Chapter 3

Incompressible Unsteady Potential Flows

In this chapter we introduce some basic concepts on incompressible potential flows, with emphasis on potential wakes. The fundamental assumptions in this chapter are that the flow is incompressible ($\frac{D\rho}{Dt} = 0$), inviscid ($\mathbf{V} = 0$), initially irrotational ($\zeta(\mathbf{x}, \mathbf{0}) = \mathbf{0}$), and supposed to be attached to the body. The frame of reference is assumed to be connected with the undisturbed air (air frame).

3.1 Governing Equations

The governing equations for an incompressible inviscid flow are the continuity equation

$$\nabla \cdot \mathbf{v} = 0 \tag{3.1}$$

and the Euler equation

$$\frac{D\mathbf{v}}{Dt} = -\frac{1}{\rho} \nabla p \tag{3.2}$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \tag{3.3}$$

denotes the substantial derivative, *i.e.*, the time derivative following the material points.

In order to complete the problem we need the boundary conditions. The surface $\mathcal{S}_B$ of the body is typically assumed to be impermeable; hence, the boundary condition on a point of $\mathcal{S}_B$

is[1]

$$(\mathbf{v} - \mathbf{v}_B) \cdot \mathbf{n} = 0 \tag{3.4}$$

where $\mathbf{v}_B$ is the velocity of the point on the surface of the body $\mathcal{S}_B$.

In addition, we have a boundary condition at infinity. Recalling that the frame of reference is assumed to be connected with the undisturbed air (air frame), the boundary condition at infinity for three-dimensional flows is

$$\mathbf{v}_\infty = O(R^{-\alpha}) \qquad (\alpha > 1) \tag{3.5}$$

where $R = \|\mathbf{x}\|$.

3.2 Incompressible Potential Flows (Non-Lifting Case)

In this section, we assume that the flow field is initially irrotational, for instance that, at time $t = 0$, the fluid is at rest. Then, according to Eq. 3.6 in Section 3.2.1, we have that, at $t = 0$, $\Gamma = 0$ for all the contours C in the field, and Kelvin's theorem yields that $\Gamma = 0$ at all times for all the material contours C that remained in the field between $t = 0$ and the current time. Therefore (see Section 3.2.2), such a field remains irrotational at all times.[2]

3.2.1 Kelvin's Theorem

Let C be a material contour, *i.e.*, a contour made of material points, and let Γ denote the circulation over C:

$$\Gamma = \oint_C \mathbf{v} \cdot d\mathbf{x} \tag{3.6}$$

where $\boldsymbol{\zeta} = \nabla \times \mathbf{v}$ is the vorticity and $\mathcal{S}$ denotes a surface bound by C. Note that according to Stokes theorem,

$$\Gamma = \iint_{\mathcal{S}} \boldsymbol{\zeta} \cdot \mathbf{n} d\mathcal{S} \tag{3.7}$$

Starting from the Euler equation one obtains Kelvin's theorem, which states that Γ is constant in time:

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \tag{3.8}$$

[1] More general boundary conditions, such as prescribed through-flow at a nacelle inlet, may be implemented by adding the flux-term on the right hand side of Eq. 3.14.

[2] This conclusion is not valid for the points that come in contact with the surface of the body. This issue is examined in Section 3.3, where it is shown that this conclusion is limited to inviscid initially-irrotational flows around non-lifting objects.

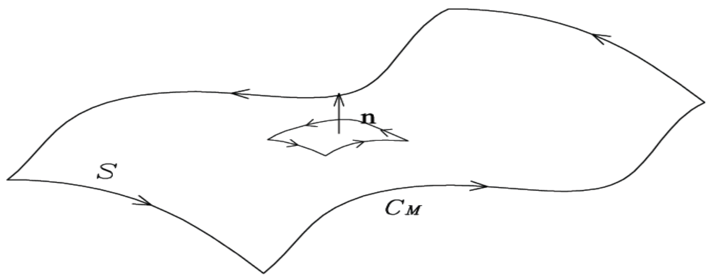

Figure 3.1: Surface and contour in the Stokes theorem

The proof is obtained by noting that if $d\mathbf{x}$ is a material element, then $D(d\mathbf{x})/Dt = d\mathbf{v}$; hence, using Eq. 3.2, and recalling that, for any f, $\nabla f \cdot d\mathbf{x} = df$, and, for any single-valued function f, $\oint_C df = 0$, we have

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\oint_C \mathbf{v}\cdot d\mathbf{x} = \oint_C \frac{D\mathbf{v}}{Dt}\cdot d\mathbf{x} + \oint_C \mathbf{v}\cdot d\mathbf{v} = -\frac{1}{\rho}\oint_C dp + \frac{1}{2}\oint_C dv^2 = 0 \tag{3.9}$$

since p and $v = \|\mathbf{v}\|$ are single-valued functions.

3.2.2 Irrotational and Potential Flows

In this Section we demonstrate that if a flow is irrotational, *i.e.*, if the vorticity is zero everywhere in the field, then such a flow is potential, *i.e.*, there exists a function φ such that $\mathbf{v} = \nabla\varphi$. The function φ is known as the velocity potential and is single-valued for simply-connected domains (*e.g.*, the three-dimensional field around an object), multi-valued for multiply-connected domains (*e.g.*, the two-dimensional field around an object, or the three-dimensional field around a doughnut-shaped object).

In order to prove the above statement, note that, according to Stokes theorem (see Fig. 3.1),

$$\oint_C \mathbf{v}\cdot d\mathbf{x} = \iint_{\mathcal{S}} \boldsymbol{\zeta}\cdot\mathbf{n}d\mathcal{S} \ = \ 0 \tag{3.10}$$

where $\boldsymbol{\zeta} = \nabla\times\mathbf{v}$ is the vorticity and $\mathcal{S}$ denotes a surface bound by C. Hence, for simply-connected domains, the integral $\int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{v}\cdot d\mathbf{x}$ is path independent. Thus, if $\mathbf{x}_0$ is treated as fixed, this integral defines a single-valued function

$$\varphi(\mathbf{x}) = \int_{\mathbf{x}_0}^{\mathbf{x}} \mathbf{v} \cdot d\mathbf{x} \tag{3.11}$$

Note that $d\varphi = \mathbf{v} \cdot d\mathbf{x}$, and hence, $\mathbf{v} = \nabla\varphi$. For multiply-connected domain the result is still valid, but the function is multi-valued.

3.2.3 Differential Formulation for the Velocity Potential

As shown in Section 3.2.2, irrotational flows are potential flows, *i.e.*, the velocity may be expressed as

$$\mathbf{v} = \nabla\varphi \tag{3.12}$$

Combining the continuity equation with Eq. 3.12 yields

$$\nabla^2\varphi = 0 \qquad\qquad \mathbf{x} \;\text{ outside }\; \mathcal{S}_B \tag{3.13}$$

Next, consider the boundary conditions. On the body, combining Eqs. 3.4 and 3.12, one obtains

$$\frac{\partial\varphi}{\partial n} = \mathbf{v}_B \cdot \mathbf{n} \qquad\qquad \mathbf{x} \;\text{ on }\; \mathcal{S}_B \tag{3.14}$$

In the air frame of reference used here, the boundary condition at infinity is given by Eq. 3.5, which, in terms of the velocity potential, may be written as

$$\varphi = O(R^{-\alpha}) \qquad\qquad (\alpha > 0) \tag{3.15}$$

Finally, for potential incompressible flows, the Euler equation may be integrated to yield Bernoulli's theorem, which, in the air frame of reference used here, is given by

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}v^2 + \frac{1}{\rho}p = \frac{1}{\rho}p_\infty \tag{3.16}$$

where ∞ indicates evaluation at infinity. The proof is obtained by noting that

$$\nabla\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}v^2 + \frac{1}{\rho}p\right) = \frac{\partial\mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + \frac{1}{\rho}\nabla p = 0 \tag{3.17}$$

The constant p_∞/ρ is obtained from the value at infinity (recall that the frame of reference is here assumed to be connected with the undisturbed air, so that $\mathbf{v}_\infty = 0$, Eq. 3.5).

3.2.4 Boundary Integral Representation for Poisson Equation*

In this Section we introduce the boundary integral representation for the equation of the velocity potential for the Poisson equation, for interior problems. The concepts introduced here are the basis of the method discussed in this chapter and will be extended to the more general cases in later chapters.

The basis for the boundary integral formulation is the second Green's identity

$$\iiint_{\mathcal{V}} (u\nabla^2 G - G\nabla^2 u) d\mathcal{V} = -\oiint_{\mathcal{S}} \left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n} \right) d\mathcal{S} \tag{3.18}$$

obtained by integrating the identity[3]

$$\nabla \cdot (u\nabla G) - \nabla \cdot (G\nabla u) = u\nabla^2 G - G\nabla^2 u \tag{3.20}$$

and using Gauss theorem with the inwardly directed normal $\mathbf{n}$.

The essence of the method consists in choosing for u any function satisfying the Poisson equation

$$\nabla^2 u = f \tag{3.21}$$

and for G the fundamental solution of the Laplace equation defined by

$$\nabla^2 G = \delta(\mathbf{x} - \mathbf{x}_*) \tag{3.22}$$

where δ is the three-dimensional Dirac delta function defined by

$$\iiint_{-\infty}^{\infty} u(\mathbf{x})\delta(\mathbf{x}) d\mathcal{V} = u(\mathbf{0}) \tag{3.23}$$

The solution to Eq. 3.22, with boundary condition $G = 0$ at infinity, is the well known potential for the unit source

$$G = \frac{-1}{4\pi r} \tag{3.24}$$

where

$$r = \|\mathbf{x} - \mathbf{x}_*\| \tag{3.25}$$

[3]Indeed, one has

$$\nabla \cdot (G\nabla\phi) = \nabla G \cdot \nabla\phi + G\nabla^2\phi \tag{3.19}$$

Similarly, developing $\nabla \cdot (\phi\nabla G)$ and subtracting it by the above Equation, one obtaing Eq. 3.20.

Introducing the function

$$\begin{aligned} E(\mathbf{x}) &= 1 \qquad\qquad \mathbf{x} \text{ inside } \mathcal{S} \\ &= 0 \qquad\qquad \mathbf{x} \text{ outside } \mathcal{S} \end{aligned} \tag{3.26}$$

we have

$$\iiint_{\mathcal{V}} u(\mathbf{x})\ \delta(\mathbf{x})\ d\mathcal{V} = \iiint_{-\infty}^{\infty} E(\mathbf{x})\ u(\mathbf{x})\ \delta(\mathbf{x})\ d\mathcal{V} = E(\mathbf{0})\ u(\mathbf{0}) \tag{3.27}$$

Combining Eqs. 3.18, 3.22, 3.21, 3.23, and 3.27, one obtains the elegant formula

$$E(\mathbf{x}_*)u(\mathbf{x}_*) = \oiint_{\mathcal{S}} \left(\frac{\partial u}{\partial n}\ G - u\frac{\partial G}{\partial n} \right) d\mathcal{S} + \iiint_{\mathcal{V}} G\nabla^2 u\ d\mathcal{V} \tag{3.28}$$

Equation 3.28 is the key to the method presented here; we will refer to it as the boundary integral representation for the Poisson equation. If $\mathbf{x}_*$ is in the field, Eq. 3.28 is an integral representation of $u(\mathbf{x}_*, t_*)$ anywhere in the field in terms of the values of u and $\partial u/\partial n$ on $\mathcal{S}$. On the other hand, if we take the limit as $\mathbf{x}_*$ tends to the surface $\mathcal{S}$,[4] then Eq. 3.28 is a compatibility condition that must be satisfied by any harmonic function.

3.2.5 Boundary Integral Formulation for Velocity Potential *

In this section we apply the above results for the interior problem of the Poisson equation to the case of interest here, *i.e.*, the problem of incompressible potential flows around a body. The only modification required is due to the fact that the domain of the problem is now external to the boundary surface $\mathcal{S}$. In order to accomplish our objective, let us apply Eq. 3.28 to a volume $\mathcal{V}$ outside the boundary surface $\mathcal{S}$ and bound by an outer surface $\mathcal{S}_\infty$ (which is a spherical surface of radius R and center $\mathbf{x}_*$), so that the boundary of the volume $\mathcal{V}$ is formed by two surfaces $\mathcal{S}_B$ and $\mathcal{S}_\infty$. Equation 3.5 implies that (see Fig. 3.2)

$$\lim_{R\to\infty} \varphi = \lim_{R\to\infty} R\frac{\partial \varphi}{\partial n} = 0 \tag{3.29}$$

Hence, the contribution of $\mathcal{S}_\infty$ goes to zero as the radius R of $\mathcal{S}_\infty$ goes to infinity. Thus, taking into account that now $f = 0$ (*i.e.*, the Poisson equation becomes the Laplace equation), Eq. 3.28 yields

$$E(\mathbf{x}_*)\varphi(\mathbf{x}_*) = \oiint_{\mathcal{S}_B} \left(\frac{\partial \varphi}{\partial n}\ G - \varphi\frac{\partial G}{\partial n} \right) d\mathcal{S} \tag{3.30}$$

where $G = -1/4\pi r$ (with $r = \|\mathbf{x} - \mathbf{x}_*\|$), and $\mathbf{n}$ is the outward normal to the surface $\mathcal{S}_B$, whereas

$$\begin{aligned} E(\mathbf{x}_*) &= 1 \qquad\qquad \mathbf{x}_* \text{ outside } \mathcal{S}_B \\ &= 0 \qquad\qquad \mathbf{x}_* \text{ inside } \mathcal{S}_B \end{aligned} \tag{3.31}$$

We will refer to Eq. 3.30 as the Green boundary integral representation for the velocity potential.

[4] This limit is discussed in Section 3.2.6.

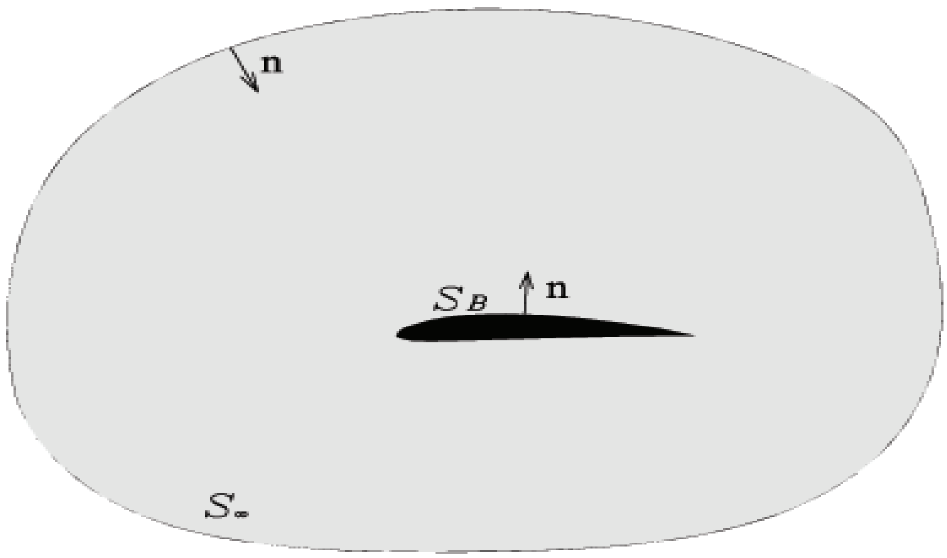

Figure 3.2: Aerodynamic domain for non-lifting boundary integral formulation

3.2.6 The Value of E on $\mathcal{S}$; Integral Equation

Equation 3.30 may be used to obtain the value of φ at any point in the field if the value of φ and $\partial\varphi/\partial n$ on the surface $\mathcal{S}_B$ are known. Note that, for the Neumann problem, $\partial\varphi/\partial n$ is known from the boundary condition on the body. Hence, in order to be able to use Eq. 3.30, one must have an equation for evaluating φ on the surface: such an equation is obtained by noting that if $\mathbf{x}_*$ approaches a point on the surface $\mathcal{S}_B$, the value of $\varphi(\mathbf{x}_*)$ approaches the value of φ at that point on the surface $\mathcal{S}_B$.

Let us consider surface $\mathcal{S}$ and a point C: furthermore, consider also an infinitesimal surface element $d\mathcal{S}$. Then the solid angle is defined as that seen from the point C to the surface $d\mathcal{S}$ projected on a sphere with center C and unitary radius. Introducing $\mathbf{n}$ as the unit normal to the surface $d\mathcal{S}$ and $\mathbf{e}_r$ as the radial unit vector, then θ represents the angle between the unit vector (see Fig. 3.3). The projection of $d\mathcal{S}$ on the sphere centred in C is the projection on a surface having the unit normal parallel to the radial direction. Therefore

$$d\mathcal{S}_p = d\mathcal{S}\cos\theta \tag{3.32}$$

where $d\mathcal{S}_p$ is the surface element projected on the plane normal to the straight line joining the points $\mathbf{x}$ and $\mathbf{x}^*$, and

$$\cos\theta = \mathbf{n}\cdot\mathbf{e}_r \tag{3.33}$$

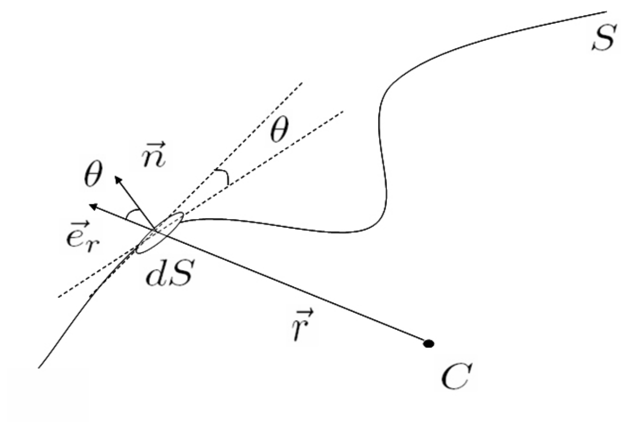

Figure 3.3: Local surface element

The solid angle $d\Omega$ is given by

$$d\Omega = \frac{\cos\theta d\mathcal{S}}{r^2} = \frac{\mathbf{n}\cdot\mathbf{e}_r}{r^2}d\mathcal{S} = \mathbf{n}\cdot\mathbf{r}\frac{d\mathcal{S}}{r^3} \tag{3.34}$$

The Green function is expressed by

$$G = -\frac{1}{4\pi r} \tag{3.35}$$

The layer of doublets is represented by the normal derivative of G,

$$\frac{\partial G}{\partial n}d\mathcal{S} = \mathbf{n}\cdot\nabla\left(-\frac{1}{4\pi r}\right)d\mathcal{S} = \mathbf{n}\cdot\mathbf{r}\frac{d\mathcal{S}}{4\pi r^3} \tag{3.36}$$

Comparing (3.34) and (3.36) yields

$$\frac{\partial G}{\partial n}d\mathcal{S} = \frac{d\Omega}{4\pi} \tag{3.37}$$

Integrating on the surface one has

$$\oiint_{\mathcal{S}} \frac{\partial G}{\partial n}d\mathcal{S} = \frac{1}{4\pi}\oiint_{\mathcal{S}} d\Omega = \begin{array}{ll} 1 & \text{with } C \text{ inside } \mathcal{S} \\ 0 & \text{with } C \text{ outside } \mathcal{S} \\ \frac{1}{2} & \text{with } C \in \mathcal{S} \end{array} \tag{3.38}$$

The domain function $E^* = E(\mathbf{x}^*)$ is defined as

$$E^* = \begin{array}{ll} 1 & \text{with } \mathbf{x}^* \text{ outside } \mathcal{S} \\ 0 & \text{with } \mathbf{x}^* \text{ inside } \mathcal{S} \\ \frac{1}{2} & \text{with } \mathbf{x}^* \in \mathcal{S} \end{array} \tag{3.39}$$

Therefore, the relationship between Ω^* and E^*

$$\oiint_{\mathcal{S}} \left.\frac{\partial G}{\partial n}\right|_{\mathbf{x}^*} d\mathcal{S} = \frac{\Omega^*}{4\pi} = 1 - E^* \tag{3.40}$$

or

$$E(\mathbf{x}_*) = 1 - \frac{1}{4\pi}\Omega(\mathbf{x}_*) \tag{3.41}$$

Note that Eq. 3.41 is valid even if $\mathbf{x}_*$ is on $\mathcal{S}_B$ (whether or not $\mathbf{x}_*$ is a regular point of $\mathcal{S}_B$ or not). In particular, if $\mathbf{x}_*$ is a regular point of $\mathcal{S}_B$, then $\Omega(\mathbf{x}_*) = 2\pi$ and $E(\mathbf{x}_*) = 1/2$. Equation 3.31 may thus be generalized as

$$\begin{aligned} E(\mathbf{x}_*) &= 1 - \Omega(\mathbf{x}_*)/4\pi & & \mathbf{x}_* \in \mathbf{R}^3 \\ &= 1 & & \mathbf{x}_* \text{ outside } \mathcal{S}_B \\ &= \frac{1}{2} & & \mathbf{x}_* \text{ on } \mathcal{S}_B \text{ (regular point)} \\ &= 0 & & \mathbf{x}_* \text{ inside } \mathcal{S}_B \end{aligned} \tag{3.42}$$

Finally, it should be emphasized that if the point $\mathbf{x}_*$ is on $\mathcal{S}_B$, Eq. 3.30 with $E(\mathbf{x}_*)$ given by Eq. 3.42 is a compatibility condition between the values of φ and $\partial\varphi/\partial n$ on $\mathcal{S}_B$. If $\partial\varphi/\partial n$ on $\mathcal{S}_B$ is known (Neumann boundary condition), Eq. 3.30 is an integral equation that may be used to evaluate φ on $\mathcal{S}_B$. This integral equation and its extensions are the key to the method used here, as well as to boundary-element methods in general.

3.2.7 Integral Equation; Discretization *

It should be emphasized that if the point $\mathbf{x}_*$ is on $\mathcal{S}_B$, Eq. 3.30 with $E(\mathbf{x}_*)$ given by Eq. 3.42 is a compatibility condition between the values of φ and $\partial\varphi/\partial n$ on $\mathcal{S}_B$. This condition is satisfied for any function that satisfies the Laplace equation.

In our case $\partial\varphi/\partial n$ on $\mathcal{S}_B$ is known from Eq. 3.14 (Neumann boundary condition). Therefore, Eq. 3.30 is an integral equation that may be used to evaluate φ on $\mathcal{S}_B$. This integral equation and its extensions are the key to the method used here (as well as to boundary integral methods in general).

In general, Eq. 3.30 cannot be solved exactly. Therefore, in order to solve (approximately) the problem, the integral formulation presented above must be discretized. The discretization process is briefly outlined here. Using a general boundary-element representation, with M nodes on the surface of the body, it is possible to approximate the potential φ as

$$\varphi(\mathbf{x}, t) = \sum_{m=1}^{M} \varphi_m(t)\ M_m(\mathbf{x}) \tag{3.43}$$

and the normalwash

$$\chi = \frac{\partial\varphi}{\partial n} \tag{3.44}$$

as

$$\chi(\mathbf{x}, t) = \sum_{m=1}^{M} \chi_m(t)\ M_m(\mathbf{x}) \tag{3.45}$$

where $\varphi_m(t) = \varphi(\mathbf{x}_m, t)$ and $\chi_m(t) = \chi(\mathbf{x}_m, t)$, whereas $M_m(\mathbf{x})$ are prescribed boundary-element shape functions, which have the property $M_m(\mathbf{x}_k) = \delta_{mk}$ (where δ_{mk} is the Kronecker delta). Note that, in order to avoid proliferation of symbols, the same shape functions are used for φ and χ, although this is not essential to the method.

Combining Eqs. 3.43 and 3.45 with Eq. 3.30, using the collocation method (*i.e.*, satisfying the equation at M collocation points), and setting the collocation points at the nodes, $\mathbf{x}_k$, one obtains

$$E_k(t)\varphi_k(t) = \sum_{m=1}^{M} B_{km}(t)\chi_m(t) + \sum_{m=1}^{M} C_{km}(t)\varphi_m(t) \tag{3.46}$$

where $E_k(t) = E(\mathbf{x}_k, t)$, whereas

$$B_{km} = \iint_{\mathcal{S}_{B_m}} M_m(\mathbf{x})\ \frac{-1}{4\pi\|\mathbf{x} - \mathbf{x}_k\|} d\mathcal{S} \tag{3.47}$$

$$C_{km} = \iint_{\mathcal{S}_{B_m}} M_m(\mathbf{x})\ \frac{\partial}{\partial n}\left(\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_k\|}\right) d\mathcal{S} \tag{3.48}$$

Note that the coefficients B_{km} and C_{km} are time independent when the surface of the body, $\mathcal{S}_B$, moves in rigid-body motion.

3.3 Incompressible potential flows (Lifting Case)

The problem considered in Section 3.2 is of limited interest in aeronautics. The reason is that the solution provided by Eq. 3.30 is limited to non-lifting configurations. For, according to the well known D'Alembert paradox, a potential-flow solution does not yield lift. Here, the potential-flow formulation is extended to lifting problems.

The objective of this Section is to exploit the fact that Kelvin's theorem cannot be applied to the points that come in contact with the boundary surface in order to develop a formulation for incompressible potential flows for lifting bodies. We will see that, in this case, the problem is complicated by the presence of the wake, which manifests itself as a surface of discontinuity in the velocity potential, φ. The theory of the wake in potential flows, as introduced by Suciu and Morino (1977), and successively refined by Morino, Kaprielian, and Sipcic (1985), and by Morino and Bharadvaj (1988), is relatively recent. Therefore, much of this chapter is devoted to this aspect of the formulation. The issues that need be addressed are: (i) the possibility of the existence of the wake; (*ii*) its generation and the trailing-edge condition; and (*iii*) its transport (*i.e.*, the boundary condition for the potential discontinuity $\Delta\varphi$ and on the continuity of the normal derivative).

3.3.1 Wakes in Potential Flows

As mentioned above, the proof that an inviscid incompressible initially-irrotational flow remains irrotational at all times fails for the fluid points that came in contact with the surface of the body. The proof of this statement is simple: in this case, Kelvin's theorem is not applicable, because it is not possible to identify a material contour, C, that surrounds the material point and that is inside the fluid region at all times (see Fig. 3.4). The points that come in contact

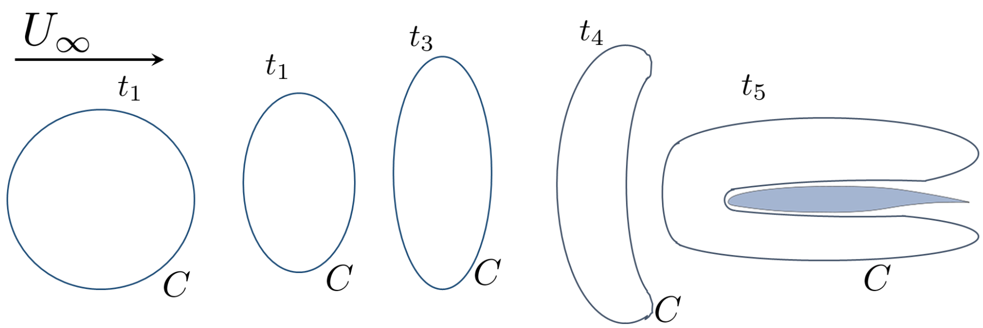

Figure 3.4: development of a generic material contour C in different time instants.

with the body form a surface called the wake. The implication of our result is that we can no longer assert that the flow is potential for the points of the wake. Indeed, we will show that on the wake there exists a discontinuity of the potential. This yields an integral representation in terms of wake doublet layers, which are equivalent to vortex layers. Therefore, there exists vorticity in the wake points, *i.e.*, the flow is not potential there.

Next, we want to emphasize the difference between the concept of the wake for potential flows that we have introduced and the more common acceptance of the word 'wake'. From a physical point of view (*i.e.*, for viscous flows), the wake is the (finite-thickness) region where the vorticity generated at the surface of the body is transported. On the other hand, the result that we have obtained may be stated as follows: in potential flows the wake exists and has zero thickness.[5] In conclusion, we now know that there exists a surface on which we need additional information (wake boundary conditions) in order to complete this problem. These conditions are obtained in the remainder of this section.

[5] A detailed analysis of the relationship between the mathematical representation of potential and viscous wakes is given in Morino (1986, 1990). It may be worth noting that the fact that the wake has zero thickness is typically taken as an independent assumption: what we have shown is that this fact is a consequence of the assumptions of inviscid, incompressible, initially-irrotational flow.

3.3.2 Wake Boundary Conditions

We have seen that Kelvin's theorem cannot be applied to the points that come in contact with the boundary surface. These points form a surface called the wake. On this surface we cannot state that the flow is potential; however, the flow is potential on both sides of the surface. Therefore, the wake is, in general, a surface of discontinuity: since φ satisfies the Laplace equation, we need boundary conditions only on the discontinuities on φ and $\partial\varphi/\partial n$.[6]

These conditions are obtained from the conservation laws across a surface of discontinuity. There exist two types of surfaces of discontinuity, shocks and wakes. In either case, applying the principle of conservation of mass across a surface of discontinuity one obtains the condition

$$\Delta[\rho(v_N - v_{\mathcal{S}})] = 0 \tag{3.49}$$

where $\Delta f = f_2 - f_1$ indicates the discontinuity across the surface, whereas $v_N = \mathbf{v} \cdot \mathbf{n}$ is the normal component of the velocity $\mathbf{v}$, and $v_{\mathcal{S}}$ is the velocity of the surface (by definition, in the direction of the normal $\mathbf{n}$). Similarly, applying the principle of conservation of momentum across a surface of discontinuity one obtains the condition $\Delta[\rho(v_N - v_{\mathcal{S}})\mathbf{v} + p\mathbf{n}] = 0$, or, combining with Eq. 3.49

$$\rho(v_N - v_{\mathcal{S}})\Delta\mathbf{v} + \Delta p\mathbf{n} = 0 \tag{3.50}$$

Noting that $\Delta v_{\mathcal{S}} = 0$, for incompressible flows Eq. 3.49 yields $\Delta v_N = 0$. Then, taking the normal component of Eq. 3.50, one obtains

$$\Delta p = 0 \tag{3.51}$$

Combining this equation with Eq. 3.50 yields $\rho(v_N - v_{\mathcal{S}})\Delta\mathbf{v} = 0$. This implies either $\Delta\mathbf{v} = 0$ (in which case everything is continuous and the surface under consideration is not a surface of discontinuity) or,

$$v_N = v_{\mathcal{S}} \tag{3.52}$$

which indicates that the fluid does not penetrate the surface of the wake.

In summary, we have seen that if there exists a surface of discontinuity in an incompressible inviscid flow, then Eqs. 3.51 and 3.52 must be satisfied.

[6]Using the Laplace equation, discontinuities on higher order derivatives along the normal may be reduced to those on φ and $\partial\varphi/\partial n$.

3.3.3 Formulation for Potential Flows

Let us now go back to the formulation for potential flows. Let us assume again that the flow is incompressible, inviscid, and initially irrotational. Then, following the same approach used in Chapter 3.2, we can apply Kelvin's theorem to all the contours that remain in the flow field at all times and obtain that the flow is potential at all times and at all points, with the possible exception of the points of the wake.[7]

Hence, except for the points of the wake, we have $\mathbf{v} = \nabla\varphi$ (Eq. 3.12). Then, combining this with the continuity equation for incompressible flows yields

$$\nabla^2\varphi = 0 \qquad \mathbf{x} \text{ outside } \mathcal{S} \tag{3.53}$$

where $\mathcal{S}$ denotes a surface that surrounds the body and the wake. Again, the Euler equation may be integrated to yield Bernoulli's theorem for incompressible potential flows, Eq. 3.16.

Next, consider the boundary conditions. On the body and at infinity, the boundary conditions are the same as those introduced for non-lifting problems (Eqs. 3.14 and 3.15).

i.e.,

$$\frac{\partial\varphi}{\partial n} = \mathbf{v}_B \cdot \mathbf{n} \tag{3.54}$$

where $\mathbf{v}_B$ is the velocity of the point on the surface of the body $\mathcal{S}_B$ and (recalling that, in the air frame of reference used here, $\mathbf{v}_\infty = 0$)

$$\varphi = 0 \tag{3.55}$$

Next, consider the boundary conditions on the wake. In the case of potential flows, Eq. 3.52 implies

$$\Delta\frac{\partial\varphi}{\partial n} = 0 \tag{3.56}$$

A boundary condition for the potential discontinuity $\Delta\varphi$ is obtained by combining Eq. 3.51 with Bernoulli's theorem (Eq. 3.16) to yield

$$\frac{\partial\varphi_2}{\partial t} - \frac{\partial\varphi_1}{\partial t} + \frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = 0 \tag{3.57}$$

where 1 and 2 denote the two sides of the wake. Then, noting that $v_2^2 - v_1^2 = (\mathbf{v}_2 + \mathbf{v}_1)\cdot(\mathbf{v}_2 - \mathbf{v}_1)$ we have[8]

$$\frac{D_W}{Dt}\Delta\varphi = 0 \tag{3.58}$$

[7]Note that the applicability of Kelvin's theorem to those points that do not come in contact with the surface of the body is not affected by the presence of the wake, since material points do not cross the wake (see Eq. 3.52).

[8]Note that we cannot say $\partial\varphi_2/\partial n - \partial\varphi_1/\partial n = \partial\Delta\varphi/\partial n$, because $\partial\Delta\varphi/\partial n$ is meaningless, since $\Delta\varphi$ is defined only on $\mathcal{S}_W$. However, $(\partial/\partial t + v_N\partial/\partial n)\,\Delta\varphi$ is meaningful, because, according to Eq. 3.52, it represents a time derivative following a point on $\mathcal{S}_W$ that moves in direction normal to $\mathcal{S}_W$.

with $D_W/Dt = \partial/\partial t + \mathbf{v}_W \cdot \nabla$, where $\mathbf{v}_W = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$. Note that D_W/Dt is the substantial derivative following a wake point, $\mathbf{x}_W$, that is defined as a point having velocity $\mathbf{v}_W$.

Equation 3.58 is the desired evolution equation for $\Delta\varphi$ and implies that $\Delta\varphi$ remains constant following a wake point $\mathbf{x}_W$, and equal to the value it had when $\mathbf{x}_W$ left the trailing edge. As shown in Section 3.3.5, the value of $\Delta\varphi$ at the trailing edge is obtained by using the Kutta-Joukowski hypothesis that no vortex filament exists at the trailing edge; this implies that the value of $\Delta\varphi$ on the wake and the value of $\Delta\varphi$ on the body are equal at the trailing edge (Eq. 3.64).

Equation 3.53 and its boundary conditions on the body (Eq. 3.54), at infinity (Eq. 3.55), on the wake (Eqs. 3.56 and 3.58), and on the trailing edge (Eq. 3.64), may be used to obtain the solution for φ. Once φ is known, the velocity is given by $\mathbf{v} = \nabla\varphi$, and the pressure may be evaluated using Bernoulli's theorem, Eq. 3.16.

3.3.4 Integral Equation; Wake Transport

Here, we extend the formulation of Section 3.2 to take into account the presence of the wake.

In order to obtain the Green boundary integral representation for lifting problems, we must use in Eq. 3.30 a surface $\mathcal{S}_{BW}$ that surrounds the volume $\mathcal{V}_B$ of the body as well as a thin layer, $\mathcal{V}_W$, which includes the wake surface $\mathcal{S}_W$ (because Eq. 3.53 is not valid in $\mathcal{V}_B$ and on $\mathcal{S}_W$).

Equation 3.30, for $f = \varphi$ and $\mathcal{S} = \mathcal{S}_{BW}$, is

$$E(\mathbf{x}_*, t_*)\varphi(\mathbf{x}_*, t_*) = \oiint_{\mathcal{S}_{BW}} \left(\frac{\partial \varphi}{\partial n}\, G \;-\; \varphi\, \frac{\partial G}{\partial n} \right) d\mathcal{S} \tag{3.59}$$

where G is given by Eq. 3.24.

Next, let the two sides of the surface $\mathcal{S}'_W$ surrounding the wake become infinitesimally close to the surface of the wake. In this process, the closed surface $\mathcal{S}'_W$ surrounding the wake is replaced by the two sides of the wake surface, $\mathcal{S}_W$. Let $\mathbf{n}$ be the normal pointing from the side 1 to the side 2 of $\mathcal{S}_W$. In the limit, one obtains

$$\oiint_{\mathcal{S}'_W} \varphi\, \frac{\partial G}{\partial n} d\mathcal{S} = \iint_{\mathcal{S}_W} \Delta\varphi\, \frac{\partial G}{\partial n} d\mathcal{S} \tag{3.60}$$

with $\Delta\varphi = \varphi_2 - \varphi_1$, whereas (see Eq. 3.56)

$$\oiint_{\mathcal{S}'_W} \frac{\partial \varphi}{\partial n}\, G\, d\mathcal{S} = \iint_{\mathcal{S}_W} \Delta \left(\frac{\partial \varphi}{\partial n} \right) G\, d\mathcal{S} = 0 \tag{3.61}$$

Therefore, in the limit, one obtains the extension of Eq. 3.30 to lifting flows

$$E(\mathbf{x}_*, t_*)\varphi(\mathbf{x}_*, t_*) = \oiint_{\mathcal{S}_B} \left(\frac{\partial \varphi}{\partial n}\, G - \varphi\, \frac{\partial G}{\partial n} \right) d\mathcal{S} - \iint_{\mathcal{S}_W} \Delta\varphi\, \frac{\partial G}{\partial n} d\mathcal{S} \tag{3.62}$$

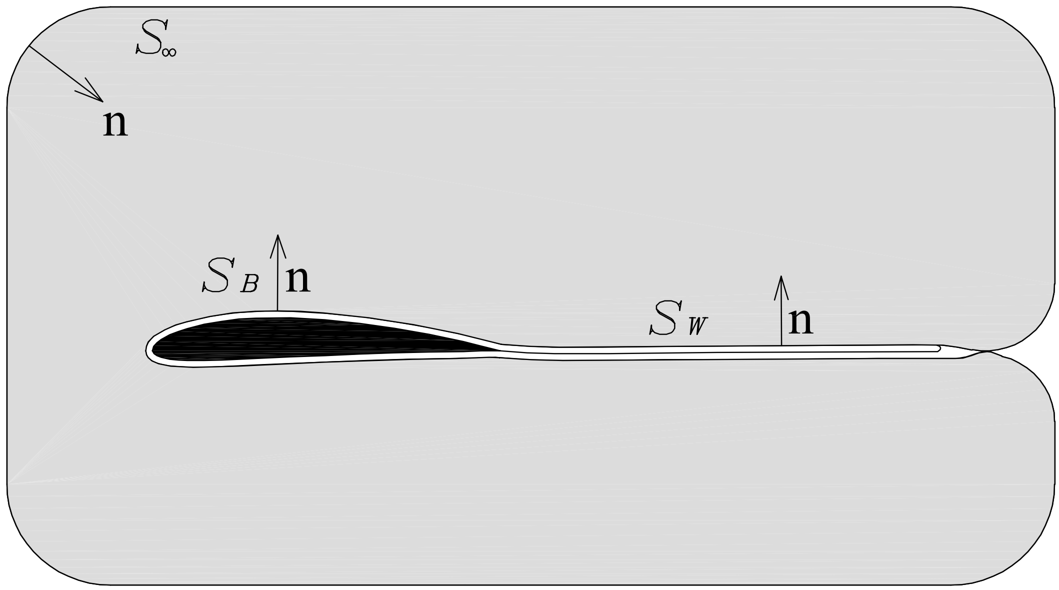

Figure 3.5: Aerodynamic domain for lifting boundary integral formulation

where $\mathcal{S}_B$ is the (closed) surface of the body and $\mathcal{S}_W$ is the (open) surface of the wake.

Equation 3.62 is the desired Green's integral representation for incompressible potential aerodynamics (lifting case). Note that the wake is represented by a doublet layer, which is known to be equivalent to a vortex layer. If $\mathbf{x}_*$ is in the field, Eq. 3.62 is an integral representation of $\varphi(\mathbf{x}_*, t_*)$ anywhere in the field in terms of the values of φ and $\partial\varphi/\partial n$ on the surface of the body, $\mathcal{S}_B$, and of $\Delta\varphi$ on the surface of the wake, $\mathcal{S}_W$. If $\mathbf{x}_*$ is on the surface $\mathcal{S}_B$, Eq. 3.62 is a compatibility condition between the values of φ and $\partial\varphi/\partial n$ on $\mathcal{S}_B$, and of $\Delta\varphi$ on $\mathcal{S}_W$. Note that $\partial\varphi/\partial n$ on $\mathcal{S}_B$ is known from the boundary condition on the surface of the body $\mathcal{S}_B$ (Eq. 3.54), and that $\Delta\varphi$ on $\mathcal{S}_W$ is known from the preceding time history (see following paragraph): hence, Eq. 3.62 is an integral equation that may be used to evaluate φ on $\mathcal{S}_B$.

Once φ on the surface $\mathcal{S}_B$ is known, Eq. 3.62 may be used to calculate φ, and hence the velocity (Eq. 3.12) and the pressure (Eq. 3.16), anywhere in the field. In particular, noting that $\nabla_* G = -\nabla_* (1/4\pi r) = -\mathbf{r}/4\pi r^3$, where $\mathbf{r} = \mathbf{x} - \mathbf{x}_*$, one obtains

$$\mathbf{v}(\mathbf{x}_*) = \oiint_{\mathcal{S}_B} \left[\frac{\partial\varphi}{\partial n} \frac{-\mathbf{r}}{4\pi r^3} - \varphi \, \frac{\partial}{\partial n} \left(\frac{-\mathbf{r}}{4\pi r^3} \right) \right] d\mathcal{S} - \iint_{\mathcal{S}_W} \Delta\varphi \, \frac{\partial}{\partial n} \left(\frac{-\mathbf{r}}{4\pi r^3} \right) d\mathcal{S} \tag{3.63}$$

This equation may also be used to calculate the velocity of the points of the wake.[9] From this,

[9]For a wake point, the contribution of the wake integral from an infinitesimal neighborhood of the wake point is excluded from the calculation: such a contribution is responsible for the velocity discontinuity across the wake, and its exclusion automatically yields the semi-sum of the values on the two sides of the wake.

one may evaluate the geometry of the wake at time $t + dt$. Note that, according to Eq. 3.58, $\Delta\varphi$ remains constant following the wake points. Thus, we need a boundary condition that gives the values of $\Delta\varphi$ as the wake point leaves the trailing edge. As shown in the following section, this is given by Eq. 3.64. At this point the value of φ at time $t + dt$ may be evaluated and the process may be repeated.

3.3.5 Wake Generation; Trailing-Edge Condition

In this section the issues of the wake generation and of the trailing-edge condition are analyzed. In order to discuss the problem of wake generation, it is convenient to consider the problem of an airfoil subject to a sudden start, *i.e.*, the case of an airfoil that at $t = 0^-$ is at rest and is surrounded by a fluid that is also a rest. At time 0^+, the airfoil is assumed to have finite velocity.

Since the fluid was at rest at $t = 0^-$, at time $t = 0^+$ the wake has not yet had time to develop. Therefore, φ is continuous everywhere in the field and the solution to the problem at time 0^+ is obtained by solving Eq. 3.62 with $\Delta\varphi = 0$. This corresponds to the solution of the Laplace equation without wake contribution. This solution, unique even for two-dimensional unsteady flows,[10] shows that the separation line may be different from the trailing edge, and that the velocity is infinite at the trailing edge (see, *e.g.*, conformal-mapping solutions for two-dimensional flows). At time $t = dt$, there exist at least two possibilities: in one, the vorticity remains at the trailing edge; in the other the vorticity is shed from the trailing edge and goes into the field. Thus, the solution is not unique, unless we introduce a specific assumption at the trailing edge. There exist physical reasons for preferring a solution with vorticity present in the field: the D'Alembert paradox states that, if the velocity is potential in the entire flow field (*i.e.*, if there is no wake), the forces acting on the body are equal to zero; on the other hand, numerical solutions obtained by assuming that the vorticity is shed in the field are in excellent agreement with experimental data. Thus, we introduce the following assumption: concentrated vortices do not exist at the trailing edge (see Morino, 1986 and 1990, for a discussion of this point).

The implication of this assumption is that the value of $\Delta\varphi$ is continuous at the trailing edge. For, the above mentioned equivalence between doublet and vortex layers (Batchelor, 1967) indicates that a discontinuity in the doublet intensity implies the presence of a concentrated

[10] In the case of unsteady two-dimensional flows around an airfoil, the solution is unique, because the intensity of the non-trivial solution (*i.e.*, the circulation around the airfoil) is determined by applying Kelvin's theorem and the initial conditions. In particular, if the fluid is initially at rest, the circulation around a sufficiently large path is always equal to zero, *i.e.*, the circulation around the airfoil is always equal and opposite to the total vorticity shed in the wake.

vortex. In mathematical terms, if $\mathbf{x}_U$ is a point on the upper surface of the body, $\mathbf{x}_L$ is a point on the lower surface of the body, $\mathbf{x}_W$ is a point on the wake, and $\mathbf{x}_{TE}$ is a point on the trailing edge, the condition is expressed as

$$\lim_{\mathbf{x}_W \to \mathbf{x}_{TE}} \Delta\varphi(\mathbf{x}_W) = \lim_{\mathbf{x}_U \to \mathbf{x}_{TE}} \varphi(\mathbf{x}_U) - \lim_{\mathbf{x}_L \to \mathbf{x}_{TE}} \varphi(\mathbf{x}_L) \tag{3.64}$$

It should be noted that the condition introduced above is fully equivalent, from the physical point of view, to the Kutta-Joukowski hypothesis that the flow be smooth at the trailing edge. However, there is a slight difference from a mathematical point of view, since the classical Kutta-Joukowski hypothesis is used to eliminate the non-uniqueness of the solution for two-dimensional steady-state flows, whereas that introduced here deals with the shedding of vorticity, in unsteady two- and three-dimensional flows.

3.3.6 Discretization of Incompressible-Flow Formulation

In order to solve the problem, the integral formulation presented above must be discretized. It should be noted that a zeroth-order formulation is used in the actual computations for all the results presented here. Hence, this type of discretization is briefly outlined here. The zeroth-order formulation is obtained by dividing the surface of the body into M elements $\mathcal{S}_m$ (see Fig. 3.6) and that of the wake into N elements $\mathcal{S}_n$.[11]

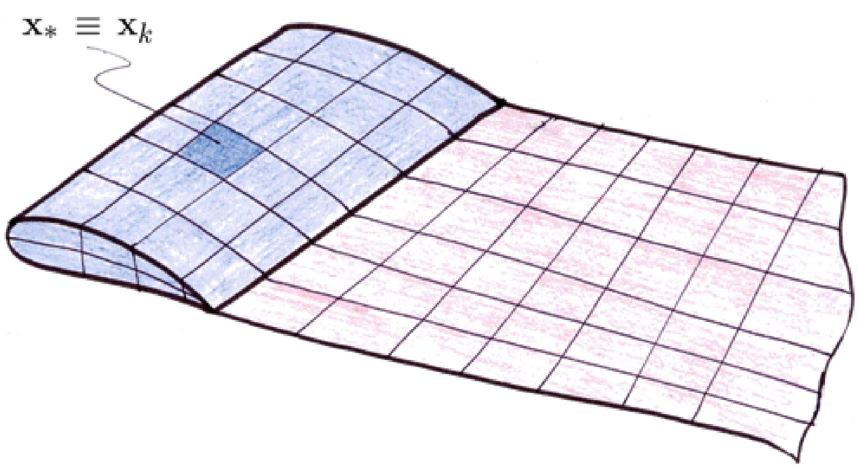

Figure 3.6: panel discretization for body and wake surfaces.

Using the collocation method (*i.e.*, satisfying the equation at M collocation points), and

[11] In particular, hyperboloidal quadrilateral elements are used for all the results presented here (see Morino, Chen, and Suciu, 1975).

setting the collocation points at the centers, $\mathbf{x}_k$, of the elements, one obtains

$$\frac{1}{2}\varphi_k(t) = \sum_{m=1}^{M} B_{km}\chi_m(t) + \sum_{m=1}^{M} C_{km}\varphi_m(t) + \sum_{n=1}^{N} F_{kn}\Delta\varphi_n(t) \tag{3.65}$$

where $E_k(t) = E(\mathbf{x}_k, t)$, and $\chi_m = (\partial\varphi/\partial n)_m$, whereas

$$B_{km} = \iint_{\mathcal{S}_m} \frac{-1}{4\pi\|\mathbf{x} - \mathbf{x}_k\|} d\mathcal{S} \tag{3.66}$$

$$C_{km} = \iint_{\mathcal{S}_m} \frac{\partial}{\partial n}\left(\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_k\|}\right) d\mathcal{S} \tag{3.67}$$

$$F_{kn} = \iint_{\mathcal{S}_n} \frac{\partial}{\partial n}\left(\frac{1}{4\pi\|\mathbf{x} - \mathbf{x}_k\|}\right) d\mathcal{S} \tag{3.68}$$

Note that the coefficients B_{km} and C_{km} are time independent if the surface $\mathcal{S}_B$ moves in rigid-body motion. In practice, constant coefficients are used when the motion consists of small oscillations around a reference configuration in the body frame. Similarly, the coefficients F_{kn} are time independent if the surface $\mathcal{S}_W$ is rigidly connected with $\mathcal{S}_B$. This occurs in the steady-state case of an airplane and of a helicopter rotor in hover, and is approximately true when the flow consists of small oscillations around a steady-state configuration.

At each time step, χ_m are known from the boundary conditions, $\Delta\varphi_n$ are known from the preceding time step, and Eq. 3.65 may be used evaluate φ_k. A similar discretization[12] is used for Eq. 3.63 to obtain the velocity of the wake points in terms of χ_m, φ_m and $\Delta\varphi_n$, as

$$\mathbf{v}_p(t) = \sum_{m=1}^{M} \mathbf{b}_{pm}\chi_m(t) + \sum_{m=1}^{M} \mathbf{c}_{pm}\varphi_m(t) + \sum_{n=1}^{N} \mathbf{f}_{pn}\Delta\varphi_n(t) \tag{3.69}$$

Once the velocity $\mathbf{v}_p(t)$ is known, the new location of a wake point is obtained from

$$\mathbf{x}_p(t + \Delta t) = \mathbf{x}_p(t) + \mathbf{v}_p(t)\Delta t \tag{3.70}$$

Note that according to Eq. 3.58, $\Delta\varphi_p$ is constant in time (as long as $\mathbf{x}_p$ indicates the same material wake point). The trailing-edge condition is used to obtain the values of $\Delta\varphi_n$ on the first row of elements from the values of φ_m at the trailing-edge points on the upper and lower surfaces. The process may then be repeated for the next time step.

In several applications, the wake may be assumed to be prescribed. In this case, the problem is linear and the solution is typically obtained in the frequency domain. In order to accomplish this, recall that Eq. 3.58 implies that $\Delta\varphi$ remains constant following a wake point $\mathbf{x}_W$, and equal to the value it had when $\mathbf{x}_W$ left the trailing edge. Thus,

$$\Delta\varphi(\mathbf{x}_W, t) = \Delta\varphi_{TE}(t - \tau) \tag{3.71}$$

[12] It may be noted that a zeroth-order discretization of the velocity representation yields "uniform doublet-gradient elements" which are identical to vortex filaments at the boundary of the elements.

where τ is the time required for a wake point to be convected from the trailing edge to $\mathbf{x}_W$.[13] Thus, Eq. 3.65 may be written as

$$\frac{1}{2}\varphi_k(t) = \sum_{m=1}^{M} B_{km}\chi_m(t) + \sum_{m=1}^{M} C_{km}\varphi_m(t) + \sum_{n=1}^{N} F_{kn}\Delta\varphi_n^{TE}(t-\tau_n) \tag{3.72}$$

where $\Delta\varphi_n^{TE}$ is the value of $\Delta\varphi^{TE}$ at the trailing-edge point from which $\mathbf{x}_W$ left the trailing edge.

In addition, according to Eq. 3.64, we have $\Delta\varphi^{TE} = \varphi_{Upper} - \varphi_{Lower}$, which may be approximated as

$$\Delta\varphi_n^{TE} = \sum_{m=1}^{M} S_{nm}\varphi_m \tag{3.73}$$

(where $S_{nm} = 1$ at the upper trailing-edge element, $S_{nm} = -1$ at the lower trailing-edge element, and $S_{nm} = 0$ otherwise). The resulting equations in the time domain are (see previous comments on the time-dependency of the source and doublet coefficients)

$$\frac{1}{2}\varphi_k(t) = \sum_{m=1}^{M} B_{km}\chi_m(t) + \sum_{m=1}^{M} C_{km}\varphi_m(t) + \sum_{n=1}^{N}\sum_{m=1}^{M} F_{kn}S_{nm}\varphi_m(t-\tau_n) \tag{3.74}$$

In the Laplace domain we have

$$Y_{kh}\tilde{\varphi}_h = B_{kh}\tilde{\chi}_h \tag{3.75}$$

where

$$Y_{kh} = \frac{1}{2}\delta_{kh} - C_{kh} - \sum_{n=1}^{N} F_{kn}e^{-s\tau_n}S_{nh} \tag{3.76}$$

[13]In particular, in the case of an airplane having velocity $-U_\infty\mathbf{i}$, the wake is typically assumed to be parallel to the x-axis). In this case, one may assume, consistently, that $\tau = (x_W - x_{TE})/U_\infty$. Indeed, if one decomposes the absolute velocity of the wake particle $\mathbf{v}_W^{(a)}$ as a function of is relative portion with respect to the body reference, *i.e.*, $\mathbf{v}_W^{(a)} = \mathbf{v}_W^{(r)} - \mathbf{i}U_\infty$, the above hypothesis consists of assuming $\mathbf{v}_W^{(a)} = 0$, *i.e.*, $\mathbf{v}_W^{(r)} = \mathbf{i}U_\infty$.

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