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055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

11 Structural Dynamics: Random Response

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Material

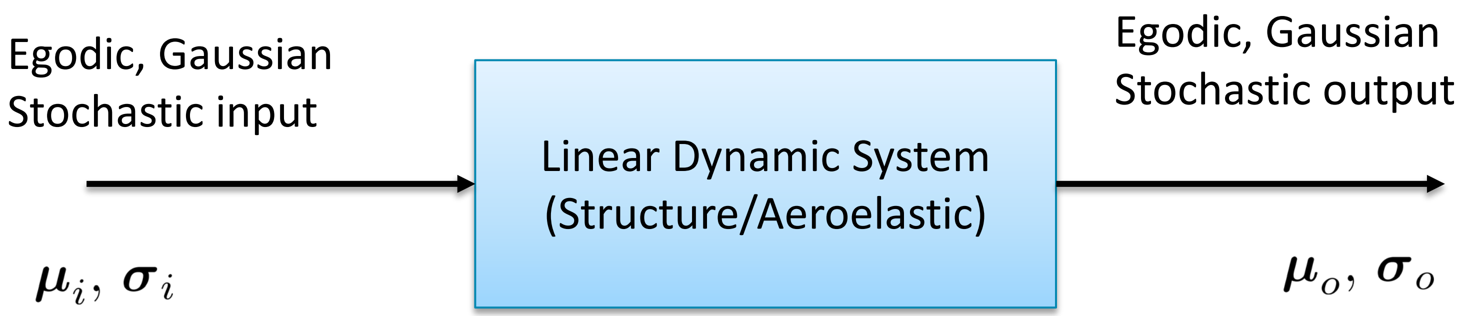
Masarati Chapter 6

Preumont Chapter 8

Additional material

Cheli Diana Chapter 7 (accessible to Polimi students through https://link.springer.com/book/10.1007%2F978-3-319-18200-1

Computation of the response of a SISO system using impulse response



Let's consider initially a single input single output (scalar) system

$$q(t) = \int_0^\infty h(\tau) F(t-\tau) \mathrm{d}\tau$$

Computation of the mean of the output knowing the mean of the input

$$\begin{aligned}
m_q &= E[Q(t)] = E\left[\int_0^{\infty} h(\tau)F(t-\tau)\mathrm{d}\tau\right] \\
m_q &= \fint \left(\int_0^{\infty} h(\tau)F(t-\tau)\mathrm{d}\tau\right) \mathrm{d}t = \int_0^{\infty} h(\tau) \fint F(t-\tau)\, \mathrm{d}t\mathrm{d}\tau \\
m_q &= \int_0^{\infty} h(\tau)E[F(t-\tau)]\mathrm{d}\tau
\end{aligned}$$

$$m_q = m_F \int_0^{\infty} h(\tau)\mathrm{d}\tau$$

Since

$$H(s) = \int_0^{\infty} h(\tau)\mathrm{e}^{-st}\mathrm{d}\tau \;\rightarrow\; \int_0^{\infty} h(\tau)\mathrm{d}\tau = H(0)$$

$$\Rightarrow m_q = H(0)m_F$$

Computation of the mean using the second order formulation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F} \qquad \mathbf{M}E[\ddot{\mathbf{q}}] + \mathbf{C}E[\dot{\mathbf{q}}] + \mathbf{K}E[\mathbf{q}] = E[\mathbf{F}]$$

$$E[\mathbf{q}] = \mathbf{m}_{\mathbf{q}}, E[\mathbf{F}] = \mathbf{m}_{\mathbf{F}}$$

$$E[\dot{\mathbf{q}}] = \fint \dot{\mathbf{q}} \mathrm{d}t = \lim_{\Delta t \to 0} \frac{\fint \mathbf{q}(t + \Delta t)\mathrm{d}t - \fint \mathbf{q}(t)\mathrm{d}t}{\Delta t} = \mathbf{0}$$

For an ergodic process $Q(t)$, the mean of a time derivative of that process $\dot{Q}(t)$ is null. Consequently also the means of $\ddot{Q}(t), \ldots$ are all null.

$$\begin{aligned} &\mathbf{K}\mathbf{m}_{\mathbf{q}} = \mathbf{m}_{\mathbf{F}} \\ &\rightarrow \mathbf{m}_{\mathbf{q}} = \mathbf{K}^{-1}\mathbf{m}_{\mathbf{F}} \end{aligned}$$

Computation of the mean using the state space formulation

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

$$\begin{cases} E[\dot{\mathbf{x}}] &= \mathbf{A}E[\mathbf{x}] + \mathbf{B}E[\mathbf{u}] \\ E[\mathbf{y}] &= \mathbf{C}E[\mathbf{x}] + \mathbf{D}E[\mathbf{u}] \end{cases}$$

$$E[\mathbf{x}] = \mathbf{m_x}, E[\mathbf{y}] = \mathbf{m_y}, E[\mathbf{u}] = \mathbf{m_u}$$

$$\begin{cases} \mathbf{0} &= \mathbf{A}\mathbf{m_x} + \mathbf{B}\mathbf{m_u} \\ \mathbf{m_y} &= \mathbf{C}\mathbf{m_x} + \mathbf{D}\mathbf{m_u} \end{cases}$$

$$\Rightarrow \mathbf{m_y} = \left(-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{m_u}$$

$$\text{Since } \mathbf{H}(s) = \left(-\mathbf{C}\left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right)$$

$$\Rightarrow \mathbf{m_y} = \mathbf{H}(0)\mathbf{m_u}$$

Computation of the variance of the output using the impulse response

$$\mathbf{q} - \mathbf{m_q} = \Delta\mathbf{q}(t) = \int_0^\infty \mathbf{h}(v)\Delta\mathbf{F}(t-v)\mathrm{d}v$$

where $\Delta\mathbf{F}(t) = \mathbf{F} - \mathbf{m_F}$

$$\mathbf{k_{qq}}(\tau) = \fint \Delta\mathbf{q}(t)\Delta\mathbf{q}^T(t+\tau)\mathrm{d}t$$

$$\mathbf{k_{qq}}(\tau) = \fint \int_0^\infty \mathbf{h}(v)\Delta\mathbf{F}(t-v)\mathrm{d}v \int_0^\infty \Delta\mathbf{F}^T(t+\tau-w)\mathbf{h}^T(w)\mathrm{d}w\ \mathrm{d}t$$

$$\mathbf{k_{qq}}(\tau) = \fint \int_0^\infty \int_0^\infty \mathbf{h}(v)\Delta\mathbf{F}(t-v)\Delta\mathbf{F}^T(t+\tau-w)\mathbf{h}^T(w)\mathrm{d}v\mathrm{d}w\ \mathrm{d}t$$

$$\mathbf{k_{qq}}(\tau) = \int_0^\infty \int_0^\infty \mathbf{h}(v) \fint \Delta\mathbf{F}(z)\Delta\mathbf{F}^T(z+v+\tau-w)\mathrm{d}z\ \mathbf{h}^T(w)\mathrm{d}v\mathrm{d}w$$

Computation of the variance of the output using the impulse response

$$\mathbf{k_{qq}} = \int_0^\infty \int_0^\infty \mathbf{h}(v) \fint \Delta\mathbf{F}(z)\Delta\mathbf{F}^T(z+v+\tau-w)\mathrm{d}z \; \mathbf{h}^T(w)\mathrm{d}v\mathrm{d}w$$

$$\fint \Delta\mathbf{F}(z)\Delta\mathbf{F}^T(z+v+\tau-w)\mathrm{d}z \;\; = \mathbf{k_{FF}}(v+\tau-w)$$

$$\mathbf{k_{qq}}(\tau) = \int_0^\infty \int_0^\infty \mathbf{h}(v)\mathbf{k_{FF}}(v+\tau-w)\mathbf{h}^T(w)\mathrm{d}v\mathrm{d}w$$

$$\boldsymbol{\sigma}^2_{\mathbf{qq}} = \mathbf{k_{qq}}(0) = \int_0^\infty \int_0^\infty \mathbf{h}(v)\mathbf{k_{FF}}(v-w)\mathbf{h}^T(w)\mathrm{d}v\mathrm{d}w$$

To compute the variance of the output we need the impulse response and the autocovariance of the input

Computation of the response in frequency domain

$$\mathbf{\Phi_{qq}}(\omega) = \int_{-\infty}^{\infty} \mathbf{k_{qq}}(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\ \mathrm{d}\tau$$

$$\mathbf{\Phi_{qq}}(\omega) = \int_{0}^{\infty}\int_{0}^{\infty} \mathbf{h}(v)\int_{-\infty}^{\infty} \mathbf{k_{FF}}(\tau+v-w)\mathrm{e}^{-\mathrm{j}\omega\tau}\ \mathrm{d}\tau\ \mathbf{h}^T(w)\ \mathrm{d}v\mathrm{d}w$$

$$t = \tau + v - w\,\tau = t - v + w$$

$$\mathbf{\Phi_{qq}}(\omega) = \int_{0}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} \underbrace{\mathbf{h}(v)\mathrm{e}^{\mathrm{j}\omega v}}_{\mathbf{H}(-\omega)}\mathbf{k_{FF}}(t)\mathrm{e}^{-\mathrm{j}\omega t}\underbrace{\mathrm{e}^{-\mathrm{j}\omega w}\mathbf{h}^T(w)}_{\mathbf{H}^T(\omega)}\ \mathrm{d}t\mathrm{d}v\mathrm{d}w$$

$$\mathbf{\Phi_{qq}}(\omega) = \mathbf{H}(-\omega)\mathbf{\Phi_{FF}}(\omega)\mathbf{H}^T(\omega)$$

Computation of the response in frequency domain

$$\mathbf{\Phi_{qq}}(\omega) = \mathbf{H}^*(\omega)\mathbf{\Phi_{FF}}(\omega)\mathbf{H}^T(\omega)$$

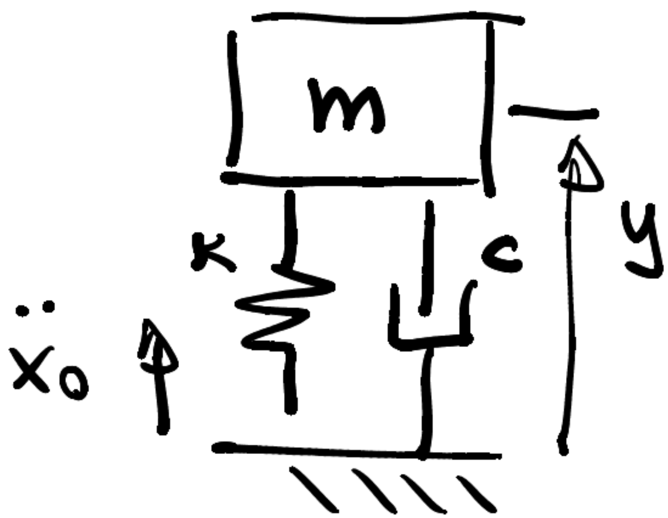
for SISO systems (1 input, 1 output no matter how many internal states)

$$\Phi_{qq}(\omega) = |H(\omega)|^2 \Phi_{FF}$$

$$\boldsymbol{\sigma}^2_{\mathbf{qq}} = \frac{1}{\pi}\int_0^{+\infty} \mathbf{\Phi_{qq}}(\omega)\mathrm{d}\omega = \frac{1}{\pi}\int_0^{+\infty} \mathbf{H}^*(\omega)\mathbf{\Phi_{FF}}(\omega)\mathbf{H}^T(\omega)\mathrm{d}\omega$$

$$\sigma^2_{q_i q_j} \begin{cases} i = j & \text{auto} - \text{variance} \\ i \neq j & \text{cross} - \text{variance} \end{cases}$$

Linear oscillator subject to a white noise input



$$m\ddot{x} + c\dot{y} + ky = 0$$

$$x = x_0 + y$$

$$m\ddot{y} + c\dot{y} + ky = -m\ddot{x}_0$$

$$\ddot{y} + 2\xi\omega_n\dot{y} + \omega_n^2 y = -\ddot{x}_0$$

Consider that the PSD of the base acceleration is a white noise with intensity $2\pi S_0$

$$\Phi_{\ddot{x}_0\ddot{x}_0} = 2\pi S_0$$

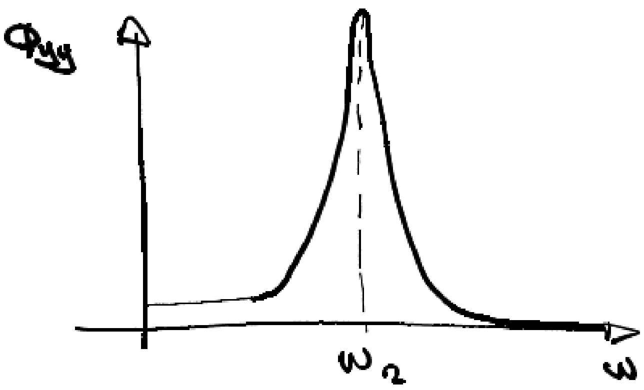
$$H(\omega) = \frac{y}{\ddot{x}_0} = \frac{1}{\omega^2 - \omega_n^2 - \mathrm{j}2\xi\omega_n\omega}$$

$$H(\omega)^* = \frac{y}{\ddot{x}_0} = \frac{1}{\omega^2 - \omega_n^2 + \mathrm{j}2\xi\omega_n\omega}$$

$$|H(\omega)|^2 = H(\omega)^* H(\omega) = \frac{1}{\left(\omega^2 - \omega_n^2\right)^2 + 4\xi^2\omega_n^2\omega^2}$$

Linear oscillator subject to a white noise input

$$\Phi_{yy} = \frac{2\pi S_0}{\left(\omega^2 - \omega_n^2\right)^2 + 4\xi^2\omega_n^2\omega^2}$$



$$\sigma_{yy}^2 = \frac{1}{\pi}\int_0^\infty \Phi_{yy}(\omega)\mathrm{d}\omega$$

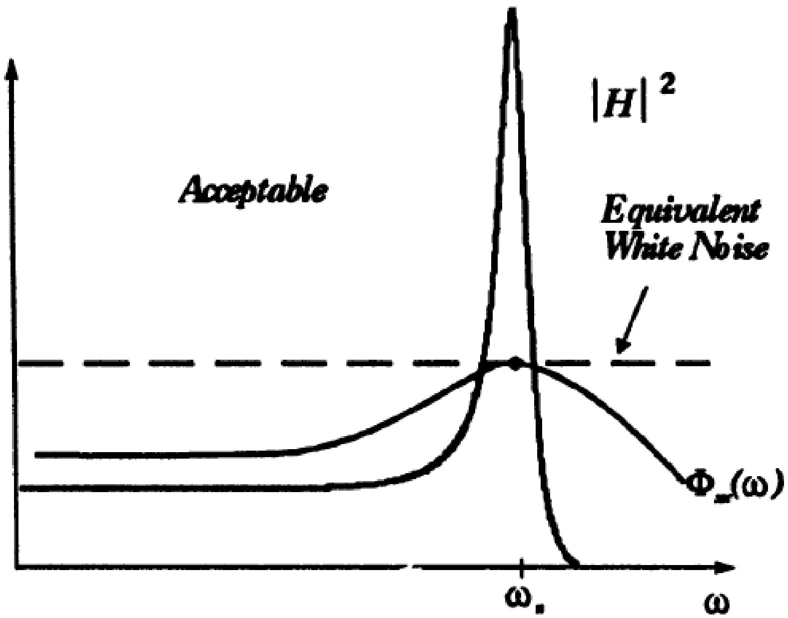
$$h(t) = \mathrm{e}^{-\xi\omega_n t}\frac{sin(\omega_d t)}{\omega_d}$$

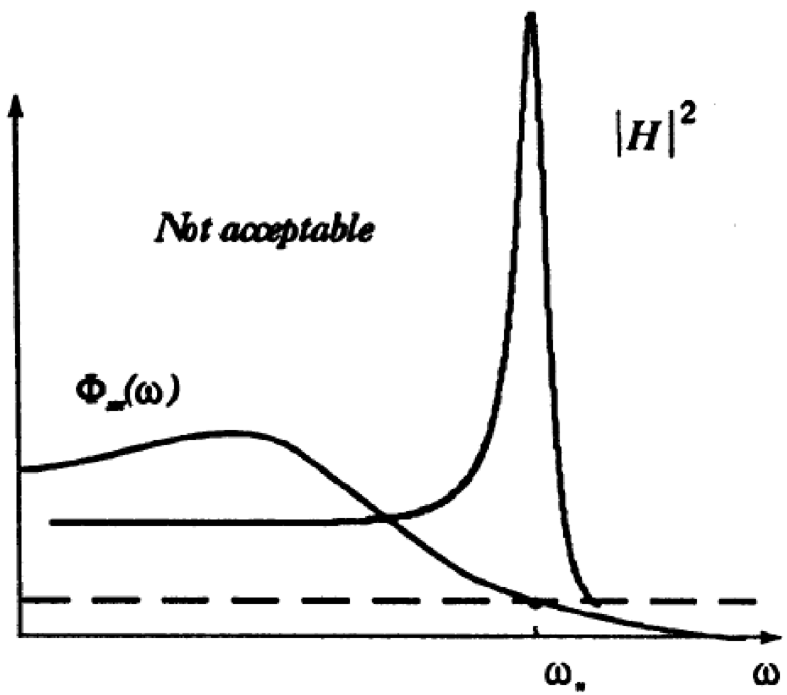
$$\sigma_{yy}^2 = S_0\int_0^\infty\int_0^\infty h(v)\delta(v-w)h(w)\mathrm{d}v\mathrm{d}w = S_0\int_0^\infty h^2(v)\mathrm{d}v$$

$$\sigma_{yy}^2 = S_0\int_0^\infty\left(\mathrm{e}^{-\xi\omega_n t}\frac{sin\omega_d t}{\omega_d}\right)^2\mathrm{d}v = \frac{S_0}{4\xi\omega_n^3}$$

Although the variance of the excitation (the base acceleration is unbounded, the variance of the displacement is finite provided that there is some damping

Applicability of the white noise approximation





State-space formulation

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

1. Find the variance matrix of the state vector
2. Find the variance of the output

$$\mathbf{k}_{\mathbf{xx}}(\tau) = \fint (\mathbf{x}(t) - \mathbf{m}_{\mathbf{x}})(\mathbf{x}(t+\tau) - \mathbf{m}_{\mathbf{x}})^T \, \mathrm{d}t = \fint \Delta\mathbf{x}(t)\Delta\mathbf{x}(t+\tau)^T \mathrm{d}t$$

$$\boldsymbol{\sigma}^2_{\mathbf{xx}} = \mathbf{k}_{\mathbf{xx}}(0)$$

$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{m}_{\mathbf{x}}$$
$$\Delta\mathbf{u}(t) = \mathbf{u}(t) - \mathbf{m}_{\mathbf{u}}$$
$$\Delta\mathbf{y}(t) = \mathbf{y}(t) - \mathbf{m}_{\mathbf{y}}$$

$$\begin{cases} \Delta\dot{\mathbf{x}} &= \mathbf{A}\Delta\mathbf{x} + \mathbf{B}\Delta\mathbf{u} \\ \Delta\mathbf{y} &= \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u} \end{cases}$$

Multiplay the first equation by $\Delta\mathbf{x}^T$ and then compute the mean integral.

$$\Rightarrow \fint \Delta\dot{\mathbf{x}}\Delta\mathbf{x}^T \mathrm{d}t = \mathbf{A} \fint \Delta\mathbf{x}\Delta\mathbf{x}^T \mathrm{d}t + \mathbf{B} \fint \Delta\mathbf{u}\Delta\mathbf{x}^T \mathrm{d}t$$

$$\Rightarrow \boldsymbol{\sigma}^2_{\dot{\mathbf{x}}\mathbf{x}} = \mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \mathbf{B}\boldsymbol{\sigma}^2_{\mathbf{ux}}$$

State space formulation

While for the scalar case it is possible to say that the cross-variance between a dof and its derivative is null, the same cannot be said for the vectorial case.

$$\fint \Delta\dot{x}\Delta x \mathrm{d}t = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\Delta\dot{x}\Delta x \mathrm{d}t$$

$$\fint \Delta\dot{x}\Delta x \mathrm{d}t = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{1}{2}\frac{\mathrm{d}(\Delta x)^2}{\mathrm{d}t}\mathrm{d}t$$

$$\lim_{T-\to\infty}\frac{1}{2T}\frac{\Delta x^2(T)-\Delta x^2(-T)}{4T} = 0$$

$$\sigma^2_{\dot{x}x} = 0$$

In this last case it is the sum of the variance between the derivatives times the states plus that of the states times the derivatives to be null.

$$\fint \Delta\dot{\mathbf{x}}\Delta\mathbf{x}^T \mathrm{d}t = \fint \frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta\mathbf{x}\Delta\mathbf{x}^T\right)\mathrm{d}t - \fint \Delta\mathbf{x}\Delta\dot{\mathbf{x}}^T \mathrm{d}t$$

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Delta\mathbf{x}\Delta\mathbf{x}^T\right)\mathrm{d}t = 0$$

$$\boldsymbol{\sigma}^2_{\dot{\mathbf{x}}\mathbf{x}} = -\boldsymbol{\sigma}^2_{\mathbf{x}\dot{\mathbf{x}}} \neq \mathbf{0}$$

Lyapunov equation

$$\boldsymbol{\sigma}^2_{\dot{\mathbf{x}}\mathbf{x}} = \mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{x}} + \mathbf{B}\boldsymbol{\sigma}^2_{\mathbf{u}\mathbf{x}} \tag{1}$$

$$\Rightarrow \fint \Delta\mathbf{x}\Delta\dot{\mathbf{x}}^T \mathrm{d}t = \fint \Delta\mathbf{x}\Delta\mathbf{x}^T \mathrm{d}t\, \mathbf{A}^T + \fint \Delta\mathbf{x}\Delta\mathbf{u}^T \mathrm{d}t\, \mathbf{B}^T$$

$$\boldsymbol{\sigma}^2_{\mathbf{x}\dot{\mathbf{x}}} = \boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{x}}\mathbf{A}^T + \boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{u}}\mathbf{B}^T \tag{2}$$

Combining (1) and (2) it results that

$$\mathbf{0} = \mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{x}} + \boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{x}}\mathbf{A}^T + \mathbf{B}\boldsymbol{\sigma}^2_{\mathbf{u}\mathbf{x}} + \boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{u}}\mathbf{B}^T$$

It is easy to verify that

$$\boldsymbol{\sigma}^2_{\mathbf{x}\mathbf{u}} = {\boldsymbol{\sigma}^2_{\mathbf{u}\mathbf{x}}}^T$$

Lyapunov equation

$$\Delta\mathbf{x}(t) = \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{B}\Delta\mathbf{u}(t-\tau)\,\mathrm{d}\tau$$

$$\rightarrow \boldsymbol{\sigma}^2_{\mathbf{xu}} = \fint \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{B}\Delta\mathbf{u}(t-\tau)\,\mathrm{d}\tau\, \Delta\mathbf{u}(t)^T\,\mathrm{d}t$$

$$\boldsymbol{\sigma}^2_{\mathbf{xu}} = \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{B} \fint \Delta\mathbf{u}(t-\tau)\, \Delta\mathbf{u}(t)^T\,\mathrm{d}t\,\mathrm{d}\tau$$

$$\boldsymbol{\sigma}^2_{\mathbf{xu}} = \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\mathbf{B}\mathbf{k}_{\mathbf{uu}}(\tau)\,\mathrm{d}\tau$$

If $\mathbf{u}$ is a white noise so that $\mathbf{k}_{\mathbf{uu}} = \mathbf{W}\delta(\tau)$ then

$$\boldsymbol{\sigma}^2_{\mathbf{xu}} = \int_{-\infty}^{+\infty} \mathbf{h}(\tau)\delta(\tau)\,\mathrm{d}\tau\,\mathbf{B}\mathbf{W} = \frac{1}{2}\mathbf{B}\mathbf{W}$$

$$\boldsymbol{\sigma}^2_{\mathbf{ux}} = {\boldsymbol{\sigma}^2_{\mathbf{xu}}}^T = \frac{1}{2}\mathbf{W}\mathbf{B}^T$$

Lyapunov equation

$$\mathbf{A}\boldsymbol{\sigma}^2_{\mathbf{xx}} + \boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{A}^T + \mathbf{B}\mathbf{W}\mathbf{B}^T = \mathbf{0}$$

The Lyapunov equation is a matrix equation that allows to compute the cross-variance of the states of a vectorial state-space system subject to a white noise input

From this cross-variance matrix, it is possible to compute the variance of the output

$$\Delta\mathbf{y} = \mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}$$

$$\rightarrow \fint \Delta\mathbf{y}\Delta\mathbf{y}^T \mathrm{d}t = \fint \left(\mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}\right)\left(\mathbf{C}\Delta\mathbf{x} + \mathbf{D}\Delta\mathbf{u}\right)^T \mathrm{d}t$$

$$\boldsymbol{\sigma}^2_{\mathbf{yy}} = \mathbf{C}\boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{C}^T + \mathbf{C}\boldsymbol{\sigma}^2_{\mathbf{xu}}\mathbf{D}^T + \mathbf{D}\boldsymbol{\sigma}^2_{\mathbf{ux}}\mathbf{C}^T + \mathbf{D}\boldsymbol{\sigma}^2_{\mathbf{uu}}\mathbf{D}^T$$

that in case of an input that is a white noise of intensity $\mathbf{W}$ becomes

$$\boldsymbol{\sigma}^2_{\mathbf{yy}} = \mathbf{C}\boldsymbol{\sigma}^2_{\mathbf{xx}}\mathbf{C}^T + \frac{1}{2}\mathbf{C}\mathbf{B}\mathbf{W}\mathbf{D}^T + \frac{1}{2}\mathbf{D}\mathbf{W}\mathbf{B}^T\mathbf{C}^T + \mathbf{D}\mathbf{W}\mathbf{D}^T$$

Shape filters

When the input is not a white noise, we can say that is the result of a shape filter system subject to a white noise as input.

Given an input vetors u with an assigned PSD Φ_{uu}, find the trasfer matrix of a state space-filter H_f that returns the same PSD when subject to a white noise input w

$$\Phi_{uu} = |H_f(\omega)|^2 w$$

Using the state space format

$$\begin{cases} \dot{\mathbf{x}}_f & = \mathbf{A}_f \mathbf{x}_f + \mathbf{B}_f n \\ u & = \mathbf{C}_f \mathbf{x}_f \end{cases}$$

The extension to a vectorial input is straightforward

$$H_f(\omega) = \mathbf{C}_f(\mathrm{j}\omega\mathbf{I} - \mathbf{A}_f)^{-1}\mathbf{B}_f$$

$$\min_{\mathbf{A}_f, \mathbf{B}_f, \mathbf{C}_f} \left(\Phi_{uu} - \left(\mathbf{C}_f(-\mathrm{j}\omega\mathbf{I} - \mathbf{A}_f)^{-1}\mathbf{B}_f\right)^T \left(\mathbf{C}_f(\mathrm{j}\omega\mathbf{I} - \mathbf{A}_f)^{-1}\mathbf{B}_f\right)\right)$$

Shape filters

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases} \qquad \begin{cases} \dot{\mathbf{x}}_f &= \mathbf{A}_f\mathbf{x}_f + \mathbf{B}_f\mathbf{n} \\ \mathbf{u} &= \mathbf{C}_f\mathbf{x}_f \end{cases}$$

$$\begin{cases} \overbrace{\begin{Bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{x}}_f \end{Bmatrix}}^{\dot{\mathbf{x}}_a} &= \overbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B}\mathbf{C}_f \\ \mathbf{0} & \mathbf{A}_f \end{bmatrix}}^{\mathbf{A}_a} \overbrace{\begin{Bmatrix} \mathbf{x} \\ \mathbf{x}_f \end{Bmatrix}}^{\mathbf{x}_a} + \overbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{B} \end{bmatrix}}^{\mathbf{B}_a} \mathbf{n} \\ \mathbf{y} &= \underbrace{\begin{bmatrix} \mathbf{C} & \mathbf{D}\mathbf{C}_f \end{bmatrix}}_{\mathbf{C}_a} \underbrace{\begin{Bmatrix} \mathbf{x} \\ \mathbf{x}_f \end{Bmatrix}}_{\mathbf{x}_a} \end{cases}$$

Usage of information for design purposes: Risk assessment

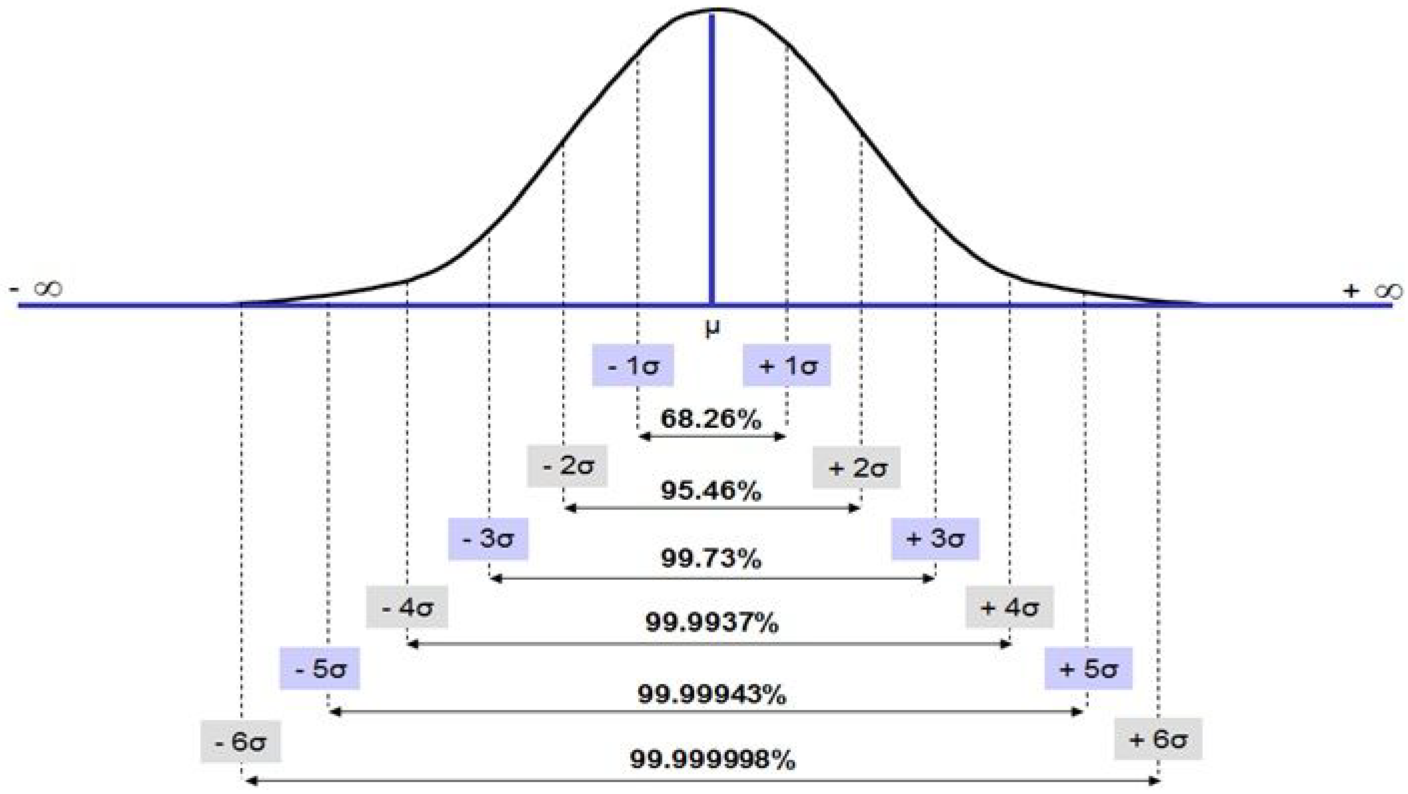
Probability is likelihood of an event

Severity is the potential effect of a hazard

Risk is the product of the two, i.e.
severity x likelihood

Likelihood \ Severity	Negligible 1	Minor 2	Major 3	Hazardous 4	Catastrophic 5
Frequent $10^{-3}<\varphi<10^{-5}$ x fl. hour		Medium Risk Revision Required			
Reasonably probable $10^{-3}<\varphi<10^{-5}$ x fl. hour				High Risk Unacceptable	
Remote $10^{-5}<\varphi<10^{-7}$ x fl. hour					
Extremely remote $10^{-7}<\varphi<10^{-9}$ x fl. hour		Low Risk Acceptable			
Extremely improbable $\varphi<10^{-9}$ x fl. hour					

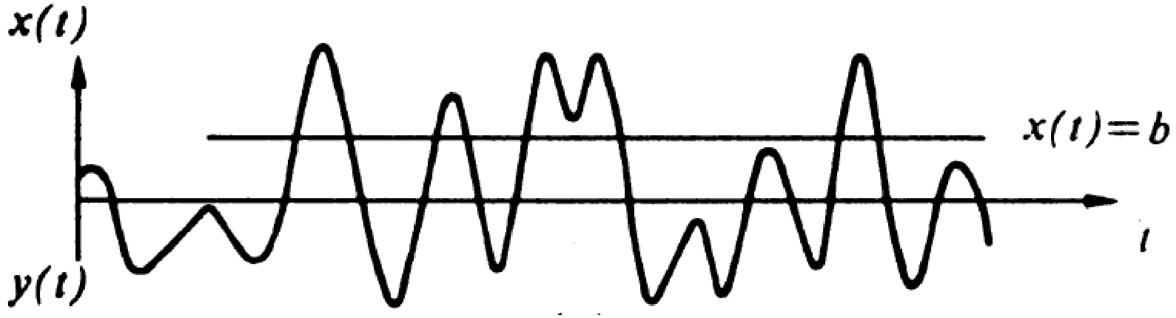
Usage of information for design purposes: Risk assessment



Rice formula

When a structure is subject to random variation for a large number of cycles failure may come from fatigue damages.

In order to compute the damage, we have to count the number of times the quantity of interest crosses a certain threshold

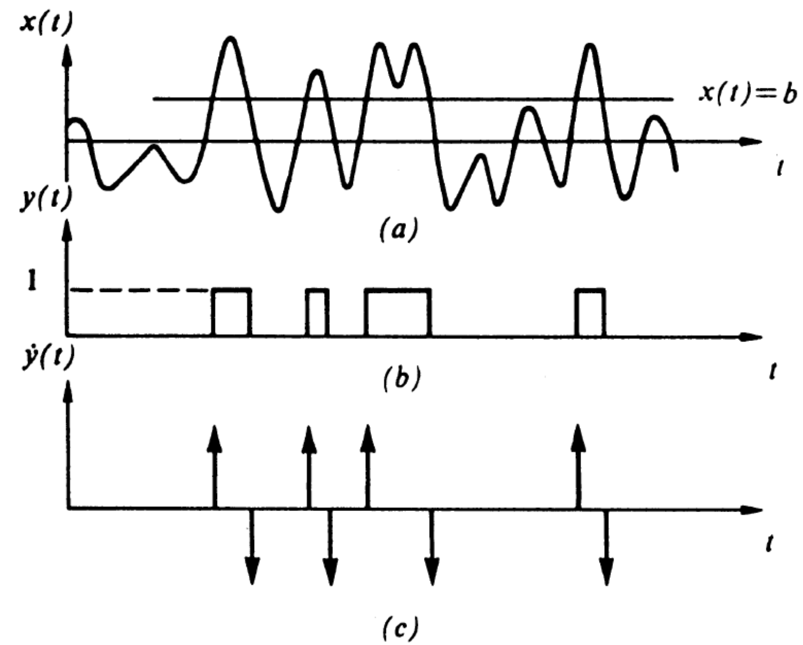


Rice Formula

We have to define a counting statistical process *N(b)*.

To di so it is possible to use the Step (Heaviside) function *H*

$$y(t) = H(x(t) - b) = \begin{cases} 1 & x \geq b \\ 0 & x < b \end{cases}$$



The derivative of the H function will be different from zero every time H is discontinuous

$$\dot{y}(t) = \dot{H}(x(t) - b) = \delta(x(t) - b)\dot{x}$$

Rice Formula

The rate of threshold crossing (independently from the direction of crossing)

$$|\dot{x}|\delta(x(t) - b)$$

$$\text{that is function of } x, \dot{x}$$

The number of threshold crossing in equal to the integral of

$$N(b, t_1, t_2) = \int_{t_1}^{t_2} |\dot{x}|\delta(x(t) - b)\mathrm{d}t$$

Rice Formula

We can compute the expected value for the number of threshold crossing per unit time

$$\bar{N} = E[N(b,t)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\dot{x}|\delta(x(t)-b)p(x,\dot{x},t)\mathrm{d}x\mathrm{d}\dot{x}$$

However the signal is ergodic and $\sigma^2_{\dot{x}x} = 0$

$$p(x,\dot{x}) = p(x)p(\dot{x}) = \frac{1}{2\pi\sigma_{xx}\sigma_{\dot{x}\dot{x}}}\mathrm{e}^{-\frac{1}{2}\frac{x^2}{\sigma^2_{xx}}}\mathrm{e}^{-\frac{1}{2}\frac{\dot{x}^2}{\sigma^2_{\dot{x}\dot{x}}}}$$

$$\bar{N}(b) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\dot{x}|\delta(x-b)p(x)p(\dot{x})\mathrm{d}x\mathrm{d}\dot{x} = \int_{-\infty}^{+\infty} |\dot{x}|p(b)p(\dot{x})\mathrm{d}\dot{x}$$

Considering also that we are interested only in the positive $\dot{x}$, upward crossings

$$\bar{N}_+(b) = \frac{1}{2\pi\sigma_{xx}\sigma_{\dot{x}\dot{x}}}\mathrm{e}^{-\frac{1}{2}\frac{b^2}{\sigma^2_{xx}}}\int_0^{+\infty} |\dot{x}|\mathrm{e}^{-\frac{1}{2}\frac{\dot{x}^2}{\sigma^2_{\dot{x}\dot{x}}}}\mathrm{d}\dot{x}$$

Rice formula

Applying the following coordinate transformation

$$\dot{x} = \sqrt{2}\sigma_{\dot{x}\dot{x}} s$$
$$\rightarrow \mathrm{d}\dot{x} = \sqrt{2}\sigma_{\dot{x}\dot{x}} \mathrm{d}s$$

$$\int_0^{+\infty} \dot{x} \mathrm{e}^{-\frac{1}{2}\frac{\dot{x}^2}{\sigma_{\dot{x}\dot{x}}^2}} \mathrm{d}\dot{x} = \int_0^{\infty} 2\sigma_{\dot{x}\dot{x}}^2 s \mathrm{e}^{-s^2} \mathrm{d}s = \sigma_{\dot{x}\dot{x}}^2 [-\mathrm{e}^{-s^2}]_0^{\infty} = \sigma_{\dot{x}\dot{x}}^2$$

$$\bar{N}_+(b) = \frac{1}{2\pi} \frac{\sigma_{\dot{x}\dot{x}}}{\sigma_{xx}} \mathrm{e}^{-\frac{1}{2}\frac{b^2}{\sigma_{xx}^2}}$$

If we compute the variance of the output signal x and the variance of its derivative it is possible to compute the expected number of crossing per unit time using this expression that is called Rice Formula

Computation of the probability of crossing a threshold level b in a finite time interval T

Probability of crossing the level b in an infinitesimal time

$$\bar{N}_+(b)\mathrm{d}t$$

So, the probability of NOT crossing is

$$\mathcal{P}(x < b, \mathrm{d}t) = 1 - \bar{N}_+(b)\mathrm{d}t$$

Considering dt = lim n->∞ T/n where T is the total interval of time considered

$$\mathcal{P}(x < b, \mathrm{d}t) = \lim_{n\to\infty}\left(1 - \bar{N}_+(b)\frac{T}{n}\right)$$

The total probability for the entire time interval is the product of the probability computed for each infinitesimal interval (considering each instant an independent event)

$$\mathcal{P}(x < b, t \in [0,T]) = \lim_{n\to\infty}\left(1 - \bar{N}_+(b)\frac{T}{n}\right)^n$$

$$\mathcal{P}(x < b, t \in [0,T]) = \mathrm{e}^{-\bar{N}_+(b)T}$$

The total probability to cross the level b in a time interval T is

$$\mathcal{P}(x > b, t \in [0,T]) = 1 - \mathrm{e}^{-\bar{N}_+(b)T}$$