POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

03 Static Aeroelasticity: Static aeroelastictiy of straight wing (MDoF models)

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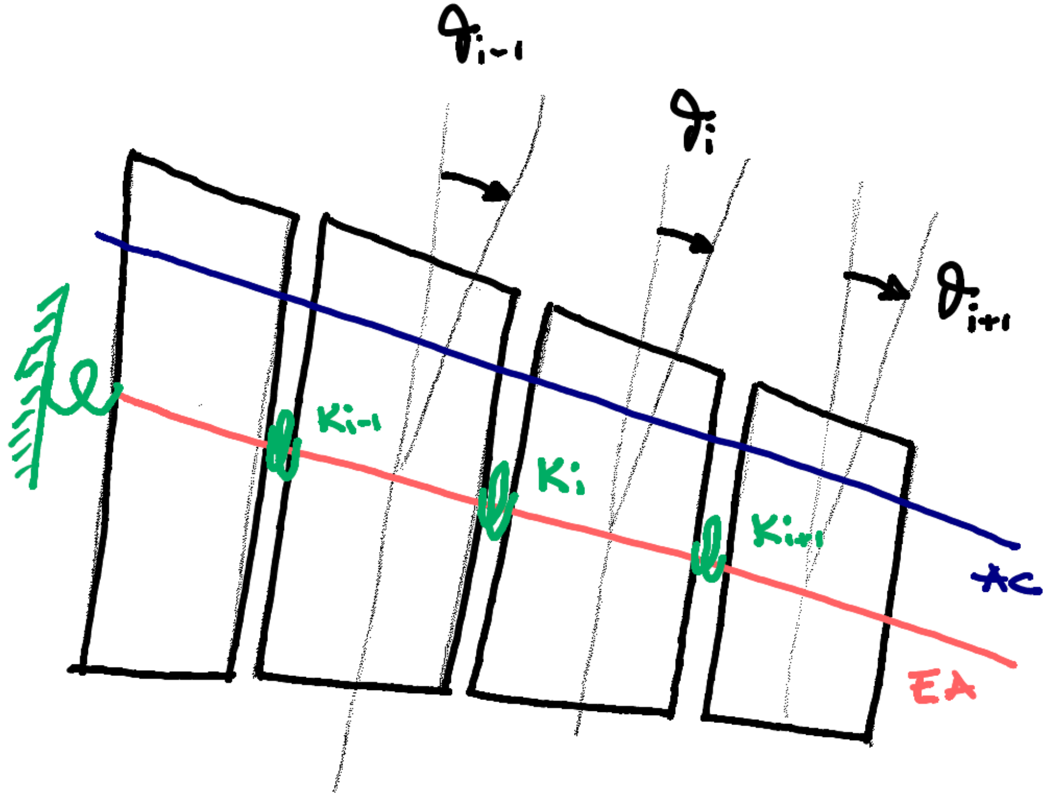
References

1) Dowell: Chapter 2 Static Aeroelasticity Sections 2.2
2) BAH Chapter 8 Sections 8.2
3) Masarati DCFA Chapter 8.1.3 8.1.4
4) Fung Chapter 3.2 – 3.4

Multi-Degrees of freedom model for torsional divergence



The model is composed by several rigid wing portions connected through torsional springs. For each rigid portion the simple pseudo-2D aerodynamic model is used. No mutual aerodynamic influence between portions is considered

Virtual Work Principle (VWP)

The model could be developed using the VIRTUAL WORK PRINCIPLE

$$\delta^* W = \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \delta \boldsymbol{r}_i + \sum_{j=1}^{m} \boldsymbol{M}_j \cdot \delta \boldsymbol{\varphi}_j$$

where $\delta \boldsymbol{r}_i, \delta \boldsymbol{\varphi}_j$ are virtual movements i.e., infinitesimal movements that do not violate constraints at fixed time.

$$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \dots, q_N)$$

$$\boldsymbol{\varphi}_j = \boldsymbol{\varphi}_j(q_1, q_2, \dots, q_N)$$

$$\delta \boldsymbol{r}_i = \sum_{k=1}^{N} \frac{\partial \boldsymbol{r}_i}{\partial q_k} \delta q_k$$

$$\delta \boldsymbol{\varphi}_j = \sum_{k=1}^{N} \frac{\partial \boldsymbol{\varphi}_j}{\partial q_k} \delta q_k$$

$$\delta^* W = \sum_{k=1} N \left(\sum_{i=1}^{n} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k} \right) \delta q_k + \sum_{k=1}^{N} \left(\sum_{j=1}^{m} \boldsymbol{M}_j \cdot \frac{\partial \boldsymbol{\varphi}_j}{\partial q_k} \right) \delta q_k$$

$$\delta^* W = \sum_{k=1}^{N} Q_k \delta q_k = 0 \qquad Q_k = \sum_{i=1}^{n} \boldsymbol{F}_i \cdot \frac{\partial \boldsymbol{r}_i}{\partial q_k} + \sum_{j=1}^{m} \boldsymbol{M}_j \cdot \frac{\partial \boldsymbol{\varphi}_j}{\partial q_k}$$

Given the arbitrariness of δq_i the result is $Q_k = 0 \ \forall k = 1, \dots, N$

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$$\boldsymbol{\varphi}_j = \boldsymbol{\varphi}_j(q_1, q_2, \ldots, q_N)$$

$$\begin{aligned} \delta \mathbf{r}_i &= \sum_{k=1}^{N} \frac{\partial \mathbf{r}_i}{\partial q_k} \delta q_k \\ \delta \boldsymbol{\varphi}_j &= \sum_{k=1}^{N} \frac{\partial \boldsymbol{\varphi}_j}{\partial q_k} \delta q_k \end{aligned}$$

If forces are conservative, it is possible to define a POTENTIAL U so that

$$\delta U = \sum_k \frac{\partial U}{\partial q_k} \delta q_k$$

$$\begin{aligned} \delta^* W &= 0 \\ \frac{\partial U}{\partial q_k} &= Q_k \end{aligned}$$

MDOF model

$$
\begin{aligned}
U &= \frac{1}{2}\sum_i k_{i+1}\left(\theta_{i+1}-\theta_i\right)^2 \\
M_i^{AER} &= qS_i\left(e_i C_{L0}(\alpha_i) + e_i C_{L\alpha_i}\theta_i + c_i C_{m_{AC_i}}\right) \\
\frac{\partial U}{\partial \theta_i} &= k_i\left(\theta_i-\theta_{i-1}\right) - k_{i+1}\left(\theta_{i+1}-\theta_i\right) \\
\delta^* W &= \sum_i M_i^{AER}\delta\theta_i \;\rightarrow\; Q_i = M_i^{AER}
\end{aligned}
$$

$$
\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots \\ -k_2 & k_2+k_3 & -k_3 & 0 \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix}\boldsymbol{\theta} - q\begin{bmatrix} \ddots & & 0 \\ & S_i e_i C_{L\alpha_i} & \\ 0 & & \ddots \end{bmatrix}\boldsymbol{\theta} = q\begin{Bmatrix} \vdots \\ S_i(e_i C_{L0_i} + c_i C_{m_{AC_i}}) \\ \vdots \end{Bmatrix}
$$

The resulting model has the same structure of the typical section case but is matricial. Divergence condition(s) could be identified solving the eigenvalue problem

$$
\left(\mathbf{K}_s - q\mathbf{K}_A\right)\boldsymbol{\theta} = \mathbf{M}_0 \quad \longrightarrow \quad \begin{aligned} &\left(\mathbf{K}_s - q\mathbf{K}_A\right)\boldsymbol{\theta} = \mathbf{0} \\ &\det\left(\mathbf{K}_s - q\mathbf{K}_A\right) = 0 \end{aligned}
$$

Power Method (for eigenvalues)

Compute the eigenvalue with maximum modulus of the matrix

- ✓ $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{z} \in \mathbb{R}^n$
- ✓ **A** possess n eigenvectors linear independent
- ✓ $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots$

$$\begin{aligned} \mathbf{z}_{k+1} &= \mathbf{A}\mathbf{z}_k \quad \text{(series)} \\ \mathbf{z}_k &= \mathbf{A}^k \mathbf{z}_0 \end{aligned}$$

Express $\mathbf{z}_0$ the initial vector as a basis for $\mathbb{R}^n$ the eigenvectors of $\mathbf{A}$ indicated by $\mathbf{x}_j$

$$\mathbf{z}_0 = \sum_{j=1}^{n} \alpha_j \mathbf{x}_j$$

Power Method (for eigenvalues)

$$\mathbf{z}_k = \sum_{j=1}^{n} \alpha_j \mathbf{A}^k \mathbf{x}_j \quad \text{but} \quad \mathbf{A}\mathbf{x}_j = \lambda_j \mathbf{x}_j$$

$$\mathbf{z}_k = \sum_{j=1}^{n} \alpha_j \lambda_j^k \mathbf{x}_j = \lambda_1^k \left(\alpha_1 \mathbf{x}_1 + \sum_{j=2}^{n} \left(\frac{\lambda_j}{\lambda_1} \right)^k \mathbf{x}_j \right)$$

since $\lambda_j / \lambda_1 < 1 \quad \forall j > 1$

$$\lim_{k \to \infty} \mathbf{z}_k = \lambda_1^k \alpha_1 \mathbf{x}_1 \qquad \mathbf{z}_k \text{ is parallel to } \mathbf{x}_1$$

The normalized vector $\mathbf{z}_k$ is equal to the first eigenvector $\mathbf{x}_1$.

Power Method (for eigenvalues)

for any eigenvector $\mathbf{x}_i$

$$
\begin{aligned}
\mathbf{x}_i^T \mathbf{A} \mathbf{x}_i &= \lambda_i \mathbf{x}_i^T \mathbf{x}_i \\
\lambda_i &= \frac{\mathbf{x}_i^T \mathbf{A} \mathbf{x}_i}{|\mathbf{x}_i|^2}
\end{aligned}
$$

So it is possible to compute the following index

$$
\sigma_k = \frac{\mathbf{z}_k^T \mathbf{A} \mathbf{z}_k}{|\mathbf{z}_k|^2}
$$

and verify that

$$
\lim_{k \to \infty} \sigma_k = \lambda_1
$$

Write in MATLAB a simple algorithm and test it

Application to the MDOF case

$$(\mathbf{K}_s - q_D \mathbf{K}_A)\,\boldsymbol{\theta} = 0$$

If the interest is to <u>identify the lower divergence dynamic pressure q_D</u>

$$\begin{aligned} q_D \mathbf{K}_A \boldsymbol{\theta} &= \mathbf{K}_s \boldsymbol{\theta} \\ \mathbf{K}_A \boldsymbol{\theta} &= \frac{1}{q_D} \mathbf{K}_s \boldsymbol{\theta} \end{aligned}$$

So, if we call $\lambda = 1/q_D$

$$\begin{aligned} \mathbf{K}_A \boldsymbol{\theta} &= \lambda \mathbf{K}_s \boldsymbol{\theta} \\ \mathbf{K}_s^{-1} \mathbf{K}_A \boldsymbol{\theta} &= \lambda \boldsymbol{\theta} \end{aligned}$$

So the matrix $\mathbf{A}$ in this case is $\mathbf{A} = \mathbf{K}_s^{-1} \mathbf{K}_A$.
Is it possible to avoid the inversion?

Application to the MDOF case

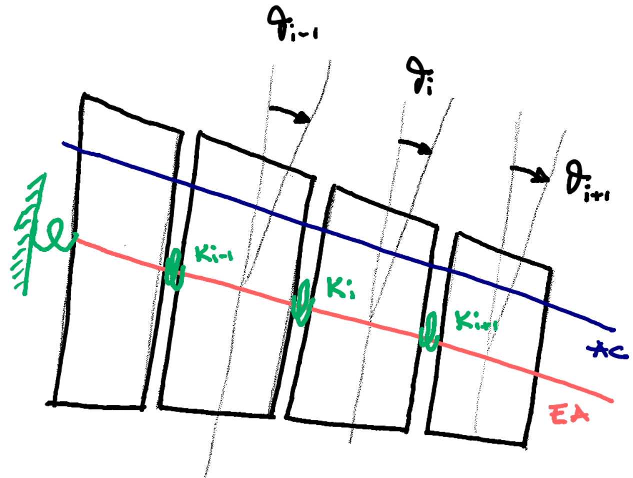
Answer: Define an intermediate vector $\mathbf{c}$ so that

$$\begin{aligned} \mathbf{c} &= \mathbf{K}_A \boldsymbol{\theta}_i \\ \mathbf{K}_s \boldsymbol{\theta}_{i+1} &= \mathbf{c} \end{aligned}$$

The solution of the second system could be obtained without the inversion of the matrix $\mathbf{K}_s$ saving computational effort.

Compute the stiffness of the springs using VWP

Integral Formulation (Flexibility influence functions)



$\hat{\theta}_i$ is the relative angle between two sections

$$\hat{\theta}_n = \frac{1}{k_n} m_n$$

$$\hat{\theta}_{n-1} = \frac{1}{k_{n-1}} \sum_{j=n-1}^{n} m_j$$

$$\vdots$$

$$\hat{\theta}_1 = \theta_1 = \frac{1}{k_1} \sum_{j=1}^{n} m_j = C_1 \sum_{j=1}^{n} m_j$$

If the number of sections increases $N \to \infty$ then the sum is transformed into an integral

$$\theta(y) = \int_0^L C^{\theta\theta}(y,\xi) m(\xi) \mathrm{d}\xi$$

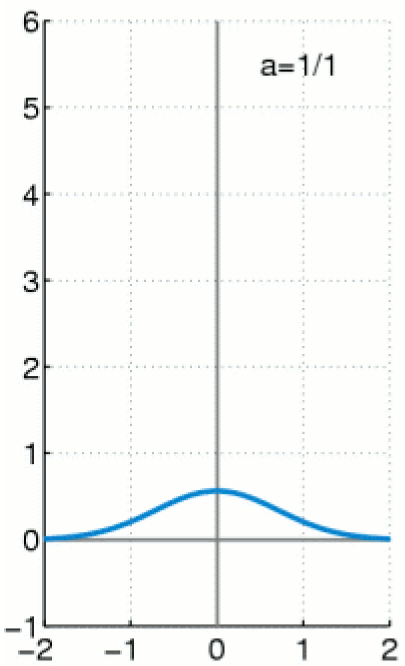
If a unit point moment is applied at $y = \gamma$ i.e., $m(\xi) = \delta(\xi - \gamma)$ with δ a Dirac delta

$$\theta(y) = \int_0^L C^{\theta\theta}(y,\xi)\delta(\xi-\gamma)\mathrm{d}\xi$$

$$\theta(y) = C^{\theta\theta}(y,\gamma)$$

$C^{\theta\theta}$ is the flexibility influence function that represents the twist at y due to a unit moment at γ.

Appendix: Dirac delta



Formal definition

$$\int_{-\infty}^{+\infty} \delta(x)\mathrm{d}x = 1 \qquad \delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

Consequently

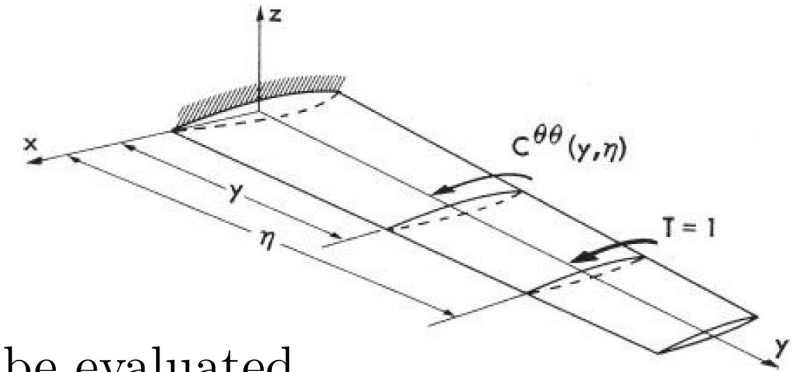
$$\int_{-\infty}^{+\infty} f(x)\delta(x)\mathrm{d}x = f(0)$$

It is not a function but a <u>generalized function</u> or a <u>distribution</u> and it is introduced to model a *narrow spike* or a pointwise function (or an *impulse* in time). The Dirac delta if applied to function has the property of sampling it.

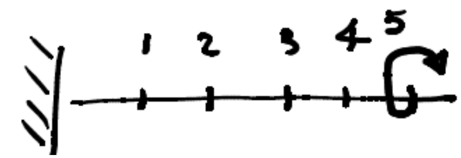
One possible representation of the Dirac delta could be

$$\delta(x) = \lim_{a \to 0} \frac{1}{|a|\sqrt{\pi}} \mathrm{e}^{-\left(\frac{x}{a}\right)^2}$$

Integral Formulation (Flexibility influence functions)



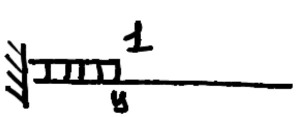
1. $C^{\theta\theta}(y_i, y_j)$ could be evaluated experimentally: apply a moment at y_i and measure twist at all other sections.

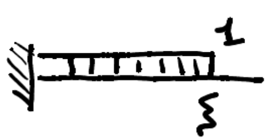


$$C^{\theta\theta}(y, \xi) = C^{\theta\theta}(\xi, y) \quad \text{Th.Maxwell} - \text{Betti}$$

2. Use VWP, compute $\theta(y)$ due to a moment at ξ

$$C^{\theta\theta}(y, \xi)\, \delta M_{tF} = \int_0^L \frac{M_{tR}}{GJ} \delta M_{tF} \mathrm{d}\eta = \int_0^{\min(y,\xi)} \frac{1}{GJ} \mathrm{d}\eta\, \delta M_{tF}$$





$$C^{\theta\theta}(y, \xi) = \begin{cases} \frac{\xi}{GJ} & y > \xi \\ \frac{y}{GJ} & y \leq \xi \end{cases}$$

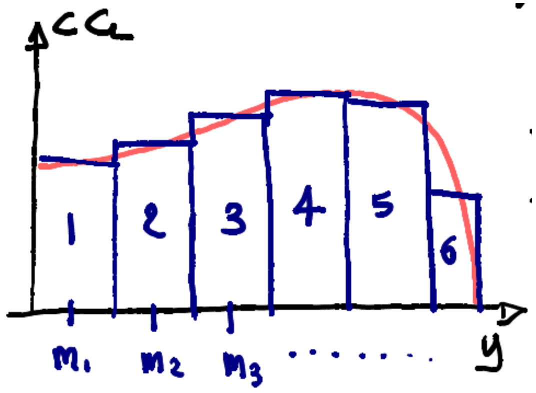
Integral Formulation (Flexibility influence functions)

$$
\begin{aligned}
m_t &= qc\left(eC_{L0} + eC_{L\alpha}\theta + cC_{m_{AC}}\right) \\
m_t &= q\left(ecC_{L0} + e(cC_L)_e + c^2 C_{m_{AC}}\right)
\end{aligned}
$$

The sectional load per unit dynamic pressure generated by the elastic deformation can be highlighted $(cC_L)_e$ and $\theta = \frac{(cC_L)_e}{cC_{L\alpha}}$

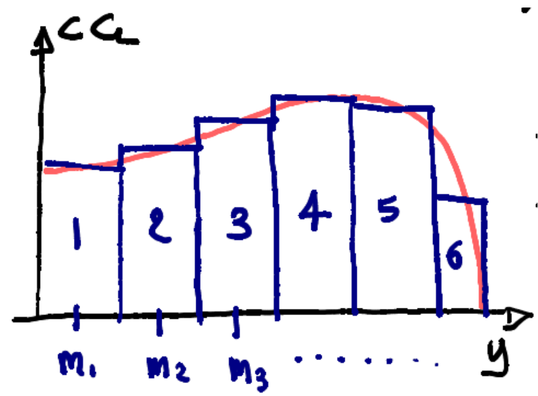
$$
\frac{(cC_L)_e}{cC_{L\alpha}} = q\int_0^L C^{\theta\theta}(y,\xi)\left(ceC_{L0} + e(cC_L)_e + c^2 C_{m_{AC}}\right)\mathrm{d}\xi
$$

(c CL)e is taken as the unknown and discretized as constant values on wing strips, making the hypothesis that the deformation does not depend on how loads are distributed on a strip.



Integral Formulation (Flexibility influence functions)

Discretization



$$\frac{(cC_L)_e}{cC_{L\alpha}} = q\int_0^L C^{\theta\theta}(y,\xi)\left(ceC_{L0} + e(cC_L)_e + c^2 C_{m_{AC}}\right)\mathrm{d}\xi$$

$$\frac{(cC_L)_{e_j}}{c_j C_{L\alpha_j}} = q\sum_i \int_{y_i}^{y_{i+1}} C^{\theta\theta}(y_j,\xi)\mathrm{d}\xi\left(c_i e_i C_{L0_i} + e_i(cC_L)_{e_i} + c^2 C_{m_{AC_i}}\right)$$

F_{ij} elements of the discretized flexibility matrix **F**

$$\mathbf{F} = \begin{bmatrix} C_{11} & C_{12} & \cdots \\ C_{21} & C_{22} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \Delta b_1 & & 0 \\ & \Delta b_2 & \\ 0 & & \ddots \end{bmatrix}$$

Integral Formulation (Flexibility influence functions)

$$\begin{bmatrix} \ddots & & \\ & \frac{1}{c_i C_{L\alpha_i}} & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ (cC_L)_{e_i} \\ \vdots \end{Bmatrix} = q\mathbf{F} \left(\begin{Bmatrix} \vdots \\ m_{0_i} \\ \vdots \end{Bmatrix} + \begin{bmatrix} \ddots & & \\ & e_i & \\ & & \ddots \end{bmatrix} \begin{Bmatrix} \vdots \\ (cC_L)_{e_i} \\ \vdots \end{Bmatrix} \right)$$

with $\mathbf{F}$ the <u>Flexibility</u> matrix i.e., $\mathbf{F} = \mathbf{K}^{-1}$

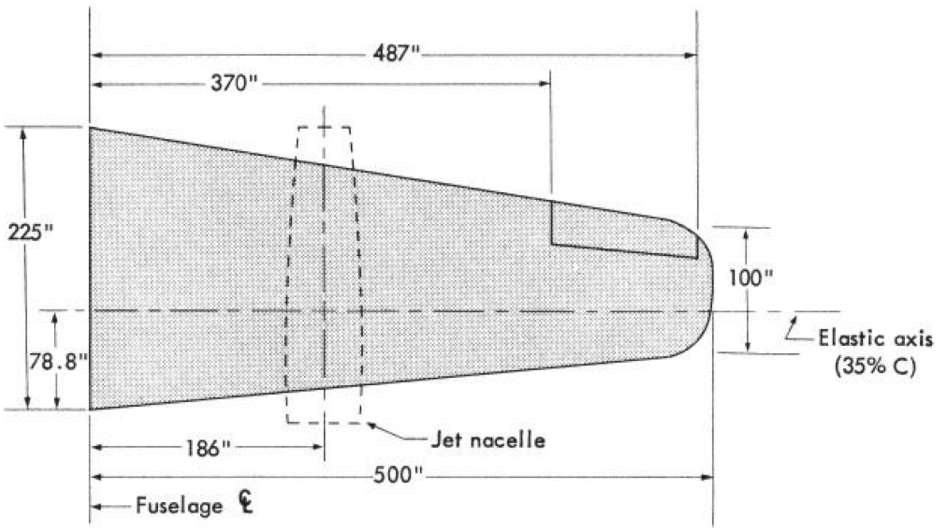
$$(\mathbf{A} - q\mathbf{B})\,(\mathbf{cC_L})_\mathbf{e} = q\mathbf{Q} \qquad \begin{aligned} \mathbf{Q} &= \mathbf{F}\mathbf{m}_0 \\ \mathbf{A} &= \mathrm{Diag}(1/c_i C_{L\alpha_i}) \\ \mathbf{B} &= \mathbf{F}\mathrm{Diag}(e_i) \end{aligned}$$

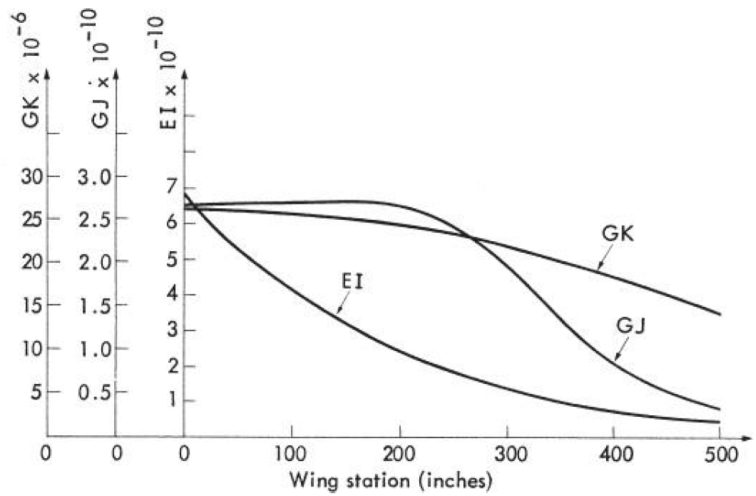
The divergence speed could be identified solving the associated homogeneous problem using the eigenvalue approach

$$(\mathbf{A} - q\mathbf{B})\,(\mathbf{cC_L})_\mathbf{e} = \mathbf{0}$$

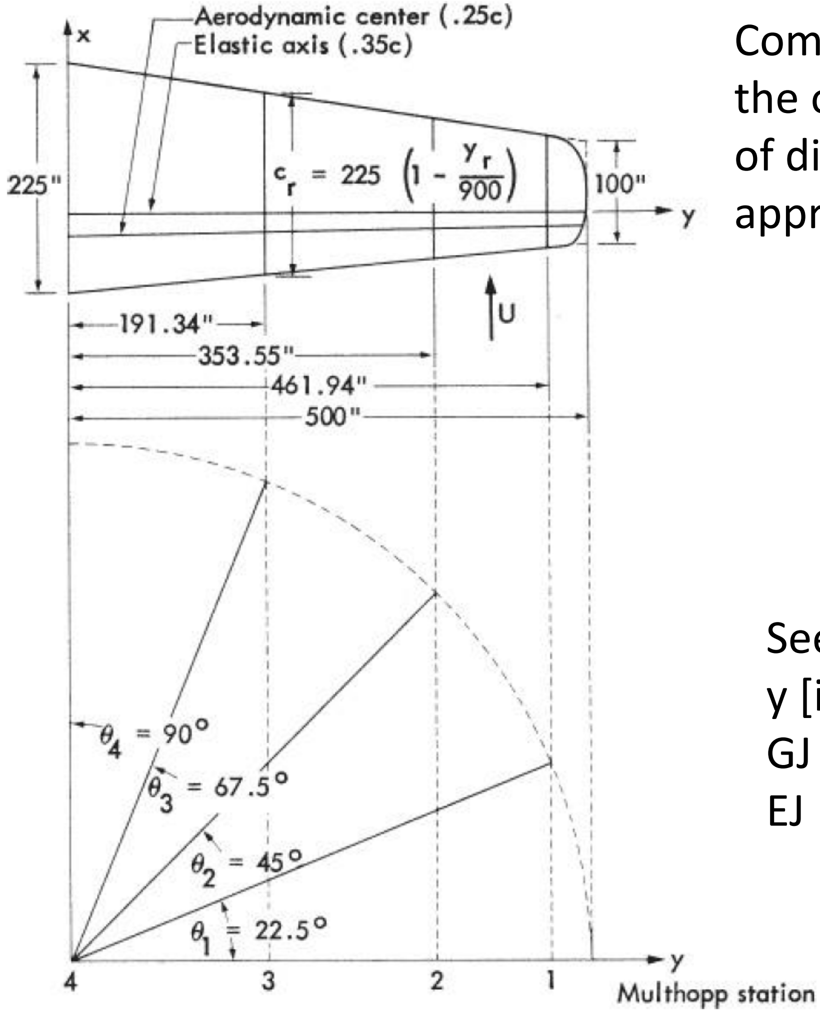
Example BAH Wing

Taken from BAH Example 2.1 page 45





Example BAH Wing



Compute the Flexibility matrix and all the other matrices to solve the problem of divergence speed using the flexibility approach

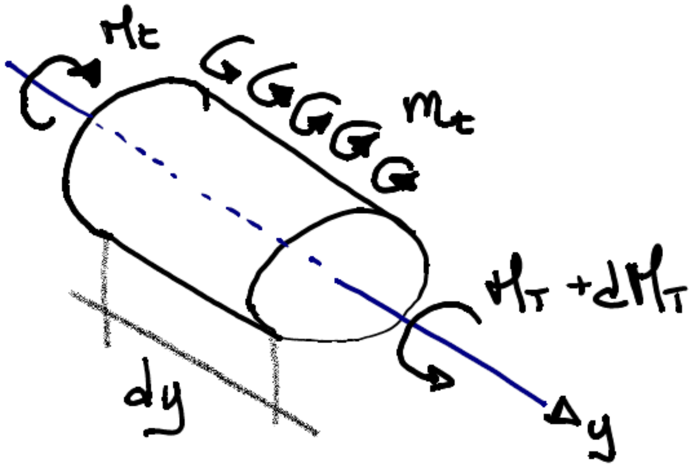
See file bah.mat for data

y [in] -> m

GJ [lb in2] -> N m2

EJ [lb in2] -> N m2

Slender straight wing: analytical solution



m_t distributed aerodynamic moment per unit span [FL/L]= [F]
M_t twisting moment [FL]

$$(M_t + \mathrm{d}M_t) - M_t + m_t \mathrm{d}y = 0$$

$$\frac{\mathrm{d}M_t}{\mathrm{d}y} + m_t = 0$$

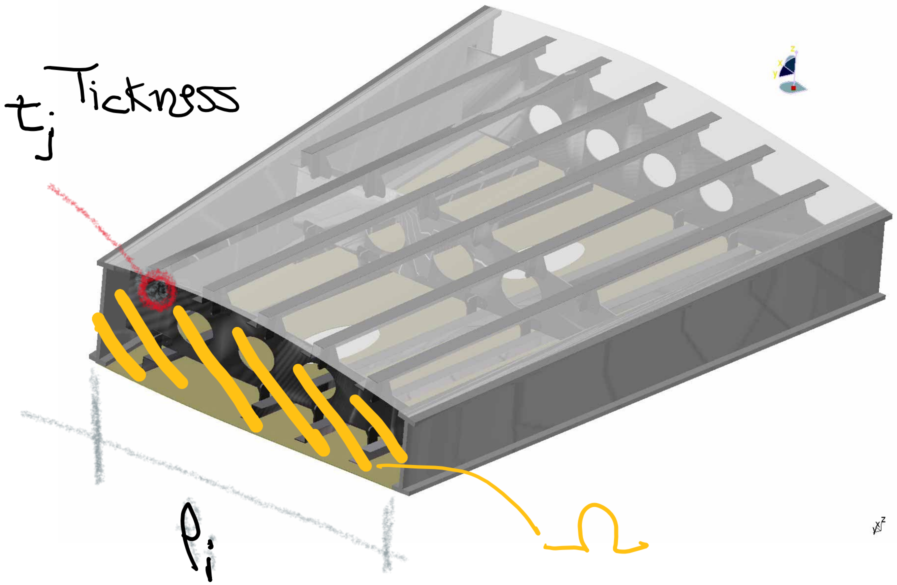
However, using the constitutive law for torsional beams the twisting moment can be expressed as $GJ\theta' = M_t$

$$(GJ\theta')' + m_t = 0$$

To this differential equation in space it is necessary to add the Boundary Conditions (BC)

$$\begin{aligned} \theta(0) &= 0 \quad \text{Root} \\ \theta'(L) &= 0 \quad \text{Free} - \text{end} \end{aligned}$$

Appendix: compute the torsional stiffness



G shear modulus of stiffness
J polar moment of cross-sectional area

TRUE ONLY FOR CIRCULAR SECTIONS!

$$GJ = 4G^*\Omega^2 \sum_{i=0}^{N} \frac{G_i}{G^*}\frac{t_i}{\ell_i}$$

G^* Reference Shear modulus

Slender straight wing: analytical solution: VWP approach

VWP for deformable structures

$$\delta W_i = \int_V \sigma_{ij} \delta\varepsilon_{ij} \mathrm{d}v = \delta W_e$$

For beams subject to torsion the internal virtual work could be expressed as function of generalized internal actions and generalized deformations

$$\delta W_i = \int_0^L \delta\theta' M_t \mathrm{d}y = \int_0^L \delta\theta' GJ\theta' \mathrm{d}y$$

Constitutive law

$$M_t = GJ\theta'$$

Using integration by parts

$$\delta W_i = \left[\delta\theta GJ\theta'\right]_0^L - \int_0^L \delta\theta (GJ\theta')' \mathrm{d}y \qquad \delta W_e = \int_0^L \delta\theta m_t \mathrm{d}y$$

Given the arbitraryness of the virtual torsion $\delta\theta$ it results

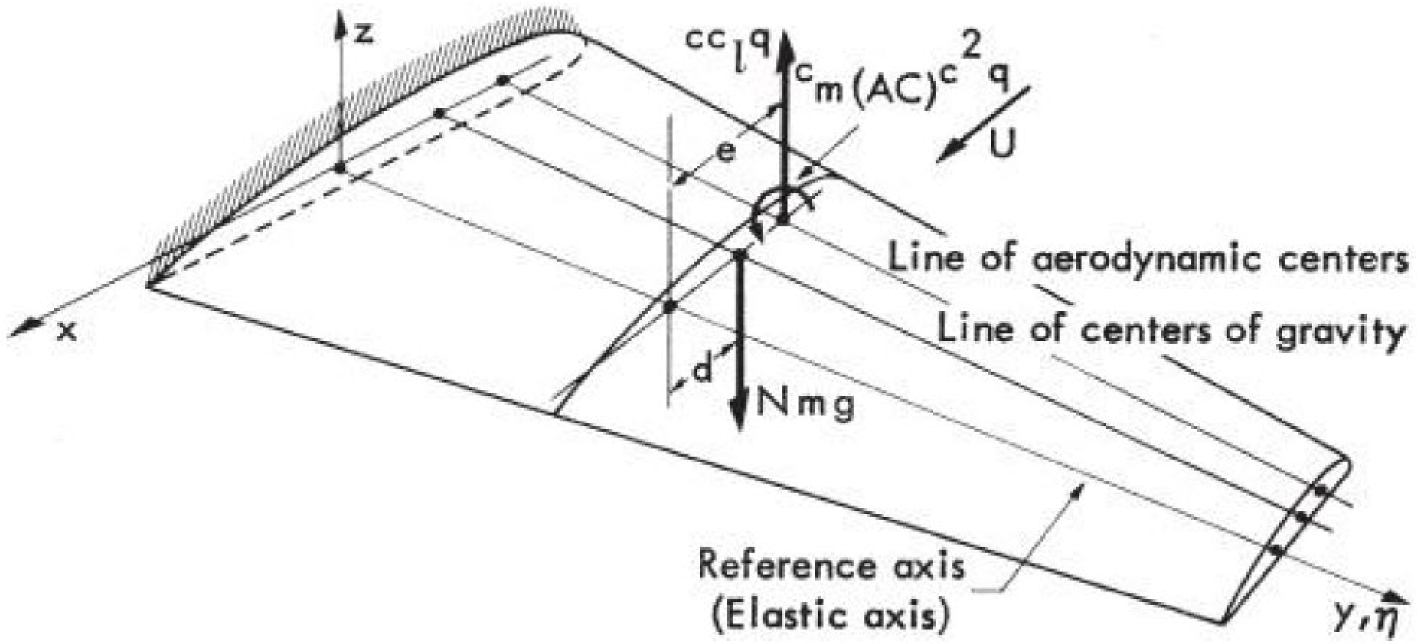
$$(GJ\theta')' + m_t = 0 \qquad \begin{cases} \theta(0) = 0 \\ \theta'(L) = 0 \end{cases}$$

Homework

Write the VWP weak formulation and the equivalent strong formulation for

1) The case with a torsional moment at the tip $\widehat{M}$
2) The case with of a torsional spring at root with stiffness Kt

Slender straight wing: analytical solution



L span of the wing

Here we will see also the effects of gravity and inertial effects through the load factor N.

Slender straight wing: analytical solution

Write the differential equation for the wing case (all properties uniform)

$$(GJ\theta')' = -qc\left(e\left(C_{L0} + C_{L\alpha}\theta\right) + cC_{m_{AC}}\right) + dNmg$$

$$\theta'' + \frac{qceC_{L\alpha}}{GJ}\theta = -\frac{qc}{GJ}\left(eC_{L0} + cC_{m_{AC}}\right) + \frac{dNmg}{GJ}$$

It is convenient to use non-dimensional variables:
$\tilde{y} = y/L$ so that $\theta'' = \mathrm{d}^2\theta/\mathrm{d}y^2 = \theta^{**}/L^2$
where $(\cdot)^* = \mathrm{d}(\cdot)/\mathrm{d}\tilde{y}$.

The non-dimensional equation becomes

$$\theta^{**} + \mu^2\theta = Q_A + Q_m, \qquad \theta(0) = 0,\ \theta^*(1) = 0.$$

where

$$\mu^2 = \frac{qecC_{L\alpha}L^2}{GJ}, \quad Q_A = -\frac{qcL^2}{GJ}\left(eC_{L0} + cC_{m_{AC}}\right), \quad Q_m = \frac{dNmgL^2}{GJ}$$

Slender straight wing: analytical solution
Solution of the homogeneous problem

$$\theta^{**} + \mu^2\theta = 0, \quad \theta(0) = 0, \theta^{*}(1) = 0 \tag{1}$$

Are there any values of μ (i.e. dynamic pressure q) for whom a non-trivial solution to Eq. (1) exist? (trivial solution is $\theta = 0 \, \forall \tilde{y}$)
This turns to be an eigenvalue problem for the continuous linear differential equation (1). For each eigenvalue $\check{\mu}$ it will be possible to find an associated eigenfunction $\check{\theta}(\tilde{y}) \neq 0$ that solves equation (1).

$$\check{\theta}(\tilde{y}) = A \sin \check{\mu}\tilde{y} + B \cos \check{\mu}\tilde{y} \tag{2}$$

Substituting expression (2) in (1), results in

$$-\check{\mu}^2 A \sin \check{\mu}\tilde{y} - B\check{\mu}^2 \cos \check{\mu}\tilde{y} + A\mu^2 \sin \check{\mu}\tilde{y} + B\mu^2 \cos \check{\mu}\tilde{y} = 0$$

And so

$$\check{\mu}^2 = \mu^2$$

Slender straight wing: analytical solution
Solution of the homogeneous problem

To identify completely the solution $\check{\theta}$ the boundary conditions must be included

$$\begin{matrix} \check{\theta}(0) = 0 \\ \check{\theta}'(1) = 0 \end{matrix} \begin{bmatrix} 0 & 1 \\ \check{\mu}\cos\check{\mu} & -\check{\mu}\sin\check{\mu} \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\mathbf{C} \begin{Bmatrix} A \\ B \end{Bmatrix} = \mathbf{0}$$

The requirement to have a nontrivial solution $(A = B = 0)$ translates into

$$\det \mathbf{C} = 0 \longrightarrow \check{\mu}\cos\check{\mu} = 0 \begin{cases} \check{\mu} = 0 \rightarrow \check{\theta} = 0 \, \text{trivial solution} \\ \cos\check{\mu} = 0 \rightarrow \check{\mu} = \pm(2n+1)\frac{\pi}{2}, n \in \mathbb{N} \end{cases}$$

Solution of the homogeneous problem: divergence

$$
\begin{aligned}
\mu^2 &= \frac{q_D e c L^2 C_{L\alpha}}{GJ} \\
\check{\mu}^2 &= (2n+1)^2 \left(\frac{\pi}{2}\right)^2
\end{aligned}
$$

Consequently, the divergence dynamic pressures are

$$
q_{D_n} = (2n+1)^2 \left(\frac{\pi}{2}\right)^2 \frac{GJ}{ecL^2 C_{L\alpha}}
$$

The divergence dynamic pressure is again obtained by solving an eigenvalue problem. However, since the problem is now continuous we have an infinite set of eigenvalues or divergence dynamic pressures.
There are also associated torsional deformation modes or eigenfunction:

$$
\check{\theta}_n = \sin\left((2n+1)\frac{\pi y}{2L}\right) \tag{1}
$$

Solution of the homogeneous problem: divergence

It is interesting to compare the first divergencer dynamic pressure of the continuous model with the one computed using the typical section

$$q_{D_0} = \left(\frac{\pi}{2}\right)^2 \frac{GJ}{ecL^2 C_{L\alpha}} \qquad q_{D_{TS}} = \frac{k_\alpha}{eSC_{L\alpha}} = \frac{k_\alpha}{ecLC_{L\alpha}}$$

The comparison says that the Typical Section returns the results of the continuous model if

$$k_\alpha = \frac{\pi^2}{4} \frac{GJ}{L}$$

Solution of the forced problem

$$\theta^{**} + \mu^2\theta = Q_A + Q_m = Q_T, \qquad \theta(0) = 0,\ \theta^*(1) = 0$$

The solution will be the sum of the generla integral (already computed) plus the particular integral

$$\theta(\tilde{y}) = A\sin\mu\tilde{y} + B\cos\mu\tilde{y} + \frac{Q_T}{\mu^2}$$

$$\begin{bmatrix} 0 & 1 \\ \mu\cos\mu & -\mu\sin\mu \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = \begin{Bmatrix} -\frac{Q_T}{\mu^2} \\ 0 \end{Bmatrix}$$

$$\begin{aligned} B &= -\frac{Q_T}{\mu^2} \\ A &= B\tan\mu = -\frac{Q_T}{\mu^2}\tan\mu \end{aligned}$$

$$\theta(\tilde{y}) = \frac{Q_T}{\mu^2}\left(1 - \tan\mu\sin\mu\tilde{y} - \cos\mu\tilde{y}\right)$$

Case 1: variation of lift distribution at constant angle of attack

If we neglect the inertial effects (i.e. $d = 0$) and consider a symmetric airfoil (i.e. $C_{m_{CA}} = 0$). Then

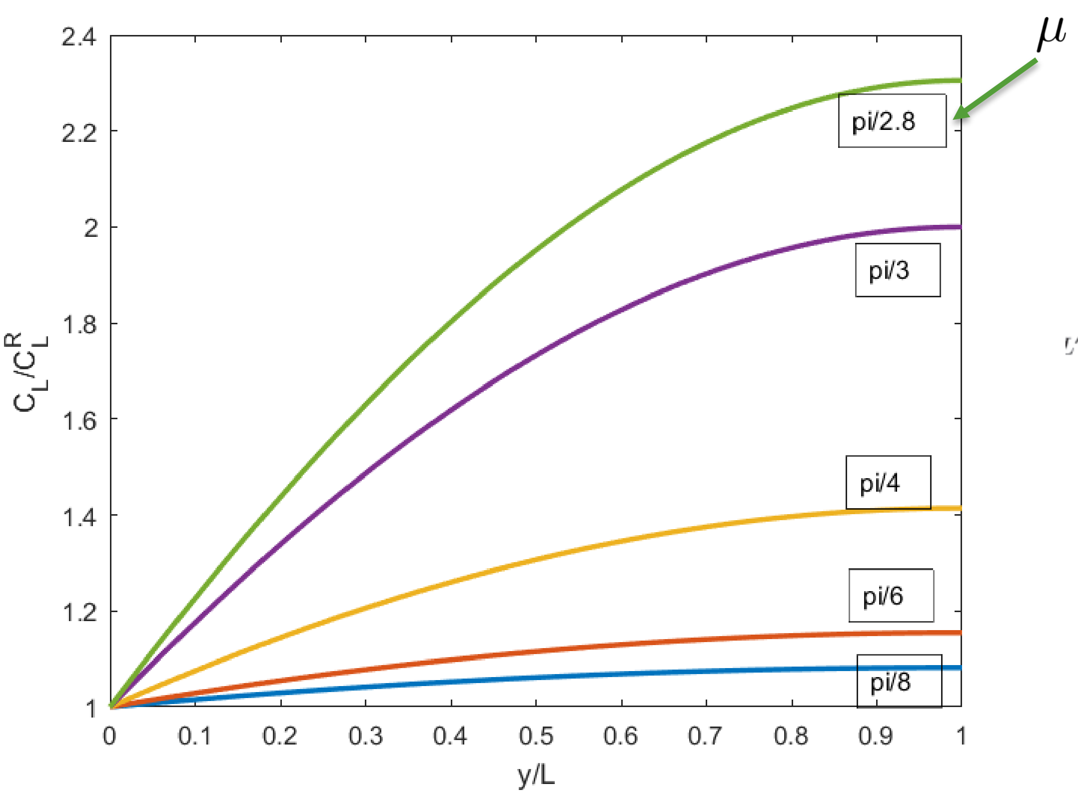
$$
\begin{aligned}
Q_T &= Q_A = -\frac{qcL^2eC_{L0}}{GJ} \\
\frac{Q_T}{\mu^2} &= -\frac{\cancel{qcL^2}eC_{L0}}{\cancel{GJ}}\frac{\cancel{GJ}}{\cancel{qcL^2}C_{L\alpha}} = -\frac{C_{L0}}{C_{L\alpha}}
\end{aligned}
$$

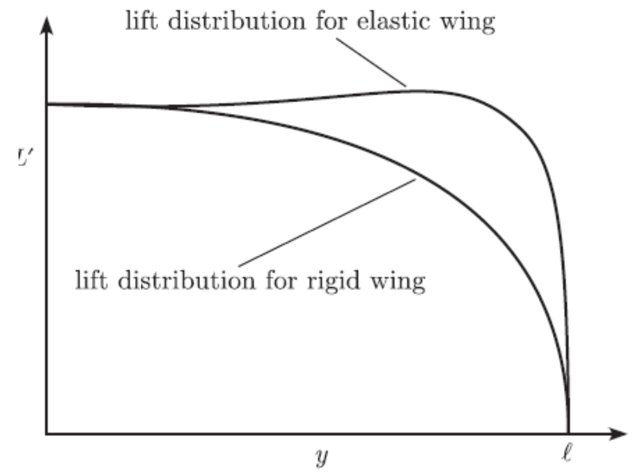
The elastic lift coefficient is

$$
(C_L)_e = C_{L\alpha}\theta = \cancel{-C_{L\alpha}}\frac{C_{L0}}{\cancel{C_{L\alpha}}}\left(1 - \tan\mu\sin\mu\tilde{y} - \cos\mu\tilde{y}\right)
$$

$$
\frac{C_L}{C_{L0}} = \frac{C_{L0} + (C_L)_e}{C_{L0}} = \tan\mu\sin\mu\tilde{y} + \cos\mu\tilde{y}
$$

Case 1: variation of lift distribution at constant angle of attack





Case 2: variation of lift distribution at constant lift

Equilibrium equation in the z direction for the rigid aircraft with $W = mg$ the weight of the aircraft

$$q \int_0^L c C_{L\alpha} \alpha_0 \mathrm{d}y = \frac{WN}{2}$$

The same equation for the elastic aircraft would read

$$q \int_0^L c (C_L)_e \mathrm{d}y + q \int_0^L c C_{L\alpha} \check{\alpha}_0 \mathrm{d}y = \frac{WN}{2}$$

Consequently, to respect the same trim equation with the elastic aircraft we have to impose that

$$q \int_0^L c (C_L)_e \mathrm{d}y = q \int_0^L c\, C_{L\alpha} \left(\alpha_0 - \check{\alpha}_0\right) \mathrm{d}y$$

So a variation of the trim angle of attack is required for the elastic model to obtain the same total lift of the rigid model $\Delta\alpha = (\alpha_0 - \check{\alpha}_0)$.

Case 2: variation of lift distribution at constant lift

$$\int_0^L cC_{L\alpha}\left(\theta - \Delta\alpha_0\right) \mathrm{d}y = 0$$

$$\cancel{cL} \int_0^1 \cancel{C_{L\alpha}} \left(\frac{Q_T}{\mu^2}\left(1 - \tan\mu \sin\mu\tilde{y} - \cos\mu\tilde{y}\right) - \Delta\alpha_0 \right) \mathrm{d}\tilde{y} = 0$$

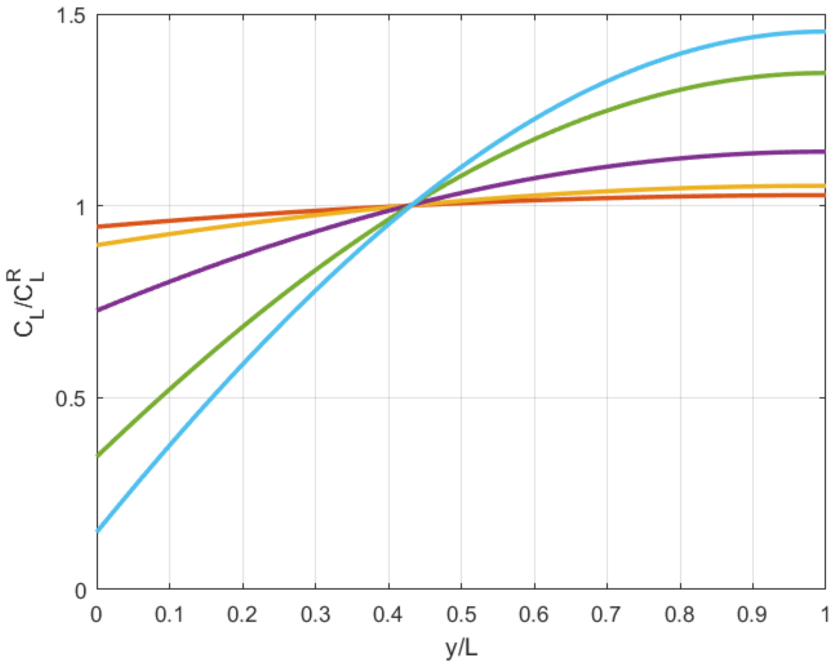
$$\begin{aligned} \frac{Q_T}{\mu^2}\left[\tilde{y} + \frac{\tan\mu\cos\mu\tilde{y}}{\mu} - \frac{\sin\mu\tilde{y}}{\mu}\right]_0^1 - \Delta\alpha_0 &= 0 \\ \frac{Q_T}{\mu^2}\left(1 + \frac{\tan\mu\cos\mu}{\mu} - \frac{\tan\mu}{\mu} - \frac{\sin\mu}{\mu}\right) - \Delta\alpha_0 &= 0 \end{aligned}$$

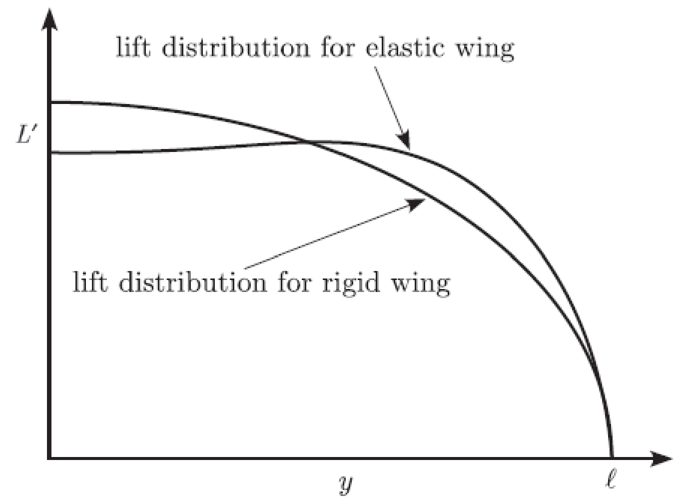
If we assume $C_{m_{AC}} = 0$ and $d = 0$

$$-\alpha_0\left(1 - \frac{\tan\mu}{\mu}\right) = \Delta\alpha_0$$

$$\frac{C_L}{C_{L0}} = \frac{C_{L\alpha}(\check{\alpha}_0 + \theta)}{C_{L\alpha}\alpha_0} = \frac{\alpha_0 + \theta - \Delta\alpha_0}{\alpha_0} = 1 + \frac{\theta - \Delta\alpha_0}{\alpha_0}$$

Case 2: variation of lift distribution at constant lift





$$\frac{C_L}{C_{L0}} = 1 + 1 - \frac{\tan\mu}{\mu} - (1 - \tan\mu \sin\mu\tilde{y} - \cos\mu\tilde{y})$$

$$\frac{C_L}{C_{L0}} = 1 - \frac{\tan\mu}{\mu} + \tan\mu \sin\mu\tilde{y} + \cos\mu\tilde{y}$$

Implement a Matlab code to compute load distribution

Goland Wing

Chord c = 1.829 m

Semispan L = 6.096 m

AC 25% of the chord

EA   33% of the chord

CG 43% of the chord

Air density ρ = 1.225 kg/m^3

Torsional Rigidity GJ = 987,600 N m^2

Wing mass per unit length m = 35.72 kg/m

Lift curve slope $C_{L\alpha}$ = 5.4

C_{mAC} = -0.05

Approximate approach (Ritz method)

$$\tilde{\theta}(y) = \sum_{i=1}^{\infty} N_i(y)\, q_i$$

with $N_i(y)$ known <u>SHAPE FUNCTIONS</u>
q_i <u>GENERALIZED COORDINATES</u>

- ✓ Express the unknown displacement as a <u>complete linear</u> combination of <u>known</u> space functions.

- ✓ The shape functions chosen must belong to a <u>complete set</u>, meaning that in the domain considered, taking a sufficient number of terms it must be possible to represent the exact solution to <u>any degree of accuracy</u>. Typical complete sets are polynomials, trigonometric functions, etc…

- ✓ The coefficient of the series q_i are a new set of coordinates (i.e. degrees of freedom) used to describe the displacement field.

Approximate Ritz approach

In practice only a limited set of n shape function is used

Truncated to n functions

$$\tilde{\theta}(y) = \sum_{i=1}^{n} N_i(y)\, q_i = \mathbf{N}(y)\mathbf{q}$$

where $\mathbf{N}$ is a row matrix and $\mathbf{q}$ is a vector of generalized degrees of freedom.

- ✓ If the method is applied to a strong formulation, (i.e. differential) the chosen shape functions must satisfy all boundary conditions.
- ✓ Each shape function should be p times differentiable where p is the highest spatial derivative present in the formulation.
- ✓ At least one of the shape function p-th order derivative should be not-zero.

Approximate Ritz Approach

$$
\begin{aligned}
F(\theta(y)) &= 0 \\
F(\tilde{\theta}(y)) &= \varepsilon(y) \leftarrow \color{red}{RESIDUAL}
\end{aligned}
$$

where $\tilde{\theta}$ is the approximate numerical of the exact solution θ. To find it, it is necessary to minimize the error $\varepsilon(y)$.

1. COLLOCATION

$$\varepsilon(y_i) = 0 \quad i = 1, 2, \cdots, n$$

2. WEIGHTED INTEGRALS

$$\int_0^L w_i(y)\varepsilon(y)\mathrm{d}y = 0 \quad w_i\; i = 1, 2, \cdots, n \text{ weight functions}$$

Approximate Ritz Approach (Collocation)

$$\mathbf{N}''(y_i)\mathbf{q} + m_t(y_i) = 0$$

$$\mathbf{K}_0\mathbf{q} = -\mathbf{m}_t$$

where

$$\mathbf{K}_0 = \begin{bmatrix} N_1''(y_1) & N_2''(y_1) & \cdots & N_n''(y_1) \\ N_1''(y_2) & N_2''(y_2) & \cdots & N_n''(y_2) \\ \vdots & \vdots & \ddots & \vdots \\ N_1''(y_n) & N_2''(y_n) & \cdots & N_n''(y_n) \end{bmatrix}$$

and

$$\mathbf{m}_t = \begin{bmatrix} m_t(y_1) \\ m_t(y_2) \\ \vdots \\ m_t(y_n) \end{bmatrix}$$

Approximate Ritz approach

More often the method is applied directly to the weak (integral) formulation, that in our case is the VWP.

Starting from an integral formulation leads naturally to a weighted integral approach, where the weight functions are the virtual displacements.

In this case there are two functionals to be approximated: the displacement field, and the virtual displacement field. If the same series is used for both the approach is called **Ritz-Galerkin.**

Approximate Ritz Galerkin

Using the VWP

$$\delta\theta \cdot M_t = \delta\theta^T M_t$$

$$\int_0^L \delta\theta'^T GJ\theta' \mathrm{d}y = \int_0^L \delta\theta^T m_t \mathrm{d}y$$

$$\delta\mathbf{q}^T \underbrace{\int_0^L \mathbf{N}'^T GJ\mathbf{N}' \mathrm{d}y}_{\mathbf{K}_s}\, \mathbf{q} = \delta\mathbf{q}^T \underbrace{\int_0^L \mathbf{N}^T m_t \mathrm{d}y}_{\mathbf{Q}}$$

Given the arbitrariness of $\delta\mathbf{q}$

$$\mathbf{K}_s\mathbf{q} = \mathbf{Q}$$

Approximate Ritz Galerkin

- ✓ Regularity required to shape function is lower since the maximum p derivative is lower
- ✓ The shape functions must satisfy boundary conditions on displacements (and rotations) often called “geometric” boundary conditions
- ✓ The boundary conditions on forces and moments, called “natural” are not necessary. Their satisfaction will be obtained naturally at convergence (by using a sufficiently large number of shape functions). However, satisfying also these conditions can speed up convergence

Approximate Ritz Galerkin

$$m_t = qc(y)\left(e(y)\left(C_{L0}(y) + C_{L\alpha}(y)\theta(y)\right) + c(y)C_{m_{AC}}(y)\right)$$

$$\mathbf{Q} = \underbrace{q\int_0^L \mathbf{N}^T c e C_{L0} \mathrm{d}y}_{\mathbf{M}_0} + \underbrace{q\int_0^L \mathbf{N}^T c e C_{L0} \mathbf{N} \mathrm{d}y}_{\mathbf{K}_s}\, \mathbf{q} + \underbrace{q\int_0^L \mathbf{N}^T c^2 C_{m_{AC}} \mathrm{d}y}_{\mathbf{M}_m}$$

$$\underbrace{\underbrace{\left(\mathbf{K}_s - q\mathbf{K}_a\right)\mathbf{q}}_{\text{STABILITY ANALYSIS}} = \mathbf{M}_0 + \mathbf{M}_m}_{\text{RESPONSE ANALYSIS}}$$

STABILITY ANALYSIS

RESPONSE ANALYSIS

Homework

Consider the following expressions for the shape function to be applied to the Goland wing model

1) $\theta = y/L$
2) $\theta = 2\,y/L - y^2/L^2$
3) $\theta = \sin\,(\pi/2\;y/L)$
4) $\theta_1 = y/L,\ \theta_2 = y/L$

Which function works best? Why?