POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

12 Unsteady Aerodynamics: Introduction to basic models

Giuseppe Quaranta

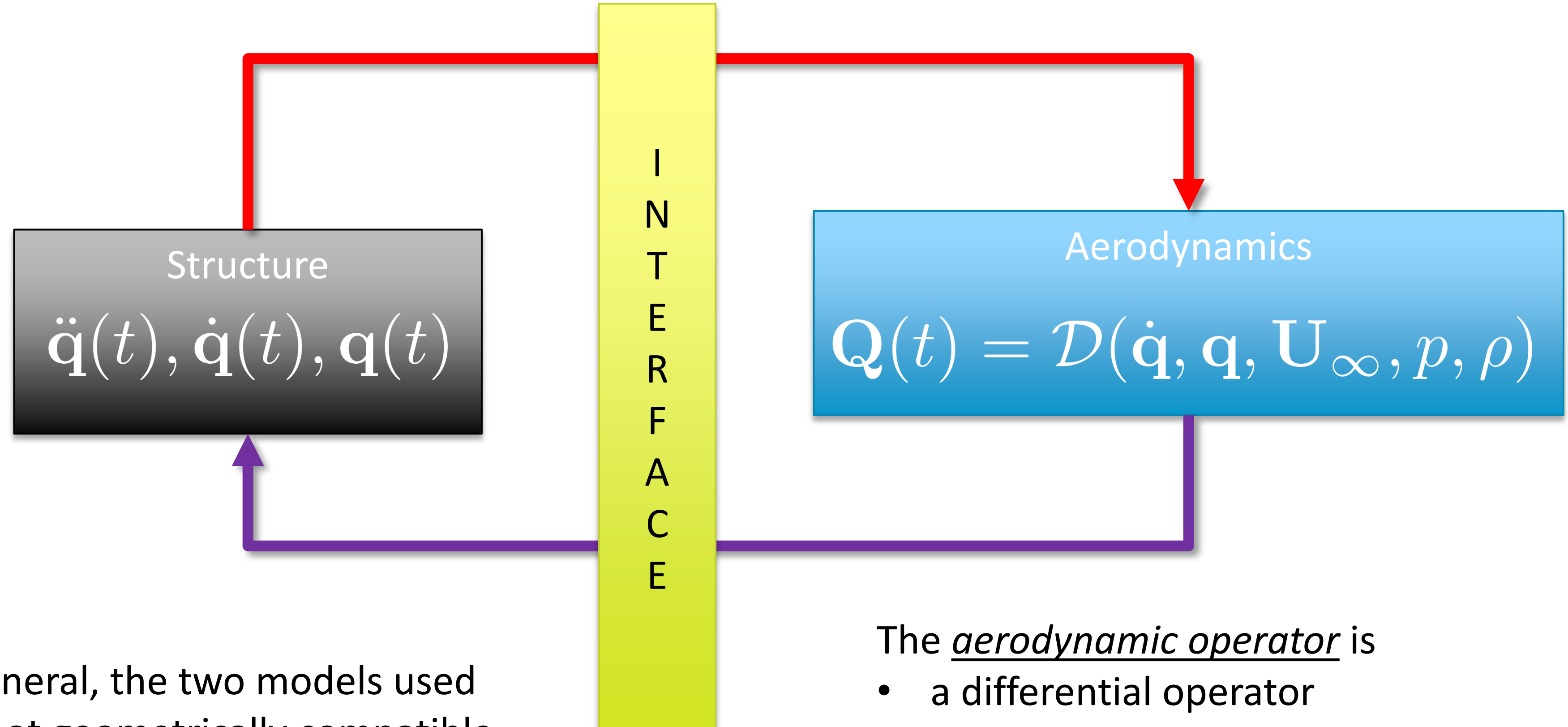
Dipartimento di Scienze e Tecnologie Aerospaziali

Material

Dowell Section 4.1

Objective

Build a tool able to perform aeroelastic predictions

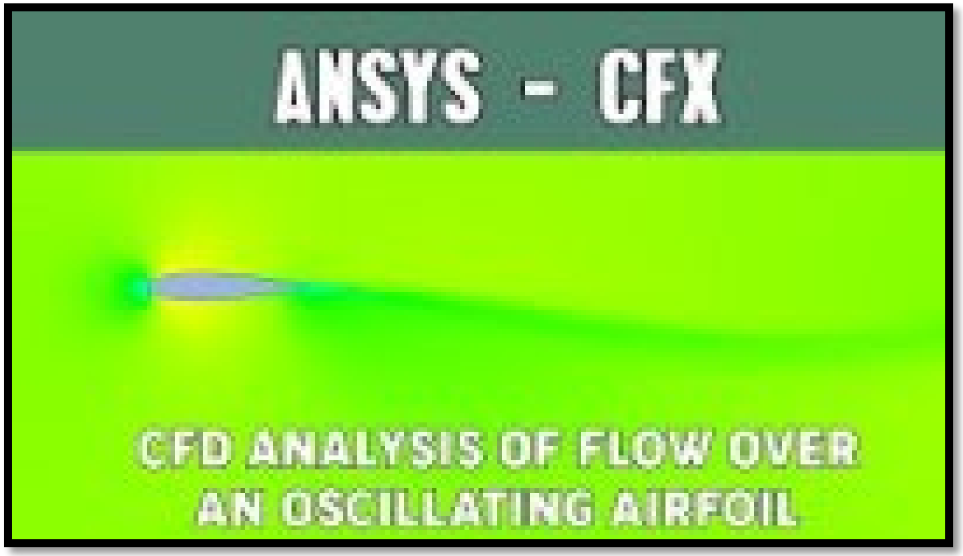


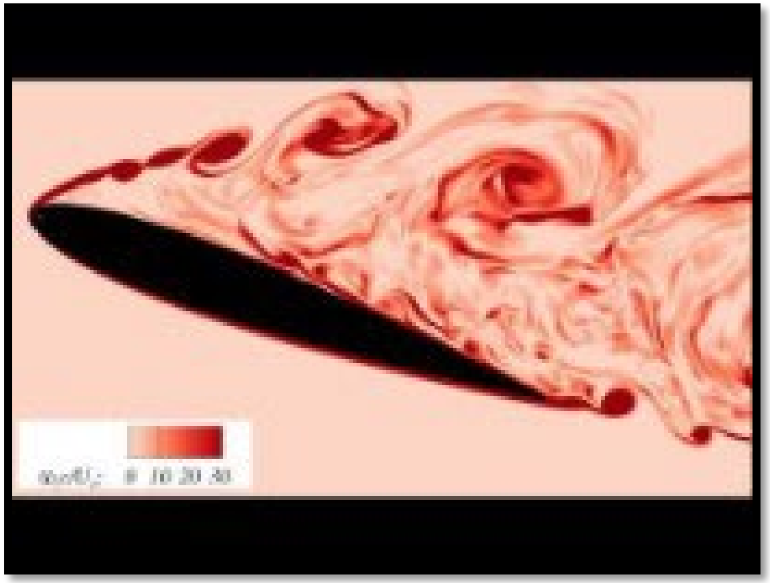
In general, the two models used are not geometrically compatible and so an ***interface*** operator is needed

The *aerodynamic operator* is

- a differential operator
- unsteady
- Effective, i.e. accurate yet computationally efficient

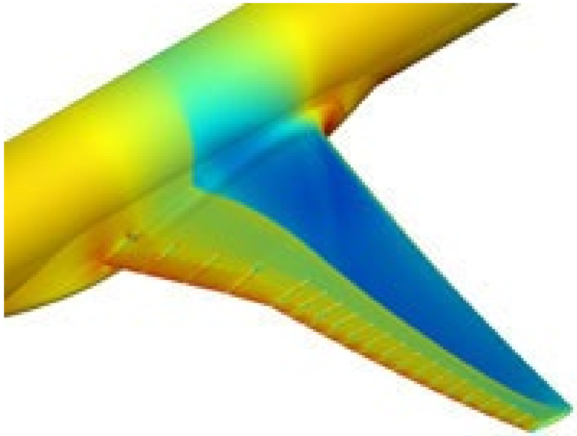
Unsteady aerodynamics phenomena

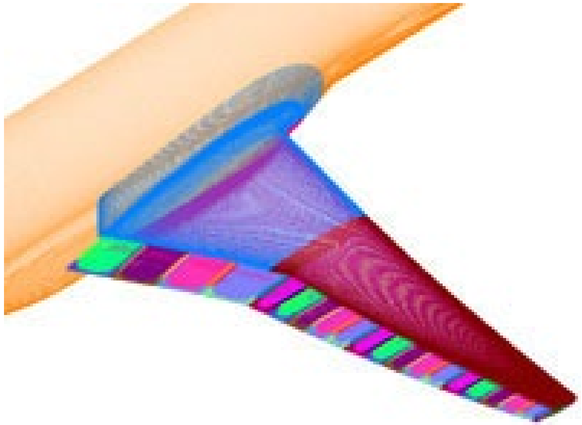




Using CFD (or WT tests) it is in principle possible to consider arbitrary complex geometry, any flow condition (M, Re,…), arbitrarily large structural motions…

CFD NASA High Fidelity Aeroelastics





65 Millions node mesh

2.5 h for each time iteration on 75 cores

Usable for verification but not for design

Downsides of using CFD

- Availability of a detailed geometry (not often available, especially at early-stage design)
- Simulation time very high: limited capability to perform parametric analysis (mass, geometry, flight conditions…)
- Often the type of analysis is limited to time marching integration. So, to go for frequency domain, eigenvalues for stability, random analysis ecc… requires a lot of post-processing
- The amount of information obtained is often very large: difficult to identify the important features that dominate the phenomena

Simplified physical models

- Very effective to understand the important features of the phenomenon under investigation (useful to identify design fixes)
- Possibility to perform massive parametric analysis (design, certification, flight test preparation, etc., …)
- Require to have a clear idea of *<u>what can be simplified</u>*
  - In many cases the model could be inviscid (excluded Buzz, buffet, stall flutter)
  - For transonic flows it is possible to use Euler instead of N-S
  - Subsonic or high supersonic flows linearized potential flow are good approximations
  - Low speed: incompressible flow models are very effective
  - Geometry is often simplified (lifting surface, lifting line, non-lifting bodies neglected) especially when the study of perturbations with respect to a trim state is sought

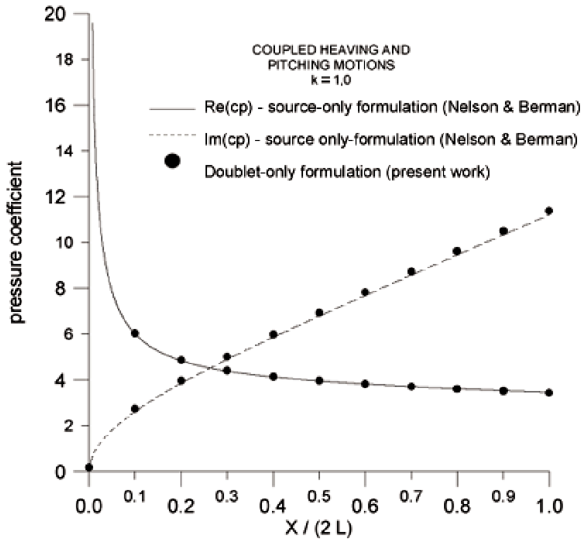
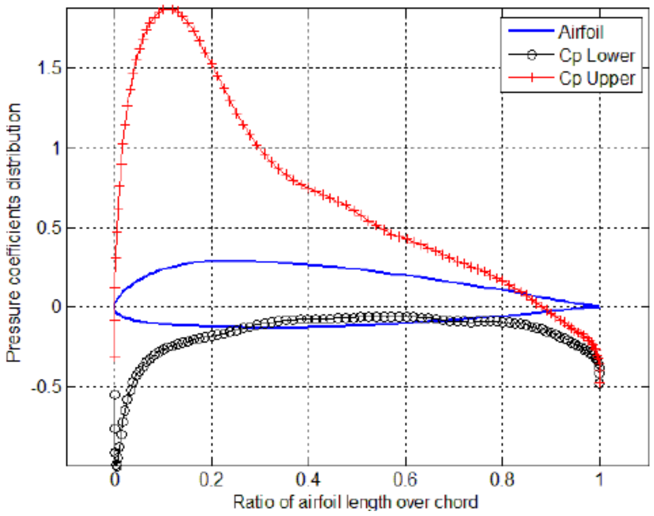


Figure 1. Pressure coeficient jump.

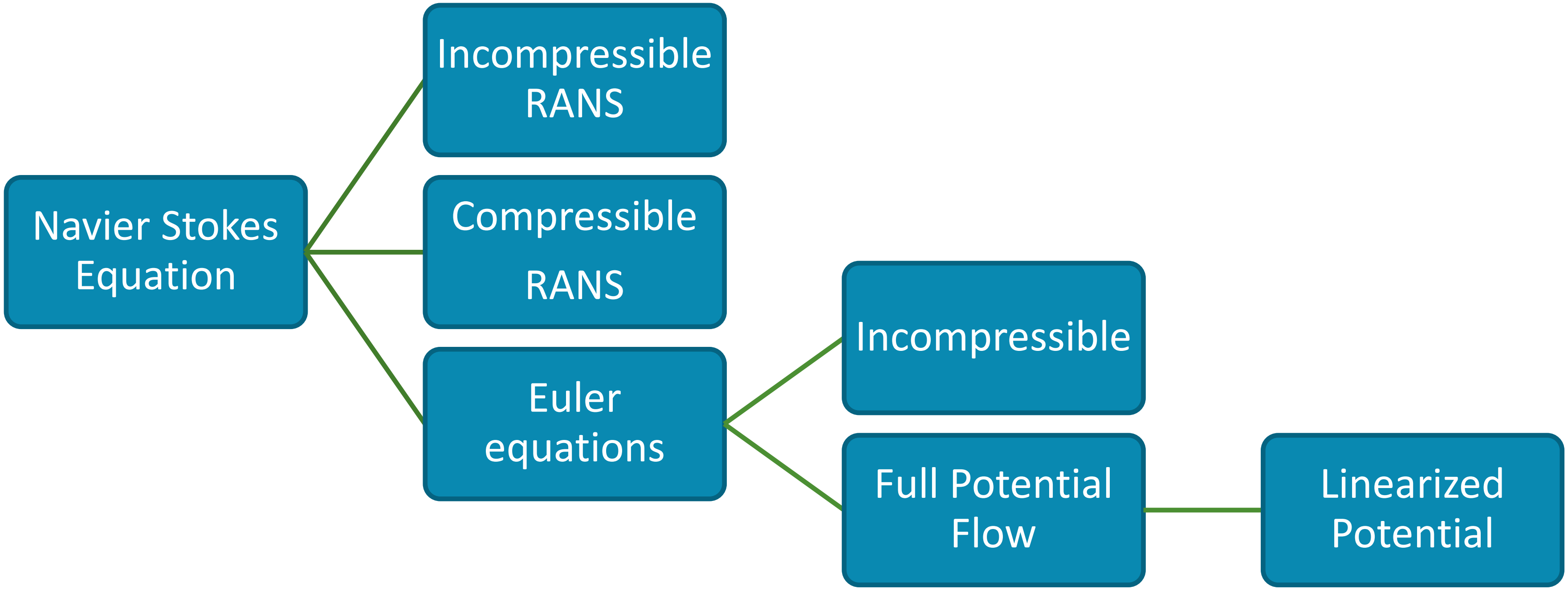


Cp distribution thick vs thin airfoils

In aeroelasticity we are interested in generalized loads (Cp integrated through a weighting function, i.e. modal form) that are not greatly influenced by local pressure distributions.

This process is equivalent to a **spatial filtering**

Basic Governing Equations: Hierarchy of models



Definitions

Consider a function f of time and space, that could be itself function of time because the reference frame is moving

$$f = f(t, \mathbf{x}(t))$$

$$\mathbf{u} = \begin{Bmatrix} \frac{\partial x_1}{\partial t} \\ \frac{\partial x_2}{\partial t} \\ \frac{\partial x_3}{\partial t} \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ w \end{Bmatrix}$$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial x_i}\frac{\partial x_i}{\partial t} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f$$

This is sometimes denominated total derivative and indicated with the symbol D/Dt

Reynolds' Transport theorem

Computation of the derivative of an integral quantity over a moving volume

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} f \,\mathrm{d}v = \underbrace{\int_{V(t)} \frac{\partial f}{\partial t}\,\mathrm{d}v}_{\text{Variation of } f \text{ in The volume } V(t)} + \underbrace{\int_{\partial V(t)} \mathbf{v}\cdot\mathbf{n}\, f \,\mathrm{d}s}_{\text{Flux of } f \text{ through the boundaries of } V(t)}$$

$$\begin{aligned} \mathbf{v} &= \mathbf{u} \quad \text{if V is fixed} \\ \mathbf{v} &= \mathbf{0} \quad \text{if V moves at the same speed of the material} \end{aligned}$$

Reynolds' Transport theorem

Computation of the derivative of an integral quantity over a fixed volume

Using the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f\,\mathrm{d}v = \int_V \left(\frac{\partial f}{\partial t} + \nabla\cdot(\mathbf{u}f)\right)\mathrm{d}v$$

Conservative form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V f\,\mathrm{d}v = \int_V \left(\frac{\mathrm{d}f}{\mathrm{d}t} + f\nabla\cdot\mathbf{u}\right)\mathrm{d}v$$

$$\nabla\cdot f\mathbf{u} = f\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla f$$

$\nabla\cdot\mathbf{u}$

is called also DILATATION. It measures the isotropic expansion/compression

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}v) = \nabla\cdot\mathbf{u}\,\mathrm{d}v$$

Mass conservation (continuity equation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \, \mathrm{d}v = 0 \quad \rightarrow \quad \int_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \mathrm{d}v = 0$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \mathbf{u} = 0$$

Balance of momentum

Newton's II law of dynamics

Rate of change of momentum = total body forces ($\mathbf{f}_b$) + sum of surface forces per unit area ($\mathbf{t}$) on the boundaries

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathbf{u}\,\mathrm{d}v = \cancel{\int_V \rho \mathbf{f_b}\,\mathrm{d}v} + \int_{\partial V} \mathbf{t}\,\mathrm{d}s$$

$$\mathbf{t} = \mathbf{n}\cdot\sigma(\mathbf{x})$$

Using the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \mathbf{u}\,\mathrm{d}v = \int_V \nabla\cdot\sigma\,\mathrm{d}s$$

Applying Reynolds' theorem and continuity equation

$$\int_V \left(\rho\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} - \nabla\cdot\sigma\right)\mathrm{d}v = 0$$

Constitutive law

Cauchy-Poisson constitutive equation for *Newtonian fluid*, **T** is related only to **thermodynamic pressure** p, to stain rate **D**, and to dilatation

$$\sigma = -p\,\mathbf{I} + \mathbf{V} = -p\,\mathbf{I} + \lambda \nabla \cdot \mathbf{u} + 2\mu \mathbf{D}$$

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

We make the hypothesis that the fluid is inviscid and so only the thermodynamic pressure remains

$$\sigma = -p\,\mathbf{I}, \qquad \mathbf{t} = -p\mathbf{n}$$

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{1}{\rho}\nabla p = 0$$

Energy Balance

I Principle of thermodynamics

Variation of the total energy = work done by external forces + heat exchange

Internal energy, kinetic energy $E = e + \frac{u^2}{2}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V E\,\mathrm{d}v = \cancel{\int_V (\mathbf{f}\cdot\mathbf{u} + Q_R)\,\mathrm{d}v} + \int_{\partial V}\frac{1}{\rho}(-p\mathbf{n}\cdot\mathbf{u} + \cancel{\mathbf{n}\cdot\mathbf{q}})\,\mathrm{d}s$$

Heat generation per unit mass, heat fluxes across the boundaries

$$\frac{\mathrm{d}E}{\mathrm{d}t} = -\frac{1}{\rho}\nabla\cdot(p\mathbf{u})$$

Euler equations

$$
\begin{cases}
\dfrac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \nabla \cdot \mathbf{u} = 0 \\
\dfrac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{1}{\rho} \nabla p = 0 \\
\dfrac{\mathrm{d}E}{\mathrm{d}t} + \frac{1}{\rho} \nabla \cdot (p\mathbf{u}) = 0
\end{cases}
$$

It is necessary to add the equation of state of the ideal gas for internal energy

$$p = \rho R T$$

$$e = C_v T$$

Sometimes it is more covenient to consider the ***hentalpy*** instead of the internal energy

$$h = C_p T = e + \frac{p}{\rho}$$

Rate of changer of kinetic energy

$$\mathbf{u} \cdot \left(\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} + \frac{1}{\rho} \nabla p \right) = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{u^2}{2} \right) = -\frac{1}{\rho} \mathbf{u} \cdot \nabla p$$

For inviscid flows the rate of change of kinetic energy is equal to the work done by pressure forces (there is no dissipation through viscosity)

Entropy (II Principle of Thermodynamics)

Using the fundamental differential equation of thermodynamics for entropy

$$\mathrm{d}e = T\mathrm{d}s - p\mathrm{d}\left(\frac{1}{\rho}\right) \qquad \frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{u^2}{2}\right) + \frac{1}{\rho}\nabla\cdot(p\mathbf{u}) = 0$$

$$T\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{p}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} - \frac{1}{\rho}\mathbf{u}\nabla p + \frac{1}{\rho}\nabla\cdot(p\mathbf{u}) = 0$$

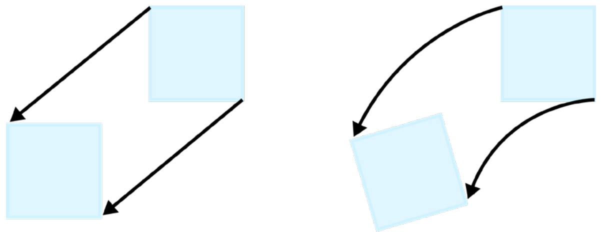
$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0$$

In an inviscid fluid flow the total time derivative of entropy in the flux is null, i.e. the derivative of entropy for each particle along its path is null

If there is no gradient of entropy at the initial instant in time the flow remains isentropic

Acceleration and vorticity

VORTICITY $\omega = \nabla \times \mathbf{u}$



$$\mathbf{a} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}$$

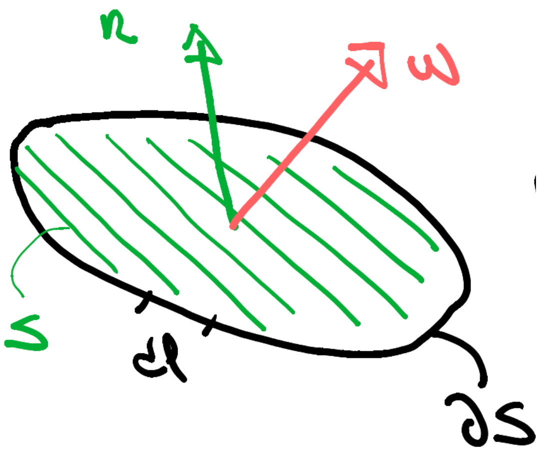
$$\mathbf{u} \cdot \nabla \mathbf{u} = (\nabla \times \mathbf{u}) \times \mathbf{u} + \nabla \mathbf{u} \cdot \mathbf{u} = \omega \times \mathbf{u} + \nabla \mathbf{u} \cdot \mathbf{u}$$

$$\mathbf{a} = \frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + \nabla \left(\frac{u^2}{2} \right)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \omega \times \mathbf{u} + \nabla \left(\frac{u^2}{2} \right) = -\frac{1}{\rho} \nabla p$$

Circulation

The circulation is the integral of the vorticity normal to a surface over the surface



CIRCULATION

$$\Gamma = \int_s \boldsymbol{\omega} \cdot \mathbf{n} \, \mathrm{d}s$$

$$\Gamma = \int_s \boldsymbol{\nabla} \times \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}s = \oint_{\partial s} \mathbf{u} \cdot \mathrm{d}\boldsymbol{\ell}$$

Circulation

$$
\begin{aligned}
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_S \boldsymbol{\nabla}\times\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}s = \int_S \boldsymbol{\nabla}\times\mathbf{a}\cdot\mathbf{n}\,\mathrm{d}s \\
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= -\int_S \boldsymbol{\nabla}\times\left(\frac{\boldsymbol{\nabla}p}{\rho}\right)\cdot\mathbf{n}\,\mathrm{d}s = -\oint_{\partial S}\frac{\boldsymbol{\nabla}p}{\rho}\cdot\mathrm{d}\boldsymbol{\ell} \\
\frac{\mathrm{d}\Gamma}{\mathrm{d}t} &= -\oint_{\partial S}\frac{\mathrm{d}p}{\rho}
\end{aligned}
$$

Kelvin's theorem

Consider a **BAROTROPIC FLUID**, i.e., a fluid for which p = p(ρ)

$$\oint_{\partial S} \frac{\mathrm{d}p}{\rho} = 0 \Rightarrow \frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \qquad \frac{\mathrm{d}\omega}{\mathrm{d}t} = 0$$

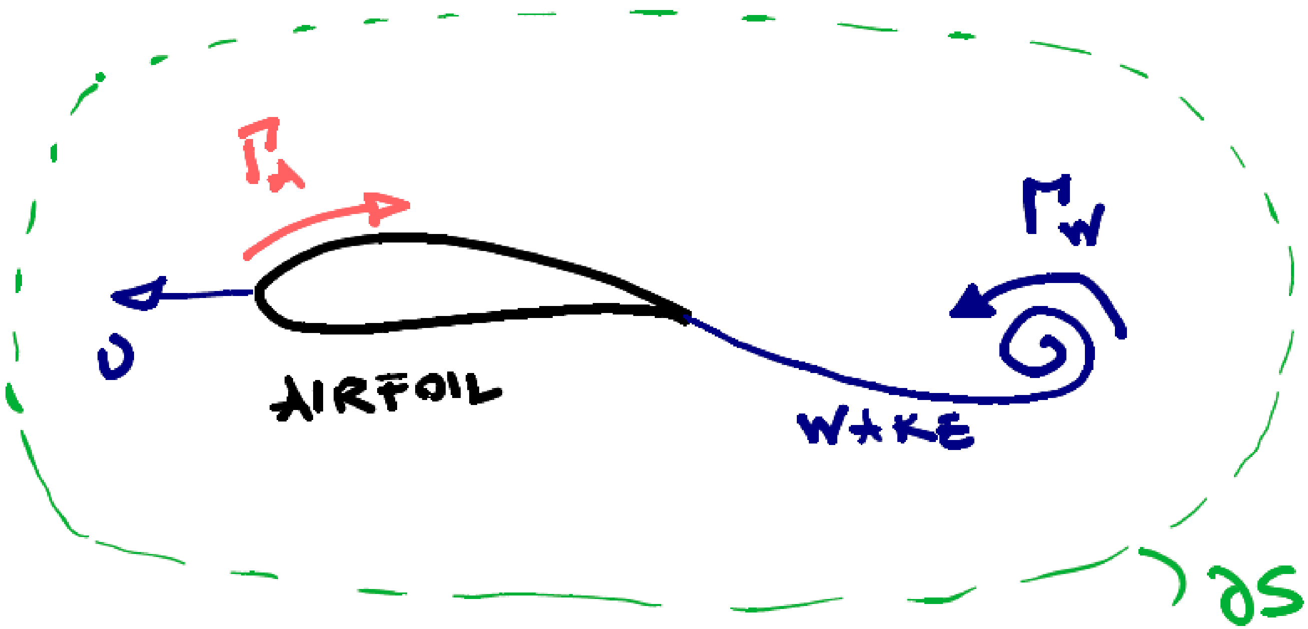
An isoentropic flow is barotropic

$$\frac{p}{\rho^\gamma} = \frac{p_\infty}{\rho_\infty^\gamma} = \text{const., with } \gamma = \frac{C_p}{C_v}$$

If the vorticity is null at the initial instant in time (the circulation is null), the flow will remain with null vorticity, i.e., **IRROTATIONAL** for all times.

Only intense strong shock waves or intense heating can spoil the hypothesis of isoentropic (and so barotropic) flow.

Sudden start of an airfoil in 2D



$$\frac{\mathrm{d}\Gamma}{\mathrm{d}t} = 0 \Rightarrow \frac{\Gamma_A + \Gamma_w}{\Delta t} = 0$$

For $t \to \infty$ Γ_w moves to infinity and the wake dissappears.

Velocity potential

If the flow is irrotational it is possible to define a VELOCITY POTENTIAL

$$\boldsymbol{\omega} = \mathbf{0} \Rightarrow \boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$$

If we define $\mathbf{u} = \boldsymbol{\nabla}\Phi$, with Φ the <u>VELOCITY POTENTIAL</u>, the velocity will be irrotational by definition, because $\boldsymbol{\nabla}\times\boldsymbol{\nabla}(\cdot) = 0$

Unsteady Bernoulli equation

For irrotational flows

$$\frac{\partial \nabla \Phi}{\partial t} + \frac{1}{2}\nabla u^2 - \frac{\nabla p}{\rho} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \left(\frac{u^2}{2} \right) = -\frac{1}{\rho}\nabla p$$

For an inviscid flow

$$T\mathrm{d}s = \mathrm{d}h - \frac{\mathrm{d}p}{\rho} = 0 \rightarrow \mathrm{d}h = \frac{\mathrm{d}p}{\rho} \rightarrow h = \int \frac{\mathrm{d}p}{\rho}$$

$$\nabla \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + h \right) = 0$$

This quantity is constant in space for an inviscid, irrotational flow

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + h = F(t)$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2 + \int \frac{\mathrm{d}p}{\rho} = F(t)$$

Unsteady Bernoulli equation: Compressible adiabatic flow

For isoentropic flow

$$\mathrm{d}\left(\frac{p}{\rho^\gamma}\right) = 0 \rightarrow \frac{\mathrm{d}p}{\rho^\gamma} - \gamma \frac{p}{\rho^{\gamma+1}} \mathrm{d}\rho = 0$$

Speed of sound

$$a^2 = \gamma R T = \frac{\gamma p}{\rho}$$

$$\frac{\mathrm{d}p}{\rho} = \gamma p \frac{\rho^{\gamma-1}}{\rho^{\gamma+1}} \mathrm{d}\rho = \gamma \rho^{\gamma-2} \frac{p}{\rho^\gamma} \mathrm{d}\rho = \gamma \rho^{\gamma-2} \frac{p_\infty}{\rho_\infty^\gamma} \mathrm{d}\rho$$

$$\int \frac{\mathrm{d}p}{\rho} = \frac{\gamma}{\gamma-1} \frac{p_\infty}{\rho_\infty^\gamma} \left(\rho^{\gamma-1} - \rho_\infty^{\gamma-1}\right)$$

$$\int \frac{\mathrm{d}p}{\rho} = \frac{\gamma}{\gamma-1} \left(\frac{p_\infty}{\rho_\infty^\gamma} \rho^{\gamma-1} - \frac{p_\infty}{\rho_\infty}\right)$$

$$\int \frac{\mathrm{d}p}{\rho} = \frac{\gamma}{\gamma-1} \left(\frac{p}{\rho} - \frac{p_\infty}{\rho_\infty}\right)$$

$$\int \frac{\mathrm{d}p}{\rho} = \frac{a^2}{\gamma-1} - \frac{a_\infty^2}{\gamma-1}$$

$$\boxed{\frac{\partial \Phi}{\partial t} + \frac{u^2}{2} + \frac{a^2}{\gamma-1} = \frac{U_\infty^2}{2} + \frac{a_\infty^2}{\gamma-1}}$$

Pressure coefficient: compressible adiabatic flow

$$
\begin{aligned}
C_p &= \frac{p - p_\infty}{\frac{1}{2}\rho U_\infty^2} \\
C_p &= \frac{1}{\frac{1}{2}\gamma \frac{U_\infty^2}{a_\infty^2}} \left(\frac{p}{p_\infty} - 1 \right) = \frac{1}{\frac{1}{2}\gamma M_\infty^2} \left(\frac{p}{p_\infty} - 1 \right) \\
a^2 &= -(\gamma - 1) \left(\frac{\partial \Phi}{\partial t} + \frac{u^2 - U_\infty^2}{2} \right) + a_\infty^2 \\
\frac{a^2}{a_\infty^2} &= 1 - \frac{\gamma - 1}{a_\infty^2} \left(\frac{\partial \Phi}{\partial t} + \frac{u^2 - U_\infty^2}{2} \right)
\end{aligned}
$$

However

$$
\frac{a^2}{a_\infty^2} = \frac{p}{\rho}\frac{\rho_\infty}{p_\infty} = \frac{p}{p_\infty}\left(\frac{\rho_\infty}{\rho}\right)^{\frac{1}{\gamma}} = \left(\frac{p}{p_\infty}\right)^{\frac{\gamma-1}{\gamma}}
$$

$$
\frac{p}{p_\infty} = \left(\frac{a^2}{a_\infty^2}\right)^{\frac{\gamma}{\gamma-1}} \rightarrow C_p = \frac{1}{\frac{1}{2}\gamma M_\infty^2} \left(\frac{a^2}{a_\infty^2}\right)^{\frac{\gamma}{\gamma-1}} - 1
$$

Pressure coefficient: compressible adiabatic flow and incompressible flow

$$\boxed{C_p = \frac{1}{\frac{1}{2}\gamma M_\infty^2}\left(\left(1 - \frac{\gamma - 1}{a_\infty^2}\left(\frac{\partial \Phi}{\partial t} + \frac{|\boldsymbol{\nabla}\Phi|^2 - U_\infty^2}{2}\right)\right)^{\frac{\gamma}{\gamma-1}} - 1\right)}$$

$$\rho_\infty = \mathrm{const.} \qquad \int \frac{\mathrm{d}p}{\rho} = \frac{1}{\rho_\infty}\int \mathrm{d}p = \frac{p - p_\infty}{\rho_\infty}$$

$$\boxed{C_p = \frac{1}{\frac{1}{2}U_\infty^2}\frac{p - p_\infty}{\rho_\infty} = -\left(\frac{2}{U_\infty^2}\frac{\partial \Phi}{\partial t} + \frac{|\boldsymbol{\nabla}\Phi|^2}{U_\infty^2} - 1\right)}$$

Full potential equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \boldsymbol{\nabla} \cdot \boldsymbol{\nabla}\Phi = 0$$

$$\rightarrow \frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho \boldsymbol{\nabla}^2\Phi = 0$$

$$a^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} \rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}p}{\mathrm{d}\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t} = a^2 \frac{\mathrm{d}\rho}{\mathrm{d}t}$$

$$\frac{\mathrm{d}p}{\rho} = \mathrm{d}h \rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = \rho \frac{\mathrm{d}h}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}p}{\mathrm{d}t} = \rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2\right)$$

$$a^2 \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Phi}{\partial t} + \frac{1}{2}u^2\right)$$

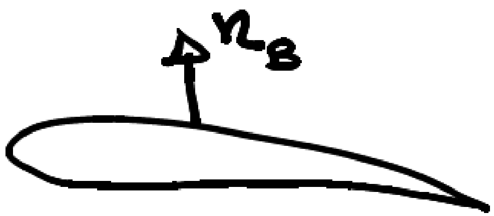
$$\begin{cases} \rho \boldsymbol{\nabla}^2\Phi = \dfrac{\rho}{a^2}\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial \Phi}{\partial t} + \dfrac{1}{2}u^2\right) \\ a^2(\Phi) = -(\gamma - 1)\left(\dfrac{\partial \Phi}{\partial t} + \dfrac{|\boldsymbol{\nabla}\Phi|^2 - U_\infty^2}{2}\right) + a_\infty^2 \end{cases}$$

Full potential equation

If we call $B(\mathbf{x}, t) = 0$ the implicit definition of the lifting surface. The normal to this surface is $\mathbf{n}_B$ so the BC is

$$\mathbf{u} \cdot \mathbf{n}_B = \mathbf{v}_B \cdot \mathbf{n}_B$$

$$\boldsymbol{\nabla}\Phi \cdot \mathbf{n}_B = \frac{\partial \Phi}{\partial \mathbf{n}_B} = \mathbf{v}_B \cdot \mathbf{n}_B$$



where $\mathbf{v}_B$ is the velocity of the body.

At the infinite boundary instead

$$\lim_{\mathbf{x} \to \infty} \mathbf{u}(\mathbf{x}) = \lim_{\mathbf{x} \to \infty} \boldsymbol{\nabla}\Phi(\mathbf{x}) = \mathbf{U}_\infty$$

$$\begin{cases} \boldsymbol{\nabla}^2 \Phi = \dfrac{1}{a^2} \dfrac{\mathrm{d}}{\mathrm{d}t} \left(\dfrac{\partial \Phi}{\partial t} + \dfrac{1}{2} u^2 \right) \\ a^2(\Phi) = -(\gamma - 1) \left(\dfrac{\partial \Phi}{\partial t} + \dfrac{|\boldsymbol{\nabla}\Phi|^2 - U_\infty^2}{2} \right) + a_\infty^2 \end{cases}$$

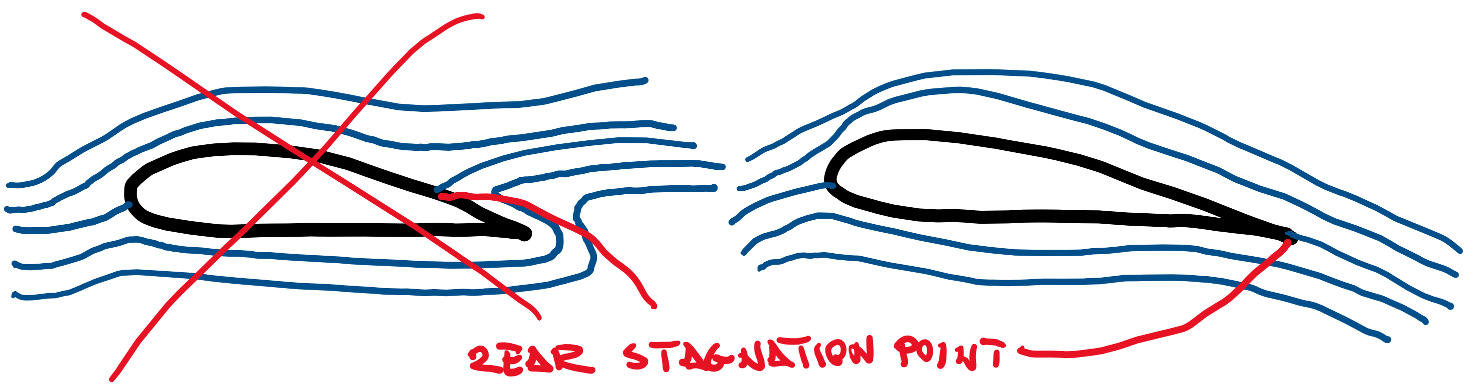
Solution of potential problems: Kutta condition

The potential problem with Neumann BC admits a unique solution only in simply connected regions

Any closed line around the airfoil (2D) cannot be contracted into a point so the region is multiconnected and the solution is not unique.

For 3D flow the region is simply connected but the wake is not a potential flow region so it cannot be crossed. So, in this case too the potential is not uniquely defined

A condition must be added to recover the effect of viscosity



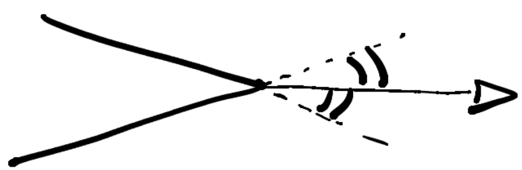
Solution of potential problems: Kutta condition

Glauert: the flow must leave from the trailing edge smoothly and velocity must be finite, i.e the TE must be a stagnation point

$$\Delta p_{TE} = 0$$

If we call γ the <u>circulation per unit length</u>, so that

$$\Gamma_A = \int_c \gamma(x)\mathrm{d}x \;\rightarrow \gamma_{TE} = 0$$



For steady flow it is possible to verify that the flow leaves the airfoil along the bisector of the TE angle

Small perturbations: Linearized potential

$$\begin{aligned} a &= a_\infty + \delta a \\ p &= p_\infty + \delta p \\ \rho &= \rho_\infty + \delta \rho \end{aligned}$$

$$\begin{aligned} \mathbf{u} &= U_\infty \mathbf{i} + \delta \mathbf{u} \\ \Phi &= U_\infty x + \varphi \\ \boldsymbol{\nabla}\Phi &= U_\infty \mathbf{i} + \boldsymbol{\nabla}\varphi \end{aligned}$$

$$\boldsymbol{\nabla}^2 \Phi = \boldsymbol{\nabla}^2 \varphi$$

$$\frac{\partial \Phi}{\partial t} = \frac{\partial \varphi}{\partial t}$$

$$\frac{|\boldsymbol{\nabla}\Phi|^2}{2} = \frac{\boldsymbol{\nabla}\Phi \cdot \boldsymbol{\nabla}\Phi}{2} \approx \frac{U_\infty^2}{2} + U_\infty \frac{\partial \varphi}{\partial x} + O(\varphi^2)$$

$$\boldsymbol{\nabla}^2 \Phi = \frac{\rho}{a^2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \Phi}{\partial t} + \frac{1}{2} u^2 \right) \to \boldsymbol{\nabla}^2 \varphi = \frac{1}{a_\infty^2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \varphi}{\partial t} + \frac{U_\infty^2}{2} + U_\infty \frac{\partial \varphi}{\partial x} \right)$$

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \approx \frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \quad \to \boldsymbol{\nabla}^2 \varphi = \frac{1}{a_\infty^2} \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathrm{d}\varphi}{\mathrm{d}t} + \frac{U_\infty^2}{2} \right) = \frac{1}{a_\infty^2} \frac{\mathrm{d}^2 \varphi}{\mathrm{d}t^2}$$

Linearized potential

$$\nabla^2\varphi = \frac{1}{a_\infty^2}\left(\frac{\partial^2\varphi}{\partial t^2} + 2U_\infty\frac{\partial\varphi}{\partial x\partial t} + U_\infty^2\frac{\partial^2\varphi}{\partial x^2}\right)$$

$$\nabla\Phi\cdot\mathbf{n}_B \approx U_\infty n_{B_x} + \frac{\partial\varphi}{\partial\mathbf{n}_B} \to U_\infty n_{B_x} + \frac{\partial\varphi}{\partial\mathbf{n}_B} = \mathbf{v}_B\cdot\mathbf{n}_B$$

$$C_p = \frac{1}{\frac{1}{2}\gamma M_\infty^2}\left(\left(1 - \frac{\gamma-1}{a_\infty^2}\left(\frac{\partial\Phi}{\partial t} + \frac{|\nabla\Phi|^2 - U_\infty^2}{2}\right)\right)^{\frac{\gamma}{\gamma-1}} - 1\right)$$

$$(1+x)^\alpha \approx 1 + \alpha x + O(x^2)$$

$$\to C_p = \frac{2}{\gamma M_\infty^2}\left(\left(1 - \frac{\gamma}{a_\infty^2}\left(\frac{\partial\varphi}{\partial t} + U_\infty\frac{\partial\varphi}{\partial x}\right)\right) - 1\right)$$

$$C_p = -\frac{2}{U_\infty^2}\left(\frac{\partial\varphi}{\partial t} + U_\infty\frac{\partial\varphi}{\partial x}\right) = -\frac{2}{U_\infty^2}\frac{\mathrm{d}\varphi}{\mathrm{d}t} \to \hat{p} = p - p_\infty = \frac{1}{2}\rho_\infty U_\infty^2 C_p = -\rho_\infty\frac{\mathrm{d}\varphi}{\mathrm{d}t}$$

Incompressible flow: Linear potential

$$\rho = \rho_\infty = \text{const.}$$
$$\rightarrow a^2 = a_\infty^2 = \frac{\mathrm{d}p}{\mathrm{d}\rho} \rightarrow \infty$$

$$\boldsymbol{\nabla}^2\varphi = \frac{1}{a_\infty^2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} \rightarrow \boldsymbol{\nabla}^2\varphi = 0$$

The equation is Linear and not time dependent. The only elements that may introduce time dependance are the boundary conditions.

Linearized boundary conditions

Compare the boundaries on the body at two instants in time t and $t + \mathrm{d}t$

$$B(\mathbf{x}, t) = 0$$
$$B(\mathbf{x} + \mathrm{d}\mathbf{x}, t + \mathrm{d}t) = 0$$

$$\Delta B = B(\mathbf{x} + \mathrm{d}\mathbf{x}, t + \mathrm{d}t) - B(\mathbf{x}, t) = 0$$
$$B(\mathbf{x} + \mathrm{d}\mathbf{x}, t + \mathrm{d}t) \approx B(\mathbf{x}, t) + \boldsymbol{\nabla} B \cdot \mathrm{d}\mathbf{x} + \frac{\partial B}{\partial t}\mathrm{d}t$$
$$\boldsymbol{\nabla} B \cdot \mathrm{d}\mathbf{x} + \frac{\partial B}{\partial t}\mathrm{d}t = 0$$
$$\boldsymbol{\nabla} B \cdot \mathbf{v}_B + \frac{\partial B}{\partial t} = 0$$

$$\mathbf{v}_B \cdot \mathbf{n}_B = \mathbf{u} \cdot \mathbf{n}_B \rightarrow \mathbf{v}_B \cdot \frac{\boldsymbol{\nabla} B}{|\boldsymbol{\nabla} B|} = \mathbf{u} \cdot \mathbf{n}_B \rightarrow -\frac{\partial B}{\partial t}\frac{1}{|\boldsymbol{\nabla} B|} = \mathbf{u} \cdot \mathbf{n}_B$$

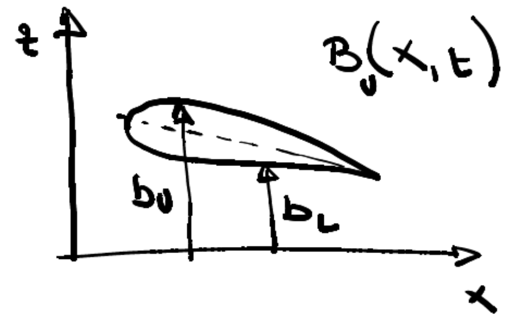
$$\rightarrow \frac{\partial B}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} B = 0, \text{on } B = 0$$

Linearized boundary conditions

Consider the case where every point of the wing is moving in the $x_3 = z$ direction in time

$$B(\mathbf{x},t) = \begin{cases} B_l(\mathbf{x},t) = x_3 - b_l(x_1,x_2,t) & \text{Lower side} \\ B_u(\mathbf{x},t) = x_3 - b_u(x_1,x_2,t) & \text{Upper side} \end{cases}$$

$$-\frac{\partial b_u}{\partial t} + \begin{bmatrix} U_\infty + \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \\ \frac{\partial \varphi}{\partial z} \end{bmatrix} \cdot \begin{bmatrix} -\frac{\partial b_u}{\partial x} \\ -\frac{\partial b_u}{\partial y} \\ 1 \end{bmatrix} = 0$$

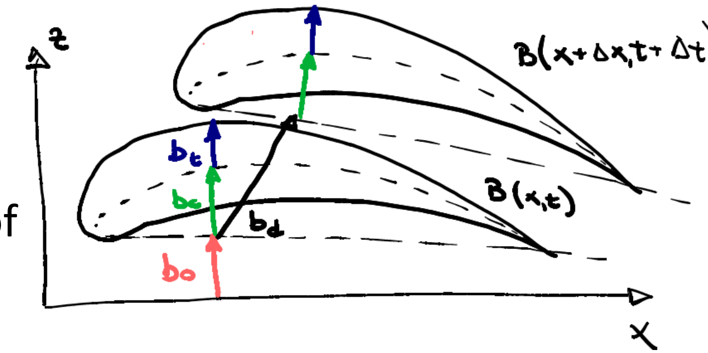


$$\frac{\partial b_u}{\partial t} + \left(U_\infty + \frac{\partial \varphi}{\partial x}\right)\frac{\partial b_u}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial b_u}{\partial y} = \frac{\partial \varphi}{\partial z}$$

This expression is linear in b_u. However, the perturbation velocity is small and so is the deformation of the boundary. Neglecting second order terms

$$\frac{\partial b_u}{\partial t} + U_\infty \frac{\partial b_u}{\partial x} = \frac{\partial \varphi}{\partial z}$$

Linearized boundary conditions



The potential problem is linear with linear boundary conditions, so superimposition of effects could be applied.
The position of the boundary could be seen as the sum of the position of the chord + the position of the camber line + the position of the thickness + the deformation

$$b_u(\mathbf{x},0) = b_{uo}(\mathbf{x}) + b_{uc}(\mathbf{x}) + b_{ut}(\mathbf{x})$$
$$b_u(\mathbf{x},0) = b_{uo}(\mathbf{x}) + b_{uc}(\mathbf{x}) + b_{ut}(\mathbf{x}) + b_{ud}(\mathbf{x},t)$$
$$b_{ud}(\mathbf{x},t) = \mathbf{N}(\mathbf{x})\mathbf{q}(t)$$

At the same time the perturbation potential can be seen as the sum of the potential due to position of the chord + …

$$\varphi = \varphi_o + \varphi_c + \varphi_t + \varphi_d$$
$$\varphi_d = \varphi_{d_{q1}} + \varphi_{d_{q2}} + \dots$$

$$U_\infty \frac{\partial b_{uo}}{\partial x} = \frac{\partial \varphi_o}{\partial z} \qquad U_\infty \frac{\partial b_{uc}}{\partial x} = \frac{\partial \varphi_c}{\partial z}$$

$$\mathbf{N}\dot{\mathbf{q}} + U_\infty \frac{\partial \mathbf{N}}{\partial x}\mathbf{q} = \frac{\partial \varphi_d}{\partial z} = w$$

Linearized boundary conditions

$$\mathbf{N}\dot{\mathbf{q}} + U_\infty \frac{\partial \mathbf{N}}{\partial x}\mathbf{q} = \frac{\partial \varphi_d}{\partial z} = w$$

$$\frac{\mathbf{N}\dot{\mathbf{q}}}{U_\infty} + \frac{\partial \mathbf{N}}{\partial x}\mathbf{q} = \frac{w}{U_\infty}$$

Kinematic change of the angle of attack

Geometric change of the angle of attack

Solution of the incompressible problem: Biot-Savart

$$\nabla \cdot \mathbf{u} = 0$$

Define a <u>Vector Potential</u> Ψ

$$\mathbf{u} = \nabla \times \boldsymbol{\Psi} \rightarrow \nabla \cdot \nabla \times (\cdot) = 0$$

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \nabla \times \nabla \times \boldsymbol{\Psi}$$

$$\nabla \times \nabla \times \boldsymbol{\Psi} = \cancel{\nabla \left(\nabla \cdot \boldsymbol{\Psi} \right)} - \nabla^2 \boldsymbol{\Psi}$$

$= 0$

u is solenoidal (i.e. null divergence), so it exists a Vector potential so that u is the rotor of it.

$$\nabla^2 \boldsymbol{\Psi} = -\boldsymbol{\omega}$$

Vectorial Poisson Equation

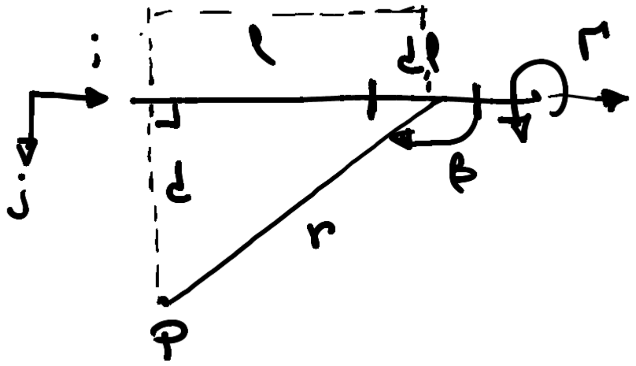
$$\mathbf{v}(\mathbf{x}) = \frac{1}{4\pi} \int_V \frac{\boldsymbol{\omega}(\boldsymbol{\xi}) \times (\mathbf{x} - \boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|^3} \, \mathrm{d}\boldsymbol{\xi}$$

Biot-Savart formula

Straight vortex in 3D

Consider a straight vorex of intensity Γ (so that $\boldsymbol{\omega} = \Gamma \mathrm{d}\ell \mathbf{i}$), and compute the velocity indeced at point P at distance $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$

$$\mathbf{v}_P = \frac{\Gamma}{4\pi} \int_L \frac{\mathbf{i} \times \mathbf{r}}{r^3} \mathrm{d}\ell$$



$$r = \frac{d}{\cos\beta}, \ell = d \tan\beta$$

$$\mathrm{d}\ell = \frac{d}{\cos^2\beta} \mathrm{d}\beta$$

$$|\mathbf{v}_P| = \frac{\Gamma}{4\pi} \int_{\beta_1}^{\beta_2} \frac{\sin\beta}{d} \mathrm{d}\beta$$

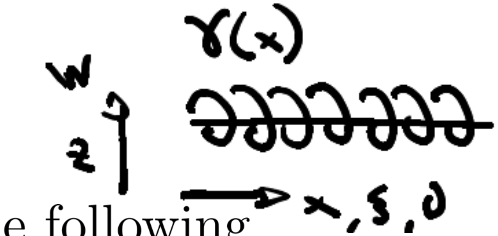
The induced velocity vector is oriented in the direction perpendicular to r and l

If the vortex is og inifinite length $L = [-\infty, \infty]$, then

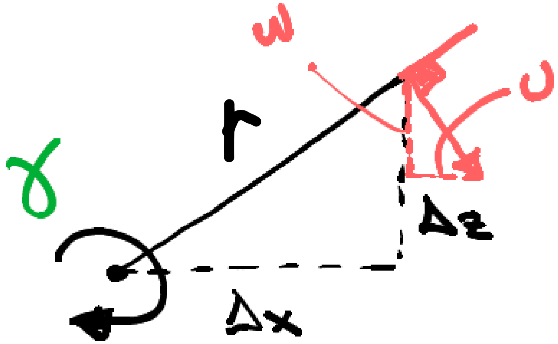
$$\begin{cases} +\infty & \beta_1 \to 0 \\ -\infty & \beta_2 \to \pi \end{cases}$$

$$\mathbf{v}_P = \frac{\Gamma}{2\pi d} \mathbf{k}$$

Velocity induced by a line of infinite length vortices



Let consider a line that goes from $x=x_{LE}$, $z=0$ to $x = x_{TE}$, $z = 0$ of vortices of infinite length with axis aligned with y. The vorticity per unit length is γ



Each vortex generates the following induced speed on a plane perpendicular to the vortex axis

$$u = \frac{1}{2\pi}\frac{\Delta z}{\Delta x^2 + \Delta z^2}\gamma$$
$$w = \frac{1}{2\pi}\frac{-\Delta x}{\Delta x^2 + \Delta z^2}\gamma$$

Since there is a continuous array of vortices the sum of all the effects is

$$\begin{aligned} u(x,z) &= \frac{1}{2\pi}\int_{x_{LE}}^{x_{TE}} \gamma(\xi)\frac{z}{(x-\xi)^2+z^2}\mathrm{d}\xi \\ w(x,z) &= \frac{1}{2\pi}\int_{x_{LE}}^{x_{TE}} \gamma(\xi)\frac{x-\xi}{(x-\xi)^2+z^2}\mathrm{d}\xi \end{aligned}$$

The induced speed along the vortical line $z = 0$ is

$$\begin{aligned} u(x,0) &= 0 \\ w(x,0) &= \frac{1}{2\pi}\int_{x_{LE}}^{x_{TE}} \frac{\gamma(\xi)}{x-\xi}\mathrm{d}\xi \end{aligned}$$