POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

09 Structural Dynamics: Dynamic response

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Material

Masarati Chapter 5 (the organization is different, and the material is more extensive, but all is contained there)

Preumont, Chapter 2 starting from 2.5

Dynamic Response

Free response: response to an initial perturbation of conditions (i.e. $\boldsymbol{q_0}, \boldsymbol{\dot{q}_0}$) without any forcing, i.e. ***F = 0***

Forced response: response to a forcing excitation when the initial conditions are zero. If the system is stable the homogeneous part dies out and so the forced repose is equal for *t* -> ∞ to the particular integral of the equation

$$\begin{cases} \mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F} \\ \mathbf{q}(0) = \mathbf{q}_0, \quad \dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_0 \end{cases}$$

Dynamic response

Box 2.3 Some Concepts of System Response

$$\begin{aligned}\text{Total response } (T) &= \underset{(H)}{\text{Homogeneous solution}} + \underset{(P)}{\text{Particular integral}} \\ &= \underset{(X)}{\text{Free response}} + \underset{(F)}{\text{Forced response}}\end{aligned}$$

Note: In general, $H \neq X$ and $P \neq F$.
With no input (no forcing excitation), by definition, $H \equiv X$
At steady state, F becomes equal to P.

How to compute dynamic response

- ✓ **Direct solution** in time domain by modal decomposition and integration
- ✓ **Direct solution** in time domain by **time marching integration**
- ✓ Solution using **Laplace Transform**
- ✓ Solution in **frequency domain** using Fourier Series/Transform

- ✓ Solution can be applied to the **full problem** or **after a coordinate reduction**

Laplace Transform

The Laplace transform is an integral transformation used to solve differential equations. It converts a function of time to a function of a complex variable *s* (denominated complex frequency or Laplace variable).

$$\mathcal{L}[f(t)] = \mathsf{f} = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t$$

$$s = \sigma + \mathrm{j}\omega, \; s \in \mathbb{C}.$$

$$\begin{aligned} \mathcal{L}[\dot{f}(t)] &= s\mathsf{f}(s) - f(0) \\ \mathcal{L}[\ddot{f}(t)] &= s^2\mathsf{f}(s) - sf(0) - \dot{f}(0) \end{aligned}$$

The Laplace transform takes a <u>Differential</u> equation and trasform it into $\rightarrow$ <u>Algebraic</u> equation.

Laplace Transform

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}$$

$$s^2\mathbf{M}\mathbf{q} - s\mathbf{M}\mathbf{q}_0 - \mathbf{M}\dot{\mathbf{q}}_0 + s\mathbf{C}\mathbf{q} - \mathbf{C}\mathbf{q}_0 + \mathbf{K}\mathbf{q} = \mathbf{F}$$

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{q} - \mathbf{M}\dot{\mathbf{q}}_0 - \left(s\mathbf{M} + \mathbf{C}\right)\mathbf{q}_0 = \mathbf{F}$$

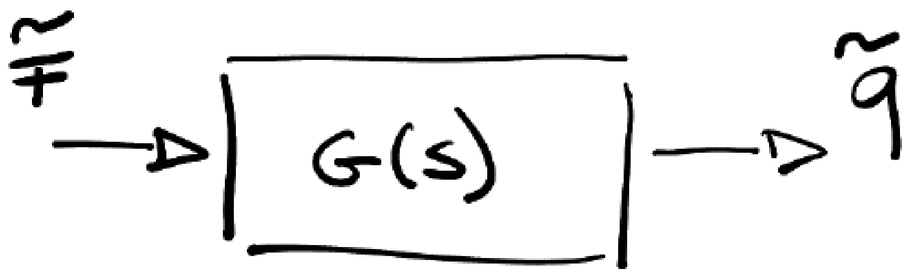
If $\mathbf{q}_0 = \dot{\mathbf{q}}_0 = \mathbf{0} \rightarrow \underline{\text{FORCED RESPONSE}}$

$$\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)\mathbf{q}(s) = \mathbf{F}(s)$$

Transfer function (or transfer matrix)

$$\mathbf{q}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{F}(s)$$

$$\mathbf{G}(s) = \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}$$



Laplace Transform State-Space format

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{q}_d \\ \mathbf{M}\dot{\mathbf{q}}_d + \mathbf{C}\mathbf{q} + \mathbf{K}\mathbf{q} = \mathbf{F} \end{cases}$$

Calling

$$\mathbf{z} = \begin{Bmatrix} \mathbf{q} \\ \mathbf{q}_d \end{Bmatrix}, \quad \mathbf{F} = \mathbf{T}\mathbf{u}(t)$$

It follows that

$$\dot{\mathbf{z}} = \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}}_{\mathbf{A}} \mathbf{z} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{M}^{-1}\mathbf{T} \end{bmatrix}}_{\mathbf{B}} \mathbf{u}(t)$$

<u>OUTPUT</u> $\mathbf{y}$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u}$$

Output equation

1. GENERALIZED STATE $\mathbf{D} = \mathbf{0}$ $\mathbf{C} = [0 \cdots 1 \cdots 0]$

2. NODAL DISPLACEMENT OR VELOCITY at position $\mathbf{x}_i$

$$y(\mathbf{x}_i) = \sum_j N_j(\mathbf{x})q_j(t) = \mathbf{N}(\mathbf{x}_i)\mathbf{q}, \quad \begin{cases} \mathbf{C} = \begin{bmatrix} \mathbf{N} & \mathbf{0} \end{bmatrix}, \mathbf{D} = 0 \text{ displacements} \\ \mathbf{C} = \begin{bmatrix} \mathbf{0} & \mathbf{N} \end{bmatrix}, \mathbf{D} = 0 \text{ velocity} \end{cases}$$

3. ACCELERATION at position $\mathbf{x}_i$

$$\ddot{\mathbf{q}} = -\mathbf{M}^{-1}\mathbf{K}\mathbf{q} - \mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{T}\mathbf{u}$$

$$\begin{aligned} y(\mathbf{x}_i) &= \sum_i N_i(\mathbf{x}_i)\ddot{q}_i(t) = \mathbf{N}(\mathbf{x}_i)\ddot{\mathbf{q}} \\ y(\mathbf{x}_i) &= \underbrace{\begin{bmatrix} -\mathbf{N}\mathbf{M}^{-1}\mathbf{K} & -\mathbf{N}\mathbf{M}^{-1}\mathbf{C} \end{bmatrix}}_{\mathbf{C}} \mathbf{z} + \underbrace{\mathbf{N}\mathbf{M}^{-1}\mathbf{T}}_{\mathbf{D}}\mathbf{u} \end{aligned}$$

State space system

$$
\begin{cases}
\dot{\mathbf{z}} & = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \\
\mathbf{y} & = \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u}
\end{cases}
$$

- ✓ n size of the matrix A (i.e. number of states)
- ✓ m number of output (size of vector y)
- ✓ r number of input (size of vector u)

- ✓ SISO Single Input Single Output r = 1, m = 1
- ✓ MIMO Multiple Input Multiple Output r > 1, m > 1

- ✓ A state matrix
- ✓ B input gain matrix
- ✓ C output gain matrix
- ✓ D direct link matrix

Laplace Transform State-Space format

$$\begin{cases} s\mathbf{z} &= \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{z} + \mathbf{D}\mathbf{u} \end{cases}$$

$$\begin{aligned} \mathbf{z} &= (s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}\,\mathbf{u} \\ \mathbf{y} &= \left(\mathbf{C}\,(s\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}\right) \mathbf{u} \end{aligned}$$

$$\mathbf{G}(s) = \frac{\mathbf{y}}{\mathbf{u}} = \mathbf{C}\,\underbrace{(s\mathbf{I} - \mathbf{A})^{-1}}_{\frac{1}{\det(s\mathbf{I} - \mathbf{A})}[\cdots]} \mathbf{B} + \mathbf{D}$$

Laplace Transform State-Space format

$$\mathsf{G}_{ij}(s) = \frac{\beta_p s^p + \beta_{m-1} s^{m-1} + \cdots + \beta_1 s + \beta_0}{\underbrace{\alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + \alpha_0}_{\text{Characteristic equation of } \mathbf{A}}}$$

$\underline{\text{poles}} \rightarrow$ eigenvalues of $\mathbf{A}$, $n = \text{size}(\mathbf{A})$

$\underline{\text{zeros}} \rightarrow \mathsf{y}_i = 0$ even when $\mathsf{u}_j \neq 0$

Solution using inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}[\mathsf{f}(s)] = \frac{1}{2\pi \mathrm{j}} \lim_{T\to\infty} \int_{\gamma-\mathrm{j}T}^{\gamma+\mathrm{j}T} \mathrm{e}^{st}\mathsf{f}(s)\,\mathrm{d}s$$

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$	Region of convergence	Reference
unit impulse	$\delta(t)$	1	all s	inspection
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$		time shift of unit impulse
unit step	$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$	integrate unit impulse
delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$	$\mathrm{Re}(s) > 0$	time shift of unit step
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$	$\mathrm{Re}(s) > 0$	integrate unit impulse twice
nth power (for integer n)	$t^n \cdot u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$ $(n > -1)$	Integrate unit step n times
qth power (for complex q)	$t^q \cdot u(t)$	$\frac{\Gamma(q+1)}{s^{q+1}}$	$\mathrm{Re}(s) > 0$ $\mathrm{Re}(q) > -1$	[29][30]
nth root	$\sqrt[n]{t} \cdot u(t)$	$\frac{1}{s^{\frac{1}{n}+1}}\Gamma\left(\frac{1}{n}+1\right)$	$\mathrm{Re}(s) > 0$	Set $q = 1/n$ above.
nth power with frequency shift	$t^n e^{-\alpha t} \cdot u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$	$\mathrm{Re}(s) > -\alpha$	Integrate unit step, apply frequency shift
delayed nth power with frequency shift	$(t-\tau)^n e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{n! \cdot e^{-\tau s}}{(s+\alpha)^{n+1}}$	$\mathrm{Re}(s) > -\alpha$	Integrate unit step, apply frequency shift, apply time shift
exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$	$\mathrm{Re}(s) > -\alpha$	Frequency shift of unit step

Frequency response

Consider the case where the input (F or u) is a harmonic function, i.e.:

$$
\begin{aligned}
F(t) &= F_1 \cos \omega t + F_2 \sin \omega t \\
F(t) &= F_0 \cos\left(\omega t + \varphi_0\right) \\
F_0 &= \sqrt{F_1^2 + F_2^2},\ \varphi_0 = \tan^{-1}\left(-\frac{F_2}{F_1}\right)
\end{aligned}
$$

<u>Euler Formulas</u>

$$
\cos \alpha = \frac{\mathrm{e}^{\mathrm{j}\alpha} + \mathrm{e}^{-\mathrm{j}\alpha}}{2}
$$

$$
\begin{aligned}
\Rightarrow F(t) &= \frac{F}{2}\mathrm{e}^{\mathrm{j}\varphi}\mathrm{e}^{\mathrm{j}\omega t} + \frac{F}{2}\mathrm{e}^{-\mathrm{j}\varphi}\mathrm{e}^{-\mathrm{j}\omega t} \\
F(t) &= \hat{F}\mathrm{e}^{\mathrm{j}\omega t} + \ldots
\end{aligned}
$$

with $\hat{F} \in \mathbb{C}$

Dynamic Stiffness

If the input is harmonic, we can compute the particular integral saying that also the generalized dofs will be harmonic

$$\mathbf{q}(t) = \hat{\mathbf{q}}e^{j\omega t}$$

$$\left(-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}\right)\hat{\mathbf{q}}e^{j\omega t} = \hat{\mathbf{F}}e^{j\omega t}$$

K$_{dyn}$ is called Dynamic Stiffness because it is the dynamic generalization of the stiffness matrix **K**

$\Rightarrow$ <u>DYNAMIC STIFFNESS</u>

$$\mathbf{K}_{\mathrm{dyn}}(\omega) = \left(-\omega^2\mathbf{M} + j\omega\mathbf{C} + \mathbf{K}\right)$$

The result obtained is the particular integral, it corresponds to the asymptotic forced response of the system if the system is stable

$$\hat{\mathbf{q}} = \mathbf{K}_{\mathrm{dyn}}^{-1}\hat{\mathbf{F}}$$

Frequency response (or Dynamic compliance)

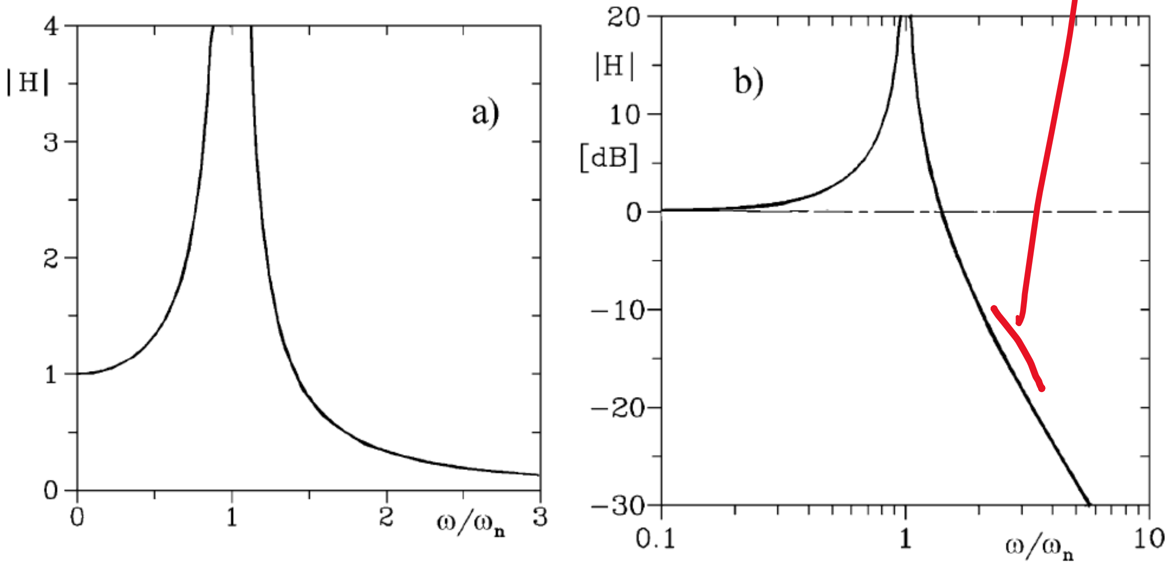
$$\hat{\mathbf{H}}(\omega) = \mathbf{K}_{\text{dyn}}^{-1}(\omega) = \frac{\hat{\mathbf{q}}}{\hat{\mathbf{F}}}$$

Sometime it is better to define a non dimensional frequency response i.e.,

$$\hat{\mathbf{H}}(\omega) = \mathbf{K}_{\text{dyn}}^{-1}(\omega)\mathbf{K} = \frac{\mathbf{K}\hat{\mathbf{q}}}{\hat{\mathbf{F}}}$$

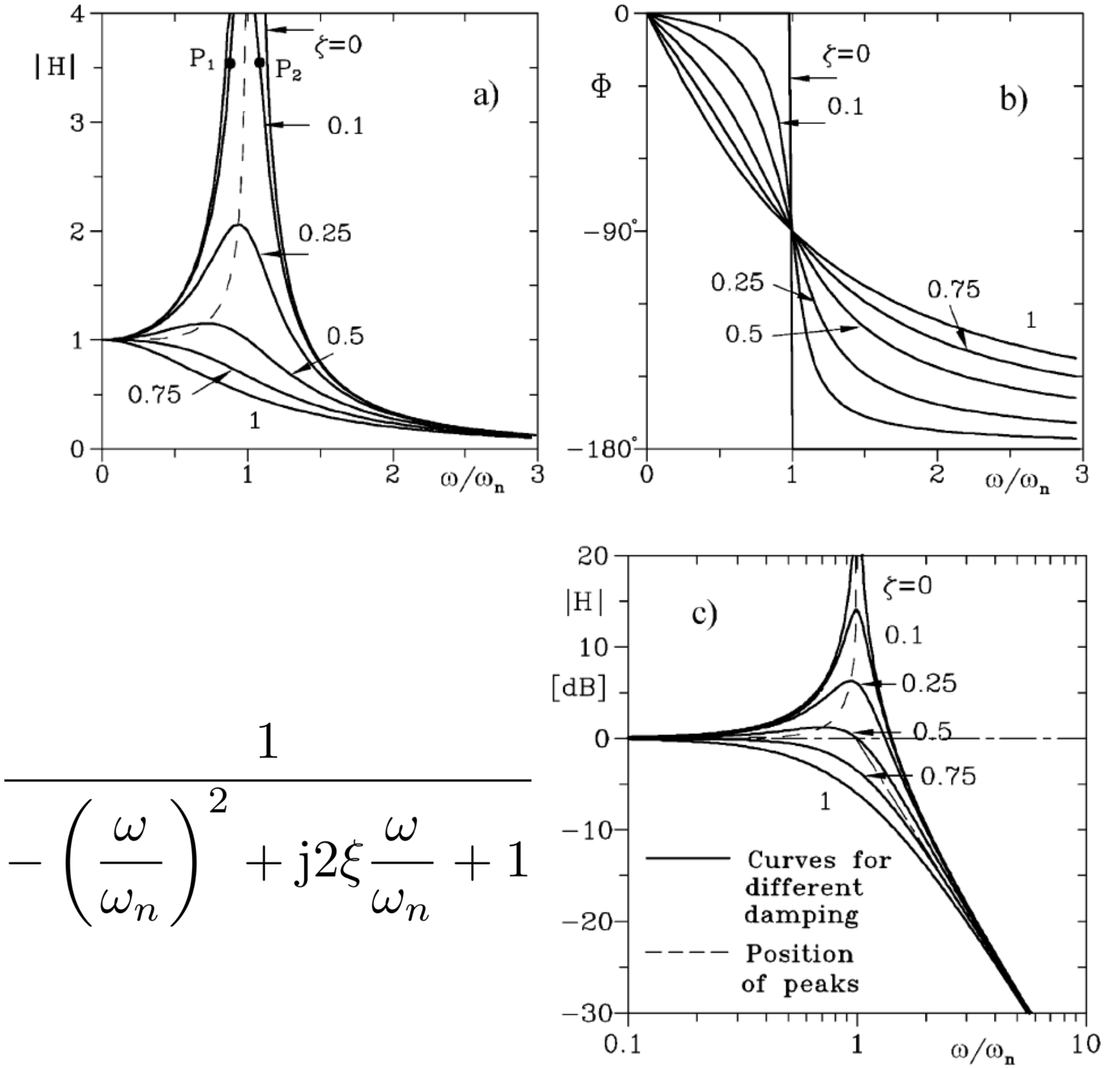
$$\hat{H}_{\text{dB}}(\omega) = 20\log_{10}|\hat{H}|$$

Slope -2 $\propto 1/\omega^2$
40 dB/dec.
12 dB/oct.



Single degree of freedom

The Frequency Response is the particular integral, it corresponds to the asymptotic forced response of the system if the system is stable

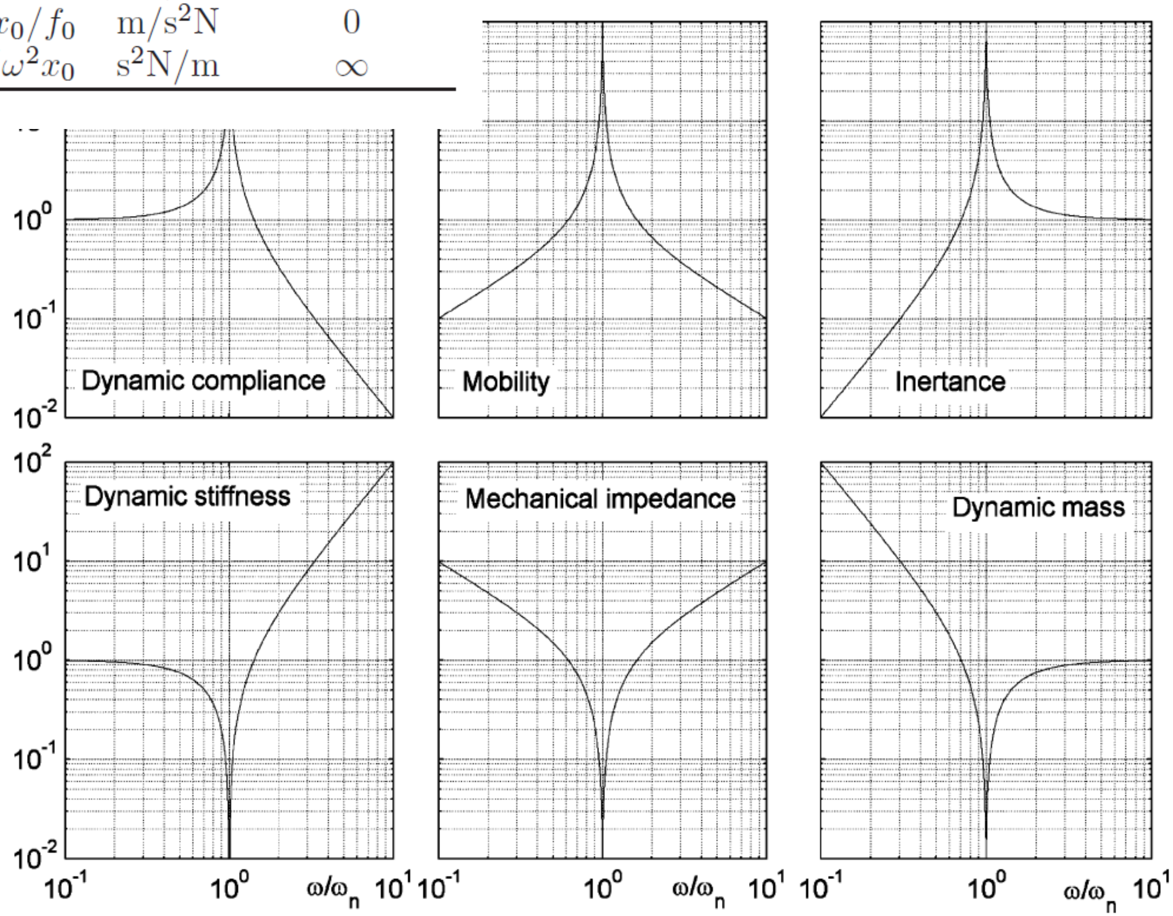


$$m\ddot{x} + c\dot{x} + kx = f$$

$$\hat{H}(\omega) = \frac{k}{-\omega^2 m + j\omega c + k} = \frac{1}{-\left(\dfrac{\omega}{\omega_n}\right)^2 + j2\xi\dfrac{\omega}{\omega_n} + 1}$$

Other noteworthy functions

Frequency response	Definition	S.I. units	Lim($\omega \to 0$)
Dynamic compliance	x_0/f_0	m/N	$1/k$
Dynamic stiffness	f_0/x_0	N/m	k
Mobility	$(\dot{x})_0/f_0 = \omega x_0/f_0$	m/sN	0
Mechanical impedance	$f_0/(\dot{x})_0 = f_0/\omega x_0$	Ns/m	∞
Inertance	$(\ddot{x})_0/f_0 = \omega^2 x_0/f_0$	m/s^{2}N	0
Dynamic mass	$f_0/(\ddot{x})_0 = f_0/\omega^2 x_0$	s^{2}N/m	∞



Modal computation of the response

$$\hat{\mathbf{q}} = \hat{\mathbf{H}}(\omega)\hat{\mathbf{F}} = \left(-\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} + \mathbf{K}\right)^{-1}\hat{\mathbf{F}} \qquad \hat{\mathbf{q}}, \hat{\mathbf{z}}, \hat{\mathbf{F}} \in \mathbb{C}^N$$

Using the modal expansion $\hat{\mathbf{q}} = \boldsymbol{\Phi}\hat{\mathbf{z}}$ and premultiplying everything by $\boldsymbol{\Phi}^T$

$$\begin{aligned}
\left(-\omega^2\boldsymbol{\Phi}^T\mathbf{M}\boldsymbol{\Phi} + \mathrm{j}\omega\boldsymbol{\Phi}^T\mathbf{C}\boldsymbol{\Phi} + \boldsymbol{\Phi}^T\mathbf{K}\boldsymbol{\Phi}\right)\hat{\mathbf{z}} &= \boldsymbol{\Phi}^T\hat{\mathbf{F}} \\
\mathrm{Diag}\left(\mu_i\left(-\omega^2 + \mathrm{j}\omega 2\xi_i\omega_i + \omega_i^2\right)\right)\hat{\mathbf{z}} &= \boldsymbol{\Phi}^T\hat{\mathbf{F}}
\end{aligned}$$

It results that

$$\begin{aligned}
\hat{\mathbf{z}} &= \mathrm{Diag}\left(\frac{1}{\mu_i\left(-\omega^2 + \mathrm{j}\omega 2\xi_i\omega_i + \omega_i^2\right)}\right)\boldsymbol{\Phi}^T\hat{\mathbf{F}} \\
\hat{\mathbf{q}} &= \boldsymbol{\Phi}\hat{\mathbf{z}} = \underbrace{\boldsymbol{\Phi}\mathrm{Diag}\left(\frac{1}{\mu_i\left(-\omega^2 + \mathrm{j}\omega 2\xi_i\omega_i + \omega_i^2\right)}\right)\boldsymbol{\Phi}^T}_{\hat{\mathbf{H}}(\omega)}\hat{\mathbf{F}}
\end{aligned}$$

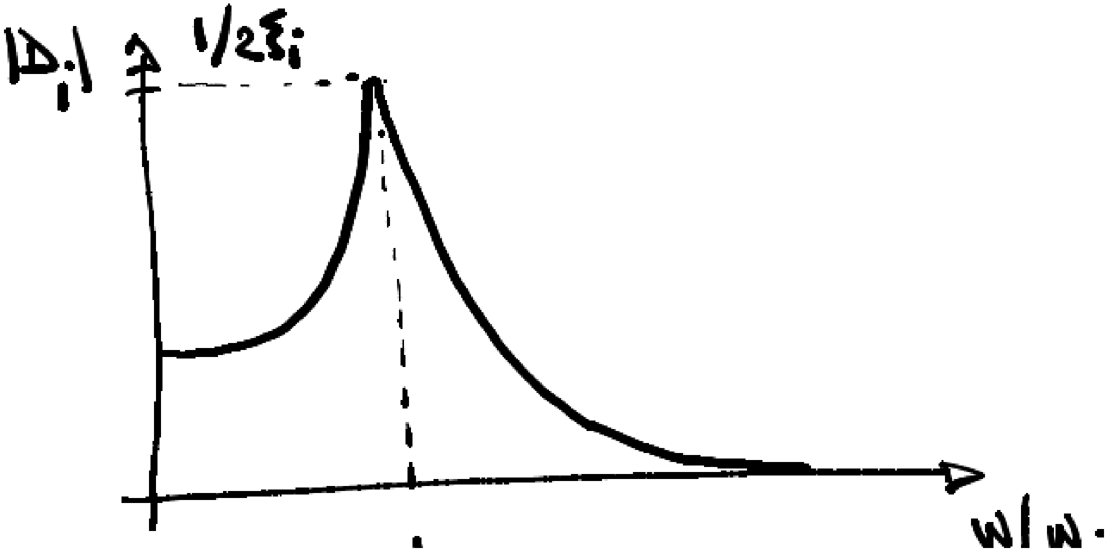
$$\hat{\mathbf{H}}(\omega) = \sum_{i=1}^{N}\frac{\boldsymbol{\varphi}_i\boldsymbol{\varphi}_i^T}{\mu_i\left(\omega_i^2 + \mathrm{j}\omega 2\xi_i\,\omega_i - \omega^2\right)}$$

Dynamic amplification factor D

$$\hat{\mathbf{H}}(\omega) = \sum_{i=1}^{N} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} D_i(\omega)$$

where $D_i(\omega)$ the dynamic amplifation factor is

$$D_i(\omega) = \frac{1}{1 + \mathrm{j}2\xi_i \dfrac{\omega}{\omega_i} - \left(\dfrac{\omega}{\omega_i}\right)^2}$$



Contribution of the different terms to the solution

Fixed ω evaluate the contribution of each mode to the solution.

Overall, each mode contribute is proportional to $1/\omega_j^2$

- for terms where $\omega << \omega_j$ the contribute is almost static
- for terms where $\omega \approx \omega_j$ the contribute is close to resonance peak
- for terms where $\omega >> \omega_j$ the contribute is not significant

Periodic signals: Fourier series

Consider a periodic signal

$$f(t,T) = f(t), \quad T = \text{PERIOD}$$

for wich the following integral over the period is finite

$$\int_t^{t+T} |f(\tau)| \mathrm{d}\tau < +\infty$$

This signal could be represented using the Fourier Series i.e.,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\Omega t + b_n \sin n\Omega t)$$

with $\Omega = 2\pi/T$ and

$$a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos n\Omega t \, \mathrm{d}t \in \mathbb{R}$$

$$b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin n\Omega t \, \mathrm{d}t \in \mathbb{R}$$

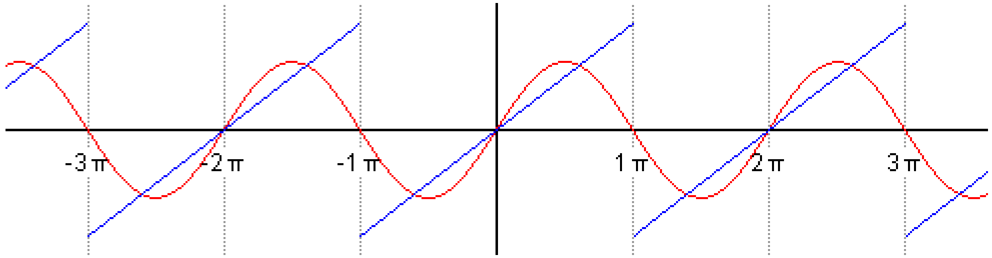
Alternatively

$$f(t) = \sum_{n=-\infty}^{+\infty} c_n \mathrm{e}^{\mathrm{j}n\Omega t}$$

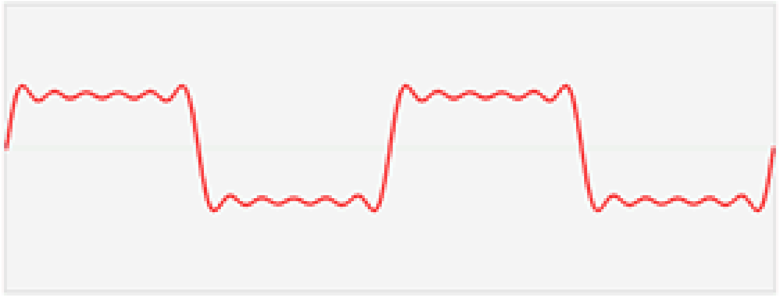
$$c_{-n} = c_n^*$$

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \mathrm{e}^{-\mathrm{j}n\Omega t} \mathrm{d}t \in \mathbb{C}$$

Periodic Signals Fourier series



Periodic Signal with superimposed the first 5 harmonics



Periodic signal: computation of the (forced) response

$$\mathbf{q}(t) = \sum_{n=-\infty}^{\infty} \mathbf{q}_n e^{j\Omega t}, \quad \mathbf{F}(t) = \sum_{n=-\infty}^{\infty} \mathbf{F}_n e^{j\Omega t}$$

$$\mathbf{q}_n = \hat{\mathbf{H}}(n\Omega)\mathbf{F}_n$$

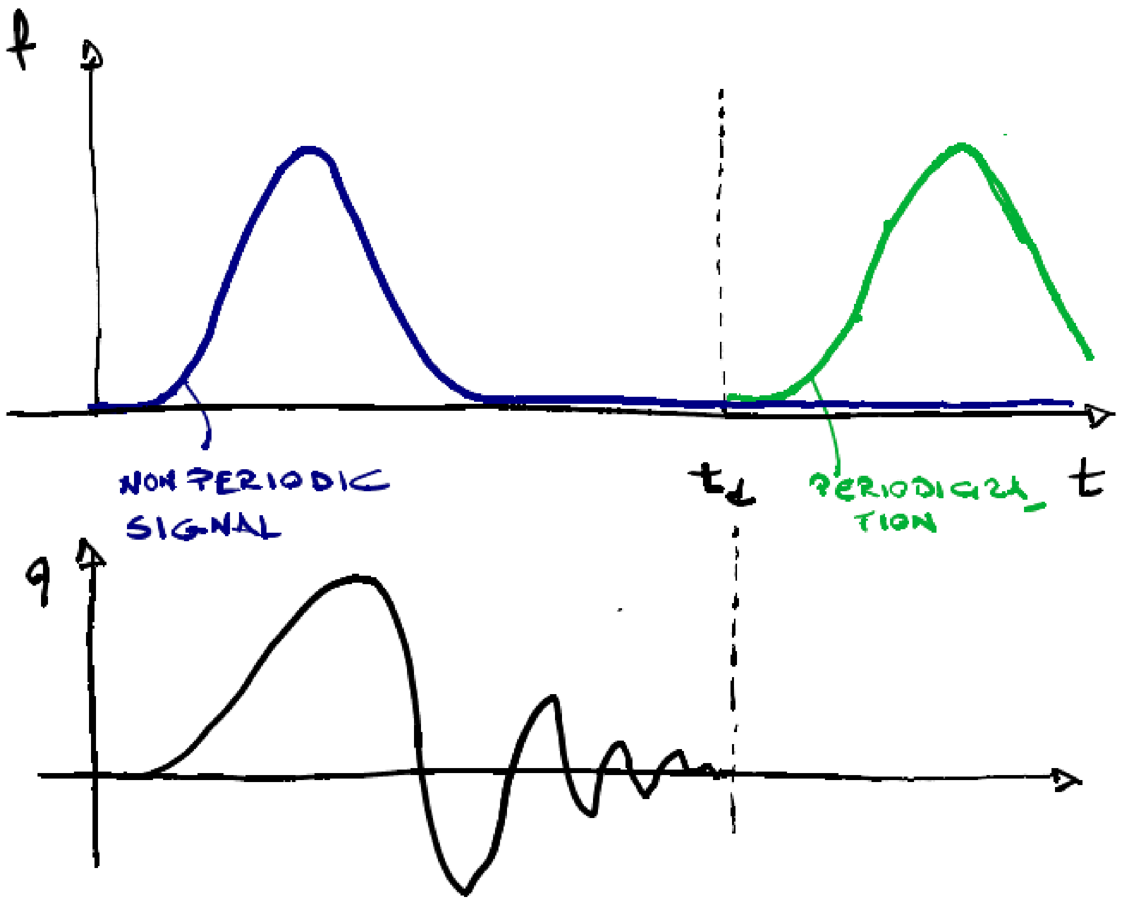
$$\mathbf{q}(t) = \sum_{n=-\infty}^{\infty} \hat{\mathbf{H}}(n\Omega)\mathbf{F}_n e^{j\Omega t}$$

If $n\Omega \to \infty$ then $\hat{\mathbf{H}}(n\Omega)$ should $\to 0$.

The approach is applicable truncating the series to a limited number of terms if H(nΩ) converge toward zero wile the frequency increases.

Typically, also F_n will be lower and lower the higher is n

Non-periodic but periodic-izable signals



t_d is the decay time, i.e. the time after which the structural dofs q will decay below a set threshold

After time t_d we can restart the excitation without introducing errors…

It is possible to apply the “Fourier series approach” to the green signal.

Fourier Transform

If $f(t)$ is not periodic but it is true that

$$\int_{-\infty}^{+\infty} |f(t)| \mathrm{d}t < \infty$$

Then it is possible to define the Fourier Transform

$$\mathcal{F}[f(t)] = \mathsf{f}(\omega) = \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{-\mathrm{j}\omega t} \mathrm{d}t$$

$$\mathcal{F}[f(t)] = \mathsf{f}(\omega) = \mathsf{f}(s)|_{s=\mathrm{j}\omega} = \mathcal{L}[f(t)]_{s=\mathrm{j}\omega}$$

It is easy to demonstrate that

$$\begin{aligned} \mathcal{F}[\dot{f}(t)] &= \mathrm{j}\omega\mathsf{f}(\omega) \\ \mathcal{F}[\ddot{f}(t)] &= (\mathrm{j}\omega)^2\mathsf{f}(\omega) - \omega^2\mathsf{f}(\omega) \end{aligned}$$

Solution using Inverse Fourier transform

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}$$

$$\Big\downarrow \mathcal{F}$$

$$\left(-\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} + \mathbf{K}\right)\mathbf{q}(\omega) = \mathbf{F}(\omega)$$

$$\mathbf{q}(\omega) = \mathbf{H}(\omega)\mathbf{F}(\omega)$$

$$\mathbf{H}(\omega) = \left(-\omega^2\mathbf{M} + \mathrm{j}\omega\mathbf{C} + \mathbf{K}\right)^{-1}$$

$$\mathcal{F}^{-1}[\mathsf{f}(\omega)] = f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\omega)\mathrm{e}^{\mathrm{j}\omega t}\mathrm{d}\omega$$

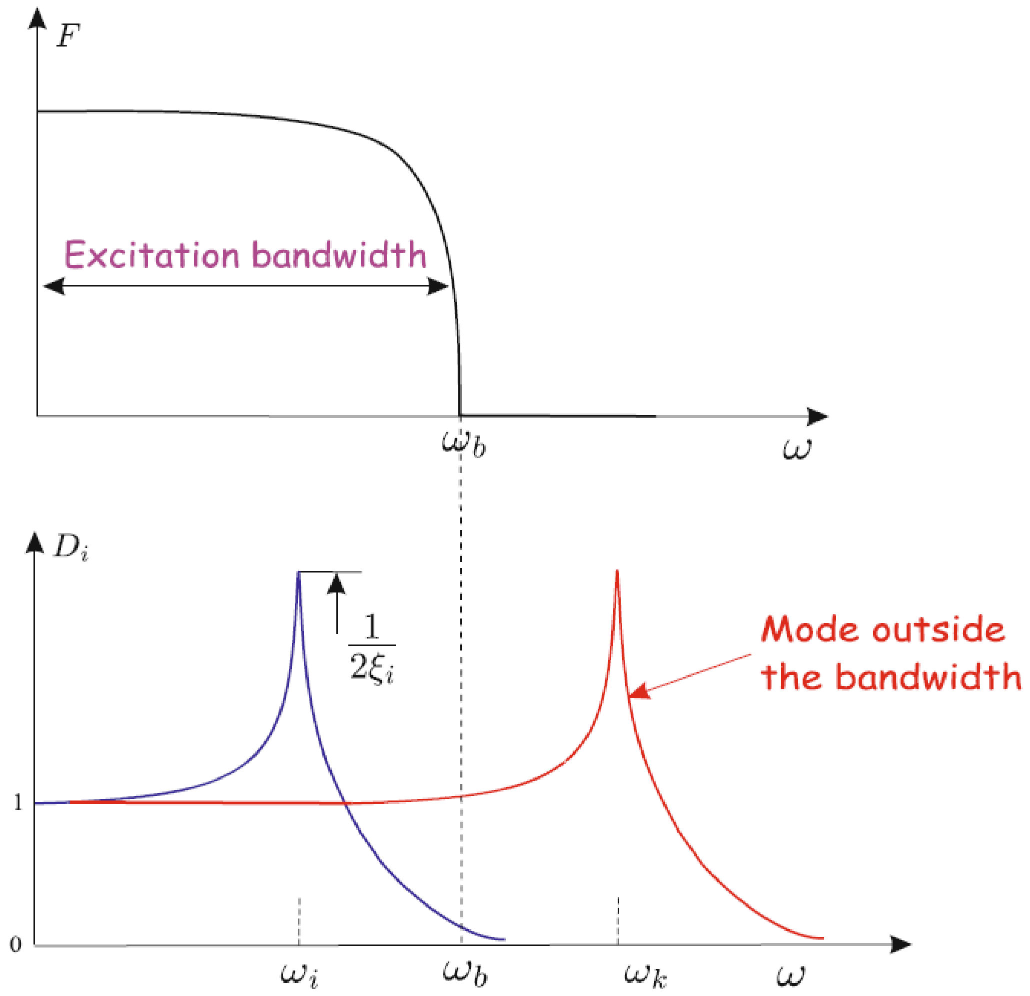
$$\mathbf{q}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathbf{H}(\omega)\mathbf{F}(\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

$$\mathbf{q}(t) = \frac{1}{\pi}\int_{0}^{+\infty}\mathbf{H}(\omega)\mathbf{F}(\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

$$\mathbf{q}(t) \approx \frac{1}{\pi}\int_{0}^{\omega_{\max}}\mathbf{H}(\omega)\mathbf{F}(\omega)\mathrm{e}^{j\omega t}\mathrm{d}\omega$$

The problem is typically solved numerically using quadrature methods limiting the integral limits to the frequencies where the transfer function and the input are above a certain threshold. Symmetry is exploited

Limited bandwidth excitation



Limited bandwidth excitation

Modes from m+1 are outside the bandwidth of excitation so respond statically

$$\mathbf{H}(\omega) \approx \sum_{i=1}^{m} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} D_i(\omega) + \sum_{i=m+1}^{N} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} 1$$

$$\mathbf{H}(0) = \sum_{i=1}^{N} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} 1 = \mathbf{K}^{-1} \quad \text{Static Gain}$$

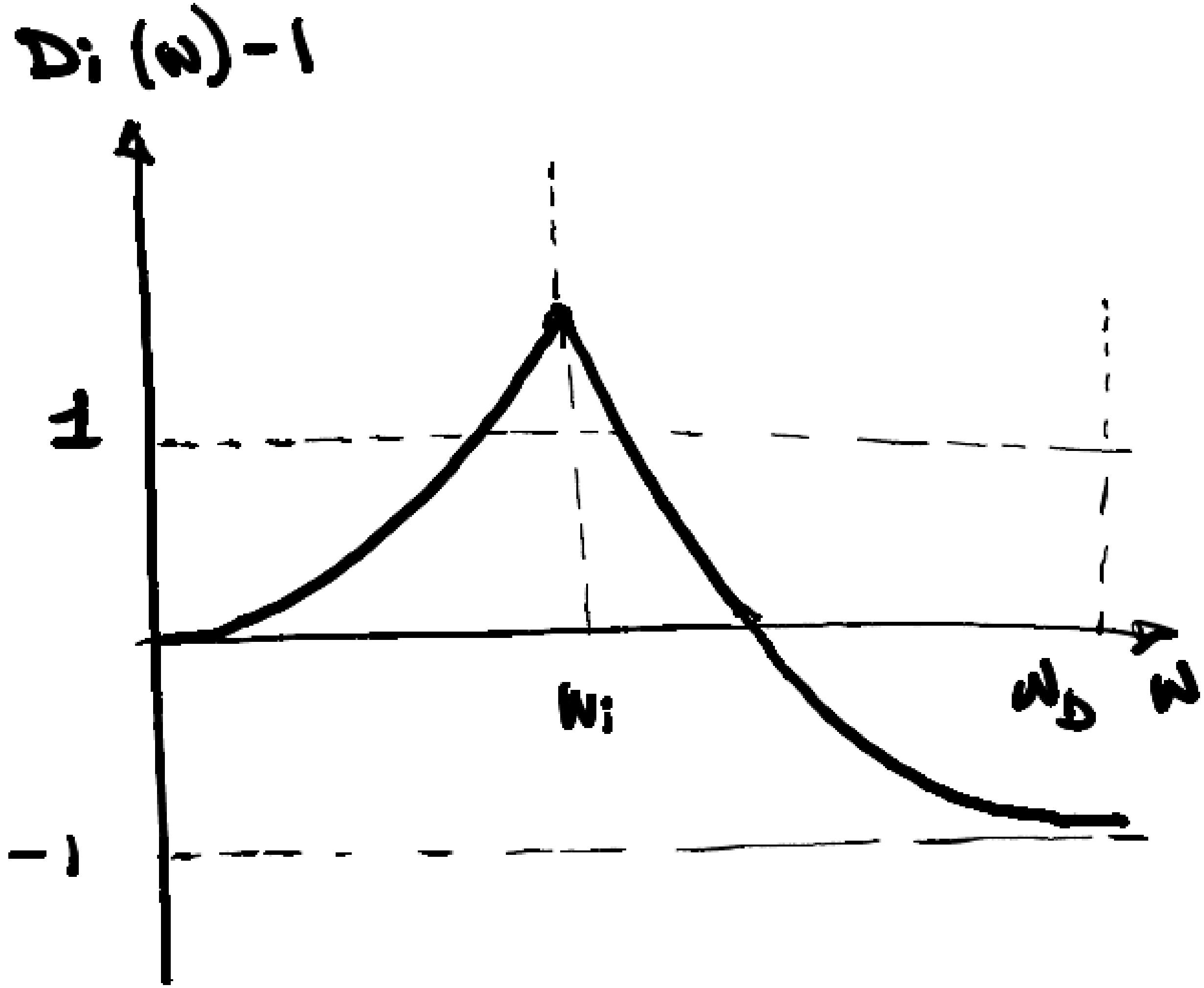
$$\mathbf{H}(\omega) \approx \sum_{i=1}^{m} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} D_i(\omega) + \mathbf{K}^{-1} - \sum_{i=1}^{m} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2}$$

$$\mathbf{q}(\omega) \approx \sum_{i=1}^{m} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} D_i(\omega) \mathbf{F}(\omega) + \underbrace{\left(\mathbf{K}^{-1} - \sum_{i=1}^{m} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2} \right) \mathbf{F}(\omega)}_{\mathbf{q}_{qs} = \mathbf{R}\mathbf{F}(\omega)}$$

QUASI-STATIC CORRECTION

The response of the system could be approximated accounting for the dynamic response of low order modes included in the bandwidth plus the static response of the others.

Limited bandwidth excitation



Inertia Relief

When there is one or more rigid body modes this procedure does not work because the matrix K is singular and not invertible

Consider to simplify expression that $C = 0$ (no damping) and that the modal forma are normalized at unit mass (so, $\mu_i = 1$ for all i), and divide the dofs in *rigid* and *elastic* dofs

$$\mathbf{q} = \mathbf{q}_R + \mathbf{q}_e = \mathbf{\Phi}_R \mathbf{z}_R + \mathbf{\Phi}_e \mathbf{z}_e$$

Inertia Relief

It is possible to compute the acceleration that are necessary to respect the equilibrium of rigid body equations (pre-multiplying the structural equation for the transposed of the rigid body modes)

$$\mathbf{M}\boldsymbol{\Phi}_R\ddot{\mathbf{z}}_R + \mathbf{M}\boldsymbol{\Phi}_e\ddot{\mathbf{z}}_e + \cancel{\mathbf{K}\boldsymbol{\Phi}_R\mathbf{z}_R} + \mathbf{K}\boldsymbol{\Phi}_e\mathbf{z}_e = \mathbf{F} \qquad (1)$$

No elastic energy is associated with rigid body movements $\mathbf{K}\boldsymbol{\Phi}_R = 0$

Premultiplying by $\boldsymbol{\Phi}_R^T$

$$\boldsymbol{\Phi}_R^T\mathbf{M}\boldsymbol{\Phi}_R\ddot{\mathbf{z}}_R = \boldsymbol{\Phi}_R^T\mathbf{F}$$

That means that if unit mass normalization is used that

$$\ddot{\mathbf{z}}_R = \boldsymbol{\Phi}_R^T\mathbf{F}$$

Inertia Relief

Substituting this in the original equation (1)

$$\begin{aligned}
\mathbf{M}\boldsymbol{\Phi}_e\ddot{\mathbf{z}}_e + \mathbf{K}\boldsymbol{\Phi}_e\mathbf{z}_e &= \mathbf{F} - \mathbf{M}\boldsymbol{\Phi}_R\ddot{\mathbf{z}}_R \\
\mathbf{M}\boldsymbol{\Phi}_e\ddot{\mathbf{z}}_e + \mathbf{K}\boldsymbol{\Phi}_e\mathbf{z}_e &= \mathbf{F} - \mathbf{M}\boldsymbol{\Phi}_R\boldsymbol{\Phi}_R^T\mathbf{F} \\
\mathbf{M}\boldsymbol{\Phi}_e\ddot{\mathbf{z}}_e + \mathbf{K}\boldsymbol{\Phi}_e\mathbf{z}_e &= \underbrace{\left(\mathbf{I} - \mathbf{M}\boldsymbol{\Phi}_R\boldsymbol{\Phi}_R^T\right)}_{\mathbf{P}^T}\mathbf{F}
\end{aligned}$$

<u>PROJECTION MATRIX</u>

$$\mathbf{P} = \left(\mathbf{I} - \boldsymbol{\Phi}_R\boldsymbol{\Phi}_R^T\mathbf{M}\right) \qquad \begin{cases} \mathbf{P}\boldsymbol{\Phi}_R = \mathbf{0} \\ \mathbf{P}\boldsymbol{\Phi}_e = \boldsymbol{\Phi}_e \end{cases}$$

The loads P^TF do not work for the rigid body modes

$$\boldsymbol{\Phi}_R^T\mathbf{P}^T\mathbf{F} = \mathbf{0} \Rightarrow \mathbf{P}^T\mathbf{F} \perp \boldsymbol{\Phi}_R$$

$$\boldsymbol{\Phi}_e^T\mathbf{P}^T\mathbf{F} = \boldsymbol{\Phi}_e^T\mathbf{F}$$

$$\boldsymbol{\Phi}_e^T\mathbf{M}\boldsymbol{\Phi}_e\ddot{\mathbf{z}}_e + \boldsymbol{\Phi}_e^T\mathbf{K}\boldsymbol{\Phi}_e\mathbf{z}_e = \boldsymbol{\Phi}_e^T\mathbf{P}^T\mathbf{F} = \boldsymbol{\Phi}_e^T\mathbf{F}$$

The work of the loads P^TF for the elastic modes is the same as the work of F for the same modes. WARNING This is true only if the elastic modes are those orthogonal to rigid body modes.

Inertia Relief

$$\mathbf{RF}(\omega) = \left(\mathbf{K}^{-1} - \sum_{i=1}^{m} \frac{\boldsymbol{\varphi}_i \boldsymbol{\varphi}_i^T}{\mu_i \omega_i^2}\right) \mathbf{F}(\omega) = \mathbf{q}^0 + \mathbf{q}^i$$

Since the system is in equilibrium as a rigid body it is possible to add dummy constraints to remove rigid body modes

$$\check{\mathbf{K}} = \mathbf{K} + \boldsymbol{\Phi}_R \mathbf{K}_g \boldsymbol{\Phi}_R^T, \quad \mathbf{K}_g \in \mathbb{R}^{6\times 6}\text{Symm., postive def.}$$

The matrix $\check{\mathbf{K}}$ is not singular so the inverse is always defined.

$$\mathbf{q}_e^0 = \check{\mathbf{K}}^{-1}\mathbf{F} = \check{\mathbf{K}}^{-1}\mathbf{P}^T\mathbf{F}$$

However any $\mathbf{q}_e = \boldsymbol{\Phi}_e \boldsymbol{\alpha}$ so $\mathbf{q}_e = \mathbf{P}\mathbf{q}_e$ so

$$\mathbf{q}_e^0 = \mathbf{P}\check{\mathbf{K}}^{-1}\mathbf{P}^T\mathbf{F}$$

$$\mathbf{P}\check{\mathbf{K}}^{-1}\mathbf{P}^T = \mathbf{P}\mathbf{K}^{-1}\mathbf{P}^T \; \forall \mathbf{K}_g \longrightarrow$$

It can be inferred that the resulting $\mathbf{q}_e^0$ is always the same no matter what is the choice of $\mathbf{K}_g$

Anti-resonance

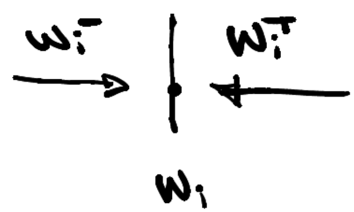
Consider an undamped system. $\xi_i = 0$

Consider the diagonal component of the transfer matrix: the harmonic response of the d.o.f. k and the harmonic excitation applied to the same d.o.f.

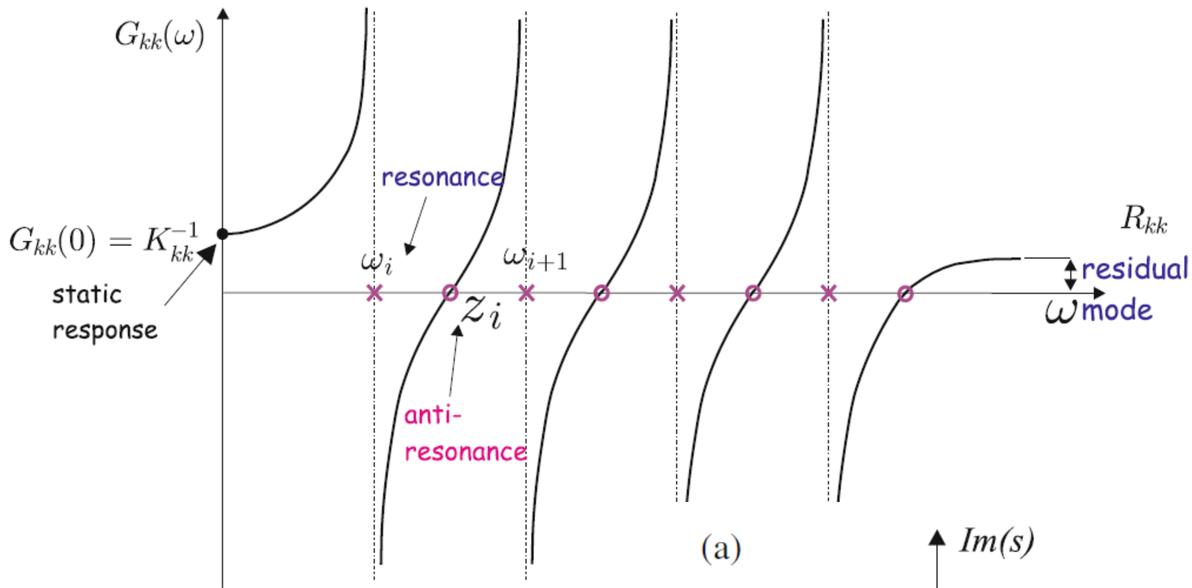
$$\mathsf{H}_{kk}(\omega) = \sum_{i=1}^{m} \frac{\varphi_{ik}^2}{\mu_i \omega_i^2} \frac{1}{1 - \left(\dfrac{\omega}{\omega_i}\right)^2} + R_{kk}$$

$\omega \rightarrow \omega_i^-$ $\quad H_{kk} \rightarrow +\infty$

$\omega \rightarrow \omega_i^+$ $\quad H_{kk} \rightarrow -\infty$

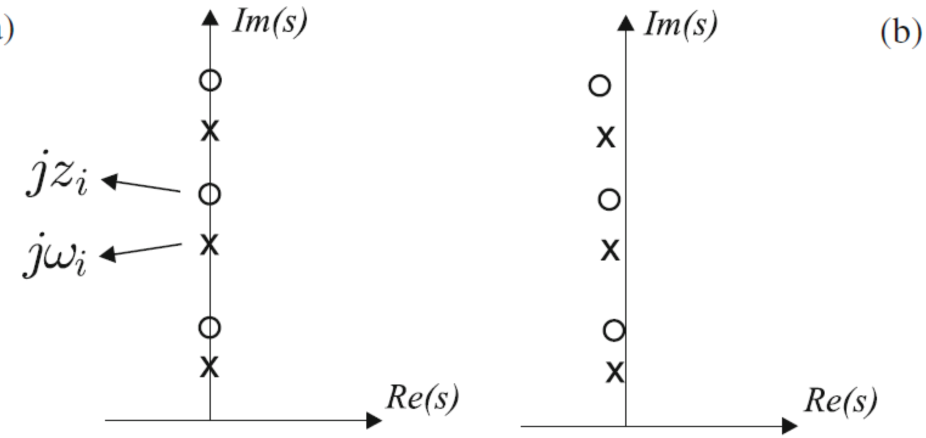


Anti-resonance

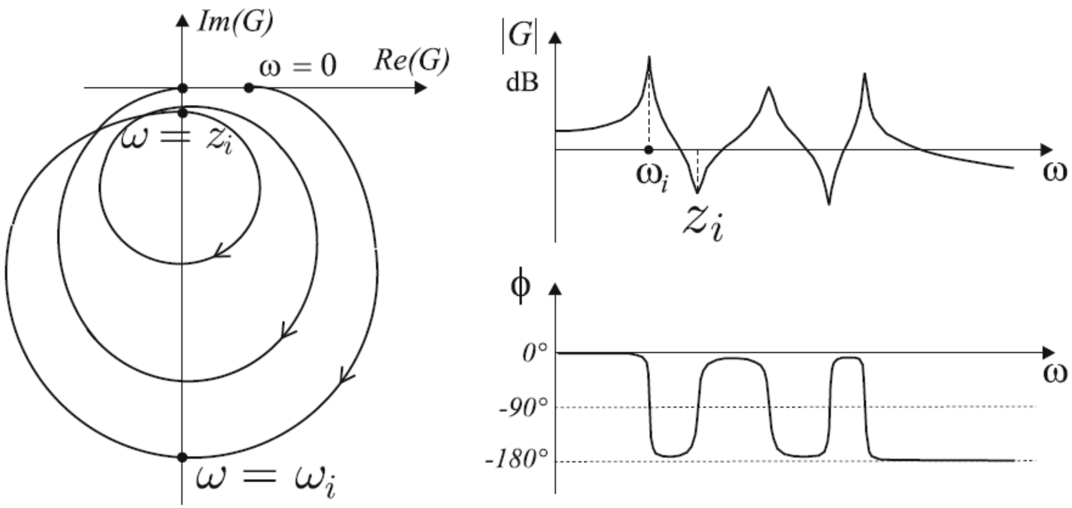


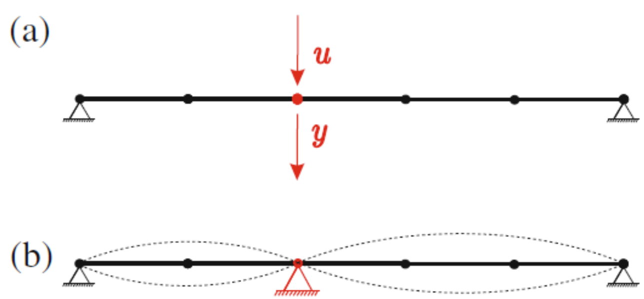
In collocated transfer functions there are anti-resonance frequencies between poles.

undamped collocated systems have alternating poles and zeros on the imaginary axis



Anti-resonance





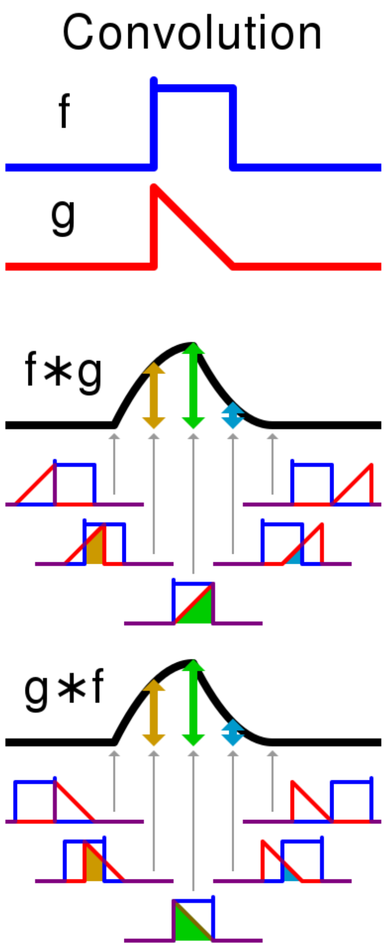
If the undamped structure is excited harmonically by the actuator at the frequency of the transmission zero, z_i , the amplitude of the response of the collocated sensor vanishes. This means that the structure oscillates at the frequency z_i according to the shape shown in dotted line

Convolution

$$p(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)\mathrm{d}\tau$$

$$|$$

$$= \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)\mathrm{d}\tau$$



If the two functions f and g are non-zero for a limited range of t, then the limits of the integral could be limited to this range

It is possible to demonstrate that

$$\mathcal{F}[f(t) * g(t)] = \mathrm{f}(\omega)\mathrm{g}(\omega)$$

or expressed in other way that

$$p(t) = \mathcal{F}^{-1}[\mathrm{f}(\omega)\mathrm{g}(\omega)]$$

Solution through the convolution

$$q(t) = \mathcal{F}^{-1}[\mathbf{H}(\omega)\mathbf{f}(\omega)]$$

$$\mathbf{q}(t) = \mathbf{h}(t) * \mathbf{f}(t) = \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{f}(t-\tau)\mathrm{d}\tau$$

What is $\mathbf{h}(t)$?

h is the Fourier anti-transform of the system transfer function H

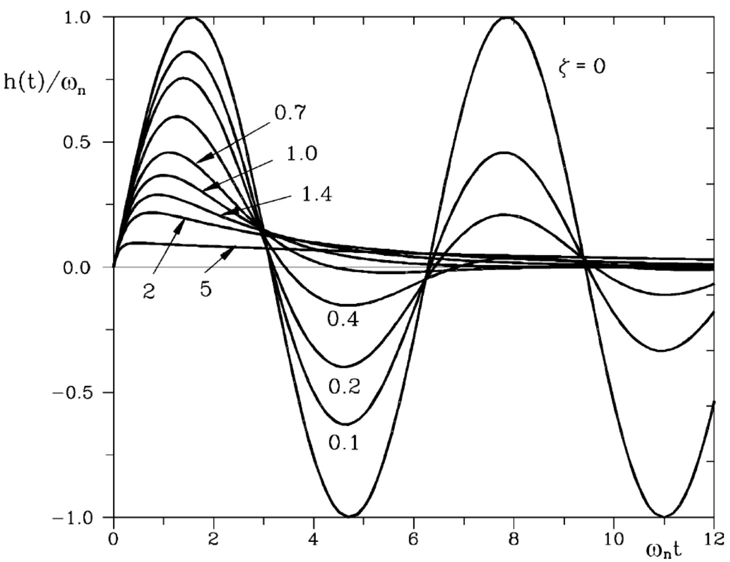
The solution obtained in this way is a basic application of the superimposition principle that is valid because the system is linear

Convolution and impulse

If $f(t)$ is a <u>DIRAC DELTA</u>, i.e. from a time point of view <u>IMPULSE</u>

$$\int_{-\infty}^{\infty} f(\tau)\delta(\tau)\mathrm{d}\tau = f(0)$$

$$\Rightarrow \; q(t) = \int_{-\infty}^{+\infty} h(\tau)\delta(t-\tau)\mathrm{d}\tau = h(t)$$



The function <u>h is the response of the system to an impulse at time t = 0.</u>
So<u>, the Transfer function is the Fourier transform of the impulse response of the system.</u>

What is the impulse response?

Let's see for a single dof system

$$\begin{cases} m\ddot{x} + c\dot{x} + kx = f_0\delta(t) \\ x_0 = \dot{x}_0 = 0 \end{cases}$$

$$\Big\downarrow \mathcal{L}$$

$$\rightarrow (s^2m + sc + k)x = f_0$$

$$(s^2m + sc + k)x - f_0 = 0$$

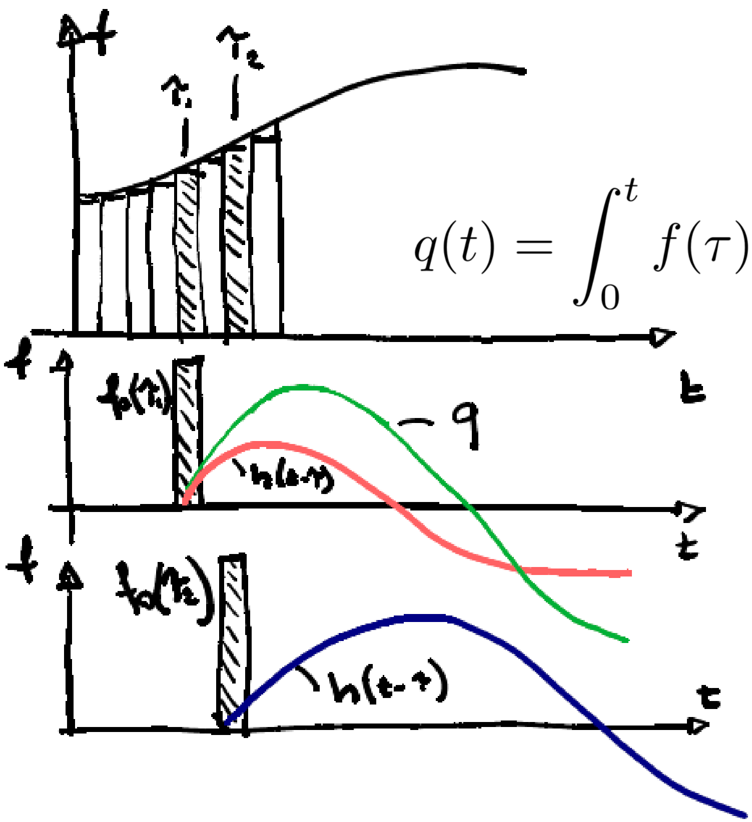
$$\Big\downarrow \mathcal{L}^{-1}$$

$$\begin{cases} m\ddot{x} + c\dot{x} + kx = 0 \\ x_0 = 0,\ \dot{x}_0 = \dfrac{f_0}{m} \end{cases}$$

The response to an impulse for at t = 0 is equal to the free response to a non-zero velocity initial condition.

It depends on the eigenvalues of the system…

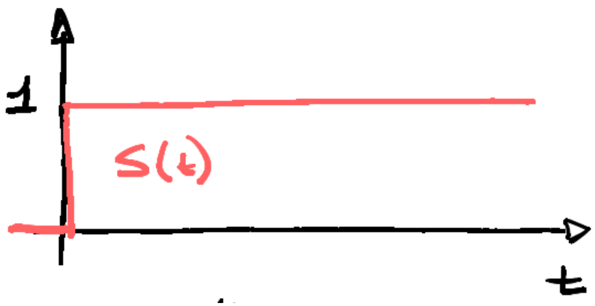
Structure time response using the impulse response



$$q(t) = \int_0^t f(\tau)h(t-\tau)\mathrm{d}\tau$$

The computation of the response using the convolution integral of the time history of the input force times the impulse response is an application of the superimposition principle. The final response is the superimposition of an infinite sequence of impulse responses with impulses of different intensity.

Response using the step response



<u>Indicial Function</u> is the response to a Step input

$$s(t) = \int_0^t \delta(t)\mathrm{d}\tau \quad = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

$$\dot{s}(t) = \delta(t) \qquad t > 0$$

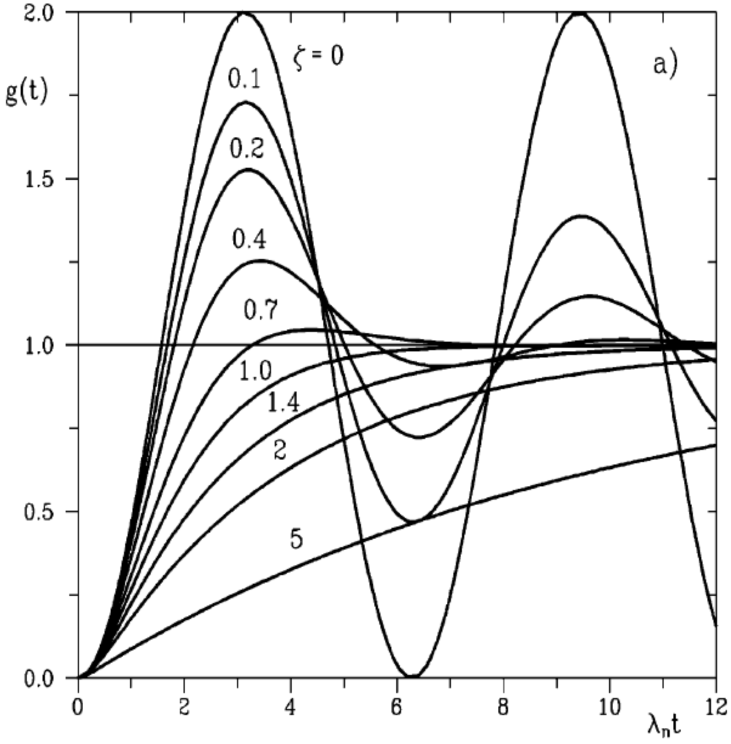
$$a(t) = \int_{-\infty}^{+\infty} h(\tau)s(t-\tau)\mathrm{d}\tau$$

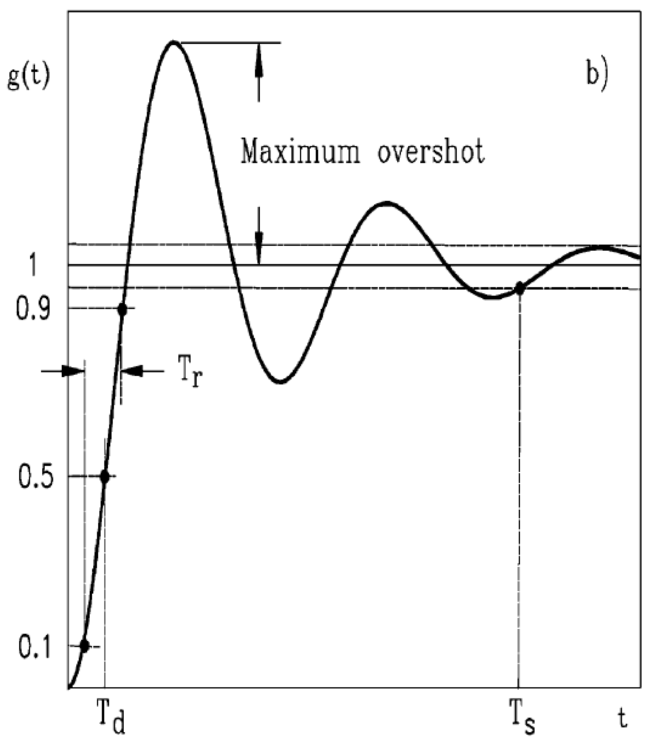
$$\dot{a}(t) = \int_{-\infty}^{+\infty} h(\tau)\delta(t-\tau)\mathrm{d}\tau = h(t)$$

The Step function *s* is the integral of the impulse function.

Using convolution, it is possible to see that the first derivative to the indicial function A is the impulse response

Response using the step response





Response using the indicial function

$$\mathbf{q}(t) = \int_{-\infty}^{+\infty} \mathbf{h}(t-\tau)\mathbf{f}(\tau)\mathrm{d}\tau = \int_0^t \dot{\mathbf{a}}(t-\tau)\mathbf{f}(\tau)\mathrm{d}\tau$$

using interation by parts

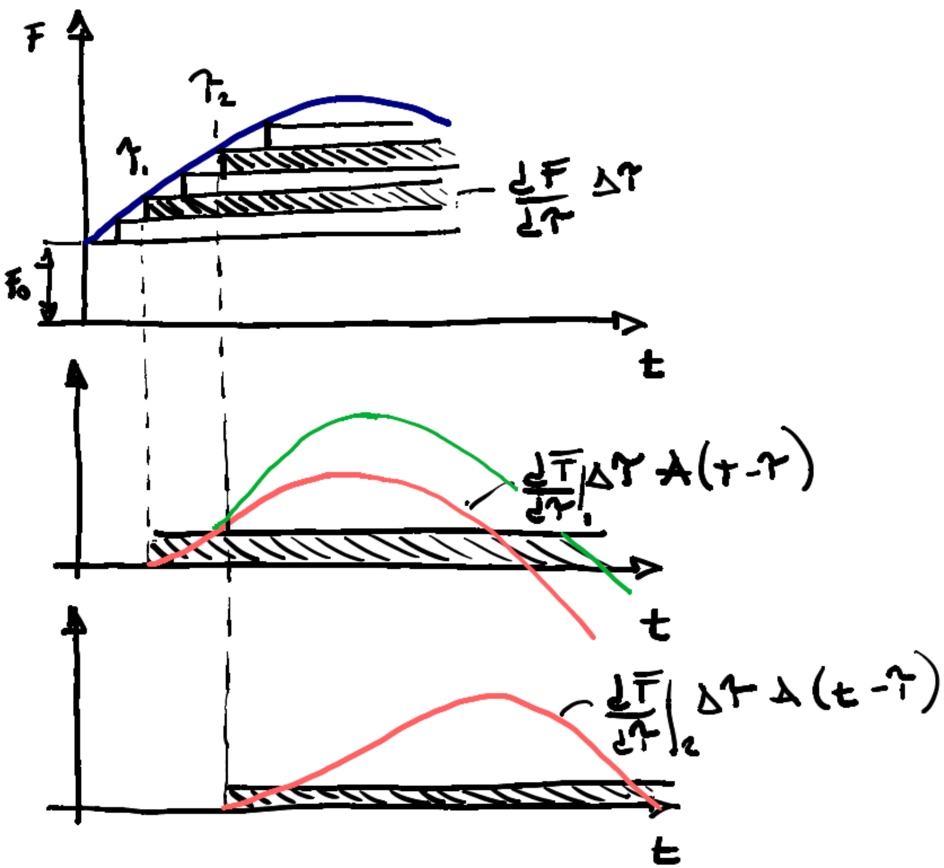
$$\mathbf{q}(t) = \left[-\mathbf{a}(t-\tau)\mathbf{f}(\tau)\right]_0^t + \int_0^t \mathbf{a}(t-\tau)\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\tau}(\tau)\mathrm{d}\tau$$

$$\mathbf{q}(t) = \mathbf{a}(t)\mathbf{f}(0) + \int_0^t \mathbf{a}(t-\tau)\frac{\mathrm{d}\mathbf{f}}{\mathrm{d}\tau}(\tau)\mathrm{d}\tau$$

DUHAMEL INTEGRAL

The Duhamel integral method uses the convolution between the indicial response and the derivative of the input force to compute the time response.
It is necessary to add the effect of the input force at the initial time

Response using the indicial function



The computation of the response using the Duhamel integral is an application of the superimposition principle. The final response is the superimposition of an infinite sequence of indicial responses with steps of different intensity.

Truncation of the modal basis

Consider to simplify expression that $C = 0$ (no damping) and that the modal form are normalized at unit mass (so, $\mu_i = 1$ for all i)

$$\mathbf{q} = \boldsymbol{\Phi} \begin{bmatrix} \ddots & & \\ & \dfrac{1}{\omega_i^2 - \omega^2} & \\ & & \ddots \end{bmatrix} \boldsymbol{\Phi}^T \mathbf{F}$$

If the mode matrix is decomposed in

$$\boldsymbol{\Phi} = [\boldsymbol{\Phi}_S \ \boldsymbol{\Phi}_F] \begin{cases} \boldsymbol{\Phi}_S \text{SLOW Modes Retained} \\ \boldsymbol{\Phi}_F \text{FAST Modes Truncated} \end{cases}$$

$$\mathbf{q} \approx \boldsymbol{\Phi}_S \begin{bmatrix} \ddots & & \\ & \dfrac{1}{\omega_i^2 - \omega^2} & \\ & & \ddots \end{bmatrix} \boldsymbol{\Phi}_S^T \mathbf{F}$$

The modes are ordered starting from the lowest frequency and increasing it.

For high modes (i large) the contribution to dofs is small because is proportional to the inverse of ω_i^2.

It is reasonable to retain only the modes that are within the bandwidth of the input plus few “guard” modes to ensure enough precision (between 5 and 10 “guard” modes)

Recovery of internal forces

The recovery of internal stresses (or internal loads) is important to employ the results of simulations for the sizing of structures

$$\begin{aligned} \boldsymbol{\sigma} &= \mathbf{D}\boldsymbol{\varepsilon} \\ \boldsymbol{\varepsilon} &= \mathcal{D}\mathbf{u} = \mathcal{D}\mathbf{N}\mathbf{q} = \mathbf{B}\mathbf{q} \\ \boldsymbol{\sigma} &= \mathbf{D}\mathbf{B}\mathbf{q} \end{aligned}$$

The internal loads are the integral of stresses. As such they are proportional to the elastic forces in the structure

$$\left.\begin{aligned} N &= \int \sigma \mathrm{d}a \\ M_b &= \int \sigma x \mathrm{d}a \\ M_t &= \int (\tau x - \ldots) \mathrm{d}a \end{aligned}\right\} \rightarrow \mathbf{K}\mathbf{q}$$

Direct recovery

In this case computed the displacements q we proceed directly to the computation of internal loads

$$\mathbf{Kq} = \underbrace{\mathbf{K\Phi}_S \mathrm{Diag}\left(\frac{1}{\omega_{iS}^2 - \omega^2}\right)\mathbf{\Phi}_S^T \mathbf{F}}_{(\mathbf{Kq})_S} + \underbrace{\mathbf{K\Phi}_F \mathrm{Diag}\left(\frac{1}{\omega_{iF}^2 - \omega^2}\right)\mathbf{\Phi}_F^T \mathbf{F}}_{(\mathbf{Kq})_F}$$

It is important to understand if $(\mathbf{Kq})_S$ is a good approximation of $\mathbf{Kq}$, when the bandwidth if $\mathbf{F}(\omega)$ is limited. Remembering that

$$\mathbf{K\Phi} = \mathbf{M\Phi}\mathrm{Diag}(\omega_i^2)$$

Since for large ω_i $\omega_i^2/(\omega_i^2 - \omega^2) \to 1$, then

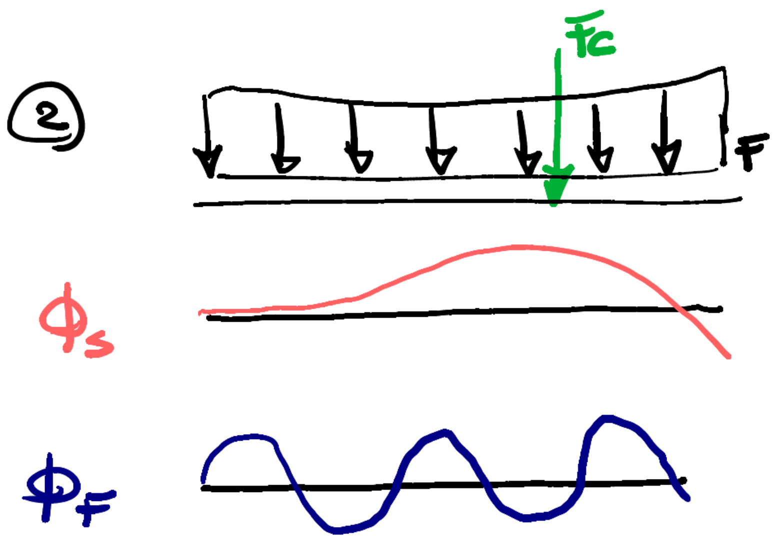
$$\mathbf{Kq} = \mathbf{M\Phi}_S \mathrm{Diag}\left(\frac{\omega_{iS}^2}{\omega_{iS}^2 - \omega^2}\right)\mathbf{\Phi}_S^T \mathbf{F} + \mathbf{M\Phi}_F \mathbf{\Phi}_F^T \mathbf{F}$$

So, using direct recovery it seems that the contribution coming from fast modes cannot be always neglected. This is since fast modes respond statically, as we saw before.

However…

If

1) the external excitation F is bandwidth limited, and
2) F is spatially distributed in a regular way



$$\Phi_F^T \mathbf{F} \approx 0$$

$$\Phi_F^T \mathbf{F}_c \neq 0$$

In this case the work equal to $\Phi^T F$ becomes negligible for FAST modes due to high spatial oscillation of the FAST modal forms.

WARNING. This is not true if local/concentrated loads are present.

Alternative method: Acceleration modes

The elastic load are equal to the sum of all other loads applied to the system (external, inertial and eventually damping if present)

$$\begin{aligned} \mathbf{Kq} &= \mathbf{F} - \mathbf{M}\ddot{\mathbf{q}} \\ \mathbf{Kq} &= \mathbf{F} + \mathbf{M}\boldsymbol{\Phi}\mathrm{Diag}\left(\frac{\omega^2}{\omega_i^2 - \omega^2}\right)\boldsymbol{\Phi}^T\mathbf{F} \end{aligned}$$

For large ω_i, $\omega^2/(\omega_i^2 - \omega^2) \to 1/\omega_i^2$ that $\to 0$. So,

$$\mathbf{Kq} = \mathbf{F} + \mathbf{M}\boldsymbol{\Phi}_S\mathrm{Diag}\left(\frac{\omega^2}{\omega_{iS}^2 - \omega^2}\right)\boldsymbol{\Phi}_S^T\mathbf{F} + \mathbf{M}\boldsymbol{\Phi}_F\mathrm{Diag}\left(\frac{1}{\omega_{iF}^2}\right)\boldsymbol{\Phi}_F^T\mathbf{F}$$

where the first part is a good approximation of the loads because the fast part tends to zero for large ω_{iF}.

This method allows to recover the static deflection (and so internal loads) of fast modes because it is equivalent to applying back to the full stiffness matrix the instantaneous loads as static loads

What is missing is only inertia (and damping) forces due to fast modes.

Alternative method: additional (static) modes

Consider the possibility to add a modal form

$$\mathsf{F} = \mathbf{F}_0 \mathsf{u}(\omega)$$

$$\boldsymbol{\varphi}_A = \mathbf{K}^{-1}\mathbf{F}_0$$

Of course, this is meaningful only if F has a fixed spatial distribution.

$$\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1 \, \boldsymbol{\varphi}_2 \, \ldots \boldsymbol{\varphi}_N \, \boldsymbol{\varphi}_a]$$

This new mode(s) is not orthogonal to proper orthogonal modes, so the M and K matrix will not be diagonal anymore.