1.1.6 Exercises on torsional problems

VWP formulation for the torsional beam where a distributed torsional moment m_t is applied together with a concentrated moment at the beam tip $\hat{M}$. To write the PVW formulation it is necessary to consider the following modified external moment function

$$\hat{m}_t = m_t + \hat{M}\delta(y - L) \tag{1.61}$$

where δ is the Dirac function. The VWP reads

$$\begin{aligned}
\int_0^L \delta\theta'^T GJ\theta' \mathrm{d}y &= \int_0^L \delta\theta^T \hat{m}_t \mathrm{d}y && (1.62)\\
\int_0^L \delta\theta'^T GJ\theta' \mathrm{d}y &= \int_0^L \delta\theta^T m_t \mathrm{d}y + \int_0^L \delta\theta^T \hat{M}\delta(y - L)\mathrm{d}y && (1.63)\\
\int_0^L \delta\theta'^T GJ\theta' \mathrm{d}y &= \int_0^L \delta\theta^T m_t \mathrm{d}y + \delta\theta(L)^T \hat{M} && (1.64)
\end{aligned}$$

Now using integration by parts, it results

$$\left[\delta\theta^T GJ\theta'\right]_0^L - \int_0^L \delta\theta^T (GJ\theta')'\mathrm{d}y = \int_0^L \delta\theta^T m_t \mathrm{d}y + \delta\theta(L)^T \hat{M} \tag{1.65}$$

Reordering the equation

$$\int_0^L \delta\theta^T \left((GJ\theta')' + m_t\right)\mathrm{d}y = \delta\theta(L)^T \left(GJ\theta'(L) - \hat{M}\right) - \delta\theta(0)^T GJ\theta'(0) \tag{1.66}$$

The virtual twist must be compatible with the constraints, so $\delta\theta(0) = 0$. Consequently, given the arbitrariness of $\delta\theta$ excluded the root section, the differential formulation will read

$$\begin{aligned}
(GJ\theta')' + m_t &= 0\\
\theta(0) &= 0\\
GJ\theta'(L) &= \hat{M}
\end{aligned}$$

VWP formulation for a uniform beam connected to the ground through a spring of characteristics k_t. The idea here is exactly the same. Consider the beam subject to the following modified external moment function

$$\hat{m}_t = m_t + k_t\theta\delta(y) \tag{1.67}$$

where δ is the Dirac function. The VWP reads

$$\int_0^L \delta\theta'^T GJ\theta' \mathrm{d}y = \int_0^L \delta\theta^T m_t \mathrm{d}y + \delta\theta(0)^T k_t\theta(0) \tag{1.68}$$

Now using integration by parts, it results

$$\left[\delta\theta^T GJ\theta'\right]_0^L - \int_0^L \delta\theta^T (GJ\theta')'\mathrm{d}y = \int_0^L \delta\theta^T m_t \mathrm{d}y + \delta\theta(0)^T k_t\theta(0) \tag{1.69}$$

Reordering the equation

$$\int_0^L \delta\theta^T \left((GJ\theta')' + m_t\right) \mathrm{d}y = \delta\theta(L)^T \left(GJ\theta'(0) - k_t\theta(0)\right) - \delta\theta(L)^T GJ\theta'(L) \tag{1.70}$$

Consequently, given the arbitrariness of $\delta\theta$ excluded the root section, the differential formulation will read

$$\begin{aligned} (GJ\theta')' + m_t &= 0 \\ GJ\theta'(0) &= k_t\theta(0) \\ GJ\theta'(L) &= 0 \end{aligned}$$