10 Unsteady Aerodynamics

10.1 Two-Dimensional Unsteady Airfoil Theory

Since the aerodynamic environment of the rotor blade in forward flight or during transient motion is unsteady, lifting-line theory requires an analysis of the unsteady aerodynamics of a two-dimensional airfoil. Consider the problem of a two-dimensional airfoil undergoing unsteady motion in a uniform free stream. Linear, incompressible aerodynamic theory represents the airfoil and its wake by thin surfaces of vorticity (two-dimensional vortex sheets) in a straight line parallel to the free stream velocity. For the linear problem the solution for the thickness and camber loads can be separated from the loads due to angle-of-attack and unsteady motion. In the development of unsteady thin-airfoil theory, the foundation is constructed for a number of extensions of the analysis for rotary wings, which are presented in later sections of this chapter.

The airfoil and shed wake in unsteady thin-airfoil theory are modeled by planar sheets of vorticity, as shown in Figure 10.1. An airfoil of chord $2b$ is in a uniform free stream with velocity U. Since the bound circulation of the section varies with time, there is shed vorticity in the wake downstream of the airfoil. The vorticity strength on the airfoil is γ_b, and in the wake γ_w. The blade motion (Figure 10.2) is described by a heaving motion h (positive downward) and a pitch angle α about an axis at $x = ab$ (positive for nose upward). The aerodynamic pitch moment is evaluated about the axis at $x = ab$. The airfoil motion produces an upwash velocity relative to the blade of

$$w_a = U\alpha + \dot{h} + (x - ab)\dot{\alpha} \tag{10.1}$$

In addition to the velocity w_a, at the blade section there is also a downwash velocity λ due to the shed wake, and w_b due to the vorticity representing the blade surface. From the strength of the vortex sheets representing the airfoil and shed wake, these induced velocities are

$$w_b(x) = \frac{1}{2\pi} \int_{-b}^{b} \frac{\gamma_b}{x - \xi}\, d\xi \tag{10.2}$$

$$\lambda(x) = \frac{1}{2\pi} \int_{b}^{\infty} \frac{\gamma_w}{x - \xi}\, d\xi \tag{10.3}$$

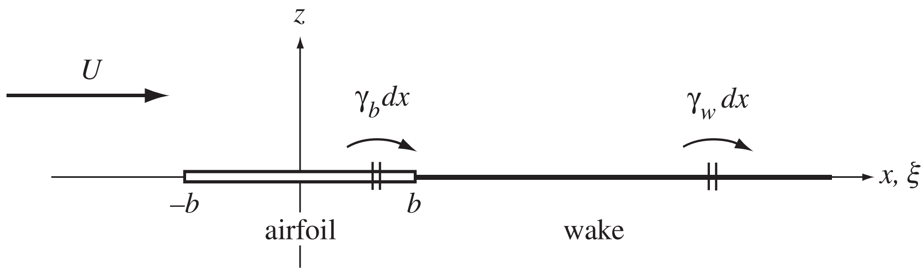

Figure 10.1. Unsteady thin airfoil theory model of the two-dimensional wing and wake.

The boundary condition of no flow through the wing surface, $w_b + \lambda - w_a = 0$, gives an integral equation for the bound vorticity γ_b:

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_b d\xi}{x-\xi} = w_a - \lambda \tag{10.4}$$

From the bound circulation γ_b the chordwise pressure loading can be found. The shed wake vorticity is given by the time rate of change in the total bound circulation $\Gamma = \int_{-b}^{b}\gamma_b dx$:

$$\gamma_w = -\frac{1}{U}\frac{d\Gamma}{dt} \tag{10.5}$$

evaluated at the time the element was shed, $t - (x-b)/U$. So the wake-induced velocity λ is also defined by the blade vorticity γ_b. The boundary condition of no pressure difference across the wake requires that the shed vorticity be convected with the free stream, so $\gamma_w = \gamma_w(x - Ut)$. Finally, the Kutta condition of finite velocity at the blade trailing edge requires $\gamma_b = 0$ at $x = b$.

With the Kutta condition, the integral equation inverts to

$$\gamma_b = -\frac{2}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b}\sqrt{\frac{b+\xi}{b-\xi}}\,\frac{w_a-\lambda}{x-\xi}\,d\xi \tag{10.6}$$

Now write for the wake-induced velocity and the upwash due to the airfoil motion,

$$\lambda = \sum_{n=0}^{\infty}\lambda_n \cos n\theta \tag{10.7}$$

$$w_a = \sum_{n=0}^{\infty} w_n \cos n\theta \tag{10.8}$$

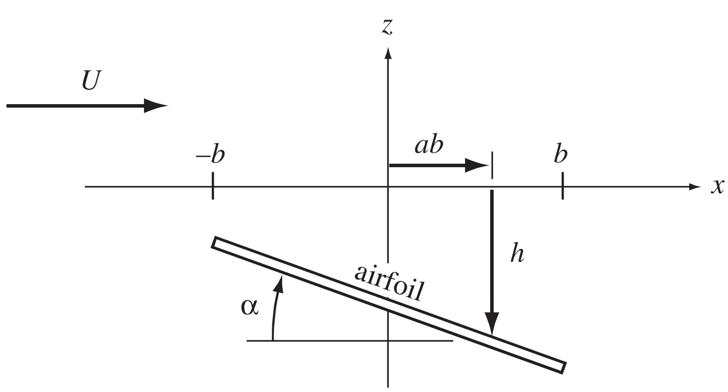

Figure 10.2. Unsteady pitching and heaving motion of the airfoil.

where $x = b\cos\theta$ ($\theta = 0$ at the trailing edge and $\theta = \pi$ at the leading edge). Then the solution for γ_b reduces to

$$\gamma_b = 2\sum_{n=0}^{\infty}\big(w_n - \lambda_n\big) f_n(\theta) \tag{10.9}$$

where f_n is the Glauert series:

$$f_n(\theta) = \begin{cases} \tan(\theta/2) & n = 0 \\ \sin n\theta & n \geq 1 \end{cases} \tag{10.10}$$

In terms of x rather than θ, the expansion of the normal velocity is

$$w_a = w_0 + w_1\big(x/b\big) + w_2\big(2x^2/b^2 - 1) + \dots \tag{10.11}$$

For the blade motion considered, $w_0 = U\alpha + \dot{h} - ab\dot{\alpha}$ (w_a at the midchord), $w_1 = b\dot{\alpha}$, and $w_n = 0$ for $n \geq 2$. The first terms in the Glauert series are

$$f_0 = \sqrt{\frac{b-x}{b+x}} \tag{10.12}$$

$$f_1 = \sqrt{1 - (x/b)^2} \tag{10.13}$$

$$f_2 = 2(x/b)\sqrt{1 - (x/b)^2} \tag{10.14}$$

The coefficients w_n can be evaluated for a particular blade motion. To complete the solution, the wake-induced velocity λ is required.

On substituting for γ_b, the airfoil bound circulation becomes

$$\Gamma = \int_{-b}^{b} \gamma_b dx = 2\pi b\left[\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right] \tag{10.15}$$

Next divide γ_b into two parts: the circulatory vorticity γ_{b_C}, which gives Γ but corresponds to $w_b = 0$ and so has no effect on the boundary conditions; and the noncirculatory vorticity $\gamma_{b_{NC}}$, which satisfies the boundary conditions but gives $\Gamma = 0$. Hence $\gamma_b = \gamma_{b_C} + \gamma_{b_{NC}}$, and the expressions

$$\gamma_{b_C} = \frac{2}{\sin\theta}\left[\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right] \tag{10.16}$$

$$\gamma_{b_{NC}} = -\frac{2}{\sin\theta}\left[(w_0 - \lambda_0)\cos\theta + \frac{1}{2}(w_1 - \lambda_1)\cos 2\theta\right] + 2\sum_{n=2}^{\infty}(w_n - \lambda_n) f_n(\theta) \tag{10.17}$$

give

$$\int_{-b}^{b} \gamma_{b_C} dx = \Gamma \tag{10.18}$$

$$\int_{-b}^{b} \gamma_{b_{NC}} dx = 0 \tag{10.19}$$

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_{b_C}}{x-\xi}\,d\xi = 0 \tag{10.20}$$

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_{b_{NC}}}{x-\xi}\,d\xi = w_a - \lambda \tag{10.21}$$

as required. The relation

$$\frac{1}{\pi}\int_{0}^{\pi}\frac{\cos n\theta\, d\theta}{\cos\theta-\cos\phi} = \frac{\sin n\phi}{\sin\phi} \tag{10.22}$$

is used to establish the last two results.

The pressure is obtained by linearizing the unsteady Bernoulli equation:

$$p = -\rho\left(U\frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial t}\right) \tag{10.23}$$

where ϕ is the velocity potential. The differential pressure on the airfoil surface is then

$$-\Delta p = \rho\left(U\frac{\partial\Delta\phi}{\partial x}+\frac{\partial\Delta\phi}{\partial t}\right) \tag{10.24}$$

where Δp is the upper surface pressure minus the lower surface pressure. The velocity parallel to the blade surface is $u = \partial\phi/\partial x$, and the blade vorticity strength is $\gamma_b = \Delta u$. Then

$$\frac{\partial\Delta\phi}{\partial x} = \Delta u = \gamma_b \tag{10.25}$$

$$\frac{\partial\Delta\phi}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{x}\Delta u\, dx = \frac{\partial}{\partial t}\int_{-b}^{x}\gamma_b dx \tag{10.26}$$

The differential pressure is thus

$$-\Delta p = \rho\left(U\gamma_b+\frac{\partial}{\partial t}\int_{-b}^{x}\gamma_{b_{NC}}dx\right) \tag{10.27}$$

Only the non-circulatory vorticity contributes pressure through the $\partial\phi/\partial t$ term. The unsteady circulatory vorticity produces pressure through the shed-wake-induced velocity λ. Substituting the expressions for γ_{b_C} and $\gamma_{b_{NC}}$ gives

$$\begin{aligned}
-\Delta p &= \rho U\gamma_b + \rho b\int_{\theta}^{\pi}\dot{\gamma}_{b_{NC}}\sin\theta\, d\theta \\
&= 2\rho U\sum_{n=0}(w_n-\lambda_n)f_n + \rho b\Bigg(2(\dot{w}_0-\dot{\lambda}_0)\sin\theta + \frac{1}{2}(\dot{w}_1-\dot{\lambda}_1)\sin 2\theta \\
&\qquad -\sum_{n=1}(\dot{w}_{n+1}-\dot{\lambda}_{n+1})\frac{\sin n\theta}{n} + \sum_{n=3}(\dot{w}_{n-1}-\dot{\lambda}_{n-1})\frac{\sin n\theta}{n}\Bigg) \\
&= \sum_{n=0}p_n f_n(\theta)
\end{aligned} \tag{10.28}$$

where

$$p_0 = 2\rho U(w_0-\lambda_0) \tag{10.29}$$

$$p_1 = 2\rho U(w_1-\lambda_1) + \rho b\left(2(\dot{w}_0-\dot{\lambda}_0) - (\dot{w}_2-\dot{\lambda}_2)\right) \tag{10.30}$$

and

$$p_n = 2\rho U(w_n - \lambda_n) + \frac{\rho b}{n}\left((\dot{w}_{n-1} - \dot{\lambda}_{n-1}) - (\dot{w}_{n+1} - \dot{\lambda}_{n+1})\right) \tag{10.31}$$

for $n \geq 2$.

The net aerodynamic forces on the airfoil are the lift L (positive upward) and moment M about the axis at $x = ab$ (positive nose upward):

$$L = \int_{-b}^{b} (-\Delta p)dx \tag{10.32}$$

$$M = \int_{-b}^{b} (-\Delta p)\big(-x + ab\big)dx \tag{10.33}$$

Substituting for Δp gives

$$L = \rho\left(U\Gamma - \frac{\partial}{\partial t}\Gamma_{NC}^{(1)}\right) \tag{10.34}$$

$$M = -\rho\left(U\Gamma^{(1)} - \frac{1}{2}\frac{\partial}{\partial t}\Gamma_{NC}^{(2)}\right) \tag{10.35}$$

where

$$\Gamma^{(n)} = \int_{-b}^{b} x^n \gamma_b dx \tag{10.36}$$

$$\Gamma_{NC}^{(n)} = \int_{-b}^{b} x^n \gamma_{b_{NC}} dx \tag{10.37}$$

The required circulations can be evaluated by substituting for γ_b:

$$\Gamma = 2\pi b\left[\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right] \tag{10.38}$$

$$\begin{aligned}\Gamma^{(1)} = 2\pi b^2\Bigg[&-\left(\frac{1}{2} + a\right)\left(\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right)\\ &+\frac{1}{4}\Big((w_1 + w_2) - (\lambda_1 + \lambda_2)\Big)\Bigg]\end{aligned} \tag{10.39}$$

$$\Gamma_{NC}^{(1)} = 2\pi b^2\left[-\frac{1}{2}\left(w_0 - \frac{1}{2}w_2\right) + \frac{1}{2}\left(\lambda_0 - \frac{1}{2}\lambda_2\right)\right] \tag{10.40}$$

$$\Gamma_{NC}^{(2)} = 2\pi b^3\left[a\left(\left(w_0 - \frac{1}{2}w_2\right) - \left(\lambda_0 - \frac{1}{2}\lambda_2\right)\right) - \frac{1}{8}\left((w_1 - w_3) - (\lambda_1 - \lambda_3)\right)\right] \tag{10.41}$$

For the blade motion considered here,

$$w_0 + \frac{1}{2}w_1 = U\alpha + \dot{h} + \left(\frac{1}{2} - a\right)b\dot{\alpha} = w_{.75c} \tag{10.42}$$

$$w_0 - \frac{1}{2}w_2 = U\alpha + \dot{h} - ab\dot{\alpha} = w_{.5c} \tag{10.43}$$

$$w_1 + w_2 = b\dot{\alpha} \tag{10.44}$$

$$w_1 - w_3 = b\dot{\alpha} \tag{10.45}$$

where $w_{.75c}$ is the upwash at the three-quarter chord and $w_{.5c}$ is the upwash at the midchord. The coefficients λ_n in the expansion of the induced velocity over the chord can be written in terms of the wake vorticity as follows:

$$\begin{aligned}
\lambda_n &= \frac{2}{\pi}\int_0^{\pi} \lambda \cos n\theta \, d\theta \\
&= \frac{2}{\pi}\int_0^{\pi}\left[\frac{1}{2\pi}\int_b^{\infty}\frac{\gamma_w d\xi}{x-\xi}\right]\cos n\theta \, d\theta \\
&= -\frac{1}{\pi}\int_b^{\infty}\gamma_w\left[\frac{1}{\pi}\int_0^{\pi}\frac{\cos n\theta}{\xi - b\cos\theta}\,d\theta\right]d\xi \\
&= -\frac{1}{\pi}\int_b^{\infty}\gamma_w\left[\frac{\left(\xi-\sqrt{\xi^2-b^2}\right)^n}{b^n\sqrt{\xi^2-b^2}}\right]d\xi
\end{aligned} \tag{10.46}$$

So

$$\lambda_0 + \frac{1}{2}\lambda_1 = -\frac{1}{2\pi b}\int_b^{\infty}\gamma_w\left(\sqrt{\frac{\xi+b}{\xi-b}}-1\right)d\xi \tag{10.47}$$

$$\lambda_1 + \frac{1}{2}\lambda_2 = -\frac{1}{\pi b^2}\int_b^{\infty}\gamma_w\left(\xi-\sqrt{\xi^2-b^2}\right)d\xi \tag{10.48}$$

$$\lambda_1 + \lambda_2 = -\frac{1}{\pi b^2}\int_b^{\infty}\gamma_w\left(\xi-\sqrt{\xi^2-b^2}\right)\left(\sqrt{\frac{\xi+b}{\xi-b}}-1\right)d\xi \tag{10.49}$$

$$\lambda_1 - \lambda_3 = -\frac{2}{\pi b^3}\int_b^{\infty}\gamma_w\left(\xi-\sqrt{\xi^2-b^2}\right)^2 d\xi \tag{10.50}$$

The circulations required for the airfoil lift are then

$$\Gamma = 2\pi b\left(U\alpha + \dot{h} + \left(\frac{1}{2}-a\right)b\dot{\alpha}\right) + \int_b^{\infty}\left(\sqrt{\frac{\xi+b}{\xi-b}}-1\right)\gamma_w d\xi \tag{10.51}$$

and

$$\begin{aligned}
\frac{\partial}{\partial t}\Gamma_{NC}^{(1)} &= \frac{\partial}{\partial t}\left[-\pi b^2\left(U\alpha + \dot{h} - ab\dot{\alpha}\right) - \int_b^{\infty}\left(\xi-\sqrt{\xi^2-b^2}\right)\gamma_w d\xi\right] \\
&= -\pi b^2\left(U\dot{\alpha} + \ddot{h} - ab\ddot{\alpha}\right) - U\int_b^{\infty}\frac{\partial}{\partial\xi}\left(\xi-\sqrt{\xi^2-b^2}\right)\gamma_w d\xi \\
&= -\pi b^2\left(U\dot{\alpha} + \ddot{h} - ab\ddot{\alpha}\right) - U\int_b^{\infty}\left(1-\frac{\xi}{\sqrt{\xi^2-b^2}}\right)\gamma_w d\xi
\end{aligned} \tag{10.52}$$

The airfoil lift now is

$$\begin{aligned}
L &= 2\pi\rho U b\left(U\alpha + \dot{h} + \left(\frac{1}{2}-a\right)b\dot{\alpha}\right) + \rho\pi b^2\left(U\dot{\alpha} + \ddot{h} - ab\ddot{\alpha}\right) \\
&\quad + \rho U\int_b^{\infty}\frac{b}{\sqrt{\xi^2-b^2}}\gamma_w d\xi \\
&= L_Q + L_{NC} + L_W
\end{aligned} \tag{10.53}$$

L_Q is the quasistatic lift, which is the only term present for the steady case ($L = 2\pi\rho U^2 b\alpha$); L_{NC} is the non-circulatory lift, which is due to $\partial\Gamma_{NC}^{(1)}/\partial t$; and L_W is the lift due to the shed-wake-induced velocity. For the unsteady case L_Q is due to the angle-of-attack at the three-quarter chord. From equations 10.34, 10.38, and 10.40, the terms in $L = L_Q + L_{NC} + L_W$ can be written

$$L_Q = 2\pi\rho U b\left(w_0 + \frac{1}{2}w_1\right) \tag{10.54}$$

$$L_{NC} = \rho\pi b^2\left(\dot{w}_0 - \frac{1}{2}\dot{w}_2\right) \tag{10.55}$$

$$L_W = -2\pi\rho U b\left(\lambda_0 + \frac{1}{2}\lambda_1\right) - \rho\pi b^2\left(\dot{\lambda}_0 - \frac{1}{2}\dot{\lambda}_2\right) \tag{10.56}$$

Now the bound circulation is

$$\Gamma = \frac{L_Q}{\rho U} + \int_b^\infty\left(\sqrt{\frac{\xi+b}{\xi-b}} - 1\right)\gamma_w d\xi \tag{10.57}$$

and conservation of vorticity requires $\Gamma = -\int_b^\infty \gamma_w d\xi$; hence

$$L_Q = -\rho U\int_b^\infty\sqrt{\frac{\xi+b}{\xi-b}}\gamma_w d\xi \tag{10.58}$$

and

$$L_C = L_Q + L_W = -\rho U\int_b^\infty\frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w d\xi \tag{10.59}$$

The lift can therefore be written as

$$L = \frac{\int_b^\infty \frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w d\xi}{\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w d\xi}L_Q + L_{NC} \tag{10.60}$$

The effect of the shed wake is to multiply the quasistatic lift L_Q by a factor that depends on γ_w, and hence on the airfoil motion. To evaluate this factor, a specific time history of motion must be considered. Assume that the airfoil has purely harmonic motion at frequency ω: $\alpha = \overline{\alpha}e^{i\omega t}$ and $h = \overline{h}e^{i\omega t}$. Then the wake vorticity γ_w must also be periodic in time and has the form $\gamma_w = \overline{\gamma}_w e^{i\omega(t-\xi/U)}$ when the requirement of convection with the free stream velocity is applied as well. Then $\gamma_w e^{i\omega t}$ factors out of the integrals over the wake, giving

$$\begin{aligned} L &= C(k)L_Q + L_{NC} \\ &= 2\pi\rho U b C(k)\left(U\alpha + \dot{h} + \left(\frac{1}{2} - a\right)b\dot{\alpha}\right) + \rho\pi b^2\left(U\dot{\alpha} + \ddot{h} - ab\ddot{\alpha}\right) \end{aligned} \tag{10.61}$$

where $C(k)$ is a function depending only on the dimensionless frequency $k = \omega b/U$. $C(k)$ is the Theodorsen lift deficiency function (Theodorsen (1935)). Since the magnitude of C varies from 1 at low frequency to 0.5 at high frequency, the effect of the shed wake is to reduce the circulatory lift below the quasistatic value.

The circulations required for the aerodynamic moment about the axis $x = ab$ are obtained by similar manipulations:

$$\Gamma^{(1)} = -b\left(\frac{1}{2}+a\right)\Gamma + \frac{1}{2}\pi b^3\dot{\alpha} + \frac{1}{2}\int_b^\infty \left(\xi - \sqrt{\xi^2-b^2}\right)\left(\sqrt{\frac{\xi+b}{\xi-b}}-1\right)\gamma_w d\xi \tag{10.62}$$

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\Gamma^{(2)}_{NC} = &-b\left(\frac{1}{2}+a\right)\frac{\partial}{\partial t}\Gamma^{(1)}_{NC} - \frac{1}{2}\pi b^3\left(U\dot{\alpha}+\ddot{h}+\left(\frac{1}{4}-a\right)b\ddot{\alpha}\right)\\ &+\frac{1}{2}U\int_b^\infty\left(\xi-\sqrt{\xi^2-b^2}\right)\left(\sqrt{\frac{\xi+b}{\xi-b}}-1\right)\gamma_w d\xi\end{aligned} \tag{10.63}$$

Thus the moment is

$$\begin{aligned}M &= b\left(\frac{1}{2}+a\right)L + M_{QC}\\ &= b\left(\frac{1}{2}+a\right)L - \frac{1}{2}\rho\pi b^3\left(2U\dot{\alpha}+\ddot{h}+\left(\frac{1}{4}-a\right)b\ddot{\alpha}\right)\\ &= b\left(\frac{1}{2}+a\right)C(k)L_Q + \rho\pi b^3\left(a\ddot{h}-\left(\frac{1}{2}-a\right)U\dot{\alpha}-\left(\frac{1}{8}+a^2\right)b\ddot{\alpha}\right)\end{aligned} \tag{10.64}$$

M_{QC} is the moment about the quarter chord, which is the aerodynamic center predicted by thin-airfoil theory. With the pitch axis at the quarter chord ($a = -\frac{1}{2}$) there is no moment due to the lift. The virtual mass terms ($\ddot{h}$ and $\ddot{\alpha}$) arise both from M_{QC} and from the non-circulatory lift L_{NC}. The non-circulatory pitch damping moment is due to lift acting at the three-quarter chord; for $a = \frac{1}{2}$ this moment is zero.

Let us now examine the Theodorsen lift deficiency function $C(k)$, which defines the influence of the shed wake on the aerodynamic loads during unsteady motion. Recall that to evaluate the wake influence, harmonic motion at frequency ω was assumed, giving $\gamma_w = \overline{\gamma}_w e^{i\omega(t-\xi/U)}$. Hence

$$C(k) = \frac{\int_b^\infty \frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w d\xi}{\int_b^\infty \sqrt{\frac{\xi+b}{\xi-b}}\gamma_w d\xi} = \frac{\int_1^\infty \frac{\xi}{\sqrt{\xi^2-1}}e^{-ik\xi}d\xi}{\int_1^\infty \sqrt{\frac{\xi+1}{\xi-1}}e^{-ik\xi}d\xi} = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k)+iH_0^{(2)}(k)} \tag{10.65}$$

where $H_n^{(2)} = J_n - iY_n$ is the Hankel function, and the reduced frequency is $k = \omega b/U$. The real and imaginary parts are defined by $C = F + iG$. Figure 10.3 shows the magnitude and phase of the lift deficiency function for reduced frequencies up to $k = 1$. For $k = 0$, $C = 1$ as is required of the static limit; for large frequencies the magnitude approaches $|C| = 0.5$, so the shed wake reduces the circulatory lift to one-half the quasistatic value. There is a moderate phase shift that has a maximum just above 15° at about $k = 0.3$ and approaches zero again at high frequencies. For small frequencies, the lift deficiency function is approximately

$$C(k) \cong \left(1-\frac{\pi}{2}k\right) + ik\left(\ln\frac{k}{2}+\gamma\right) \tag{10.66}$$

where $\gamma = 0.5772156\ldots$ is Euler's constant. For a rotor, the frequency of the blade motion can be expressed in terms of the rotational speed Ω. Consider n/rev motion,

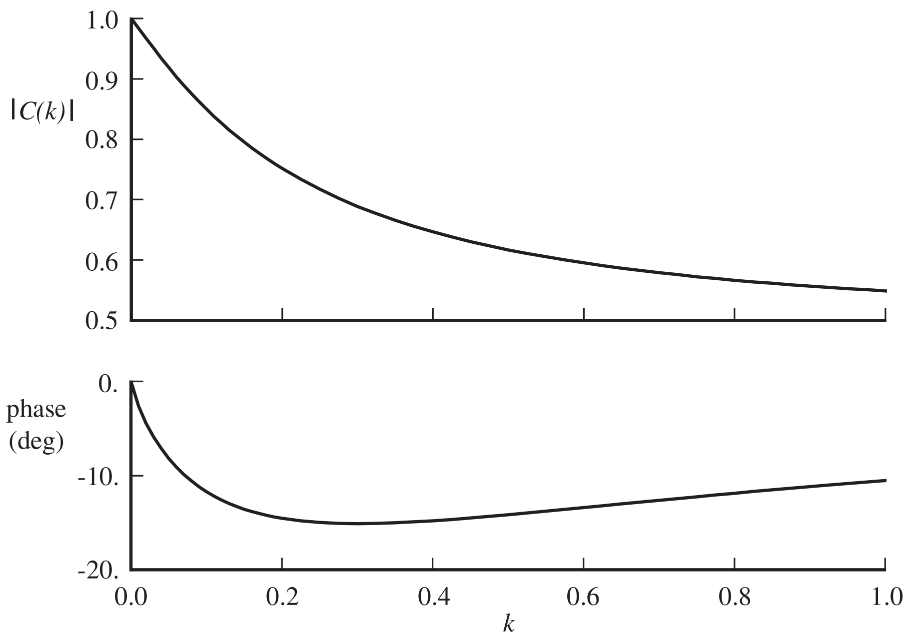

Figure 10.3. Theodorsen lift deficiency function.

where $\omega = n\Omega$. The free stream velocity in hover is Ωr, and the semi-chord is $c/2$, so the reduced frequency becomes $k = nc/2r$. For the high-aspect-ratio blades of rotors, typically $k \cong 0.05n$. For the lower harmonics, the reduced frequency is small, and the lift deficiency function is near unity. For 1/rev motion there is perhaps a 5% reduction in the lift due to the shed wake. Thus the neglect of the shed wake and other unsteady aerodynamic effects in the analysis of the rotor performance and flap motion of the earlier chapters is justified. For the higher harmonics, the reduced frequency is large enough that the shed wake effects must be accounted for to obtain an accurate estimate of the loads.

An alternative form of the unsteady thin-airfoil result is a Glauert series for the pressure, developed by Cicala (1951):

$$-\Delta p = \rho U^2 e^{i\omega t} \sum_{n=0}^{\infty} a_n f_n(\theta) \tag{10.67}$$

where $x = b\cos\theta$. Expand the upwash due to the blade motion as a cosine series:

$$w_a = U e^{i\omega t} \left(A_0 + 2 \sum_{n=1}^{\infty} A_n \cos n\theta \right) \tag{10.68}$$

Then the solution can be written as

$$a_0 = 2\big(A_0 + A_1\big)C(k) - 2A_1 \tag{10.69}$$

$$a_n = -\frac{2ik}{n}\,(A_{n+1} - A_{n-1}) + 4A_n \tag{10.70}$$

with the lift and moment given by

$$L = \rho U^2 b\pi \left(a_0 + \frac{1}{2} a_1 \right) e^{i\omega t} \tag{10.71}$$

$$M_{QC} = -\rho U^2 b^2 \frac{\pi}{4} \big(a_1 + a_2\big) e^{i\omega t} \tag{10.72}$$

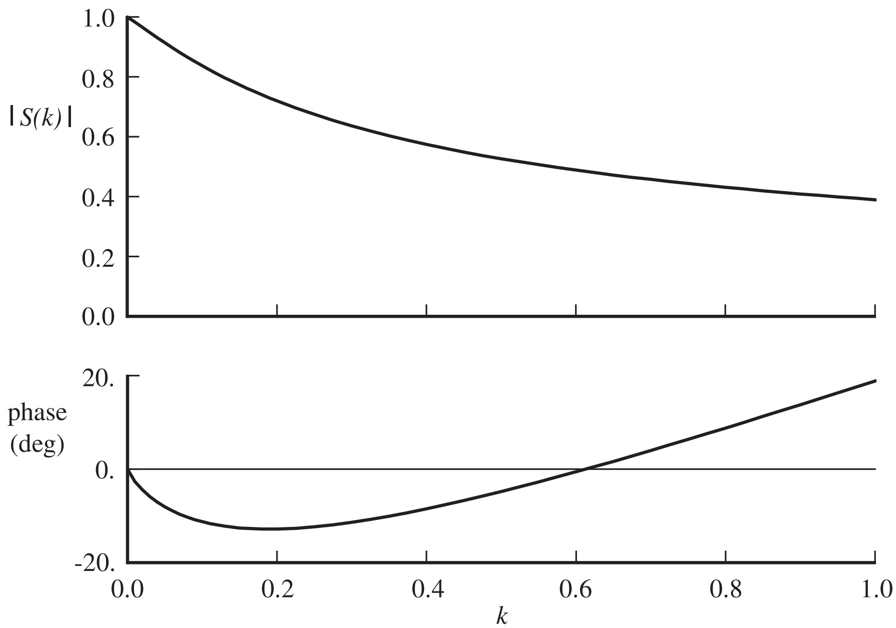

Figure 10.4. Sears function for sinusoidal gust loading.

For example, consider an encounter with a sinusoidal gust of wavelength $2\pi b/k$, so the airfoil sees the upwash velocity

$$w_a = w_0 e^{i\omega(t-x/U)} = w_0 e^{i\omega t} e^{-ikx/b} = w_0 e^{i\omega t} e^{-ik\cos\theta} \tag{10.73}$$

Expanding $e^{-ik\cos\theta}$ as a cosine series in θ gives

$$A_n = \frac{w_0}{U}(-1)^n J_n(k) \tag{10.74}$$

where J_n is the Bessel function. Thus

$$a_0 = \frac{w_0}{U} 2\Big[(J_0(k) - iJ_1(k))\, C(k) + iJ_1(k)\Big] \tag{10.75}$$

$$a_n = \frac{w_0}{U}\frac{2ik}{n}(-1)^{n-1}\left[J_{n+1} + J_{n-1} - \frac{2n}{k}J_n\right] = 0 \tag{10.76}$$

The pressure then has only the first term in the Glauert series:

$$-\frac{\Delta p}{\rho U^2} = e^{i\omega t}\frac{w_0}{U} 2S(k)\sqrt{\frac{b-x}{b+x}} \tag{10.77}$$

The lift is

$$\frac{L}{\rho U^2 b} = e^{i\omega t}\frac{w_0}{U} 2\pi S(k) \tag{10.78}$$

and $M_{QC} = 0$. Here $S(k)$is the Sears function,

$$S(k) = (J_0(k) - iJ_1(k))\, C(k) + iJ_1(k) \tag{10.79}$$

which is shown in Figure 10.4; see Sears (1941). Since any gust can be Fourier analyzed, the resulting aerodynamic lift always acts at the quarter chord. At $k = 0$, the Sears function is $S = 1$. For large frequency, S is approximately

$$S(k) \sim \frac{1}{\sqrt{2\pi k}} e^{i(k-\pi/4)} \tag{10.80}$$

so the magnitude approaches zero (in contrast to the Theodorsen function), while the phase is linear with k.

10.2 Lifting-Line Theory and Near Shed Wake

Two-dimensional airfoil theory shows that the shed wake is an important factor in determining the unsteady aerodynamic loading at frequencies characteristic of rotor blade motion. Unlike the two-dimensional model, the rotary-wing shed wake is in a helical sheet behind the blade, but the major effects are produced by the shed wake nearest to the trailing edge. The near shed wake (extending 15° to 45° in wake age behind the blade) must be modeled appropriately in a calculation of induced velocity and airloading on the rotating wing. Helicopter airloads analyses generally use lifting-line theory to calculate the wake-induced velocity at the bound vortex. Although for the trailed vorticity the wing in lifting-line theory is collapsed to a bound vortex and the induced velocity evaluated at a single point on the chord, the near shed vorticity properly is part of the wing problem. The two-dimensional loads due to the shed wake are obtained from the distribution of induced velocity over the chord (in terms of the inflow coefficients λ_0, λ_1, and λ_2), the evaluation of which requires calculation of the inflow at many points along the chord. In a numerical implementation of lifting-line theory, treating both the shed and trailed vorticity in the wake problem is desirable, by evaluating the induced velocity from all wake elements at a single chordwise collocation point.

Miller (1964) considered a lifting-line theory approximation for the near shed wake. Since the lifting-line assumption of high aspect ratio also implies low reduced frequency, the result is expected to be equivalent to a low-frequency approximation. The approach is to determine what treatment of the shed wake in the lifting-line evaluation of the induced velocity correctly gives the unsteady loads on the two-dimensional airfoil, particularly the lift deficiency function. Evaluating the induced velocity at a single point on the airfoil, equation 10.56 becomes $L_W = -2\pi\rho U b\lambda$, and the circulatory lift is

$$L_C = L_Q - 2\pi\rho U b\lambda \tag{10.81}$$

where the induced velocity is obtained from the shed wake vorticity:

$$\lambda = \frac{1}{2\pi}\int_b^\infty \frac{\gamma_w}{x-\xi}\, d\xi \tag{10.82}$$

In the lifting-line approximation, the airfoil is collapsed to a bound vortex at the quarter chord ($x = -b/2$), and the wake vorticity is extended up to bound vortex. So

$$\lambda = -\frac{1}{2\pi}\int_{-\frac{b}{2}}^\infty \frac{\gamma_w}{\xi+\frac{b}{2}}\, d\xi \tag{10.83}$$

The wake vorticity is given by the time variation of the bound circulation:

$$\gamma_w = -\frac{1}{U}\frac{d\Gamma}{dt} \tag{10.84}$$

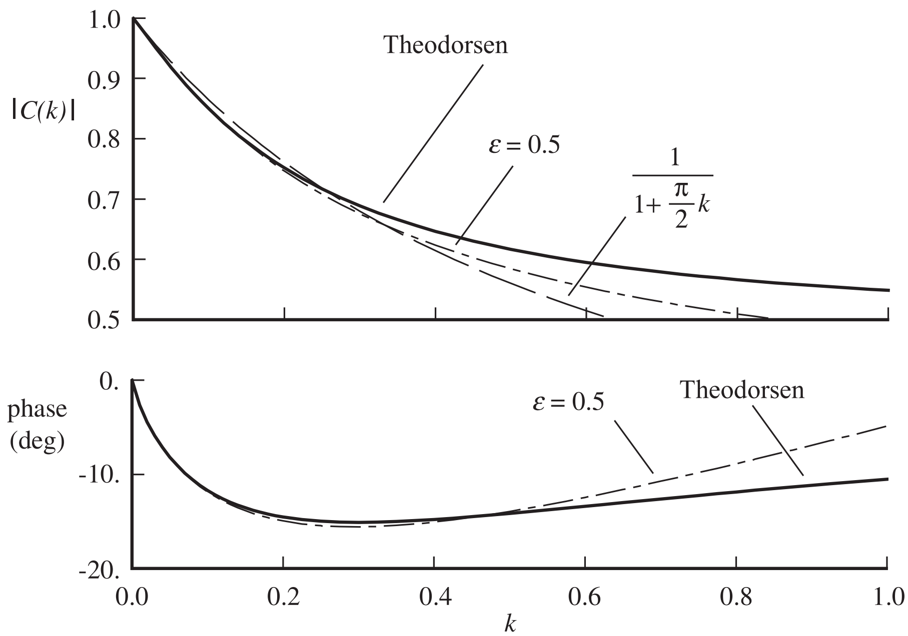

Figure 10.5. Lifting-line approximations for the Theodorsen lift deficiency function.

now at $t - (\xi + \frac{b}{2})/U$. Assuming harmonic motion so that $\Gamma = \overline{\Gamma} e^{i\omega t}$, γ_w is

$$\gamma_w = -\frac{i\omega}{U}\,\Gamma e^{-i\omega(\xi+\frac{b}{2})/U} \tag{10.85}$$

and the induced velocity becomes

$$\begin{aligned} \lambda &= \frac{i\omega}{U}\,\Gamma\frac{1}{2\pi}\int_{-\frac{b}{2}}^{\infty}\frac{e^{-i\omega(\xi+\frac{b}{2})/U}}{\xi+\frac{b}{2}}\,d\xi \\ &= \Gamma\frac{ik}{2\pi b}\int_{0}^{\infty}\frac{e^{-ik\xi}}{\xi}\,d\xi \\ &= \Gamma\frac{k}{2\pi b}\left(\int_{0}^{\infty}\frac{\sin k\xi}{\xi}\,d\xi + i\int_{0}^{\infty}\frac{\cos k\xi}{\xi}\,d\xi\right) \end{aligned} \tag{10.86}$$

The cosine integral is not finite, so is omitted for now. The remaining integral (the real part) is

$$\lambda = \Gamma\frac{k}{2\pi b}\int_{0}^{\infty}\frac{\sin k\xi}{\xi}\,d\xi = \Gamma\frac{k}{4b} = \frac{L_C}{2\pi\rho U b}\,\frac{\pi}{2}k \tag{10.87}$$

Then the unsteady lift is $L_C = L_Q - 2\pi\rho U b\lambda = L_Q - L_C\frac{\pi}{2}k$, or

$$L_C = \frac{L_Q}{1+\frac{\pi}{2}k} \tag{10.88}$$

Thus an approximate lift deficiency function has been obtained:

$$C = \frac{1}{1+\frac{\pi}{2}k} \tag{10.89}$$

which is a correct approximation to order k for the Theodorsen lift deficiency function. Figure 10.5 compares this result with Theodorsen's function. The approximation

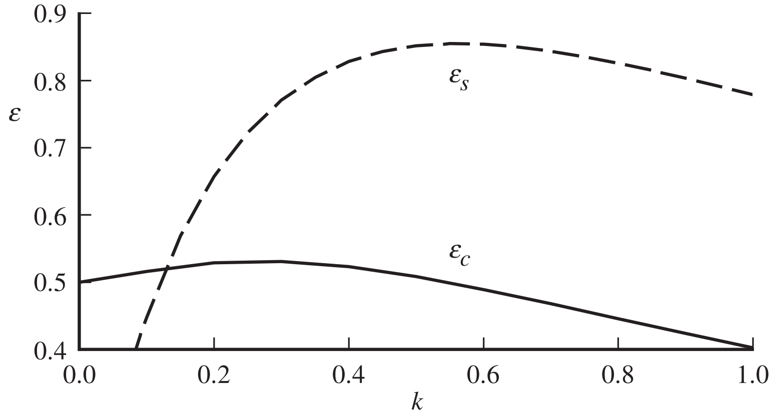

Figure 10.6. Integration limit for near shed wake model in lifting-line theory.

is good even for fairly large values of reduced frequency, but at high k the correct value for $C(k)$ is significantly underestimated.

The lifting-line approximation gives the proper results, except that actually the integral over the wake vorticity is divergent. The difficulty is due to the singularity in the induced velocity at the edge of the vortex sheet, which was extended up to the quarter chord. To correct the model, consider stopping the shed wake a distance $b\epsilon$ behind the quarter chord (where the induced velocity is evaluated). Hence λ is

$$\lambda = \frac{i\omega}{U}\Gamma\frac{1}{2\pi}\int_{-\frac{b}{2}+b\epsilon}^{\infty}\frac{e^{-i\omega(\xi+\frac{b}{2})/U}}{\xi+\frac{b}{2}}\,d\xi = \Gamma\frac{ik}{2\pi b}\int_{\epsilon}^{\infty}\frac{e^{-ik\xi}}{\xi}\,d\xi$$

$$= \Gamma\frac{k}{2\pi b}I = \frac{L_C}{2\pi\rho U b}kI \tag{10.90}$$

where

$$I = \int_{\epsilon}^{\infty}\frac{\sin k\xi}{\xi}\,d\xi + i\int_{\epsilon}^{\infty}\frac{\cos k\xi}{\xi}\,d\xi \tag{10.91}$$

The resulting lift deficiency function is

$$C = \frac{1}{1+kI} \tag{10.92}$$

Requiring that this approximation give exactly the Theodorsen function determines the parameter ϵ; actually there are two values, ϵ_c and ϵ_s, for the cosine and sine integrals, respectively. The important parameter is ϵ_c, which prevents the divergence of the cosine integral. The limit for low frequency is $\epsilon_c = \frac{1}{2}$. Figure 10.6 shows the results for ϵ_c and ϵ_s over a range of frequencies. For $k = 0$ to 1, $\epsilon = \frac{1}{2}$ is a good approximation, particularly for the cosine integral.

Figure 10.5 shows the lift deficiency function obtained from equation 10.92 using $\epsilon = \frac{1}{2}$. To first order in k, equation 10.92 gives

$$C = \frac{1}{1+kI} \cong 1 - kI \cong \left(1 - \frac{\pi}{2}k\right) + ik\,(\ln k\epsilon + \gamma) \tag{10.93}$$

which matches the expansion of Theodorsen's function (equation 10.66) if $\epsilon = \frac{1}{2}$. It is concluded that the near shed wake in the lifting-line model should be extended to a quarter chord ($b\epsilon = b/2 = c/4$) behind the point where the induced velocity is being calculated.

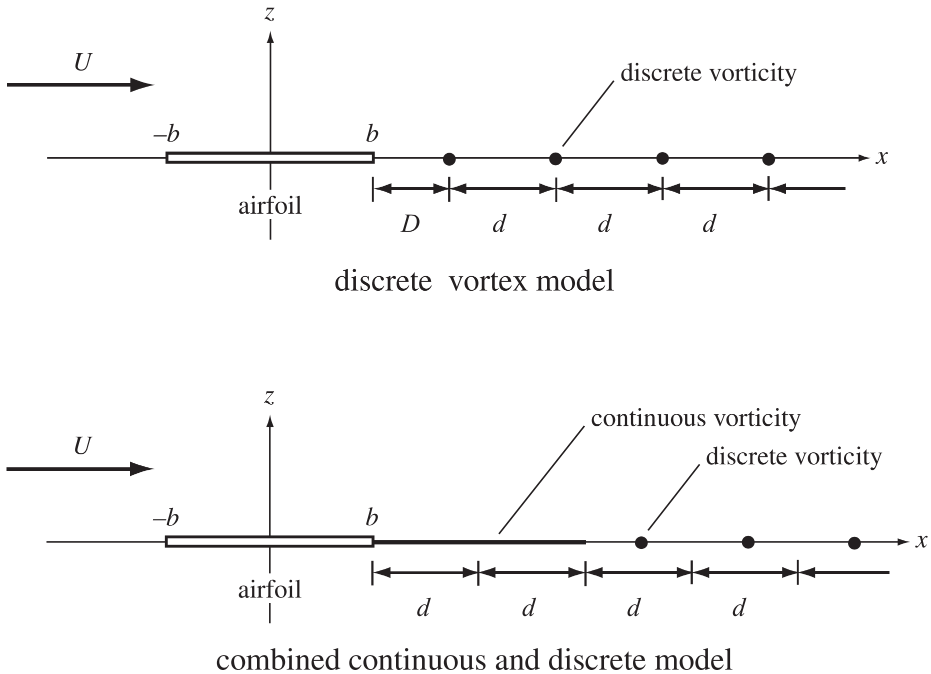

Figure 10.7. Models for shed wake of a two-dimensional airfoil.

For a sinusoidal gust, the lifting-line approximation uses the downwash at the three-quarter chord. Equation 10.73 becomes $w_a = w_0 e^{i\omega t} e^{-ik/2}$, and $A_0 = \frac{w_0}{U} e^{-ik/2}$ for the cosine series. Then

$$S = \frac{1}{2}\left(a_0 + \frac{1}{2}a_1\right)\frac{U}{w_0} = \frac{1}{2}\Big(2A_0 C + ikA_0\Big)\frac{U}{w_0} = A_0\left(C + \frac{ik}{2}\right)\frac{U}{w_0}$$
$$= e^{-ik/2}\left(C(k) + \frac{ik}{2}\right) \tag{10.94}$$

is the lifting-line approximation for the Sears function.

Piziali (1966) considered the effect of a discrete vortex representation of the rotor wake on the shed wake influence. The spirals of the rotary-wing wake are most easily represented by a lattice of finite-strength line vortex segments. In two-dimensional airfoil theory, the corresponding shed wake model is a series of point vortices (Figure 10.7). The distance between the vortices in the wake is $d = 2\pi U/N\omega$, for N vortices per cycle of oscillation. The induced velocity is calculated N times per cycle. The discrete shed vortices correspond to a step change in the airfoil bound circulation. The distance of the first vortex behind the trailing edge is D. Piziali calculated the ratio of the unsteady lift and moment to their quasisteady counterparts for this model and then compared this ratio with the Theodorsen function for pitch and heaving of the airfoil at various frequencies. It was found that $D = d$ does not give good results even with a large number of points per cycle. However, if the entire discrete wake model is advanced closer to the trailing edge so that $D = d/3$, reasonable results are obtained over the frequency range of interest. The conclusion was that with a vortex lattice wake model in a rotary-wing airloads analysis, the shed wake elements should be advanced by about 70% of the azimuthal spacing, so the first elements are closer to the blade trailing edge. A linear variation of bound circulation between the azimuthal collocation points generates vortex sheet elements in the wake. Collapsing each sheet element to a point vortex at the center of the element implies $D = d/2$. Daughaday and Piziali (1966) found that the calculation of the lift and moment at high frequency can be improved by representing the shed wake just behind the blade as a continuous distribution of vorticity, replacing the first few discrete elements in the vortex lattice (Figure 10.7).

Lifting-line theory calculates the blade loads from the velocity induced at the section by the shed and trailed vorticity in the wake. For the inflow calculation, the blade is modeled by the bound vortex at the quarter chord, and the trailed vorticity (due to the spanwise lift variation) is extended up to the bound vortex. The induced velocity is then evaluated along the bound vortex or at the three-quarter chord. The simplest and most economical representation of the complex wake structure is a lattice of finite-strength vortex-line elements. A line vortex is in fact a good representation of the rolled-up tip vortices of the blades. Based on the two-dimensional airfoil analyses discussed in this section, such a lifting-line model can be used for the shed wake as well, and the shed-wake-induced velocity can be calculated at a single chordwise point. To correct for the neglect of the chordwise variation of the induced velocity, the shed wake is not extended all the way to collocation point, but is stopped a quarter chord behind it.

10.3 Reverse Flow

Extending thin-airfoil theory to reverse flow ($U < 0$) introduces several sign changes in the results. The variable θ still runs from $\theta = 0$ at the trailing edge to $\theta = \pi$ at the leading edge, so $x = \pm b \cos\theta$. The convention is that in the double sign ($\pm$) the upper sign is for normal flow, and the lower sign is for reverse flow. The differential pressure $-\Delta p$ is still positive upward, but the vorticity (γ_b) changes direction in reverse flow. In reverse flow, equation 10.27 becomes

$$
\begin{aligned}
-\Delta p &= \rho\left(-U\gamma_b + \frac{\partial}{\partial t}\int_x^b \gamma_{b_{NC}}\,dx\right) \\
&= -\rho U\gamma_b + \rho b\int_\theta^\pi \dot{\gamma}_{b_{NC}}\sin\theta\,d\theta \\
&= \sum_{n=0} p_n f_n(\theta)
\end{aligned}
\tag{10.95}
$$

and the absolute value of U is used in equations 10.29, 10.30, and 10.31. The circulation and section loads are now

$$
\Gamma = \pm\int_{-b}^{b}\gamma_b dx = \pm 2\pi b\left[\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right] \tag{10.96}
$$

$$
\begin{aligned}
L &= \pm\int_{-b}^{b}(-\Delta p)dx = \pm\pi b\left(p_0 + \frac{1}{2}p_1\right) \\
&= \pm 2\pi\rho|U|b\left[\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right] \\
&\quad \pm \pi\rho b^2\left[\left(\dot{w}_0 - \frac{1}{2}\dot{w}_2\right) - \left(\dot{\lambda}_0 - \frac{1}{2}\dot{\lambda}_2\right)\right]
\end{aligned}
\tag{10.97}
$$

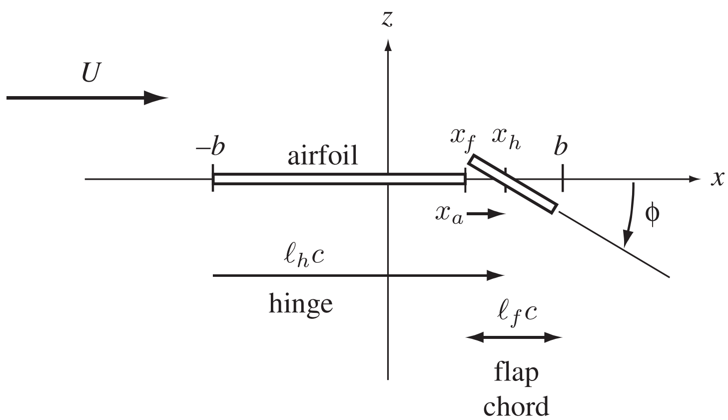

Figure 10.8. Trailing-edge flap geometry.

$$
\begin{aligned}
M_{QC} &= \int_{-b}^{b} (-\Delta p)\left(-x - \frac{b}{2}\right) dx \\
&= -\frac{\pi}{4} b^2 \Big(p_0 (2 \mp 2) + p_1 \pm p_2 \Big) \\
&= -\frac{1}{2}\pi\rho |U| b^2 \Big[(w_0 (2 \mp 2) + w_1 \pm w_2) - (\lambda_0 (2 \mp 2) + \lambda_1 \mp \lambda_2) \Big] \\
&\quad - \frac{1}{2}\pi\rho b^3 \left[\left(\dot{w}_0 - \frac{1}{2}\dot{w}_2 \right) - \left(\dot{\lambda}_0 - \frac{1}{2}\dot{\lambda}_2 \right) \right] \\
&\quad \mp \frac{1}{8}\pi\rho b^3 \left[(\dot{w}_1 - \dot{w}_3) - (\dot{\lambda}_1 - \dot{\lambda}_3) \right]
\end{aligned}
\tag{10.98}
$$

These unsteady loads are used with static airfoil characteristics from a wind-tunnel test, in which the lift is positive upward as the angle-of-attack varies from -180° to 180°. Positive upwash w with reverse flow corresponds to a positive angle-of-attack near 180°, for which the lift is downward; hence the $\pm$ signs on L. M_{QC} is the moment about the theoretical aerodynamic center (quarter chord), positive nose up.

10.4 Trailing-Edge Flap

For an airfoil with a trailing-edge flap, the unsteady loads are derived following Küssner and Schwarz (1941) and Theodorsen and Garrick (1942). Figure 10.8 shows the geometry. The flap chord is $\ell_f c$, with the hinge axis $\ell_h c$ aft of the leading edge. The flap angle is ϕ. The flap leading-edge coordinate is $x_f = b(1 - 2\ell_f)$, so $\cos\theta_f = (1 - 2\ell_f)$. The flap hinge coordinate is $x_h = b(-1 + 2\ell_h)$. The distance of the hinge aft of the leading edge is $x_a = x_h - x_f = 2b(\ell_f + \ell_h - 1)$. An aerodynamically balanced flap (flap leading edge forward of the hinge) has $x_a > 0$.

Thin airfoil theory is linear, so the effects of the trailing-edge flap can be examined separately from the other airfoil motion. The vertical deflection of the airfoil is $z = -(x - x_h)\phi = -(x - x_f)\phi + x_a\phi$ for $x > x_f$. So the upwash along the section is

$$
w = -\left(U\frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} \right) = U\phi + (x - x_h)\dot{\phi} - U x_a \delta(x_f)\phi \tag{10.99}
$$

The last term is present only with a sealed leading-edge gap. For an open gap, the step change in displacement at $x = x_f$ is assumed not to contribute to the upwash.

The coefficients in the cosine Fourier series for w are

$$w_0 = \frac{1}{\pi}\left[U\theta_f\phi + \left(bS_1 - x_h\theta_f\right)\dot{\phi} - U\frac{x_a/b}{S_1}\phi\right] \tag{10.100}$$

$$w_1 = \frac{2}{\pi}\left[US_1\phi + \left(\frac{b}{2}\left(\theta_f + \frac{S_2}{2}\right) - x_hS_1\right)\dot{\phi} - U\frac{(x_a/b)C_1}{S_1}\phi\right] \tag{10.101}$$

$$w_n = \frac{2}{\pi}\left[U\frac{S_n}{n}\phi + \left(\frac{b}{2}\left(\frac{S_{n-1}}{n-1} - \frac{S_{n+1}}{n+1}\right) - x_h\frac{S_n}{n}\right)\dot{\phi} - U\frac{(x_a/b)C_n}{S_1}\phi\right] \tag{10.102}$$

where $S_k = \sin k\theta_f$ and $C_k = \cos k\theta_f$. Reverse flow is not considered here. The total airfoil loads are obtained from equations 10.96 to 10.98. The flap lift and hinge moment are

$$\begin{aligned} L_f &= \int_{x_f}^{b}(-\Delta p)dx \\ &= b\left[p_0(\theta_f - S_1) + \frac{1}{2}p_1\left(\theta_f - \frac{S_2}{2}\right) + \sum_{n=2}\frac{1}{2}p_n\left(\frac{S_{n-1}}{n-1} + \frac{S_{n+1}}{n+1}\right)\right] \end{aligned} \tag{10.103}$$

$$\begin{aligned} M_f &= -\int_{x_f}^{b}(-\Delta p)(x - x_h)dx \\ &= b^2\left[p_0\left(\left(\frac{1}{2} + x_h/b\right)\theta_f + S_1\left(1 - x_h/b\right) + \frac{1}{4}S_2\right)\right. \\ &\quad + p_1\left(\frac{1}{4}\left(S_1 - \frac{S_3}{3}\right) + \frac{x_h/b}{2}\left(\theta_f - \frac{S_2}{2}\right)\right) \\ &\quad + p_2\left(\frac{1}{4}\left(\theta_f - \frac{S_4}{4}\right) + \frac{x_h/b}{2}\left(S_1 - \frac{S_3}{3}\right)\right) \\ &\quad \left. + \sum_{n=3}p_n\left(\frac{1}{4}\left(\frac{S_{n-2}}{n-2} - \frac{S_{n+2}}{n+2}\right) + \frac{x_h/b}{2}\left(\frac{S_{n-1}}{n-1} - \frac{S_{n+1}}{n+1}\right)\right)\right] \end{aligned} \tag{10.104}$$

with p_n given by equations 10.29 to 10.31. The series converge with about 10 pressure terms, except that the aerodynamic balance terms produced by static flap deflection ($x_a\phi$ terms) do not converge at all for flap lift and hinge moment. These static loads for a flap with sealed gap must be obtained from tests or a more appropriate theory.

10.5 Unsteady Airfoil Theory with a Time-Varying Free Stream

The rotating blade of a helicopter rotor in forward flight sees a periodically varying free stream velocity: $u_T = r + \mu\sin\psi = r(1 + (\mu/r)\sin\psi)$. For either high advance ratio or the inboard sections, the 1/rev variation of the velocity is a significant fraction of the mean. In such cases the time-varying free stream must be included in the unsteady airfoil theory, both for its direct effects and for its influence through the shed wake. Only the case $\mu/r < 1$ is considered. If $\mu/r > 1$, the blade section passes through the reverse flow region, and a simple wake model is not applicable.

Consider the two-dimensional airfoil and wake model described in section 10.1. Only a few modifications are required to account for a time-varying free stream velocity U. The time derivative acts on the velocity now, so equations 10.52 amd 10.63 become

$$\frac{\partial}{\partial t}\Gamma_{NC}^{(1)} = -\pi b^2\left(\frac{d}{dt}(U\alpha+\dot{h}) - ab\ddot{\alpha}\right) - U\int_b^\infty\left(1-\frac{\xi}{\sqrt{\xi^2-b^2}}\right)\gamma_w d\xi \quad (10.105)$$

$$\frac{1}{2}\frac{\partial}{\partial t}\Gamma_{NC}^{(2)} = -b\left(\frac{1}{2}+a\right)\frac{\partial}{\partial t}\Gamma_{NC}^{(1)} - \frac{1}{2}\pi b^3\left(\frac{d}{dt}(U\alpha+\dot{h}) + \left(\frac{1}{4}-a\right)b\ddot{\alpha}\right)$$
$$+\frac{1}{2}U\int_b^\infty\left(\xi-\sqrt{\xi^2-b^2}\right)\left(\sqrt{\frac{\xi+b}{\xi-b}}-1\right)\gamma_w d\xi \quad (10.106)$$

Then the lift and moment are

$$\begin{aligned} L &= L_C + L_{NC} \\ &= L_Q + \rho U\int_b^\infty\frac{b}{\sqrt{\xi^2-b^2}}\gamma_w d\xi + \rho\pi b^2\left(\frac{d}{dt}(U\alpha+\dot{h}) - ab\ddot{\alpha}\right) \end{aligned} \quad (10.107)$$

$$\begin{aligned} M &= b\left(\frac{1}{2}+a\right)L + M_{QC} \\ &= b\left(\frac{1}{2}+a\right)L - \frac{1}{2}\rho\pi b^3\left(\frac{d}{dt}(U\alpha+\dot{h}) + U\dot{\alpha} + \left(\frac{1}{4}-a\right)b\ddot{\alpha}\right) \end{aligned} \quad (10.108)$$

with the quasistatic lift still

$$L_Q = 2\pi\rho U b\left(U\alpha+\dot{h}+\left(\frac{1}{2}-a\right)b\dot{\alpha}\right) \quad (10.109)$$

The only changes are $\dot{U}\alpha$ terms added to the non-circulatory lift and moment. The quasistatic lift and circulatory lift are still given in terms of the wake vorticity:

$$L_Q = -\rho U\int_b^\infty\sqrt{\frac{\xi+b}{\xi-b}}\gamma_w d\xi \quad (10.110)$$

$$L_C = -\rho U\int_b^\infty\frac{\xi}{\sqrt{\xi^2-b^2}}\gamma_w d\xi \quad (10.111)$$

Relating L_Q and L_C in terms of a lift deficiency function requires a knowledge of the dependence of γ_w on ξ. The criterion that there be no pressure difference across the vortex sheet gives

$$-\Delta p = \rho\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial\xi}\right)\Delta\phi = 0 \quad (10.112)$$

which implies

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial\xi}\right)\frac{\partial\Delta\phi}{\partial\xi} = \left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial\xi}\right)\gamma_w = 0 \quad (10.113)$$

the solution of which has the form

$$\gamma_w = \gamma_w\left(\xi - \int^t U\,dt\right) \quad (10.114)$$

If the free stream velocity is constant, the wake vorticity is convected at a constant rate and γ_w is a function of $(\xi - Ut)$ as before. Considering the rotor blade in forward flight, the dimensionless free stream velocity is $U = r + \mu \sin\psi$, so

$$\gamma_w = \gamma_w\big(\xi - r\psi + \mu\cos\psi\big) \tag{10.115}$$

Here $\psi = \Omega t$ is the dimensionless time variable. Now assume periodic motion of the blade. For the flow field to be entirely periodic, the blade motion can consist only of harmonics of the fundamental frequency Ω of the free stream variation. The period of the flow is then $2\pi/\Omega$. The wake vorticity must be a periodic function of ξ, with a wavelength equal to the distance the wake is convected during the period: $\int_0^{2\pi} U\, d\psi = 2\pi r$. Next, write the periodic function γ_w as a Fourier series in ξ with period $2\pi r$:

$$\gamma_w = \sum_{m=-\infty}^{\infty} \gamma_{w_m}(\psi) e^{-im\xi/r} \tag{10.116}$$

Since γ_w must be a function of the quantity $(\xi - r\psi + \mu\cos\psi)$ alone,

$$\gamma_w = \sum_{m=-\infty}^{\infty} \overline{\gamma}_m e^{im(\psi-(\mu/r)\cos\psi)-im\xi/r} \tag{10.117}$$

where $\overline{\gamma}_m$ are constants. For $\mu = 0$ this reduces to $\gamma_w = \overline{\gamma}_w e^{i\omega(t-\xi/U)}$ as before.

With the structure of the wake vorticity established, the relation between the quasistatic and circulatory lift can be constructed. Substituting for γ_w gives

$$L_Q = -\rho U \sum_{m=-\infty}^{\infty} \overline{\gamma}_m e^{im(\psi-(\mu/r)\cos\psi)} \int_b^{\infty} \sqrt{\frac{\xi+b}{\xi-b}} e^{-im\xi/r} d\xi \tag{10.118}$$

$$L_C = -\rho U \sum_{m=-\infty}^{\infty} \overline{\gamma}_m e^{im(\psi-(\mu/r)\cos\psi)} \int_b^{\infty} \frac{\xi}{\sqrt{\xi^2-b^2}} e^{-im\xi/r} d\xi \tag{10.119}$$

Noting that

$$\frac{1}{2\pi}\int_0^{2\pi} \big(1 + (\mu/r)\sin\psi\big) e^{in(\psi-(\mu/r)\cos\psi)} d\psi = 1 \tag{10.120}$$

if $n = 0$ and is zero otherwise, the harmonics γ_m can be evaluated:

$$\gamma_m = \frac{-\int_0^{2\pi}\big(1+(\mu/r)\sin\psi\big)e^{-im(\psi-(\mu/r)\cos\psi)} L_Q d\psi}{2\pi\rho U \int_b^{\infty}\sqrt{\frac{\xi+b}{\xi-b}} e^{-im\xi/r} d\xi} \tag{10.121}$$

Thus the circulatory lift is

$$L_C = 2\pi\rho U b \sum_{m=-\infty}^{\infty} e^{im(\psi-(\mu/r)\cos\psi)} C(mb/r)$$

$$\frac{1}{2\pi}\int_0^{2\pi} \big(1 + (\mu/r)\sin\psi\big) e^{-im(\psi-(\mu/r)\cos\psi)} Q\, d\psi \tag{10.122}$$

where $C(mb/r)$ is the Theodorsen lift deficiency function at reduced frequency $k = mb/r$ ($\omega = m\Omega$ and an average velocity $\overline{U} = \Omega r$), and

$$Q = \frac{L_Q}{2\pi\rho U b} = U\alpha + \dot{h} + \left(\frac{1}{2} - a\right) b\dot{\alpha} \tag{10.123}$$

If now the quasistatic circulation is written as a Fourier series, $Q = \sum_{n=-\infty}^{\infty} Q_n e^{in\psi}$, then the lift can be written in a form analogous to the constant velocity result:

$$L_C = 2\pi \rho U b \sum_{n=-\infty}^{\infty} Q_n e^{in\psi} C_\mu(n, \psi) \tag{10.124}$$

where $C_\mu(n, \psi)$ is the modified lift deficiency function for the n-th harmonic of the blade motion with free stream velocity $U = r + \mu \sin\psi$:

$$C_\mu(n, \psi) = \sum_{m=-\infty}^{\infty} e^{im(\psi - (\mu/r)\cos\psi) - in\psi} C(mb/r)$$
$$\frac{1}{2\pi}\int_0^{2\pi} \big(1 + (\mu/r)\sin\psi\big) e^{-im(\psi - (\mu/r)\cos\psi) + in\psi} d\psi \tag{10.125}$$

For a constant free stream velocity ($\mu = 0$), the integral is non-zero only for $m = n$, and hence $C_{\mu=0}(n, \psi) = C(nb/r)$ as required. An alternative form is

$$L_C = 2\pi \rho U b \sum_{n=-\infty}^{\infty} Q_n \left[\sum_{\ell=-\infty}^{\infty} C_{\ell n} e^{i\ell\psi}\right] \tag{10.126}$$

where

$$C_{\ell n} = \sum_{m=-\infty}^{\infty} \left[\frac{1}{2\pi}\int_0^{2\pi} e^{im(\psi - (\mu/r)\cos\psi) - i\ell\psi} d\psi\right] C(mb/r)$$
$$\left[\frac{1}{2\pi}\int_0^{2\pi} \big(1 + (\mu/r)\sin\psi\big) e^{-im(\psi - (\mu/r)\cos\psi) + in\psi} d\psi\right] \tag{10.127}$$

The coefficients $C_{\ell n}$ are the harmonics in a Fourier series expansion of $e^{in\psi} C_\mu(n, \psi)$. This form shows that the time-varying free stream couples the harmonics of the circulation and lift through the influence of the shed wake.

The integrals appearing in the lift deficiency function for a time-varying free stream can be evaluated in terms of Bessel functions:

$$\begin{aligned} I_{mn} &= \frac{1}{2\pi}\int_0^{2\pi} \big(1 + (\mu/r)\sin\psi\big) e^{-im(\psi - (\mu/r)\cos\psi) + in\psi} d\psi \\ &= \frac{n}{m}\frac{1}{\pi}\int_0^{\pi} e^{im(\mu/r)\cos\psi} \cos(n-m)\psi \, d\psi \\ &= \begin{cases} \frac{n}{m} i^{|n-m|} J_{|n-m|}(m\mu/r) & m > 0 \\ \frac{n}{m} (-i)^{|n-m|} J_{|n-m|}(|m\mu/r|) & m < 0 \end{cases} \end{aligned} \tag{10.128}$$

and

$$\begin{aligned} I_{mn} &= \frac{1}{2\pi}\int_0^{2\pi} \big(1 + (\mu/r)\sin\psi\big) e^{in\psi} d\psi \\ &= \begin{cases} 1 & n = 0 \\ \frac{i}{2}\mu/r & n = 1 \\ 0 & n \geq 2 \end{cases} \end{aligned} \tag{10.129}$$

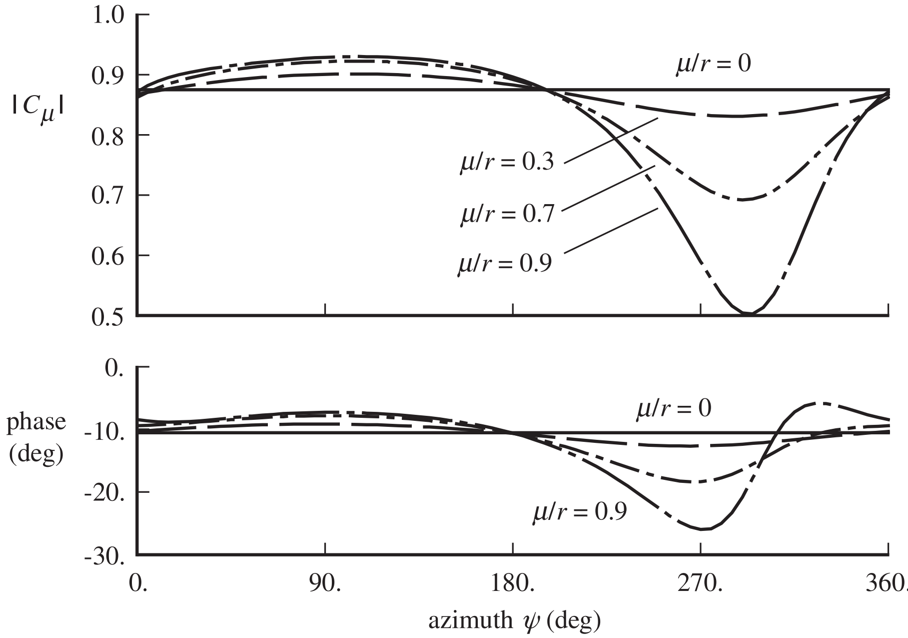

Figure 10.9. Lift deficiency function with a time-varying free stream, for the second harmonic ($n = 2$) and $b/r = 0.04$.

for $m = 0$. Figure 10.9 shows typical results for $C_\mu(n, \psi)$, with $n = 2$ and $b/r = 0.04$. The 1/rev variation in the free stream velocity produces a basic 1/rev variation of C with ψ. The largest influence occurs nearest the reverse flow boundary, at $\psi = 270°$. Most of the range of velocities and radial stations of interest are covered by $0 < \mu/r < 0.7$. The model breaks down for $\mu/r > 1$, when the section passes through the reverse flow region. For small μ/r, the lift deficiency function is approximately

$$\begin{aligned} C_\mu(n, \psi) &\cong \sum_{m=-\infty}^{\infty} e^{i(m-n)\psi}\big(1 - im(\mu/r)\cos\psi\big)C(mb/r) \\ &\quad \frac{1}{2\pi}\int_0^{2\pi}\big(1 + (\mu/r)(\sin\psi + im\cos\psi)\big)e^{i(n-m))\psi}\,d\psi \\ &= C_n + (\mu/r)\frac{in}{2}\Big[\cos\psi\,(C_{n+1} + C_{n-1} - 2C_n) + i\sin\psi\,(C_{n+1} - C_{n-1})\Big] \end{aligned} \tag{10.130}$$

where C_n means $C(nb/r)$. Assuming as well small values of b/r gives

$$\begin{aligned} C_\mu(n, \psi) &\cong C(nb/r) - \Big[(nb/r)(\mu/r)\sin\psi\Big]C'(nb/r) \\ &\cong C(nb/(r + \mu\sin\psi)) \end{aligned} \tag{10.131}$$

Thus for small variations in the free stream velocity (small $nb\mu/r^2$), the lift deficiency function is nearly the same as the Theodorsen function $C(k)$, with the reduced frequency based on the local velocity, $k = \omega b/U$. This approximation works well for moderate n. Figure 10.9 shows the basic dependence on the local reduced frequency. On the advancing side, the increased velocity lowers the reduced frequency, and hence the lift deficiency function is nearer unity. On the retreating side there is the greatest accumulation of shed vorticity in the wake near the trailing edge, and thus the greatest reduction in the lift.

In summary, a time-varying free stream has the following influence on the unsteady aerodynamics of a two-dimensional airfoil: there are additional non-circulatory lift and moment terms due to $d(U\alpha)/dt$; there is coupling by the wake of all the harmonics of the quasistatic and unsteady circulation; and there is a significant influence on the lift deficiency function due to stretching and compressing of the vorticity in the shed wake. For the free stream variation of the rotor blade in forward flight, all these effects basically produce 1/rev variations of the loads. The non-circulatory lift and moment terms are valid for a general time variation of U. The simple approximation $C_\mu(n, \psi) \cong C(k)$ using the local reduced frequency is good up to $\mu/r \cong 0.7$. For small enough μ/r the cruder approximation $C_\mu(n, \psi) \cong C(nb/r)$ using the mean reduced frequency can be chosen, neglecting entirely the influence of a time-varying free stream on the shed wake.

10.6 Unsteady Airfoil Theory for the Rotary Wing

Application of unsteady airfoil theory to rotary wings is often done in the context of lifting-line theory (see section 9.2). Thus the steady two-dimensional loads are obtained from measured airfoil data as a function of angle-of-attack and Mach number. The angle-of-attack is evaluated from the blade motion at the quarter chord. The influence of the near shed wake is in the induced velocity calculated from a vortex wake model, along with the influence of the tip vortices, trailed wake, and far shed wake. What is required from thin-airfoil theory are expressions for the unsteady loads in attached flow.

The airfoil upwash velocity for thin air-foil theory is written in terms of the upwash at the quarter chord, w_{QC}, and the gradient of the upwash along the chord, $w' = dw/dx$:

$$w_a = w_{QC} + \left(x + \frac{c}{4}\right) w' = w_0 + w_1 \cos\theta \tag{10.132}$$

with $x = \pm b\cos\theta$. Thus $w_0 = w_{QC} + \frac{c}{4}w'$ and $w_1 = \pm\frac{c}{2}w'$. Equations 10.97 and 10.98 for the lift and moment, which include reverse flow, become

$$L = \pm 2\pi\rho|U|b\left[w_{QC} - \lambda + \left(\frac{b}{2} \pm \frac{b}{2}\right) w'\right] \pm \pi\rho b^2\left(\dot{w}_{QC} + \frac{b}{2}\dot{w}'\right) \tag{10.133}$$

$$M = b\left(\frac{1}{2} + a\right) L - \frac{1}{2}\pi\rho|U|b^2\left[\left(w_{QC} - \lambda\right)(2 \mp 2) + \frac{b}{4}w'\right] - \frac{1}{2}\pi\rho b^3\left(\dot{w}_{QC} + \frac{3b}{4}\dot{w}'\right) \tag{10.134}$$

where the upper sign in $\pm$ or $\mp$ is for normal flow, and the lower sign is for reverse flow. The lift term in brackets is the quasistatic lift L_Q, and the $\dot{w}$ terms are the non-circulatory lift L_{NC}. Multiplying by $a/2\pi$ introduces the real lift-curve slope a. The chord $c = 2b$ is used instead of the semi-chord, and the section velocity is written $U = u_T$. The distance of the aerodynamic center (quarter chord for thin-airfoil theory) aft of the pitch axis is $x_A = -b\left(\frac{1}{2} + a\right)$. The lift and moment about