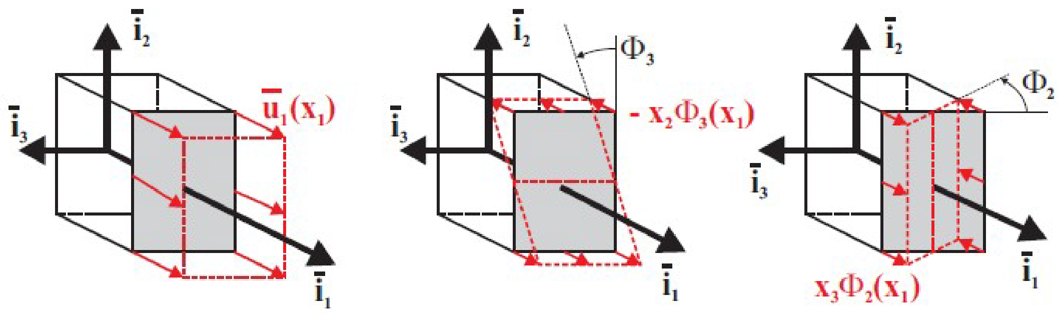
1. The cross-section is infinitely rigid in its own plane (does not change shape)
2. The cross section of a beam remains plain after deformation
3. The cross-section remains normal to the deformed axis of the beam

Thanks to assumption #1 the displacements in the plane of the section are rigid, i.e. not function of the position on the section

$$u_2(x_1, x_2, x_3) = v(x_1), \quad u_3(x_1, x_2, x_3) = w(x_1)$$

Euler-Bernoulli assumptions

Thanks to assumption #2 the axial displacement could be only composed by one translation and two rotations

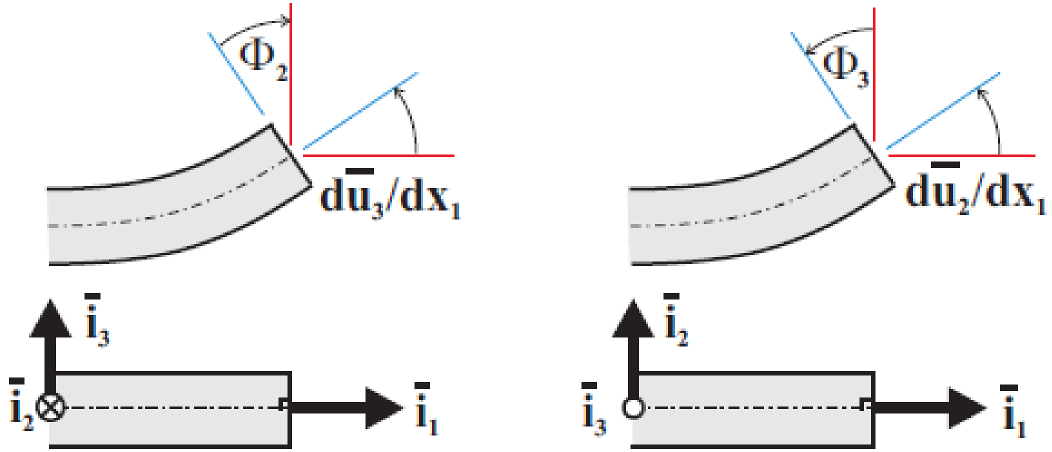


$$u_1(x_1, x_2, x_3) = u(x_1) + x_3\phi_2(x_1) - x_2\phi_3(x_1)$$

Euler-Bernoulli assumptions

Thanks to the assumption #3

$$\phi_2 = -\frac{\mathrm{d}w}{\mathrm{d}x_1} = -w'$$
$$\phi_3 = \frac{\mathrm{d}v}{\mathrm{d}x_1} = v'$$



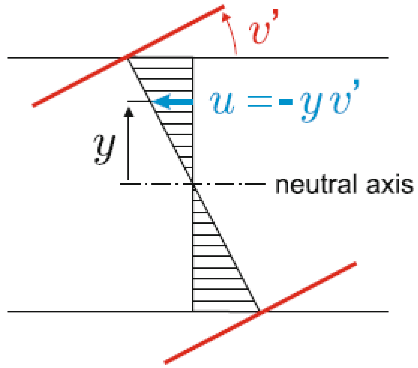
Strain in the beam

$$\begin{cases} u_1 & = u(x_1) - x_3 w'(x_1) - x_2 v'(x_1) \\ u_2 & = v(x_1) \\ u_3 & = w(x_1) \end{cases}$$

Consequently

$$\varepsilon_2 = \frac{\partial v}{\partial x_2} = 0, \quad \varepsilon_3 = \frac{\partial w}{\partial x_3} = 0$$

$$\gamma_{13} = \frac{\partial u_1}{\partial x_3} + \frac{\partial w}{\partial x_1} = 0, \quad \gamma_{12} = \gamma_{23} = 0$$



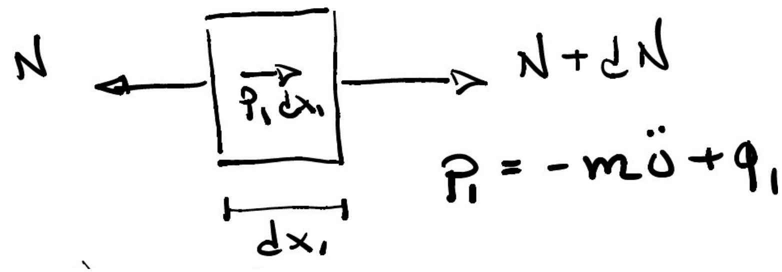
$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = u' - x_3 w'' - x_2 v''$$

SECTIONAL CURVATURES

The strain is linear over the section and this is a direct consequence of assumption #3

Beam subject to axial load

$$N(x_1) = \int_A \sigma_1 \mathrm{d}a = \int_A E(x_1, x_2, x_3)\varepsilon_1 \mathrm{d}a$$



Let's consider the case where only axial deformation is present i.e., $\varepsilon_1 = u'(x_1)$

p_1 distributed axial load per unit length N/m

$$\begin{aligned} N &= \int_A E \mathrm{d}a u' \rightarrow EA^* = \int_A E \mathrm{d}a \\ EA^* &= EA \,\text{only if}\, E(\bar{x}_1, x_2, x_3) = \text{const.} \\ N &= EA^*(x_1)u' \end{aligned}$$

Constitutive law

Equilibrium

$$(N + \mathrm{d}N) - N + p_1 \mathrm{d}x_1 = 0$$

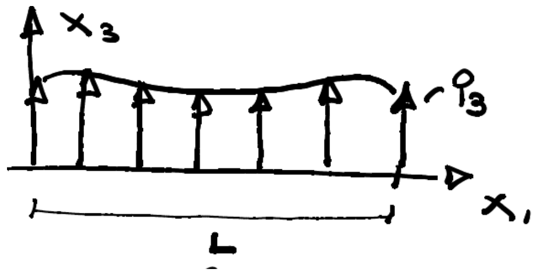
$$\frac{\mathrm{d}N}{\mathrm{d}x_1} = -p_1, \quad N' = -p_1$$

$$\boxed{\begin{array}{l} (EA^*u')' - m\ddot{u} = -q_1, \\ +2\mathrm{BC} \ + 2\mathrm{IC} \end{array}}$$

Virtual work due to axial load

$$\begin{aligned} \delta W_i &= \int_V \delta \underline{\boldsymbol{\varepsilon}}^T \underline{\boldsymbol{\sigma}} \mathrm{d}v = \int_L \delta u'^T \mathrm{d}x_1 \int_A \sigma_1 \mathrm{d}a \\ \delta W_i &= \int_L \delta u'^T EA^* u' \mathrm{d}x_1 \end{aligned}$$

(Pure) Beam bending



Let's suppose only transverse displacements w are not null

$$
\begin{aligned}
u_1 &= -x_3 w' \\
\varepsilon_1 &= -x_3 w'' \\
\sigma_1 &= -E x_3 w''
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= M_y = \int_A x_3 \sigma_1 \mathrm{d}a = -\int_A E x_3^2 \mathrm{d}a \, w'' \\
EI_2^* &= \int_A E x_3^2 \mathrm{d}a \rightarrow M_2 = -EI_2^* w''
\end{aligned}
$$

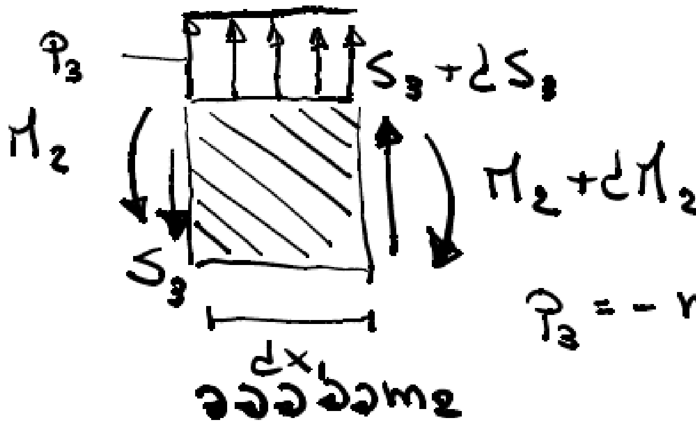
Constitutive law

The reference system about which we measure the bending moment to decouple from axial load cannot be generic, but <u>it must be placed in the centroid of the area section (weighted by E)</u>

$$N = \int_A \sigma_1 \mathrm{d}a = -\int_A E x_3 \mathrm{d}a \, w''$$

$$N = 0 \text{ only if } \int_A E x_3 \mathrm{d}a = 0$$

Beam Bending



$$p_3 = -m\ddot{u}_3 + q_3$$

Equilibrium

$$(\cancel{S_3} + \mathrm{d}S_3) - \cancel{S_3} + p_3 \mathrm{d}x_1 = 0$$

$$(\cancel{M_2} + \mathrm{d}M_2) - \cancel{M_2} - S_3 \mathrm{d}x_1 + \cancel{p_3 \mathrm{d}x_1 \frac{\mathrm{d}x_1}{2}} + m_2 \mathrm{d}x_1 = 0$$

$$\begin{cases} \dfrac{\mathrm{d}S_3}{\mathrm{d}x_1} = -p_3 \\ \dfrac{\mathrm{d}M_2}{\mathrm{d}x_1} - S_3 + m_2 = 0 \end{cases} \rightarrow M_2'' + m_2' = -p_3 \rightarrow (EI_2^* w'')'' = p_3 + m_2'$$

Beam Bending

$$\boxed{(EI_2^* w'')'' + m\ddot{w} = q_3 + m_2' \quad + 4BC \ + 2IC}$$

Internal virtual work

$$\delta W_i = \int_V \delta \underline{\boldsymbol{\varepsilon}}^T \underline{\boldsymbol{\sigma}} \mathrm{d}v = \int_L \delta w''^T \int_A E x_3^2 \mathrm{d}a w'' \mathrm{d}x_1$$

$$\delta W_i = \int_L \delta w''^T EI_2^* w'' \mathrm{d}x_1$$

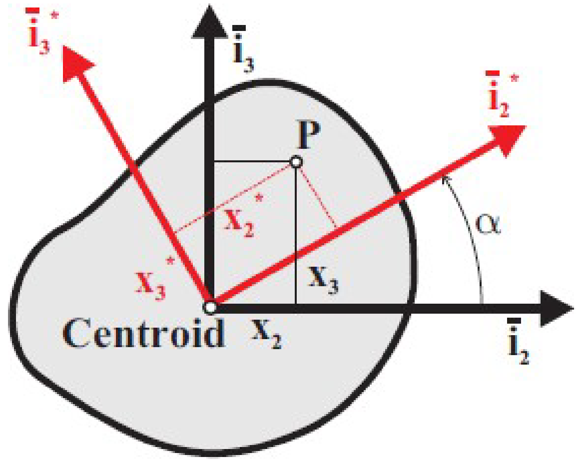
$$EI_3^* = \int_A E x_2^2 \mathrm{d}x_1 \rightarrow M_3 = M_y = EI_3^* v''$$

$$\boxed{(EI_3^* v'')'' + m\ddot{v} = q_2 - m_3' \quad + 4BC \ + 2IC}$$

Internal virtual work

$$\delta W_i = \int_L \delta v'' EI_3^* v'' \mathrm{d}x_1$$

However, in the most general case



$$\sigma_1 = E\varepsilon_1 = E\left(u' - x_3 w'' - x_2 v''\right)$$

$$N = \int_A \sigma_1 \mathrm{da} = EA^* u' - V_3 w'' - V_2 v''$$

where

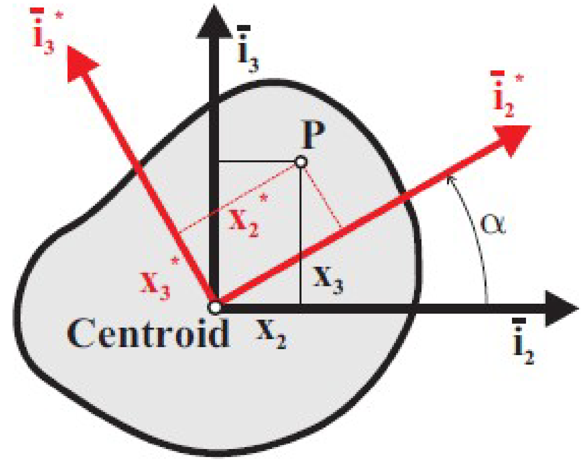
$$V_2^* = \int_A E x_2 \mathrm{da}, \quad V_3^* = \int_A E x_3 \mathrm{da}$$

$$M_3 = -\int_A x_2 \sigma_1 \mathrm{da} = -V_2^* u' + EI_{23}^* w'' + EI_2^* v''$$

where

$$EI_{23}^* = \int_A E x_2 x_3 \mathrm{da}$$

However, in the most general case



Most general form of the constitutive law (fully coupled)

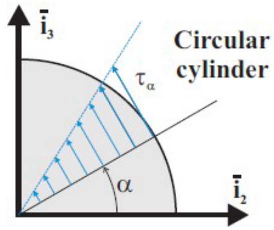
$$\begin{Bmatrix} N \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} EA^* & V_3^* & -V_2^* \\ V_3^* & EI_2^* & -EI_{23}^* \\ -V_2^* & -EI_{23}^* & EI_3^* \end{bmatrix} \begin{Bmatrix} u' \\ -w'' \\ v'' \end{Bmatrix}$$

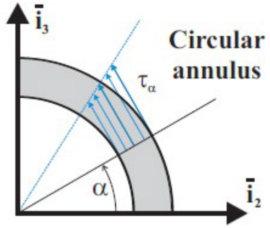
The decoupling between the axial and bending problems is obtained if the reference line of the beam is at the E-weighted centroid of the section i.e., $V_2^* = V_3^* = 0$.
The two bending problems are decoupled if the axes $\mathbf{i}_2$ and $\mathbf{i}_3$ are oriented so that $EI_{23}^* = 0$

$$\begin{Bmatrix} N \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} EA^* & 0 & 0 \\ 0 & EI_2^* & 0 \\ 0 & 0 & EI_3^* \end{bmatrix} \begin{Bmatrix} u' \\ -w'' \\ v'' \end{Bmatrix}$$

Torsion

Pure rotation of the section with an angle θ



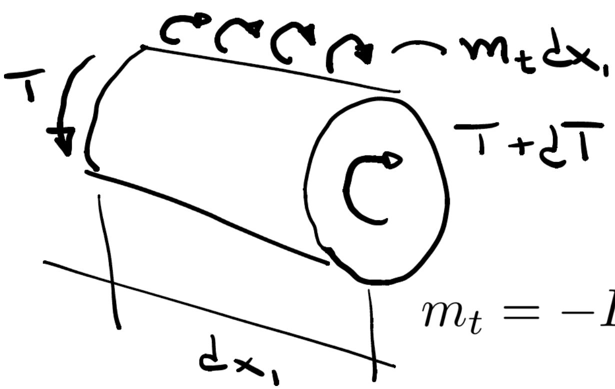


$$
\begin{cases} u_1 &= 0 \\ u_2 &= -x_3\theta(x_1) \\ u_3 &= x_2\theta(x_1) \end{cases} \Rightarrow \begin{cases} \varepsilon_1 &= \varepsilon_2 = \varepsilon_3 = 0 \\ \gamma_{12} &= \dfrac{\partial u_2}{\partial x_1} = -x_3\theta' \\ \gamma_{13} &= \dfrac{\partial u_3}{\partial x_1} = x_2\theta' \\ \gamma_{23} &= \dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2} = -\theta + \theta = 0 \end{cases}
$$

$$
\begin{aligned} \tau_{12} &= G\gamma_{12} = -Gx_3\theta' \\ \tau_{13} &= G\gamma_{13} = Gx_2\theta' \end{aligned}
$$

$$
\begin{aligned} T &= \int_A (x_2\tau_{13} - x_3\tau_{12})\,\mathrm{d}a \\ T &= \int_A G\left(x_2^2 + x_3^2\right)\theta'\mathrm{d}a \\ T &= \int_A Gr^2\mathrm{d}a\psi' = GJ^*\theta' \end{aligned}
$$

Constitutive law



$$m_t = -I_\theta\ddot{\theta} + q_t$$

$$(T + \mathrm{d}T) - T + m_t\mathrm{d}x_1 = 0$$

$$T' + m_t = 0$$

Equilibrium

Torsion

$$\boxed{(GJ^*\theta')' - I_\theta\ddot{\theta} = -q_t \; + 2BC, \; +2IC}$$

$$\delta W_i = \int_V \delta\underline{\boldsymbol{\varepsilon}}^T \underline{\boldsymbol{\sigma}} \mathrm{d}v = \int_v \left(\delta\gamma_{12}^T \tau_{12} - \gamma_{13}^T \tau_{13}\right) \mathrm{d}v$$

$$\delta W_i = \int_L \delta\theta'^T GJ^* \theta' \mathrm{d}x_1$$

Virtual work due to torsion

Summary Euler-Bernoulli

Equilibrium

$$\begin{cases} N' - m\ddot{u} + q_1 = 0 \\ M_2'' - m\ddot{w} + q_3 + m_2' = 0 \\ M_3'' + m\ddot{v} - q_2 + m_3' = 0 \\ T' - I_\theta \ddot{\theta} + q_t = 0 \end{cases}$$

Constitutive law

$$\begin{Bmatrix} N \\ M_2 \\ M_3 \\ T \end{Bmatrix} = \begin{bmatrix} EA^* & 0 & 0 & 0 \\ 0 & EI_2^* & 0 & 0 \\ 0 & 0 & EI_3^* & 0 \\ 0 & 0 & 0 & GJ^* \end{bmatrix} \begin{Bmatrix} u' \\ -w'' \\ v'' \\ \theta' \end{Bmatrix}$$

$$\begin{cases} (EA^* u')' - m\ddot{u} + q_1 = 0 \\ (EI_2^* w'')'' + m\ddot{w} - q_3 - m_2' = 0 \\ (EI_3^* v'')'' + m\ddot{v} - q_2 + m_3' = 0 \\ (GJ^* \theta')' - I_\theta \ddot{\theta} + q_t = 0 \end{cases}$$

$$\delta W_i = \int_L \delta u'^T EA^* u' \mathrm{d}x_1 + \int_L \delta w'^T EI_2^* w'' \mathrm{d}x_1 + \int_L \delta v''^T EI_3^* v'' \mathrm{d}x_1 + \int_L \delta \theta'^T GJ^* \theta' \mathrm{d}x_1$$

Limits of the Euler Bernoulli beam model: warping

Real beams are free to warp to comply with boundary conditions

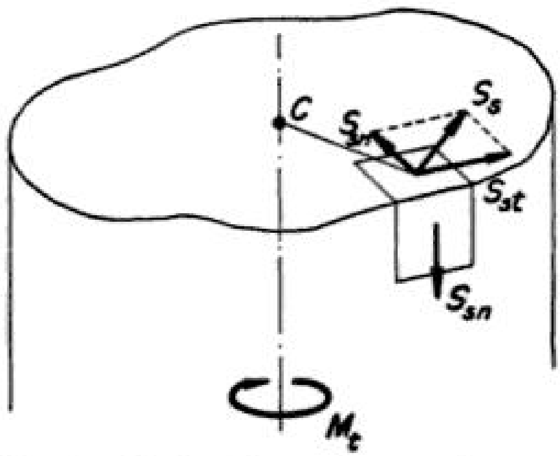


FIG. 1. If the shear stress s_s in a peripheral point of the cross section is perpendicular to a radius from the center of twist C, then it can be resolved into tangential s_{st} and normal s_{sn} components. The normal component must have a companion stress on the free outside surface, which does not exist. Hence the normal component s_{sn} must be absent.

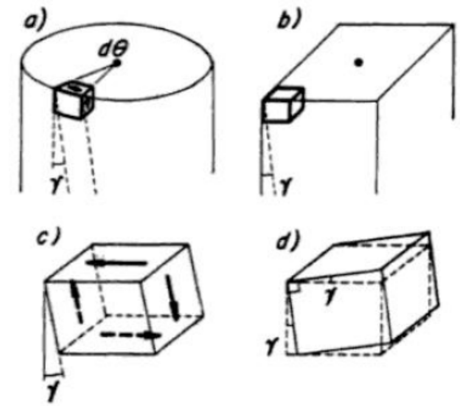
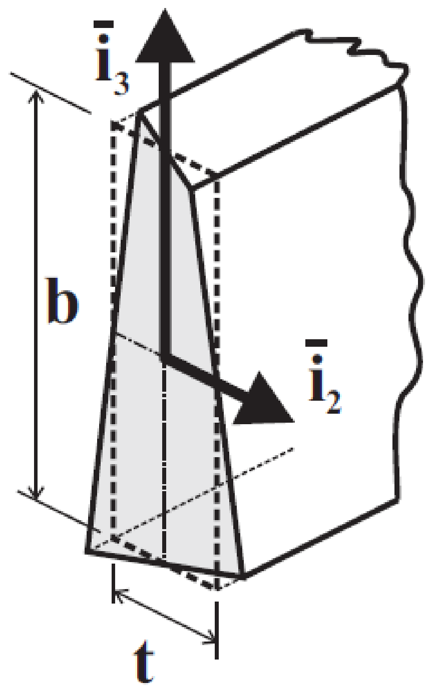


FIG. 2. If in a twisted bar of square cross section a plane cross section should remain plane, there would be shear stresses in the corner, as shown in (c); a zero shear stress in the corner is possible only when the upper surface of (d), that is, the normal cross section, tilts up locally. Only with a circle (a) is a plane cross section possible.

The shear stress at the corner must go to zero. No local twist at the corner is possible. Remember that $\tau_{st} = G\,\gamma_{st}$.
Warping depends on the geometry of the section. When an idealized 1D description of the beam is considered, the info related to section geometry is lost.

Warping

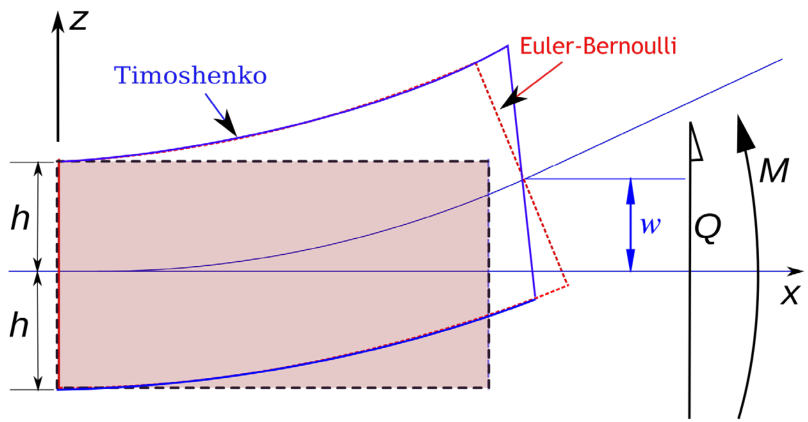
If the twist moment varies along the length of the beam, warping displacement varies along the beam, and torsion is accompanied by tension or compression of longitudinal fibers.

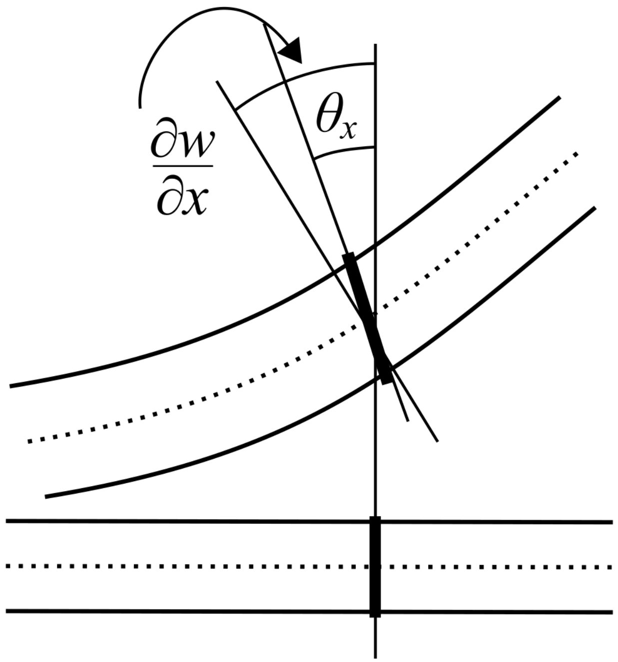


Timoshenko beam

The simplest model to recover an "average" effect of warping is to consider that sections do not remain perpendicular to the beam axis

An independent rotation angle for the section is defined θ





Timoshenko Beam (bending about z)

$$\begin{cases} u_1 &= x_3\theta_y(x_1) \\ u_2 &= 0 \\ u_3 &= w(x_1) \end{cases} \qquad \begin{cases} \varepsilon_1 &= x_3\theta_y' \\ \gamma_{13} &= w' + \theta_y \\ \varepsilon_2 &= \varepsilon_3 = \gamma_{12} = \gamma_{23} = 0 \end{cases}$$

$$\begin{aligned} M_2 &= \int_A x_3\sigma_1 \mathrm{d}a = \int_A E x_3^2 \mathrm{d}a\, \theta_y' = EJ_3^*\theta_y' \\ S_3 &= \int_a \tau_1 3 \mathrm{d}a = \int_A G\gamma_{13}\mathrm{d}a = \int_A G(w' + \theta_y)\mathrm{d}a \\ S_3 &= GA^*(w' + \theta_y) \end{aligned}$$

$$\begin{cases} \dfrac{\mathrm{d}S_3}{\mathrm{d}x_1} = -p_3 \\ \dfrac{\mathrm{d}M_2}{\mathrm{d}x_1} - S_3 + m_2 = 0 \end{cases} \rightarrow \begin{cases} (GA^*(w' + \theta_y))' = -p_3 \\ (EJ_3^*\theta_y')' - GA^*(w' + \theta_y) + m_2 = 0 \end{cases}$$

Virtual work due to bending and shear loads

$$\delta W_i = \int_L \delta\theta_y'^{\,T} EI_3^*\theta_y' \mathrm{d}x_1 + \int_L \delta(w' + \theta_y)^T GA^*(w' + \theta_y)\mathrm{d}x_1$$

Summary

$$\begin{cases} N' - m\ddot{u} + q_1 = 0 \\ S_2' - m\ddot{v} + q_2 = 0 \\ S_3' - m\ddot{w} + q_3 = 0 \\ M_2' + S_3 + m_2 = 0 \\ M_3' - S_2 + m_3 = 0 \\ T' - I_\theta \ddot{\theta} + q_t = 0 \end{cases}$$

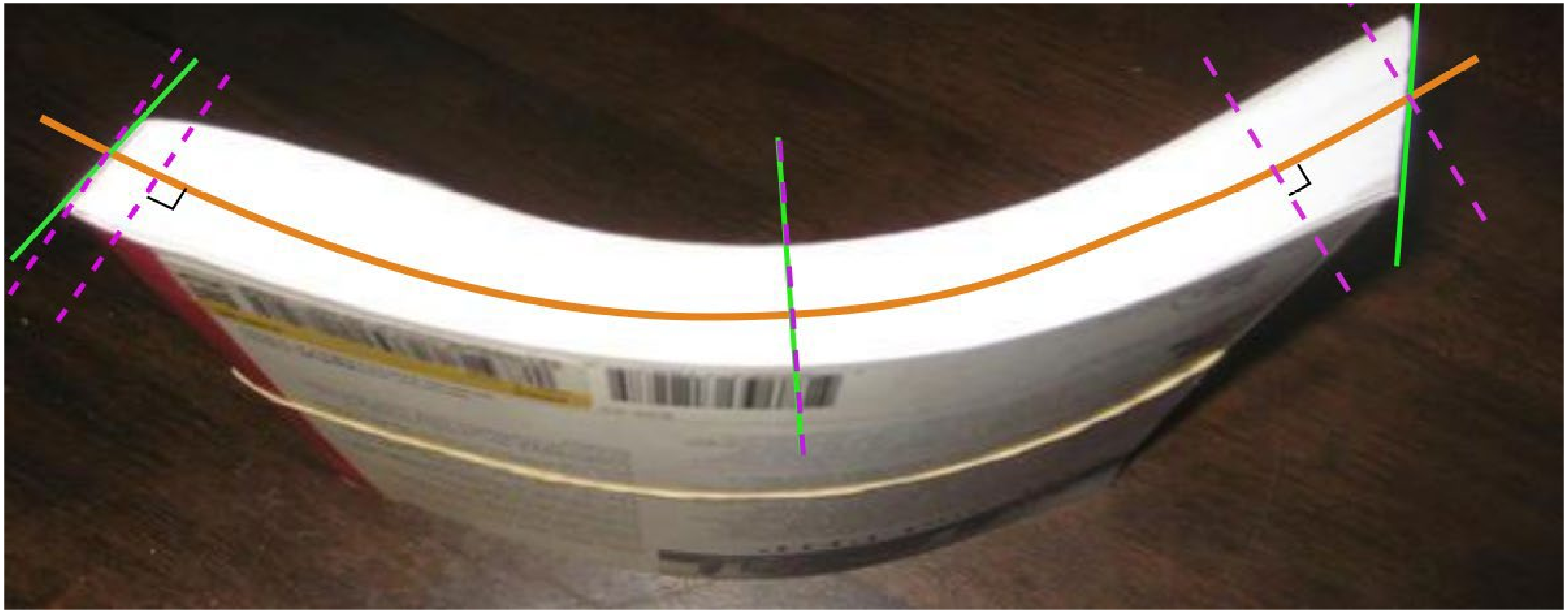
Equilibrium

$$\begin{Bmatrix} N \\ M_2 \\ M_3 \\ S_2 \\ S_3 \\ T \end{Bmatrix} = \begin{bmatrix} EA^* & 0 & 0 & 0 & 0 & 0 \\ 0 & EI_2^* & 0 & 0 & 0 & 0 \\ 0 & 0 & EI_3^* & 0 & 0 & 0 \\ 0 & 0 & 0 & GA^* & 0 & 0 \\ 0 & 0 & 0 & 0 & GA^* & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ^* \end{bmatrix} \begin{Bmatrix} u' \\ \theta_y' \\ \theta_z' \\ \gamma_z \\ \gamma_y \\ \theta' \end{Bmatrix}$$

Constitutive law

$$\begin{aligned} \gamma_y &= w' + \theta_y \\ \gamma_z &= v' - \theta_z \end{aligned}$$

Timoshenko beam example



Free vibrations for the beam bending

Consider the homogeneous problem for a uniform beam

$$EIw'''' + m\ddot{w} = 0$$

This is a partial differential equation in space and time that could be resolved using the approach of separation of variables add the boundary and initial conditions

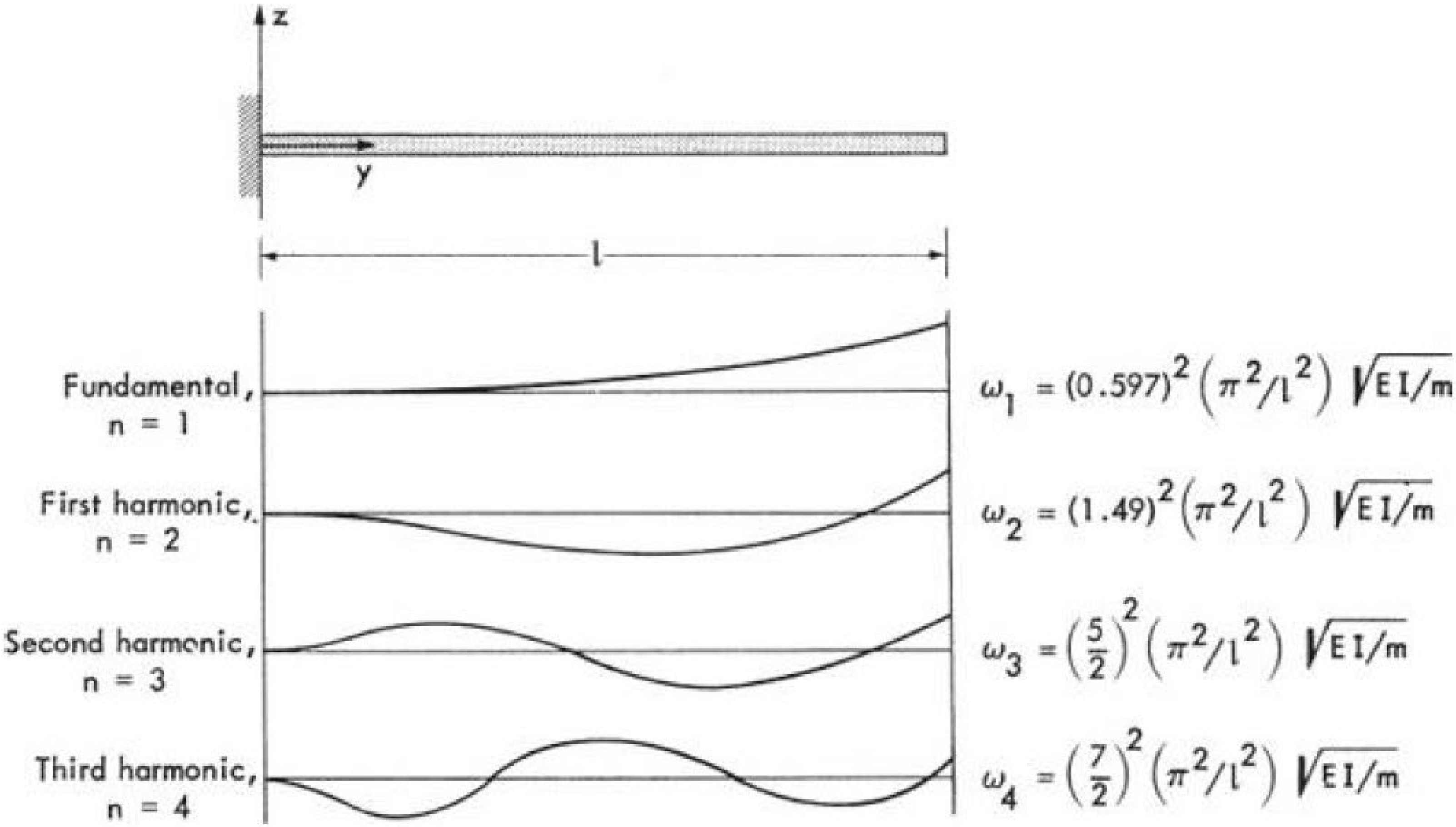
$$\begin{aligned} w(x,0) &= f_1(x) \\ \dot{w}(x,0) &= f_2(x) \\ &+ 4BC \end{aligned}$$

$$w(x,t) = f(x)q(t)$$

$$f(x) = a \sinh \beta x + b \cosh \beta x + c \sin \beta x + d \cos \beta x$$

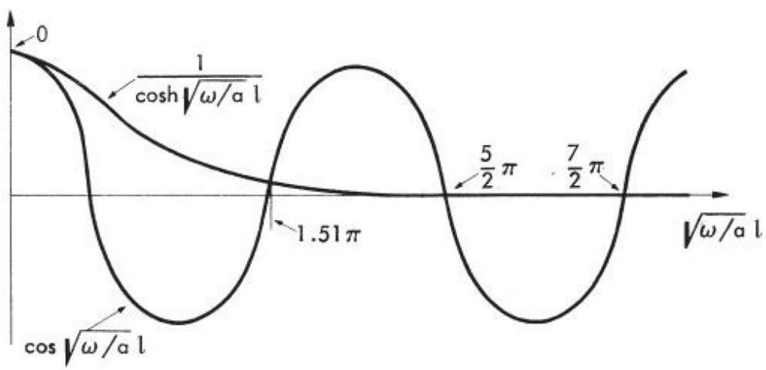
$$q(t) = g \sin \omega t + h \cos \omega t$$

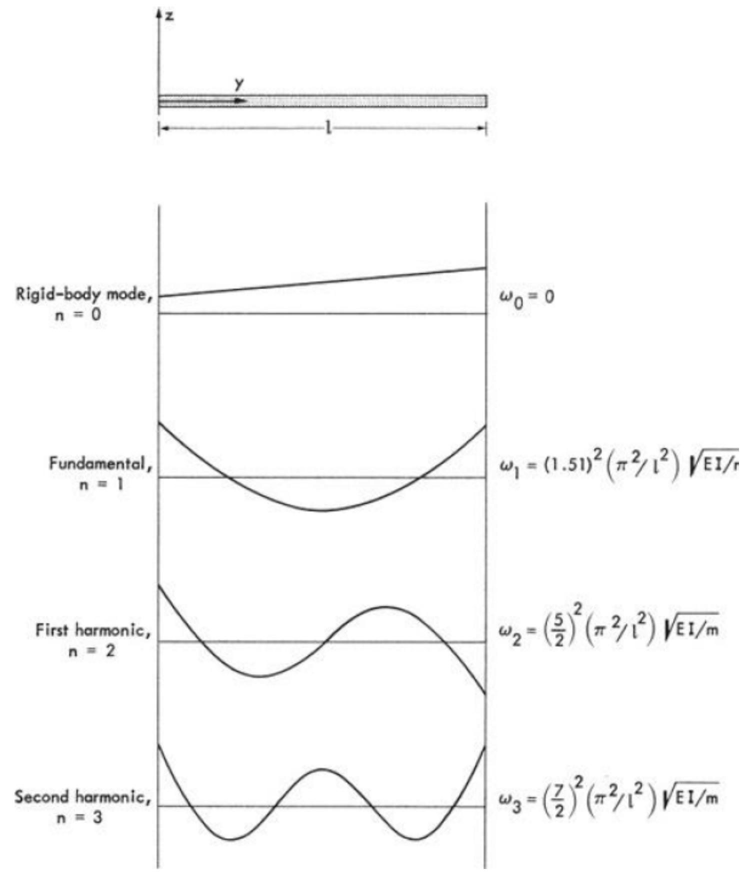
Cantilever


z
y
l
Fundamental, n = 1
First harmonic, n = 2
Second harmonic, n = 3
Third harmonic, n = 4
$\omega_1 = (0.597)^2 (\pi^2/l^2) \sqrt{EI/m}$
$\omega_2 = (1.49)^2 (\pi^2/l^2) \sqrt{EI/m}$
$\omega_3 = \left(\frac{5}{2}\right)^2 (\pi^2/l^2) \sqrt{EI/m}$
$\omega_4 = \left(\frac{7}{2}\right)^2 (\pi^2/l^2) \sqrt{EI/m}$

Unrestrained beam

$$M_2 = f''(0) = f''(L) = 0$$
$$S_3 = f'''(0) = f'''(L) = 0$$





Free vibrations for the beam torsion

$$GJ\theta'' - I_\theta \ddot{\theta} = 0$$

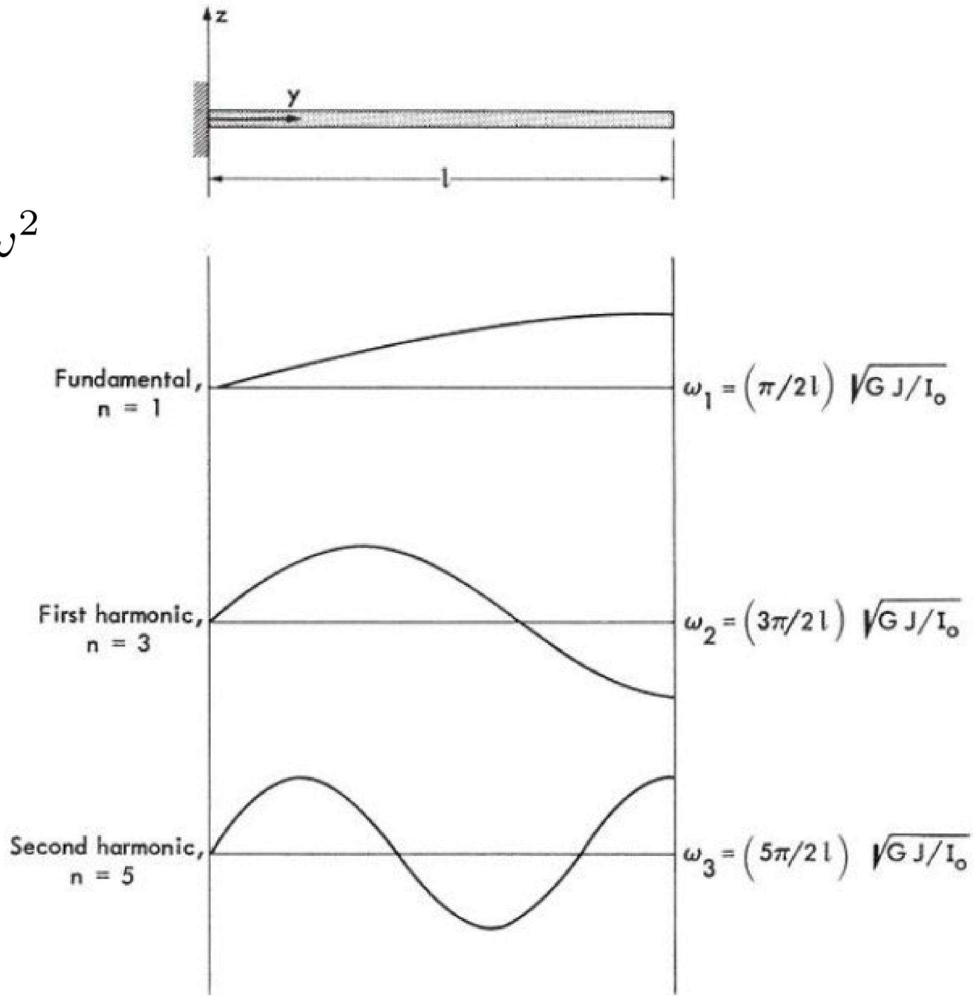
$$\theta = f(x)q(t), \; \frac{\ddot{q}}{q} = \frac{GJ}{I_\theta}\frac{f''}{f} = \omega^2$$

$$\begin{aligned} f(x) &= a\sin\beta x + b\cos\beta x \\ q(x) &= c\sin\omega x + d\cos\omega x \end{aligned}$$

with $\beta^2 = \dfrac{I_\theta \omega^2}{GJ}$.

The boudary conditions are

$$f(0) = 0, f'(L) = 0$$



Properties of modal forms

The generic bending modal solution is

$$w(x,t) = w_i(x)\mathrm{e}^{\mathrm{j}\omega_i t}$$

with the modal frequency ω_i and the eigenfunction (modal shape) w_i. Let's take two modal solution w_i and w_j for whom

$$\begin{aligned}(EJ_2 w_i'')'' - m\omega_i^2 w_i &= 0\\(EJ_2 w_j'')'' - m\omega_j^2 w_j &= 0\end{aligned}$$

Multiply the first by w_j and integrate over the domain

$$\int_0^L w_j\left((EJ_2 w_i'')'' - m\omega_i^2 w_i\right)\mathrm{d}x_1 = 0$$

$$\int_0^L w_j (EJ_2 w_i'')''\mathrm{d}x_1 = \left[(EJ_2 w_i'')' w_j\right]_0^L - \left[EJ_2 w_i'' w_j'\right]_0^L + \int_0^L EJ_2 w_j'' w_i'' \mathrm{d}x_1$$

$$\int_0^L \left(EJ_2 w_j'' w_i'' - m\omega_i^2 w_j w_i\right)\mathrm{d}x_1 = 0$$

Properties of modal forms

$$\int_0^L \left(EJ_2 w_j'' w_i'' - m\omega_i^2 w_j w_i \right) \mathrm{d}x_1 = 0$$

1. Orthogonality with respect to the mass distribution

$$\begin{aligned}\int_0^L EJ_2 w_i'' w_j'' \mathrm{d}x_1 &= \omega_i^2 \int_0^L w_j m w_i \mathrm{d}x_1 \\ \int_0^L EJ_2 w_j'' w_i'' \mathrm{d}x_1 &= \omega_j^2 \int_0^L w_i m w_j \mathrm{d}x_1\end{aligned}$$

$$\Rightarrow (\omega_i^2 - \omega_j^2) \int_0^L m w_i w_j \mathrm{d}x_1 = 0, \Rightarrow \int_0^L m w_i w_j \mathrm{d}x_1 = 0, \quad \forall i \neq j$$

Properties of modal forms

2. Orthogonality with respect to stiffness distribution

$$\Rightarrow 2\int_0^L EJ_2 w_i'' w_j'' \mathrm{d}x_1 = (\omega_i^2 + \omega_j^2)\int_0^L m w_i w_j \mathrm{d}x_1 = 0 \qquad \forall i \neq j$$

$$\Rightarrow \int_0^L EJ_2 w_i'' w_j'' \mathrm{d}x_1 = 0, \quad \forall i \neq j$$

Properties of modal forms

$$\int_0^L m w_i w_j \mathrm{d}x_1 = \mu_i \delta_{ij}$$

$$\int_0^L EJ w_i'' w_j'' \mathrm{d}x_1 = \mu_i \omega_i^2 \delta_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

μ_i Modal Mass

$$\omega_i^2 = \frac{\displaystyle\int_0^L EJ_2 (w_i'')^2 \mathrm{d}x_1}{\displaystyle\int_0^L m w_i^2 \mathrm{d}x_1}$$

RAYLEIGH QUOTIENT

Orthogonality relationships:

- The mode shapes corresponding to distinct frequencies are orthogonal with respect to the mass distribution
- The mode shapes are also orthogonal with respect to the stiffness distribution

Properties of modal forms

- Proper orthogonal models are form functions that mutually orthogonal with respect to mass and stiffness distributions, i.e. to scalar products defined as

$$< w_i, w_j >_m = \int_0^L w_i m w_j \mathrm{d}x_1, \qquad < w_i, w_j >_s = \int_0^L w_i'' EJ w_j'' \mathrm{d}x_1$$

- Consequently, proper orthogonal models can be used to decouple the equations of motion for systems where only elastic and inertia forces are considered

- When other fields are considered, as aerodynamic forces, the equation decoupling capability becomes less important

Modal decomposition

Consider the case where the solution is represented as superimpostion of all modal forms i.e.,

$$w(x,t) = \sum_{i=1}^{\infty} w_i(x) q_i(t)$$

Using the PVW

$$\int_0^L \delta w''^T EJ_2 w'' \mathrm{d}x_1 = \int_0^L \delta w^T \left(-m\ddot{w} + p_3\right) \mathrm{d}x_1$$

Using the modal forms, for each δq_i

$$\delta q_i \sum_{j=1}^{\infty} \int_0^L EJ_2 w_i'' w_j'' \mathrm{d}x_1\, q_j + \delta q_i \sum_{i=1}^{\infty} \int_0^L m w_i w_j \mathrm{d}x_1 \ddot{q}_j = \delta q_i \int_0^L w_i p_3 \mathrm{d}x_1$$

$$\delta q_i \left(\mu_i \left(\ddot{q}_i + \omega_i^2 q_i\right) - \int_0^L w_i p_3 \mathrm{d}x_1\right) = 0$$

This equation states that every mode behaves like a single d.o.f. oscillator of mass μ_i and frequency ω_i; the *generalized force* is once again the work of the external distributed force p on the mode w_i.

if p is a point force

$$p_3(x,t) = \delta(x_0 - x) F(t)$$

δ is the Dirac's delta.

Ritz-Galerkin approximation

For any generic unknown displacement field $u(x,t)$, $x \in [a,b] \subset \mathbb{R}$ and $t \in \mathbb{R}^+$

$$u(x,t) \approx \hat{u}(x,t) = \sum_{i=1}^{n} N_i(x) q_i(t)$$

Properties of the shape functions N_i

1. Linear independence, that means that exist a linear combination with coefficient a_i so that $\forall x \in (a,b)$

$$\sum_i a_i N_i(x) \neq 0$$

2. Completeness Defining the error as

$$\varepsilon_n = \|u(x,t) - \hat{u}(x,t)\|, \quad \text{with } \|a\| = < a, a >^{1/2}$$

$$\lim_{n \to \infty} \varepsilon_n = 0$$

The scalar product could be with respect to mass or stiffness

To be complete the set of functions N_i must satisfy the kinematic boundary conditions (ESSENTIAL BC)

Ritz-Galerkin approximation

If we apply the Ritz- Galerkin approach to our model considering as shape function the proper orthogonal modes and starting from the weak PVW formulation the result will be

$$\int_0^L \delta w''^T EJ_2 \delta w'' \mathrm{d}x_1 + \int_0^L \delta w^T m \ddot{w} \mathrm{d}x_1 = \int \delta w^T p \mathrm{d}x_1$$

$$w(x,t) = \sum_i N_i(x) q_i(t), \quad < N_i, N_j >= 0$$

It is possible to solve the problem as *n* independent harmonic oscillators

$$\delta q_i \sum_{j=1}^n \int_0^L N_i'' EJ_2 N_j'' \mathrm{d}x_1 \, q_j + \delta q_i \sum_{j=1}^n \int_0^L N_i m N_j \mathrm{d}x_1 \, \ddot{q}_j = \delta q_i \int N_i p \mathrm{d}x_1$$

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}_p$$

$$\mathbf{M} = \begin{bmatrix} \ddots & & \\ & \mu_i & \\ & & \ddots \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} \ddots & & \\ & \mu_i \omega_i^2 & \\ & & \ddots \end{bmatrix}, \quad \mathbf{F}_p = \begin{Bmatrix} \vdots \\ \int N_i p \mathrm{d}x_1 \\ \vdots \end{Bmatrix}$$

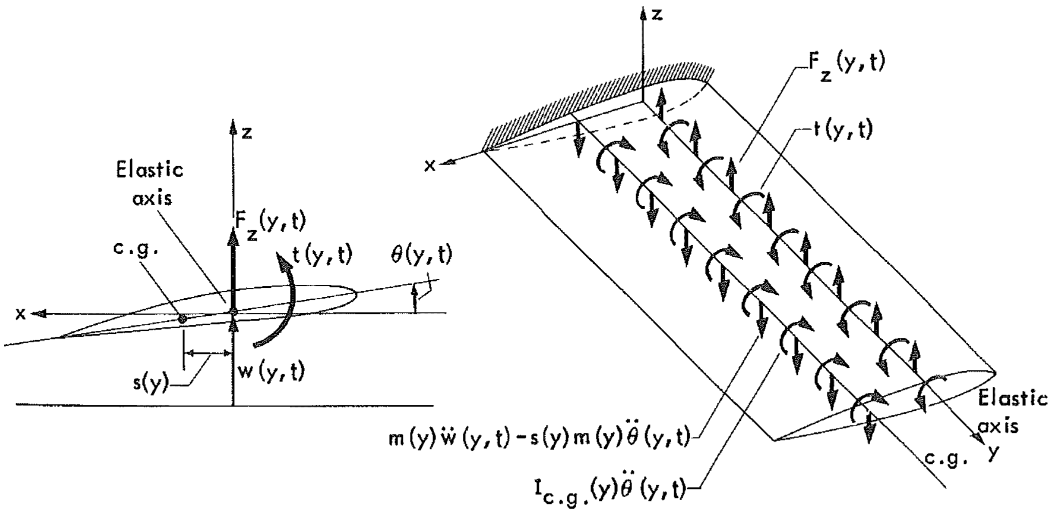
Exercise: Coupled Flexion and Torsion

Write the coupled equations.

Use the Ritz-Galerkin approach to compute the mass matrix and the stiffness matrix of this system

Use

the uncoupled bending modes and the first uncoupled torsion modes



Finite Element Method

$$\hat{u}(x,t) = \sum_i \psi_i(x) q_i(t)$$

Ritz-Galerkin global shape function

FE Local shape function chosen so that

$q_i(t) = \hat{u}(x_i, t)$

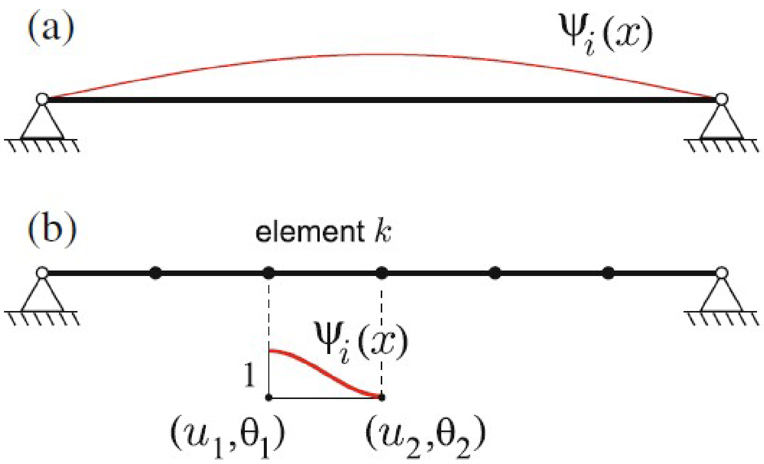
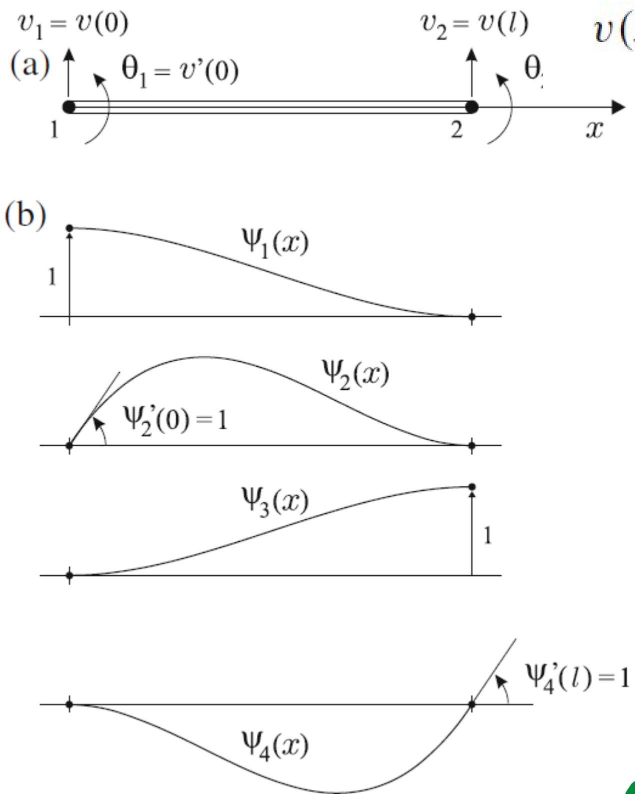


Fig. 6.1 Comparison between the Rayleigh-Ritz method and the finite element method. (a) In the Rayleigh-Ritz method, the shape functions $\psi_i(x)$ are defined globally and must satisfy the kinematic boundary conditions. (b) In the finite element method, the shape functions are defined within the element, in such a way that the generalized coordinates are the nodal displacements u_i and rotations θ_i of the element.

Finite Element Method

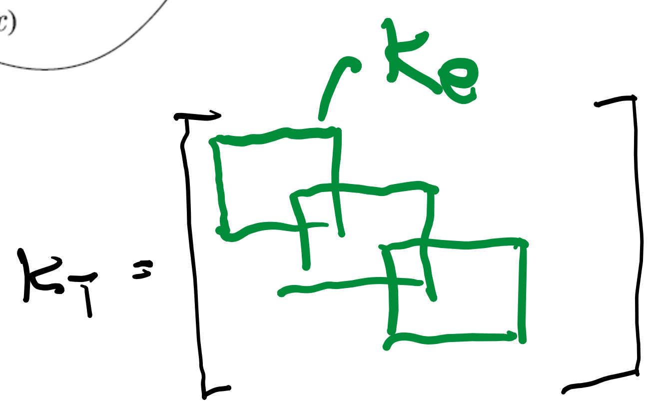


$$v(x,t) = v_1(t)\psi_1(x) + \theta_1(t)\psi_2(x) + v_2(t)\psi_3(x) + \theta_2(t)\psi_4(x)$$

$$\psi_1(x) = 1 - 3x^2/l^2 + 2x^3/l^3$$
$$\psi_2(x) = x - 2x^2/l + x^3/l^2$$
$$\psi_3(x) = 3x^2/l^2 - 2x^3/l^3$$
$$\psi_4(x) = x^3/l^2 - x^2/l$$

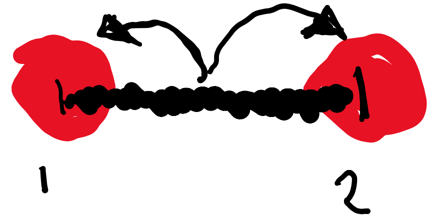
$$K_e^{ij} = \int_0^l EI\,\psi_i''(x)\psi_j''(x)\,dx$$

$$K_e = \frac{EI}{l^3}\begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix}$$



Finite Element Method

$$M_e^{ij} = \int_0^l \varrho A \, \psi_i(x) \psi_j(x) \, dx$$



Lumped matrix

$$M_e = \frac{\varrho A l}{2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_e = \frac{\varrho A l}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ 54 & 13l & 156 & -22l \\ -13l & -3l^2 & -22l & 4l^2 \end{bmatrix}$$

consistent matrix

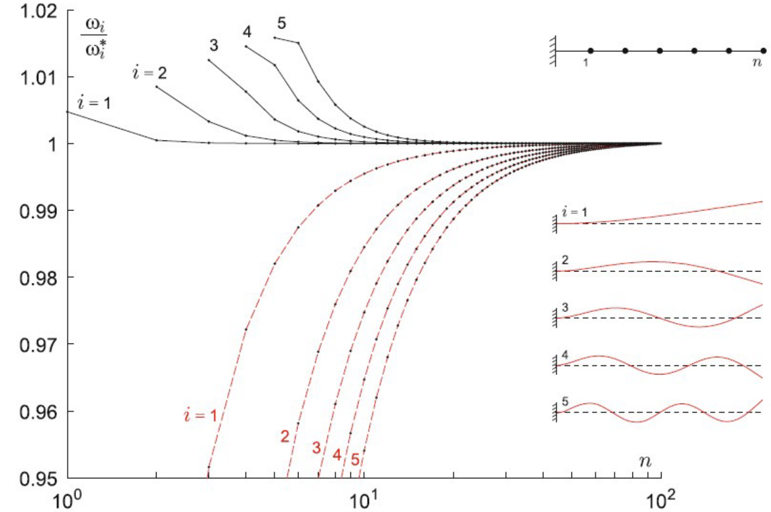


Fig. 6.6 Convergence analysis: uniform clamped-free beam; evolution of the natural frequencies ω_i normalized to the analytical values $\omega_i^\star$ as a function of the number n of elements in the model. Full line: consistent mass matrix with convergence from above. Dashed lines: lumped mass model which converges from below.

Ritz-Galerkin vs FEM

Ritz-Galerkin

- Shape functions with global support
- Matrices M,K are full
- When geometry is complex the definition of appropriate function may be difficult
- The prediction of local effects may require the inclusion of local shape functions who may be difficult to define
- With well defined shape functions the convergence is fast (with few dofs)

FEM

- Shape functions with local support
- Matrices M,K are sparse
- Easy(-er) to apply to complex geometries
- The prediction of local effects may be obtained with local refinement of the number of elements
- Availability of lots of tool (commercial/free/open-source software)