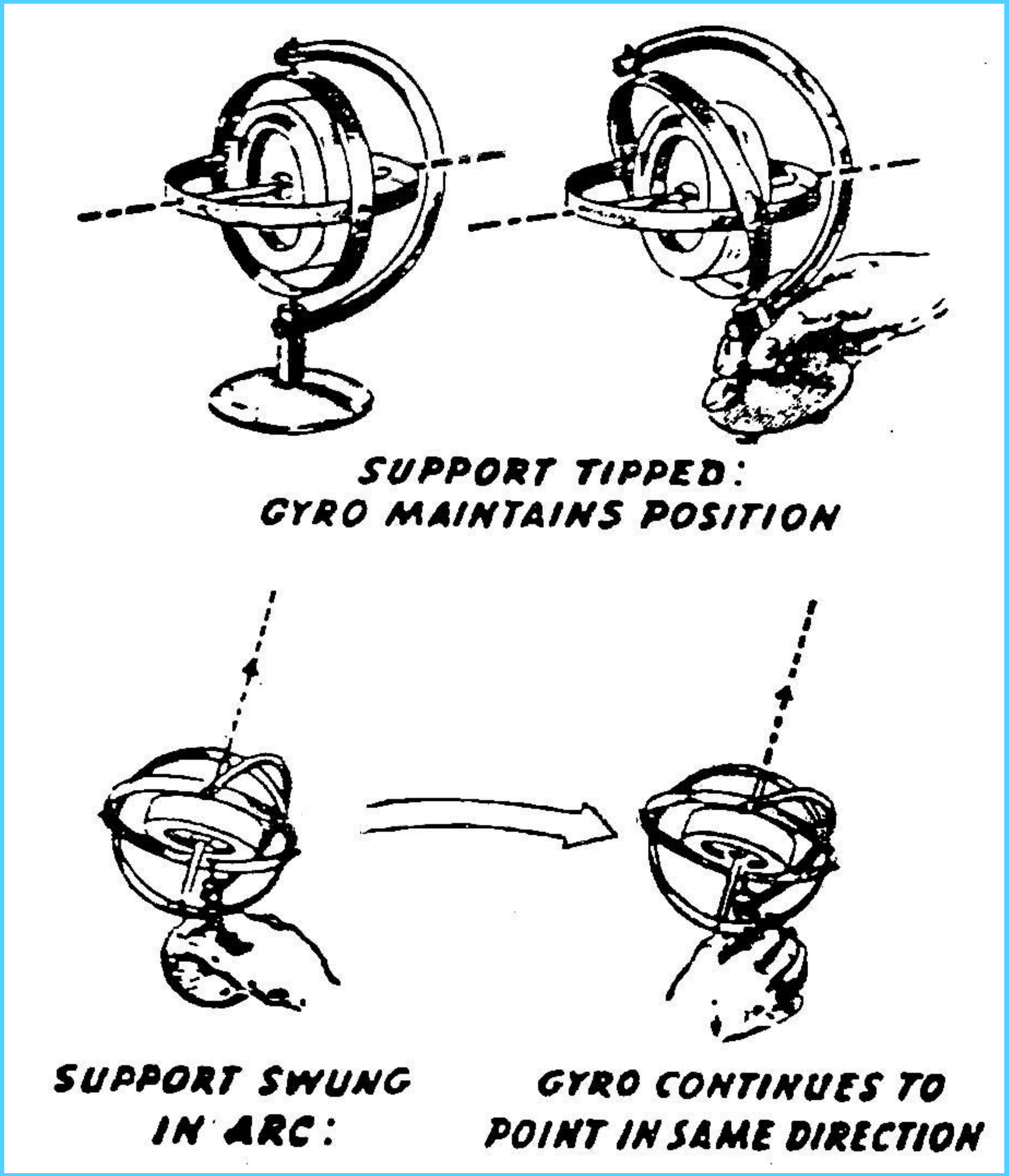
Proprietà del giroscopio



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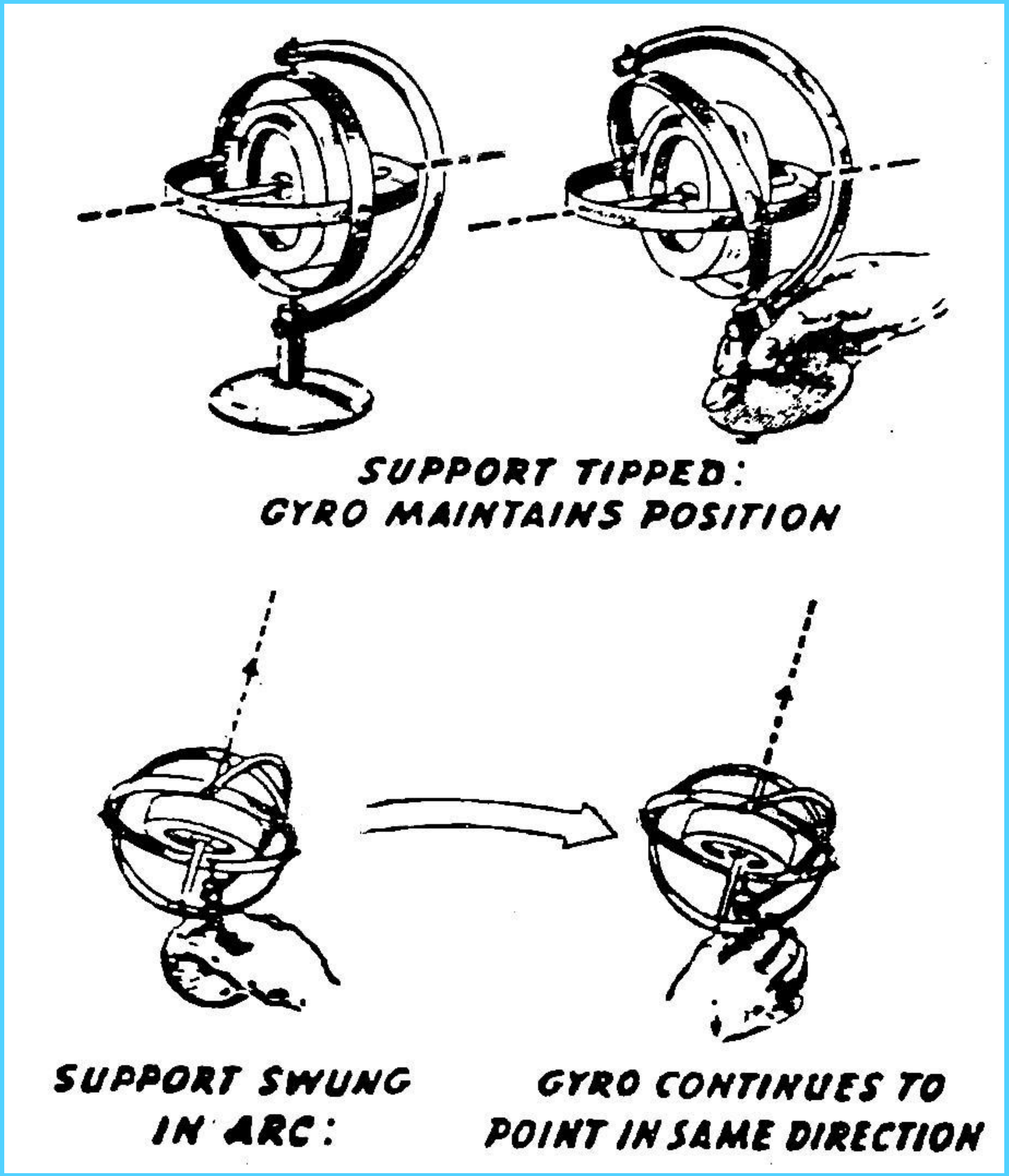
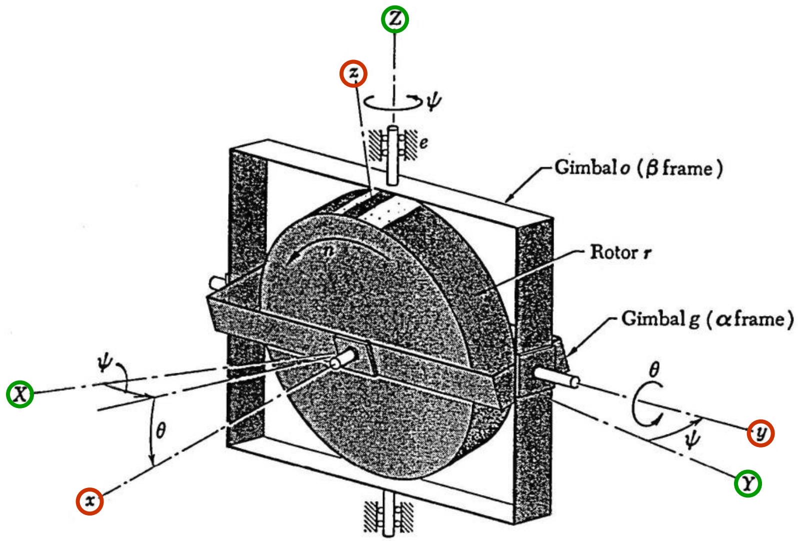
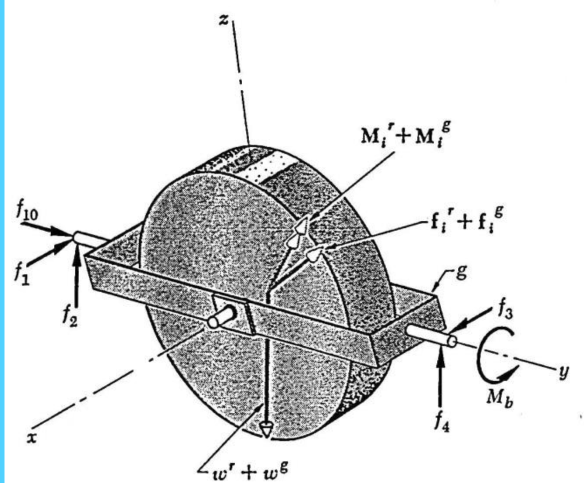
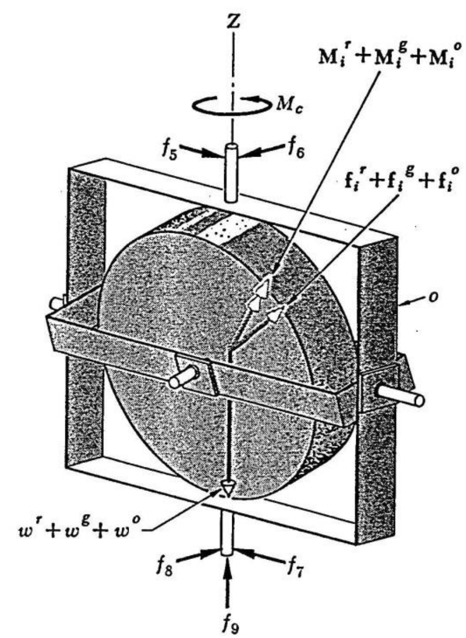




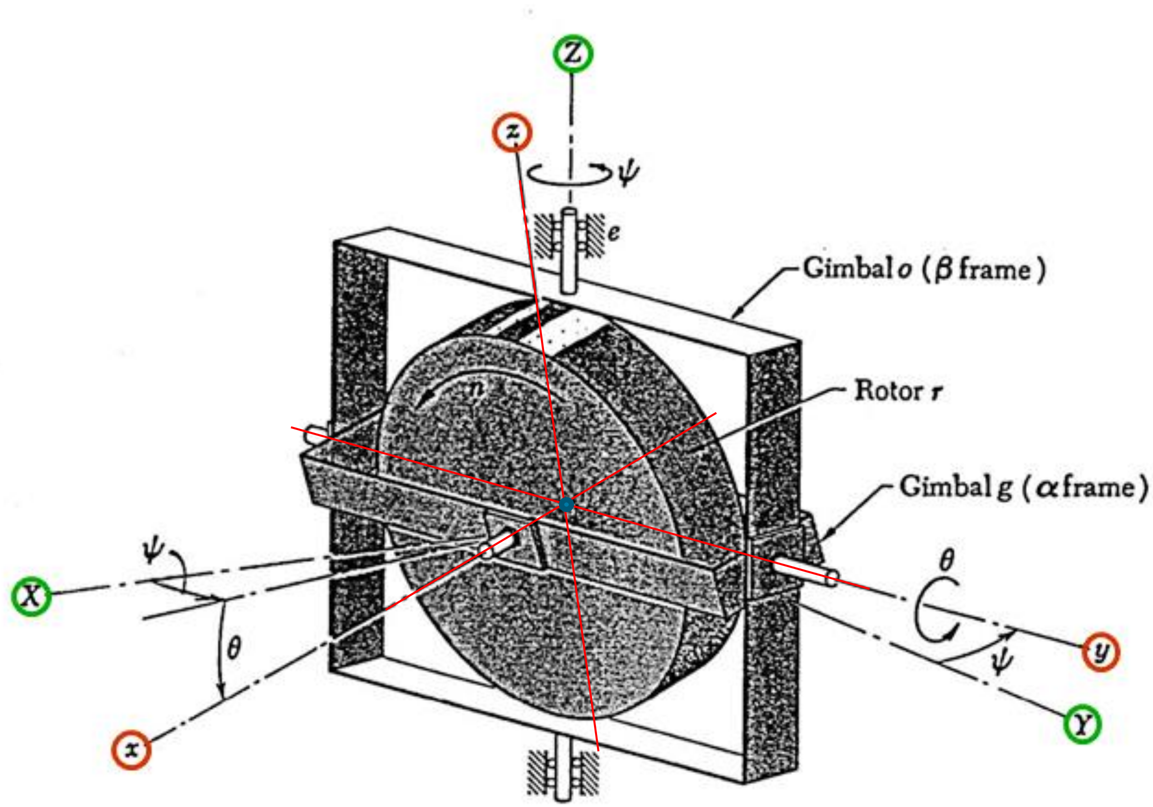
(a) *Arrangement and coordinates*



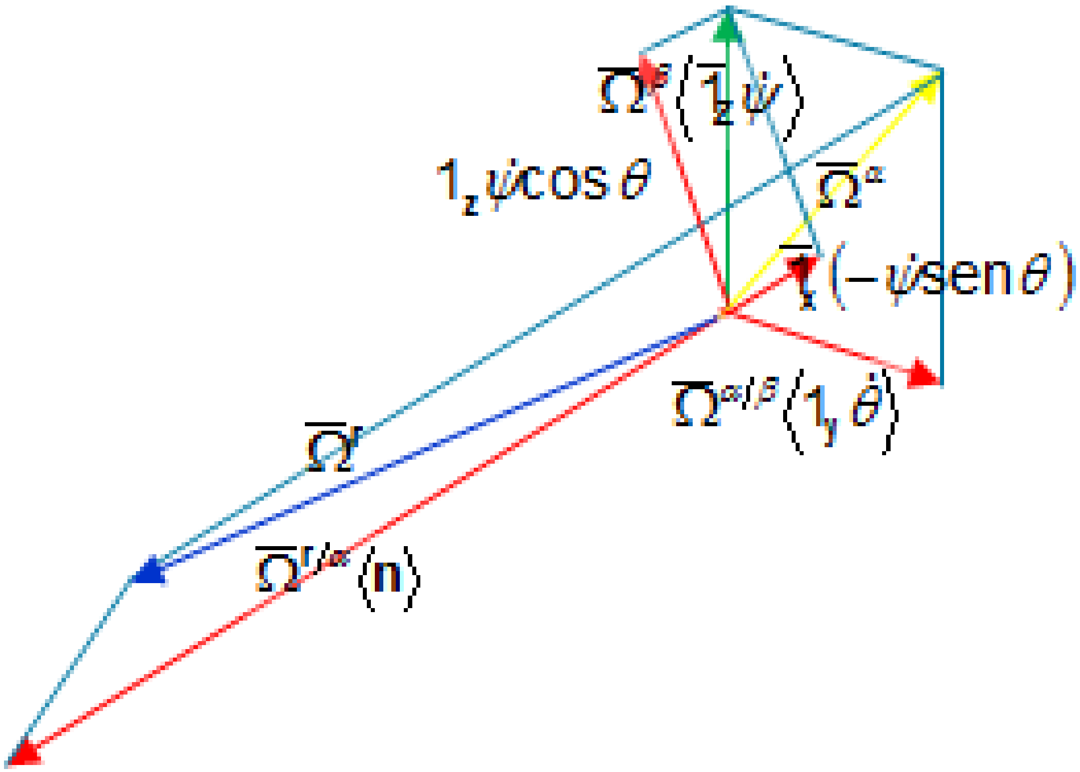
(b) *Free-body diagram of system 1*



(c) *Free-body diagram of system 2*



(a) *Arrangement and coordinates*



$\overline{\Omega}^{\beta}\langle \overline{1}_z \dot{\psi} \rangle$
$1_z \dot{\psi} \cos\theta$
$\overline{\Omega}^{\alpha}$
$\overline{1}_x \langle -\dot{\psi} \operatorname{sen}\theta \rangle$
$\overline{\Omega}^{\alpha/\beta}\langle 1_y \dot{\theta} \rangle$
$\overline{\Omega}^{I}$
$\overline{\Omega}^{I/\alpha}\langle n \rangle$

$$\overline{\Omega}^{\beta} = \overline{1}_{\mathbf{z}}\dot{\psi}$$

$$\overline{\Omega}^{\alpha} = \overline{\Omega}^{\alpha/\beta} + \overline{\Omega}^{\beta} = \overline{1}_{\mathbf{y}}\dot{\theta} + \overline{1}_{\mathbf{z}}\dot{\psi}$$

$$\overline{\Omega}^{\alpha} = \overline{1}_{\mathbf{x}}\left(-\dot{\psi}\,\mathrm{sen}\,\theta\right) + \overline{1}_{\mathbf{y}}\dot{\theta} + \overline{1}_{\mathbf{z}}\dot{\psi}\cos\theta$$

$$\overline{\Omega}^{\mathbf{r}} = \overline{\Omega}^{\mathbf{r}/\alpha} + \overline{\Omega}^{\alpha}$$

$$\overline{\Omega}^{\mathbf{r}} = \overline{1}_{\mathbf{x}}\mathbf{n} + \overline{1}_{\mathbf{y}}\dot{\theta} + \overline{1}_{\mathbf{z}}\dot{\psi} = \overline{1}_{\mathbf{x}}\left(\mathbf{n} - \dot{\psi}\,\mathrm{sen}\,\theta\right) + \overline{1}_{\mathbf{y}}\dot{\theta} + \overline{1}_{\mathbf{z}}\dot{\psi}\cos\theta$$

$$\left(\Sigma\mathbf{M}\right)_{\mathbf{y}} = 0$$

$$\left(\Sigma\mathbf{M}\right)_{\mathbf{z}} = 0$$

$$\begin{cases} \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}} + \bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}}\right)_{\mathbf{y}} + \bar{\mathbf{M}}_{\mathbf{b}} = 0 \\ \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}} + \bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}}\right)_{\mathbf{z}} \cos\theta - \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}} + \bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}}\right)_{\mathbf{x}} \operatorname{sen}\theta + \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{o}}\right)_{\mathbf{z}} + \bar{\mathbf{M}}_{\mathbf{c}} = 0 \end{cases}$$

ricaviamo ora:

$$\left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}}\right)_{\mathbf{x}}, \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}}\right)_{\mathbf{y}}, \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}}\right)_{\mathbf{z}}, \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}}\right)_{\mathbf{x}} \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}}\right)_{\mathbf{y}}, \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}}\right)_{\mathbf{z}}, \left(\bar{\mathbf{M}}_{\mathbf{i}}^{\mathbf{o}}\right)_{\mathbf{z}}$$

$$\begin{cases} \dot{\overline{\mathbf{H}}}^{\mathbf{r}} + \overline{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}} = 0 \\ \dot{\overline{\mathbf{H}}}^{\mathbf{g}} + \overline{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}} = 0 \\ \dot{\overline{\mathbf{H}}}^{\mathbf{o}} + \overline{\mathbf{M}}_{\mathbf{i}}^{\mathbf{o}} = 0 \end{cases}$$

$\overline{\mathbf{H}}$ è il momento della quantità di moto

$$\begin{cases} \dot{\overline{\mathbf{H}}}^{\mathbf{r}} = {}^{\alpha}\dot{\overline{\mathbf{H}}}^{\mathbf{r}} + \overline{\Omega}^{\alpha} \times \overline{\mathbf{H}}^{\mathbf{r}} = -\overline{\mathbf{M}}_{\mathbf{i}}^{\mathbf{r}} \\ \dot{\overline{\mathbf{H}}}^{\mathbf{g}} = {}^{\alpha}\dot{\overline{\mathbf{H}}}^{\mathbf{g}} + \overline{\Omega}^{\alpha} \times \overline{\mathbf{H}}^{\mathbf{g}} = -\overline{\mathbf{M}}_{\mathbf{i}}^{\mathbf{g}} \\ \dot{\overline{\mathbf{H}}}^{\mathbf{o}} = {}^{\beta}\dot{\overline{\mathbf{H}}}^{\mathbf{o}} + \overline{\Omega}^{\beta} \times \overline{\mathbf{H}}^{\mathbf{o}} = -\overline{\mathbf{M}}_{\mathbf{i}}^{\mathbf{o}} \end{cases}$$

ma in generale è:

$$\bar{H}_c = \bar{1}_{x_p}\left(J_{x_p}\Omega_{x_p}\right) + \bar{1}_{y_p}\left(J_{y_p}\Omega_{y_p}\right) + \bar{1}_{z_p}\left(J_{z_p}\Omega_{z_p}\right)$$

dove appaiono i momenti di inerzia attorno
a tre assi principali d'inerzia

e quindi

$$\bar{H}^r = \bar{1}_x\left[J_x{}^r\left(n - \dot{\psi}\,\mathrm{sen}\,\theta\right)\right] + \bar{1}_y\left[J_y{}^r\dot{\theta}\right] + \bar{1}_z\left[J_z{}^r\dot{\psi}\cos\theta\right]$$

$$\bar{H}^g = \bar{1}_x\left[J_x{}^g\left(-\dot{\psi}\,\mathrm{sen}\,\theta\right)\right] + \bar{1}_y\left[J_y{}^g\dot{\theta}\right] + \bar{1}_z\left[J_z{}^g\dot{\psi}\cos\theta\right]$$

$$\bar{H}^o = \bar{1}_z\left[J_z{}^o\dot{\psi}\right]$$

da cui

$$\bar{\Omega}^{\alpha}\times\bar{\mathbf{H}}^{r}=\begin{bmatrix}\bar{1}_x & \bar{1}_y & \bar{1}_z\\ -\dot{\psi}\,\mathrm{sen}\,\theta & \dot{\theta} & \dot{\psi}\cos\theta\\ \mathbf{J}_x{}^{r}\left(\mathbf{n}-\dot{\psi}\,\mathrm{sen}\,\theta\right) & \mathbf{J}_y{}^{r}\dot{\theta} & \mathbf{J}_z{}^{r}\dot{\psi}\cos\theta\end{bmatrix}$$

$$\bar{\Omega}^{\alpha}\times\bar{\mathbf{H}}^{g}=\begin{bmatrix}\bar{1}_x & \bar{1}_y & \bar{1}_z\\ -\dot{\psi}\,\mathrm{sen}\,\theta & \dot{\theta} & \dot{\psi}\cos\theta\\ \mathbf{J}_x{}^{g}\left(-\dot{\psi}\,\mathrm{sen}\,\theta\right) & \mathbf{J}_y{}^{g}\dot{\theta} & \mathbf{J}_z{}^{g}\dot{\psi}\cos\theta\end{bmatrix}$$

ponendo

$\mathbf{h} = \mathbf{J}_x{}^r\mathbf{n}$ rigidità del giroscopio o spin momentum

$\mathbf{J}_x \equiv \mathbf{J}_x{}^r + \mathbf{J}_x{}^g \qquad \mathbf{J}_y \equiv \mathbf{J}_y{}^r + \mathbf{J}_y{}^g \qquad \mathbf{J}_z \equiv \mathbf{J}_z{}^r + \mathbf{J}_z{}^g$

e sostituendo si ricava:

$$\begin{cases} -\left(\bar{\mathbf{M}}_i{}^r + \bar{\mathbf{M}}_i{}^g\right)_y = \mathbf{J}_y\ddot{\theta} + \mathbf{h}\,\dot{\psi}\cos\theta + \left(\mathbf{J}_z - \mathbf{J}_x\right)\dot{\psi}^2\mathrm{sen}\,\theta\cos\theta \\ -\left(\bar{\mathbf{M}}_i{}^r + \bar{\mathbf{M}}_i{}^g\right)_z \cos\theta = \mathbf{J}_z\left(\ddot{\psi}\cos^2\theta - \dot{\psi}\dot{\theta}\mathrm{sen}\,\theta\cos\theta\right) - \mathbf{h}\,\dot{\theta}\cos\theta + \left(\mathbf{J}_x - \mathbf{J}_y\right)\dot{\psi}\dot{\theta}\mathrm{sen}\,\theta\cos\theta \\ -\left(\bar{\mathbf{M}}_i{}^r + \bar{\mathbf{M}}_i{}^g\right)_x \mathrm{sen}\,\theta = -\mathbf{J}_x\left(\ddot{\psi}\mathrm{sen}^2\theta + \dot{\psi}\dot{\theta}\mathrm{sen}\,\theta\cos\theta\right) + \left(\mathbf{J}_z - \mathbf{J}_y\right)\dot{\psi}\dot{\theta}\mathrm{sen}\,\theta\cos\theta \end{cases}$$

Sostituendo le espressioni trovate nelle due equazioni di equilibrio scritte inizialmente si ottiene:

$$\begin{cases} \mathbf{J_y}\ddot{\theta} + \mathbf{h}\,\dot{\psi}\cos\theta + \left(\mathbf{J_z} - \mathbf{J_x}\right)\dot{\psi}^2 \operatorname{sen}\theta\cos\theta = \mathbf{M_b} \\ \left(\mathbf{J_z}^{\mathbf{o}} + \mathbf{J_z}\cos^2\theta + \mathbf{J_x}\operatorname{sen}^2\theta\right)\ddot{\psi} - \mathbf{h}\,\dot{\theta}\cos\theta + 2\left(\mathbf{J_x} - \mathbf{J_z}\right)\dot{\psi}\dot{\theta}\operatorname{sen}\theta\cos\theta = \mathbf{M_c} \end{cases}$$

se inoltre consideriamo piccole rotazioni $\theta \simeq 0$ e $\mathbf{n} \gg \dot{\psi}$

si ha $\operatorname{sen}\theta \equiv \theta$ e $\cos\theta \equiv 1$

e $\mathbf{J}\,\dot{\psi}^2 \mathbf{sen}\,\theta << \mathbf{Jn}\,\dot{\psi}$ e $\mathbf{J}\,\dot{\psi}\dot{\theta}\mathbf{sen}\,\theta << \mathbf{Jn}\,\dot{\theta}$, quindi trascurabili

per cui si ottiene:

$$\begin{cases} \mathbf{J_y}\ddot{\theta} + \mathbf{h}\,\dot{\psi} = \mathbf{M_b} \\ \mathbf{J_z}\ddot{\psi} - \mathbf{h}\,\dot{\theta} = \mathbf{M_c} \end{cases}$$

dove $\mathbf{J_z} = \mathbf{J_z}^{\mathbf{o}} + \mathbf{J_z}$

Utilizzando la trasformata di Laplace

$$\mathbf{X}(\mathbf{s}) = \mathrm{L}_{-}\left[\mathbf{x}(\mathbf{t})\right] = \int_{0^-}^{\infty} \mathbf{x}(\mathbf{t}) e^{-st} dt$$

$$\Theta(\mathbf{s}) = \mathrm{L}_{-}\left[\theta(\mathbf{t})\right] = \int_{0^-}^{\infty} \theta(\mathbf{t}) e^{-st} dt$$

$$\mathrm{L}_{-}\left[\dot{\theta}(\mathbf{t})\right] = -\theta\left(0^-\right) + \mathbf{s}\Theta(\mathbf{s})$$

$$\mathrm{L}_{-}\left[\ddot{\theta}(\mathbf{t})\right] = -\mathbf{s}\theta\left(0^-\right) - \dot{\theta}\left(0^-\right) + \mathbf{s}^2\Theta(\mathbf{s})$$

si ottiene:

$$\begin{bmatrix} \mathbf{s}^2 & \dfrac{\mathbf{h}}{\mathbf{J_y}}\mathbf{s} \\ -\dfrac{\mathbf{h}}{\mathbf{J}_Z}\mathbf{s} & \mathbf{s}^2 \end{bmatrix} \begin{bmatrix} \Theta \\ \Psi \end{bmatrix} = \begin{bmatrix} \dfrac{M_{\mathbf{y}}(\mathbf{s})}{\mathbf{J_y}} + \mathbf{s}\theta\left(0^-\right) + \dot{\theta}\left(0^-\right) + \dfrac{\mathbf{h}}{\mathbf{J_y}}\psi\left(0^-\right) \\ \dfrac{M_Z(\mathbf{s})}{\mathbf{J}_Z} - \dfrac{\mathbf{h}}{\mathbf{J}_Z}\theta\left(0^-\right) + \mathbf{s}\psi\left(0^-\right) + \dot{\psi}\left(0^-\right) \end{bmatrix}$$

l'equazione caratteristica è:

$$\mathbf{s}^2\left(\mathbf{s}^2+\frac{\mathbf{h}^2}{\mathbf{J_y J_z}}\right)=0 \quad \text{da cui} \quad \begin{cases} \mathbf{s}=0,0 \\ \mathbf{s}=\pm\mathbf{i}\dfrac{\mathbf{h}}{\sqrt{\mathbf{J_y J_z}}} \end{cases} \quad \Rightarrow \quad \omega=\frac{\mathbf{h}}{\sqrt{\mathbf{J_y J_z}}}$$

$$\text{se } \mathbf{J_y}=\mathbf{J_z}=\frac{\mathbf{J_r}}{2} \quad \Rightarrow \quad \omega=2\mathbf{n} \quad \text{frequenze naturali}$$

$$\begin{cases} \Theta=\dfrac{\left[\dfrac{M_y}{\mathbf{J_y}}+\dot{\theta}\left(0^-\right)\right]}{\mathbf{s}^2+\omega^2}+\dfrac{\theta\left(0^-\right)}{\mathbf{s}} \\ \Psi=\sqrt{\dfrac{\mathbf{J_y}}{\mathbf{J}_Z}}\,\dfrac{\left[\dfrac{M_y}{\mathbf{J_y}}+\dot{\theta}\left(0^-\right)\right]\omega}{\mathbf{s}\left(\mathbf{s}^2+\omega^2\right)}+\dfrac{\psi\left(0^-\right)}{\mathbf{s}} \end{cases} \quad \text{con } M_z=0 \text{ e } \dot{\psi}\left(0^-\right)=0$$

Si consideri il caso in cui $\mathbf{M_y}$ sia una funzione impulsiva

$$\mathbf{M_y}(\mathbf{t}) = \mu\delta(\mathbf{t}) \quad \Rightarrow \quad M_{\mathbf{y}}(\mathbf{s}) = \mu$$

essendo $\delta(\mathbf{t})$ la funzione impulsiva unitaria, detto $\Omega_{yo} = \dfrac{\mu}{\mathbf{J_y}}$ si ottiene:

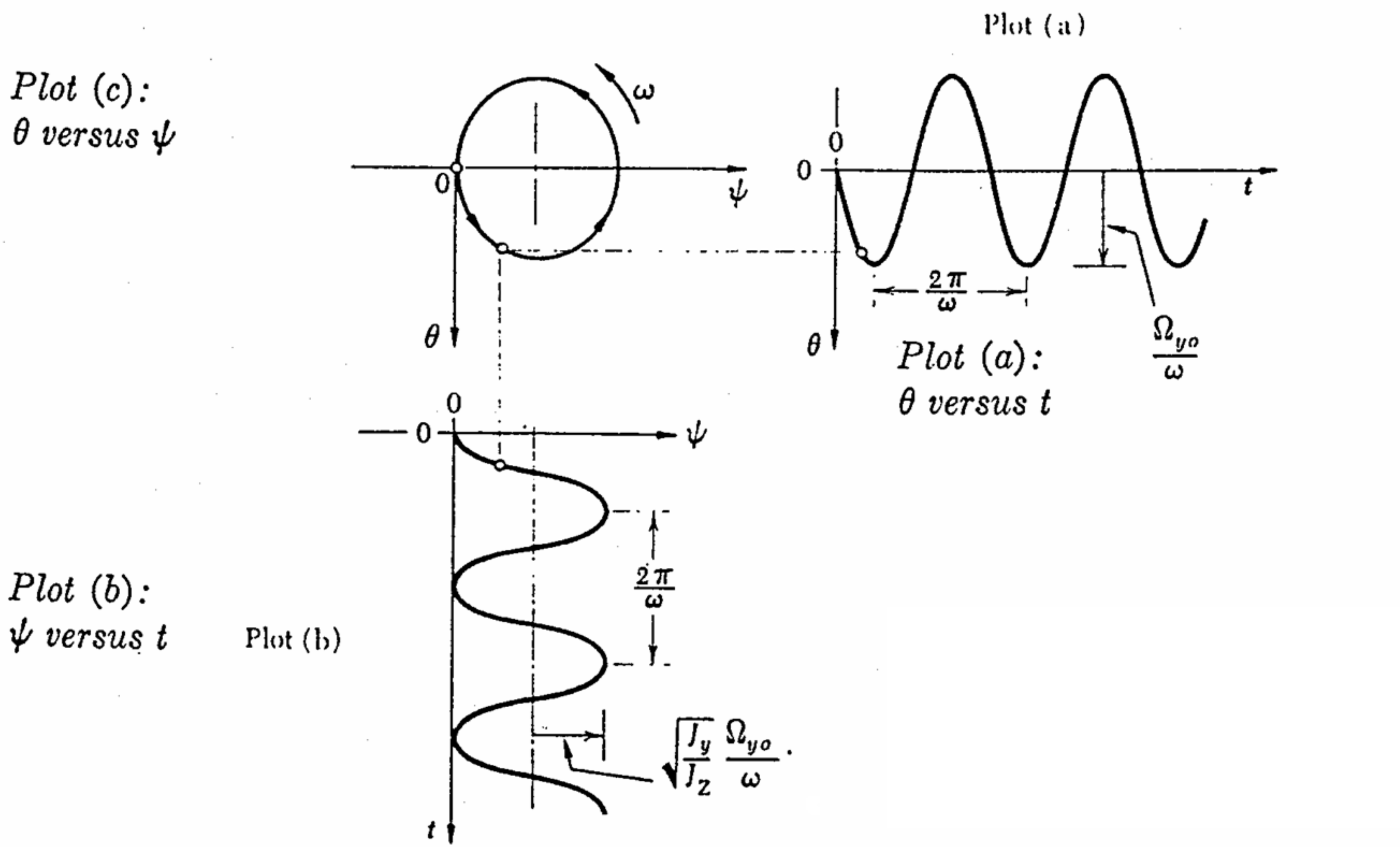
$$\begin{cases} \Theta = \dfrac{\Omega_{yo}}{\mathbf{s}^2 + \omega^2} \\ \Psi = \sqrt{\dfrac{\mathbf{J_y}}{\mathbf{J}_Z}}\Omega_{yo} \dfrac{\omega}{\mathbf{s}(\mathbf{s}^2 + \omega^2)} \end{cases}$$

Ritornando nel dominio del tempo si ottiene:

$$\theta(\mathbf{t}) = \frac{\Omega_{yo}}{\omega}\mathrm{sen}\,\omega\mathbf{t}\,\mathrm{u}(\mathbf{t}) \qquad \text{e} \qquad \psi(\mathbf{t}) = \sqrt{\frac{\mathbf{J_y}}{\mathbf{J}_Z}}\Omega_{yo}(1 - \cos\omega\mathbf{t})\mathrm{u}(\mathbf{t})$$

dove $\mathrm{u}(\mathbf{t})$ è la funzione unitaria a gradino

Coning motion of a two-axis gyro



caso in cui $\mathbf{M_y}$ è una funzione a gradino $\mathbf{M_y}(\mathbf{t}) = \mathbf{M_{y_0}} u(\mathbf{t}) \Rightarrow \mathrm{M_y}(\mathbf{s}) = \dfrac{\mathbf{M_{y_0}}}{\mathbf{s}}$

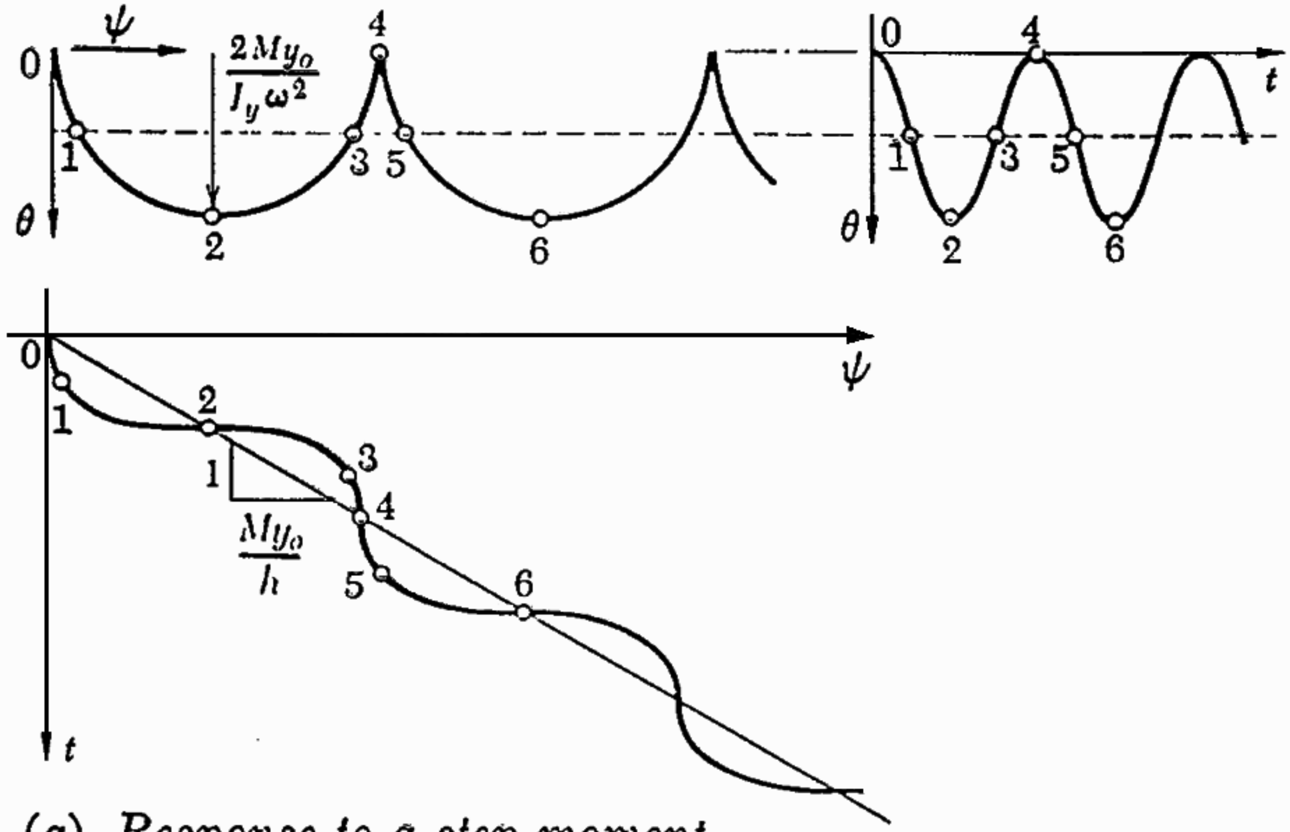
$$
\begin{cases}
\Theta = \dfrac{\mathbf{M_{y_0}} / \mathbf{J_y}}{\mathbf{s}\left(\mathbf{s}^2 + \omega^2\right)} \\[2ex]
\Psi = \sqrt{\dfrac{\mathbf{J_y}}{\mathbf{J_Z}}} \dfrac{\left(\mathbf{M_{y_0}} / \mathbf{J_y}\right)\omega}{\mathbf{s}^2\left(\mathbf{s}^2 + \omega^2\right)}
\end{cases}
$$

ritornando nel dominio del tempo si ottiene:

$$\theta(\mathbf{t}) = \frac{\mathbf{M}_{\mathbf{y}_0}}{\mathbf{J}_\mathbf{y}\omega^2}(1-\cos\omega\mathbf{t})\mathrm{u}(\mathbf{t}) \qquad \text{e} \qquad \psi(\mathbf{t}) = \frac{\mathbf{M}_{\mathbf{y}_0}}{\mathbf{h}}\left(\mathbf{t} - \frac{\mathrm{sen}\,\omega\mathbf{t}}{\omega}\right)\mathrm{u}(\mathbf{t})$$

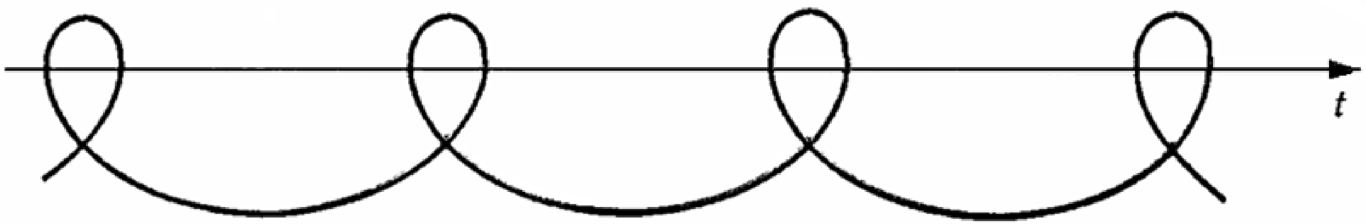
La deriva del giroscopio è data dal termine $\psi(\mathbf{t}) = \frac{\mathbf{M}_{\mathbf{y}_0}}{\mathbf{h}}\mathbf{t}$ che cresce nel tempo con una velocità di precessione pari a $\dot{\psi}(\mathbf{t}) = \frac{\mathbf{M}_{\mathbf{y}_0}}{\mathbf{h}}$

Response to a step moment



(a) *Response to a step moment*

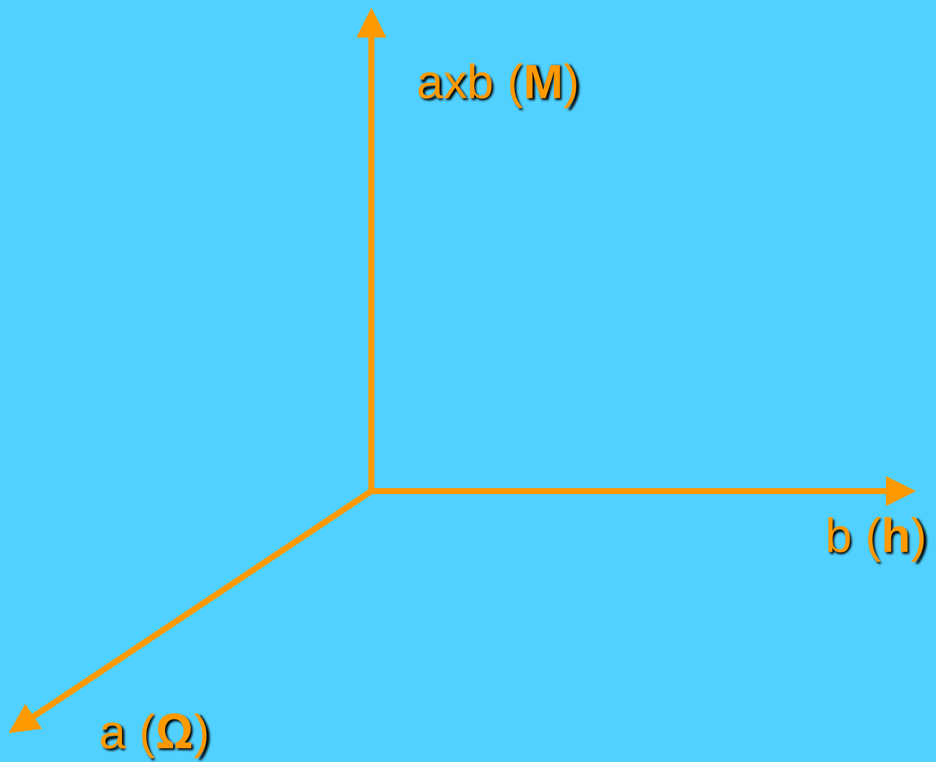
Response to a combination of step and impulse moment



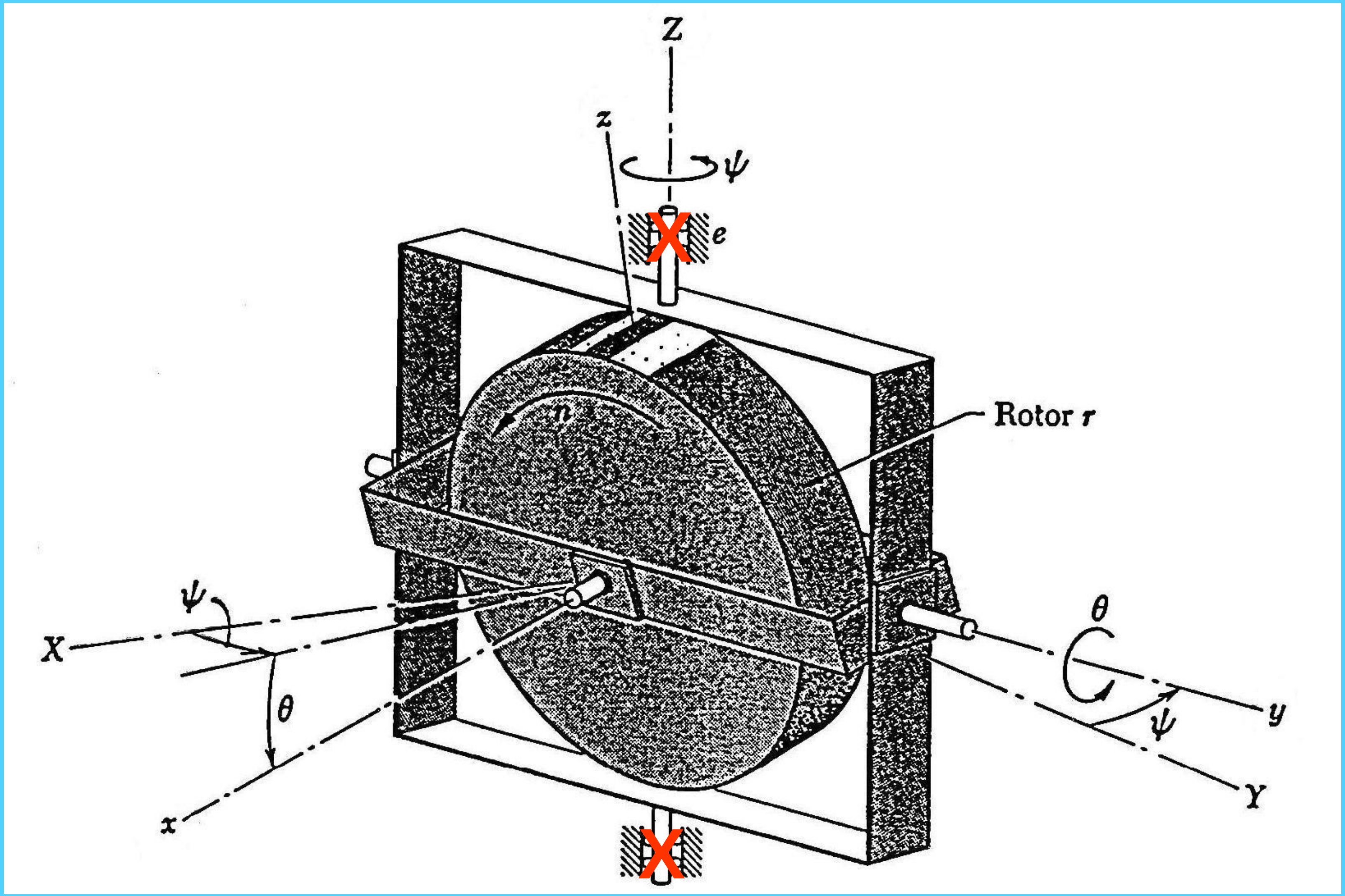
(b) *Response to a combination of step and impulse moment*

In termini vettoriali la relazione diventa: $\bar{\mathbf{M}} = \bar{\Omega} \times \bar{\mathbf{h}}$ dove

$$\bar{\mathbf{M}} = 1_{\mathbf{y}} \mathbf{M}_{\mathbf{y}_0} \qquad \bar{\mathbf{h}} = 1_{\mathbf{x}} \mathbf{h} \qquad \bar{\Omega} = 1_Z \dot{\psi}$$



Soppressione di un grado di libertà



Qualora un grado di libertà fosse soppresso, ψ ad esempio, l'asse di spin sarebbe costretto a seguire un movimento imposto cioè una rotazione attorno all'asse Z, come se precessionasse con velocità $\dot{\psi}$ per effetto di un momento $\mathbf{h}\dot{\psi}$ attorno all'asse y per cui avremmo:

$$\mathbf{J_y}\ddot{\theta} = -\mathbf{h}\dot{\psi} + \mathbf{M_b}$$

ed esprimendo $\mathbf{M_b}$ come: $\mathbf{M_b} = -\mathbf{k}\theta - \mathbf{b}\dot{\theta} + \mathbf{M_u}$, dove $\mathbf{M_u}$ rappresenta le incertezze della realizzazione del giroscopio reale, si ha:

$$\mathbf{J_y}\ddot{\theta} + \mathbf{b}\dot{\theta} + \mathbf{k}\theta = -\mathbf{h}\dot{\psi} + \mathbf{M_u}$$