POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

13 Unsteady Aerodynamics: 2D unsteady airfoil theory

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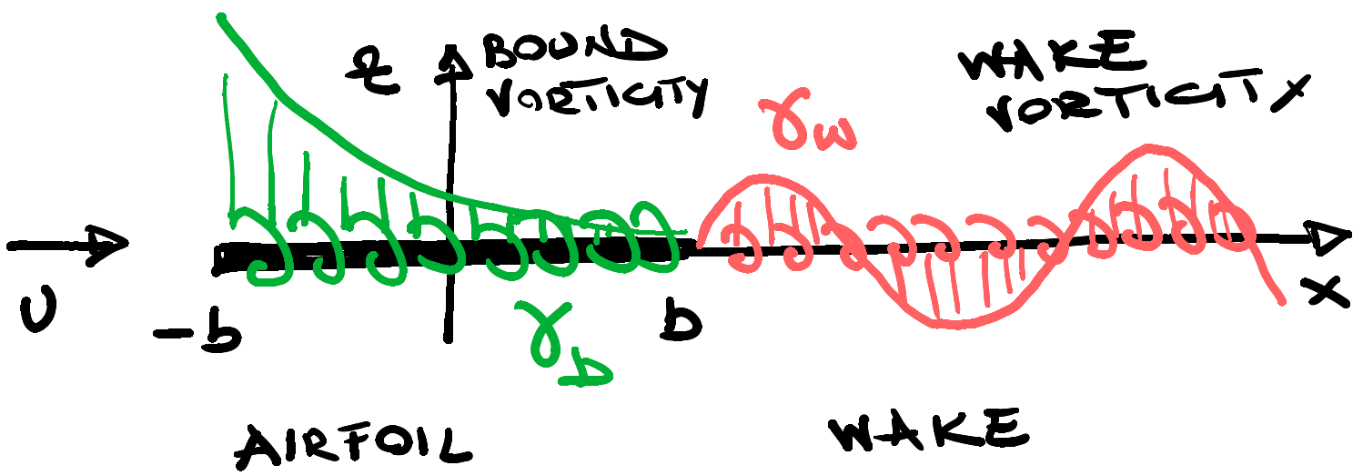
Material

Wayne Johnson Rotorcraft Aeromechanics, Chapter 10

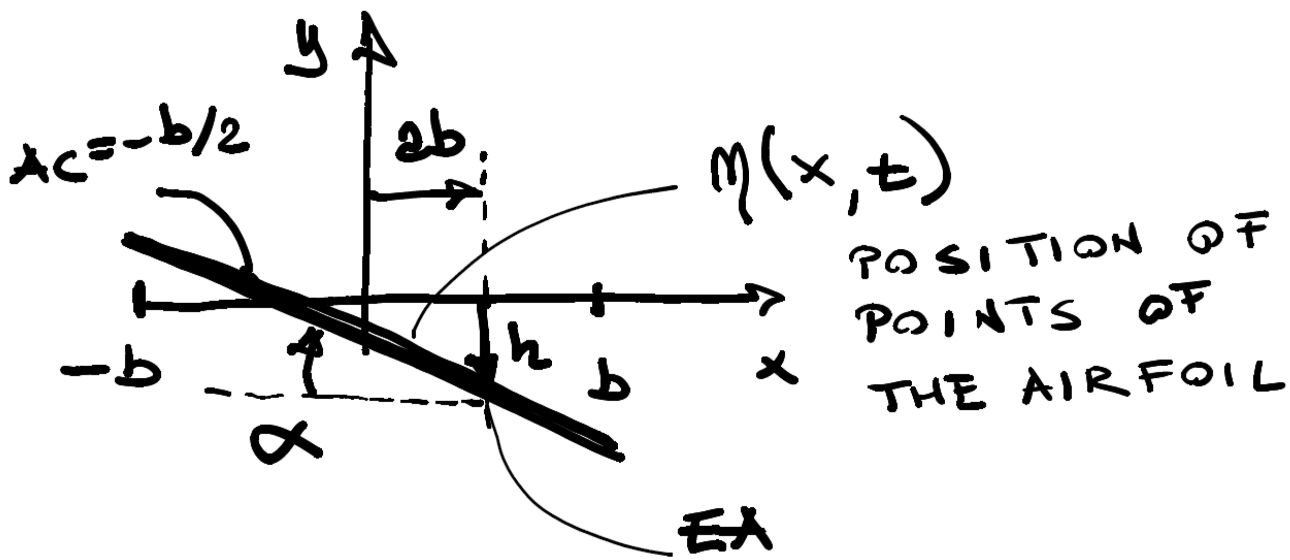
Hypothesis

- 2D irrotational, incompressible flow undergoing unsteady motion in uniform flow
- Airfoil represented by a thin flap-plate (no thickness or camber effects)
- The shed wake is geometrically represented by a straight line from the TE to infinity in the direction of the flow field
- Both airfoil and wake are planar sheets of vorticity



Hypothesis

- The airfoil chord is equal to 2b
- The position of the elastic axis is at ab (a non dimensional position of the EA)



Boundary conditions

For the detailed derivation of Theodorsen models see Johnson. Here are highlighted only some explanations

$$V_{b_z} = w_a = \frac{\partial \eta}{\partial t} + U_\infty \frac{\partial \eta}{\partial x}$$

$$w_a(x) = w_b(x) + \lambda(x)$$

VELOCITY INDUCED BY WAKE VORTEXES

$\forall t$

VELOCITY INDUCED BY BOUND VORTEXES

$$w_b(x,0) = \frac{1}{2\pi} \int_{-b}^{b} \frac{\gamma_b(\xi,t)}{x-\xi} \, d\xi$$

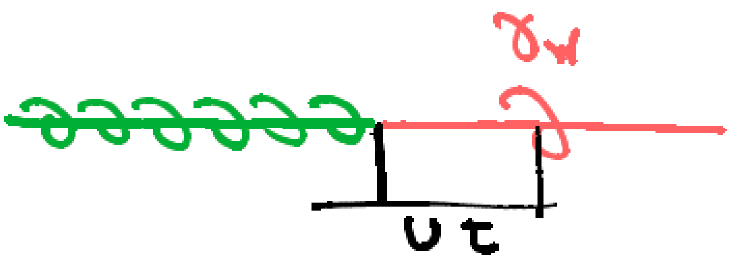
$$\lambda(x,0) = \frac{1}{2\pi} \int_{b}^{\infty} \frac{\gamma_w(\xi,t)}{x-\xi} \, d\xi$$

Wake vorticity

BOUND CIRCULATION $\Gamma = \int_{-b}^{b} \gamma_b(\xi)\, d\xi$

$$\frac{d\Gamma_T}{dt} = 0 \qquad -\frac{d\Gamma}{dt} = +\frac{d\Gamma_w}{dt}$$

$$\Gamma_w = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} \gamma_w(t) \underbrace{U_\infty dt}_{dx} \Rightarrow \frac{d\Gamma_w}{dt} = U_\infty \gamma_w$$



$$U_\infty \gamma_w(x,t) = -\frac{d\Gamma}{dt}\left(t - \frac{(x-b)}{U}\right)$$ KELVIN'S THEOREM

The vorticity in the wake at (x, t) is function of the variation of vorticity on the airfoil at time t – (x-b)/U_∞, i.e., when the vortex who travel at U_∞ speed was at the TE

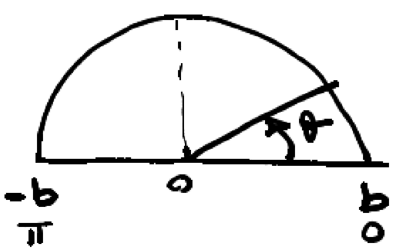
Method of solution

$$\frac{1}{2\pi}\int_{-b}^{b}\frac{\gamma_b}{x-\xi}\,d\xi = w_a - \lambda \qquad \underbrace{\gamma_b(b,t)=0}_{\text{KUTTA}}$$

Fedholm integral equation of I kind

$$\gamma_b = -\frac{2}{\pi}\sqrt{\frac{b-x}{b+x}}\int_{-b}^{b}\sqrt{\frac{b+\xi}{b-\xi}}\,\frac{w_a-\lambda}{x-\xi}\,d\xi$$

Solution



$$x = b\cos\vartheta$$

$$\lambda = \sum_{n=0}^{\infty}\lambda_n\cos n\vartheta$$

$$w_a = \sum_{n=0}^{\infty}w_n\cos n\vartheta$$

$$\Rightarrow \gamma_b = 2\sum_{n=0}^{\infty}(w_n-\lambda_n)f_n(\vartheta)$$

$$f_n(\vartheta) = \begin{cases}\tan(\vartheta/2) & n=0\\ \sin n\vartheta & n\geq 1\end{cases}$$

Decomposition of bound vorticity

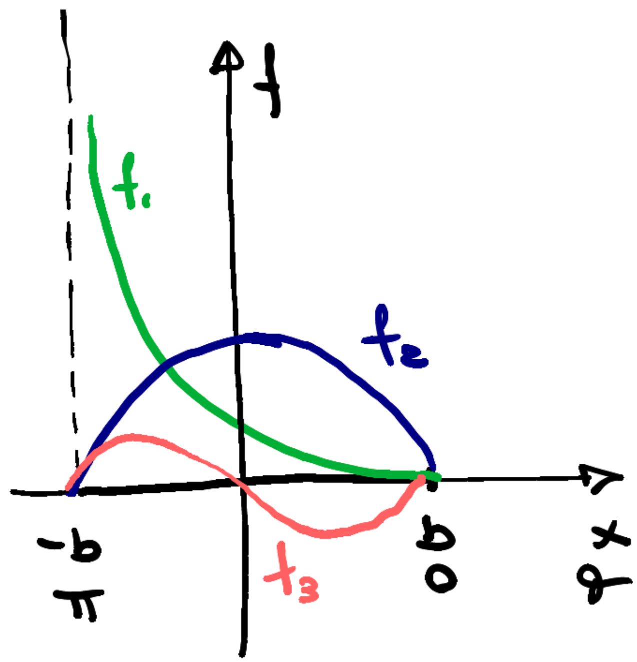
$$\Gamma = \int_{-b}^{b} \gamma_b \, dx = + \int_0^{\pi} \gamma_b(\vartheta)\, b \sin\vartheta \, d\vartheta$$

$$\Gamma = 2\pi b \left[\left(w_0 + \frac{1}{2} w_1 \right) - \left(\lambda_0 + \frac{1}{2} \lambda_1 \right) \right]$$

$$\gamma_b = \gamma_{b_C} + \gamma_{b_{NC}}$$

γ_{b_C} = CIRCULATORY BOUND VORTICITY
is the one that gives Γ
but $w_{b_C} = 0$ so no effects on BC

$\gamma_{b_{NC}}$ = NON-CIRCULATORY BOUND VORTICITY
gives $\Gamma = 0$
but $w_{b_{NC}} = w_2 - \lambda$



Decomposition of bound vorticity

$$\gamma_{bc} = \frac{2}{\sin\theta}\left[\left(w_0 + \frac{1}{2}w_1\right) - \left(\lambda_0 + \frac{1}{2}\lambda_1\right)\right] \qquad \frac{1}{2\pi}\int_{-b}^{b} \frac{\gamma_{bc}}{x-\xi}\,d\xi = 0$$

$$\frac{1}{\pi}\int_0^{\pi} \frac{\cos n\theta'}{\cos\theta' - \cos\theta}\,d\theta' = \frac{\sin n\theta}{\sin\theta}$$

$$\gamma_{b_{NC}} = -\frac{2}{\sin\theta}\left[(w_0 - \lambda_0)\cos\theta + \frac{1}{2}(w_1 - \lambda_1)\cos 2\theta\right]$$

$$+ 2\sum_{n=2}^{\infty}(w_n - \lambda_n)\sin n\theta$$

$$\int_{-b}^{b} \gamma_{b_{NC}}\,dx = 0 \qquad \frac{1}{2\pi}\int_{-b}^{b} \frac{\gamma_{b_{NC}}}{x-\xi}\,d\xi = w_2 - \lambda$$

Computation of loads

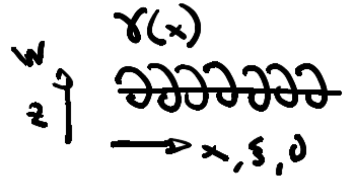
$$p = -\rho\left(U_\infty \frac{\partial \varphi}{\partial x} + \frac{\partial \varphi}{\partial t}\right) \qquad \frac{p_U(x)}{p_L(x)}$$

$$\Delta p = p_U - p_L$$

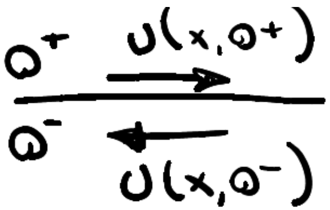
$$\Delta p = -\rho\left(U_\infty \frac{\partial \Delta\varphi}{\partial x} + \frac{\partial \Delta\varphi}{\partial t}\right)$$

$$\Delta\varphi = \varphi(x, 0^+, t) - \varphi(x, 0^-, t)$$

Circulation and potential

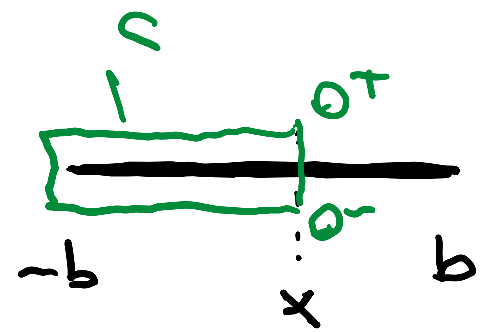


$$u(x,z) = \frac{1}{2\pi}\int_{x_1}^{x_2} \gamma(\xi)\,\frac{z}{(x-\xi)^2+z^2}\,d\xi$$



$$\Gamma(x) := \int_{-b-\varepsilon}^{x} u(\xi,0^+)\,d\xi + \int_{x}^{-b-\varepsilon} u(\xi,0^-)\,d\xi := \int_{-b-\varepsilon}^{x} \gamma_b(\xi)\,d\xi$$

$$u(x,0^\pm) = \pm\frac{\gamma_b(x)}{2} = \frac{\partial\varphi(x,0^\pm)}{\partial x}$$



$$\Gamma(x) = \int_{-b-\varepsilon}^{x} \frac{\partial\varphi}{\partial\xi}(\xi,0^+)\,d\xi - \int_{-b-\varepsilon}^{x} \frac{\partial\varphi}{\partial\xi}(\xi,0^-)\,d\xi$$

$$\Gamma(x) = \Delta\varphi(x)$$

Computation of Loads

$$\frac{\partial \Delta\varphi}{\partial x} = \Delta u = \gamma_b(x)$$

$$\frac{\partial \Delta\varphi}{\partial t} = \frac{\partial}{\partial t}\Gamma(x) = \frac{\partial}{\partial t}\int_{-\infty \to -b}^{x} \gamma_b(\xi)\, d\xi$$

$$\Rightarrow -\Delta p = \rho\left(\underbrace{U_\infty \gamma_b}_{\text{KUTTA JOUKOVSKY}} + \frac{\partial}{\partial t}\int_{-b}^{x} \gamma_b \, d\xi\right)$$

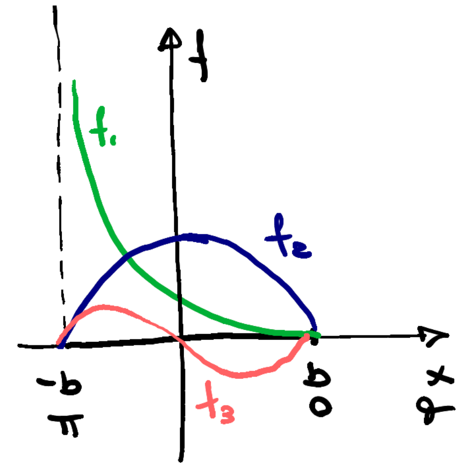
It is possible to verify that only the non-circulatory components of bound vorticity led to unsteady loads

Loads

$$-\Delta p = \sum_{n=0}^{\infty} P_n f_n(\theta)$$

$$P_0 = 2\rho U(w_0 - \lambda_0)$$

$$P_n = 2\rho U(w_n - \lambda_n) + \frac{\rho b}{n}\left(2(\dot{w}_{n-1} - \dot{\lambda}_{n-1}) - (\dot{w}_{n+1} - \dot{\lambda}_{n+1})\right)$$



This dynamic system connects the distributed pressure to the dynamic of the velocity on the airfoil (sum of the velocity of the body and the velocity induced by the wake or *inflow velocity*)

Loads

$$L = \int_{-b}^{b} (-\Delta p)\, dx$$

$$L = \rho \left(U\Gamma - \frac{\partial}{\partial t} \Gamma^{(1)}_{NC} \right)$$

$$\Gamma = 2\pi b \left[\left(w_0 + \frac{1}{2} w_1 \right) - \left(\lambda_0 + \frac{1}{2} \lambda_1 \right) \right]$$

$$\Gamma^{(1)}_{NC} = 2\pi b^2 \left[-\frac{1}{2} \left(w_0 - \frac{1}{2} w_2 \right) + \frac{1}{2} \left(\lambda_0 - \frac{1}{2} \lambda_2 \right) \right]$$

Loads

$$L = L_Q + L_W + L_{NC}$$

$$L_Q = 2\pi b \rho U_\infty \left(w_0 + \frac{1}{2} w_1\right)$$

QUASI-STATIC LIFT when $\Gamma_{NC} = 0$

$$L_{NC} = \rho \pi b^2 \left(\dot{w}_0 - \frac{1}{2} \dot{w}_1\right)$$

NON-CIRCULATORY LIFT when $\Gamma = 0$

it is called also 'ADDED MASS'

$$L_W = -2\pi \rho U b \left(\lambda_0 + \frac{1}{2} \lambda_1\right) + \\ -\rho \pi b^2 \left(\dot{\lambda}_0 - \frac{1}{2} \dot{\lambda}_2\right)$$

WAKE LIFT

due to shed-wake induced velocity

$\lambda = f(\gamma_w)$ $\gamma_w = f(\Gamma)$

Theodorsen: pitch and plunge harmonic oscillation

$$\eta = h + \alpha(x - ab)$$

$$w_a = \frac{\partial \eta}{\partial t} + U_\infty \frac{\partial \eta}{\partial x}$$

$$w_a(x) = \dot{h} + \dot{\alpha}(x - ab) + U_\infty \alpha$$

$$w_a(\vartheta) = \dot{h} + \dot{\alpha}(\cos\vartheta - ab) + U_\infty \alpha$$

$$w_0 = \dot{h} - \dot{\alpha}\, ab + U_\infty \alpha = \alpha_{b/4} U_\infty$$

$$w_1 = \dot{\alpha}$$

$$w_n = 0 \qquad n \geq 2$$

$$h = \bar{h} e^{i\omega t} \implies \gamma_W \text{ is harmonic}$$

$$\alpha = \bar{\alpha} e^{i\omega t}$$

Theodorsen for harmonic motion

The integrals may be computed in the case of harmonic motions

$$L = \pi\rho b^2\left(\ddot{h} + U\dot{\theta} - ba\ddot{\theta}\right) + 2\pi\rho U b C(k)\left(\dot{h} + U\theta + b\left(\frac{1}{2} - a\right)\dot{\theta}\right) \quad (8)$$

$$M_{1/4} = -\pi\rho b^2\left(\frac{1}{2}\ddot{h} + U\dot{\theta} + b\left(\frac{1}{8} - \frac{a}{2}\right)\ddot{\theta}\right) \quad (9)$$

$C(k)$ *Theodorsen function* or *Lift deficiency function.*

$$C(k) = F(k) + iG(k) = \frac{H_1^{(2)}(k)}{H_1^{(2)}(k) + iH_0^{(2)}(k)} \quad (10)$$

$H_n^{(2)}$ Henkel function of second kind.

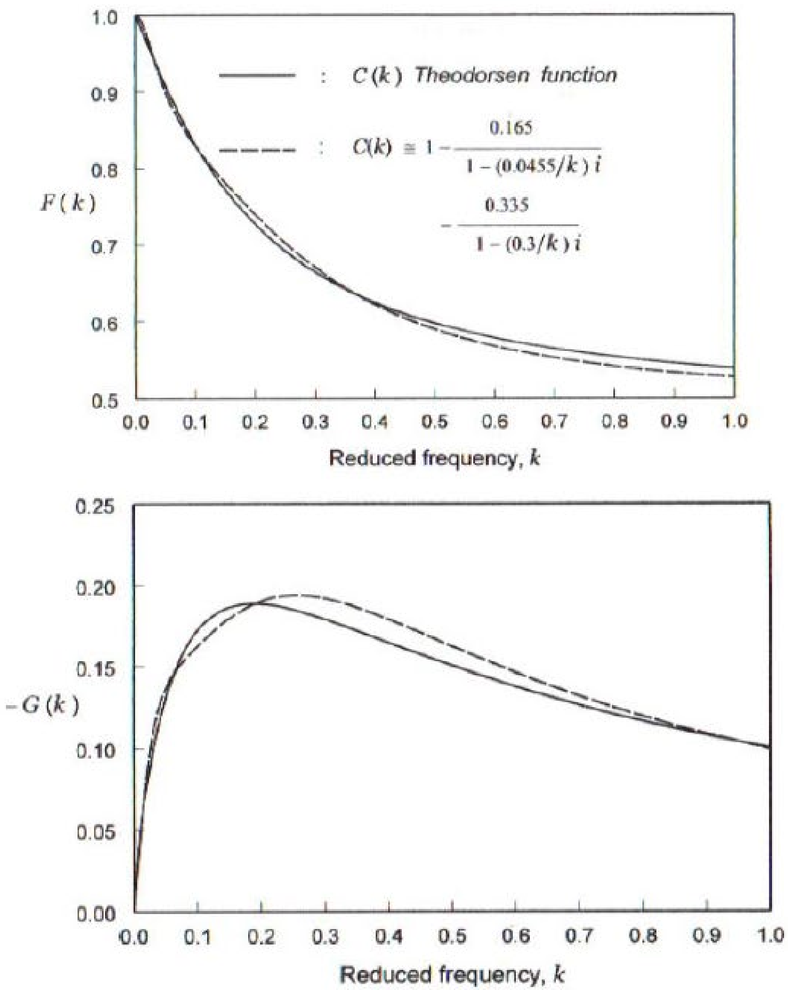
The circulatory term looks like a static airfoil coefficient expect that is multiplied by the lift deficiency.

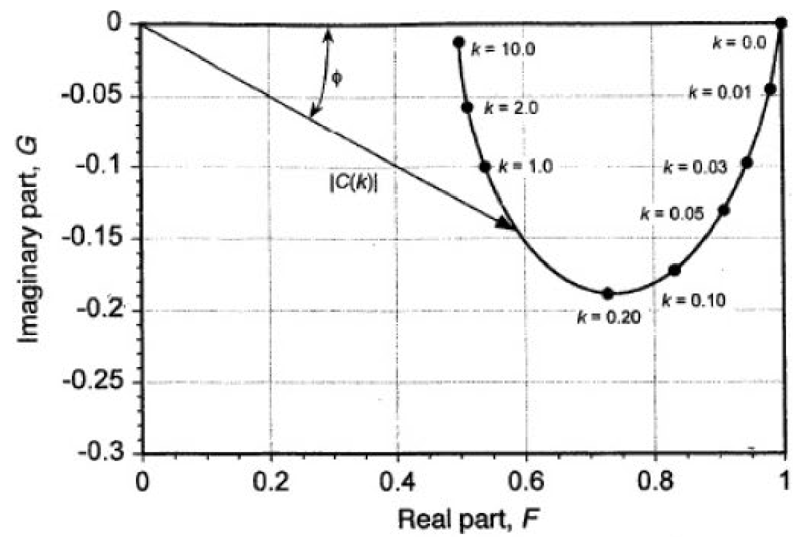
$$\alpha_{3/4} = \frac{\dot{h}}{U} + \theta + \left(\frac{b}{2} - ab\right)\frac{\dot{\theta}}{U} \quad (14)$$

$$L_C = C_{L/\alpha}\rho U^2 b C(k)\alpha_{3/4} \quad (15)$$

The unsteady airfoil behaves like a steady ones if the angle of attack is taken at the three-quarter-chord point.

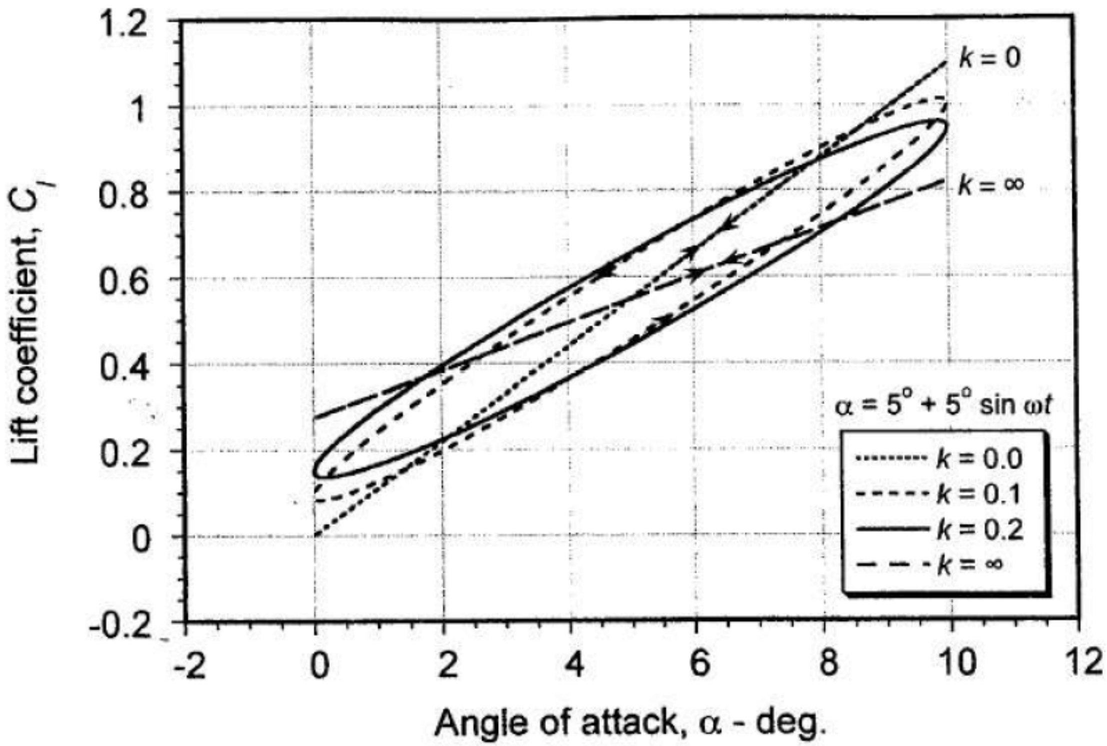
Theodorsen function





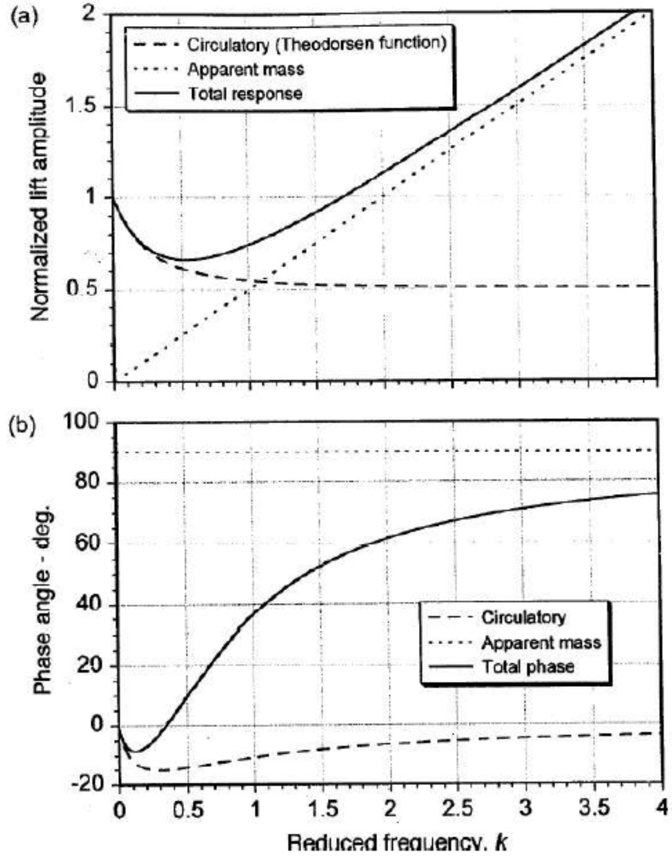
The maximum phase shift is at k close to 0.20

Lift variation due to a harmonic oscillation in pitch



Effects on the C_L for the circulatory part: the lift is lower than the static value when α increases, and higher when α decreases. The building of circulatory lift lags behind the instantaneous angle of attack.

Lift variation due to a harmonic oscillation in pitch



Effects on the normalized C_L for a pure pitch oscillation around the leading edge.

Cicala, Küssner-Schwarz approach

$$w_a(x,t) = f(x)\, e^{j(\omega t + \varphi(x))}$$

$$x = b\cos\vartheta$$

$$w_a(\vartheta,t) = f(\vartheta)\, e^{j(\omega t + \varphi(\vartheta))}$$

Using FOURIER SERIES

$$w_a(\vartheta,t) = U_\infty\left(A_0 + 2\sum_{n=1}^{\infty} A_n \cos n\vartheta\right) e^{j\omega t}$$

$$A_n = \frac{1}{U_\infty \pi}\int_0^{\pi} f(\vartheta)\, e^{j\varphi(\vartheta)} \cos n\vartheta \, d\vartheta$$

This approach can be used for any arbitrary change of shape of the flat-plate

- Movable surface (see notes)
- Morphing airfoils

$$\Delta c_p(\vartheta,t) = \left(4 a_0 \tan\frac{\vartheta}{2} + 8\sum_{n=1}^{\infty} a_n \sin n\vartheta\right) e^{j\omega t}$$

$$\begin{cases} a_0 = -A_1 + C(k)\,(A_0 + A_1) \\ a_n = -\dfrac{jk}{2n}\left(A_{n+1} - A_{n-1}\right) + A_n \quad n \geq 1 \end{cases}$$

Wagner approach

Response of the airfoil to a step change in angle of attack, i.e. computation of the *indicial function* or Wagner function $\phi(\tau)$ known exactly

$$L_c(\tau) = \rho b U^2 \alpha \int_{-\infty}^{\infty} \frac{C(k)}{ik} \exp{(ik\tau)} \mathrm{d}\tau = 2\pi\rho b U^2 \alpha \phi(\tau) \tag{17}$$

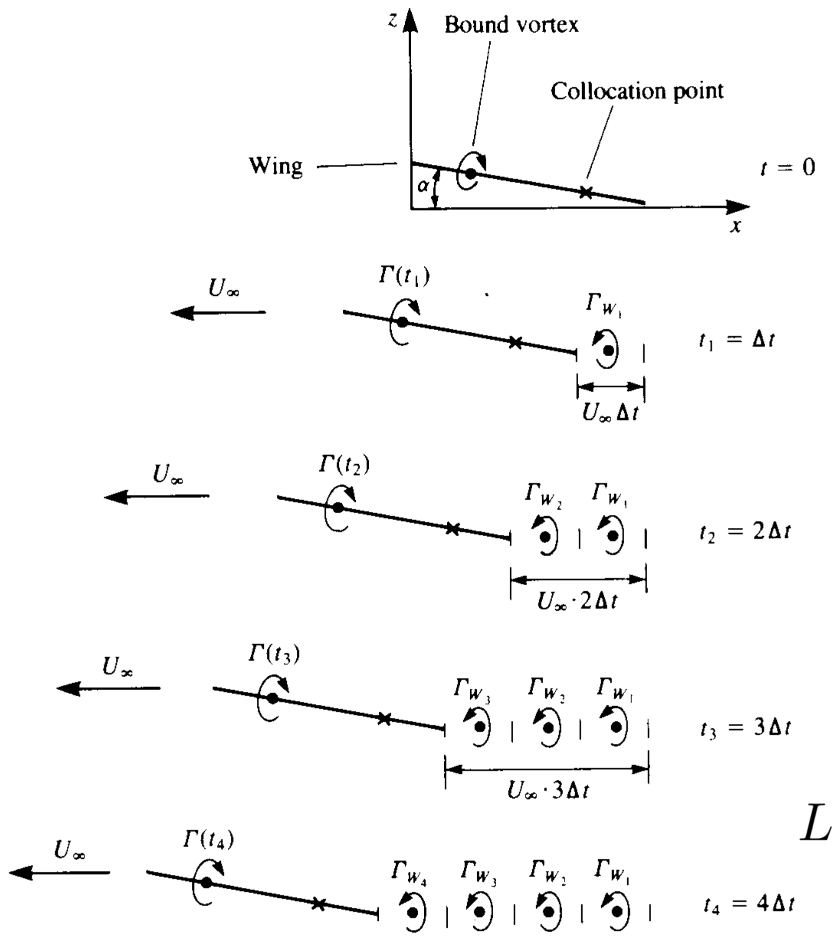
with $\tau = Ut/b$ the non-dimensional time. $\alpha\phi(\tau)$ is the effective instantaneous angle of attack.
An approximation of the indicial function (RT Jones, 1940) is

$$\phi(\tau) = 1 - 0.165 \exp(-0.455\tau) - 0.335 \exp{(-0.3\tau)} \tag{18}$$

Then in general, using the convolution

$$L = \pi\rho b^2 \left(\ddot{h} + U\dot{\theta} - ba\ddot{\theta}\right) + 2\pi\rho b U^2 \left(\alpha_0 \phi(\tau) + \int_0^{\tau} \frac{\mathrm{d}(\alpha(\sigma))}{\mathrm{d}\sigma} \phi(\tau - \sigma) \mathrm{d}\sigma\right) \tag{19}$$

Lumped vortex



$$\frac{d\Gamma}{dt} = 0 \rightarrow \Gamma(t_i) + \sum_{k=1}^{i} \Gamma_{w_k} = 0$$

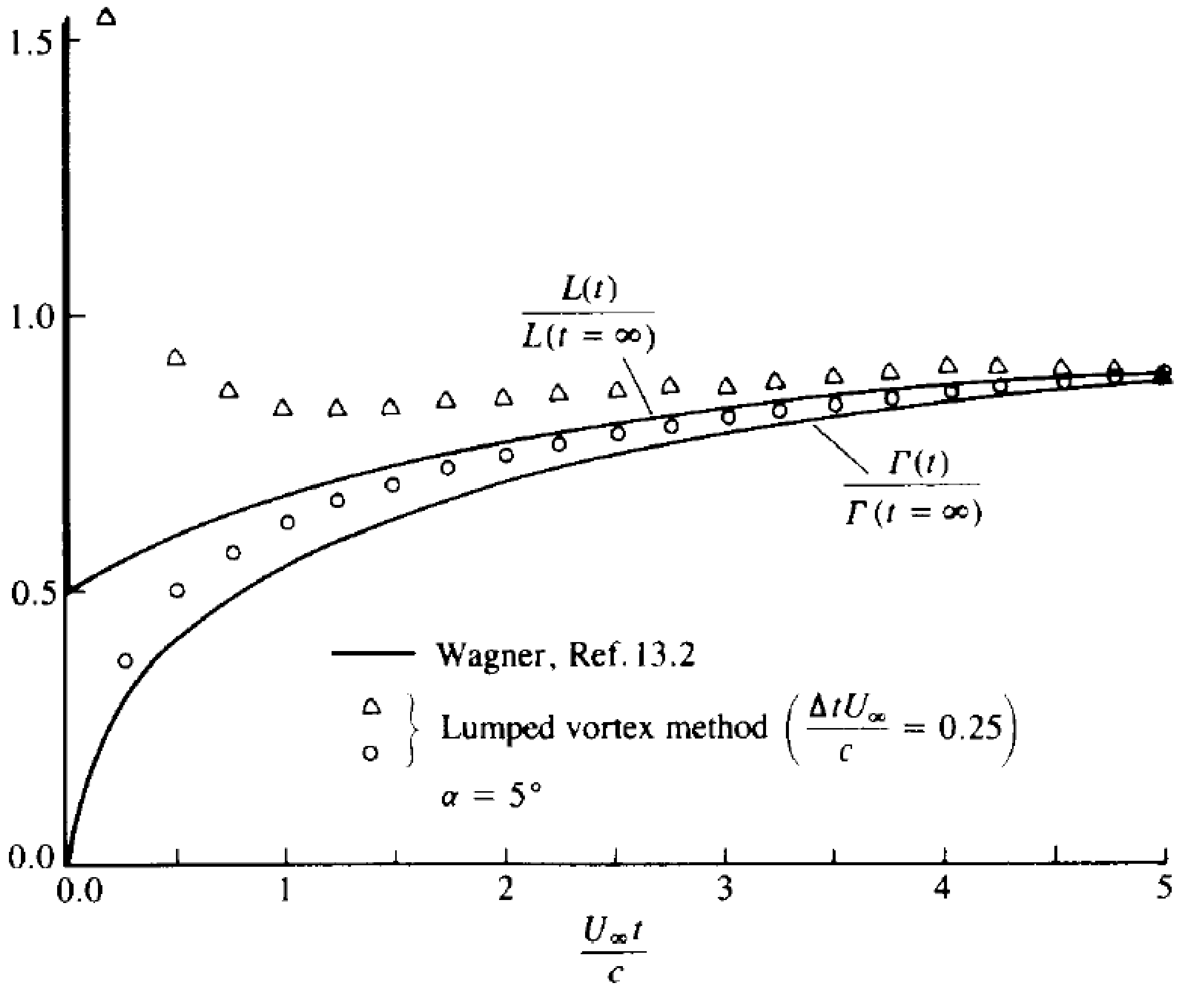
$$|\mathbf{v}_i| = \frac{\Gamma_i}{2\pi d}$$

$$L = \int_{-b}^{b} -\Delta p \mathrm{d}x$$

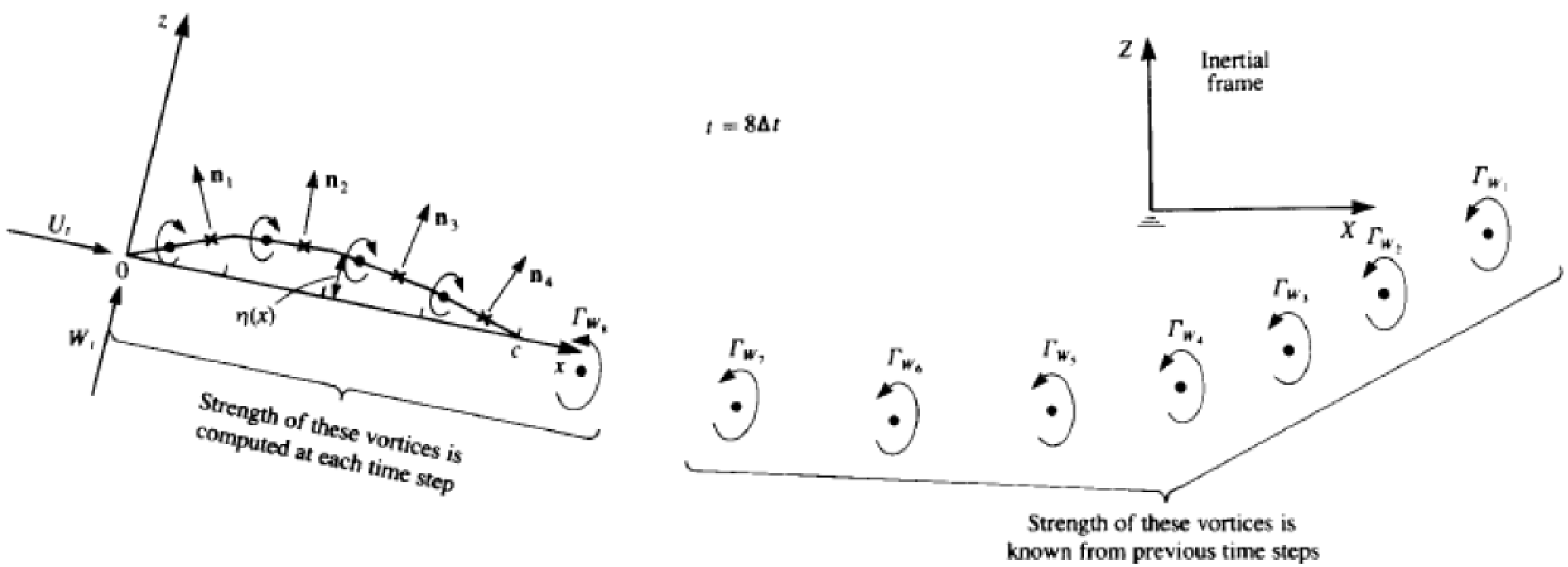
$$L = \int_{-b}^{b} \rho \left(U_\infty \gamma_b(x) + \frac{\partial}{\partial t} \int_{-b}^{x} \gamma_b(\xi) \mathrm{d}\xi \right) \mathrm{d}x$$

$$L = \rho \left(U_\infty \Gamma(t) + \frac{\partial \Gamma(t)}{\partial t} 2b \right)$$

2D Profile sudden acceleration

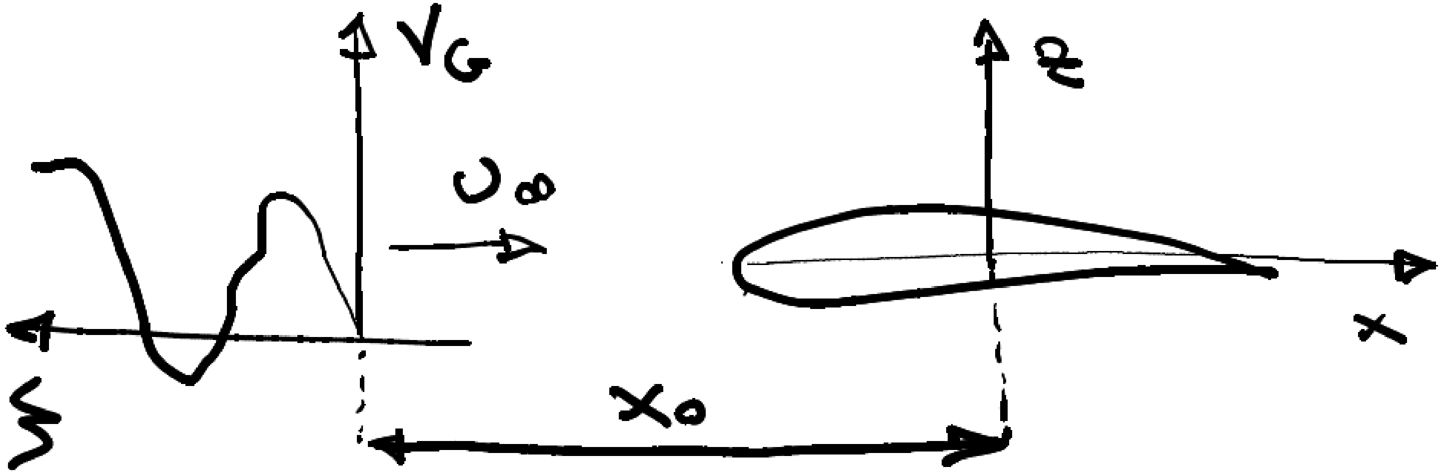


Lumped vortexes



Gust

Frozen gust front



$$V_G = V_G(\xi)$$

$$\xi = -(x + x_0) + U_\infty t$$

$$V_G = V_G((x + x_0) - U_\infty t) = V_G(x, t)$$

Gust

$$V_G(\omega) = \int_{-\infty}^{+\infty} V_G(U_\infty t - (x + x_0))\, e^{-j\omega t}\, dt$$

$$V_G(\omega) = \int_{-\infty}^{+\infty} V_G(\xi)\, e^{-j\omega \frac{\xi + (x + x_0)}{U_\infty}}\, \frac{1}{U_\infty}\, d\xi$$

$\tilde{x} = x/b \qquad \bar{\xi} = \xi/b$

Gust

$$V_G(\omega) = \underbrace{e^{-j\frac{\omega b}{U_\infty}(\tilde{x}+\tilde{x}_0)}}_{\text{SPATIAL DELAY}} \underbrace{\frac{b}{U_\infty}\int_{-\infty}^{+\infty} V_G(\bar{\xi})\, e^{-jk\bar{\xi}}\, d\bar{\xi}}_{\text{INPUT SPECTRUM}}$$

for a single harmonic at freq k

$\tilde{x} = \cos\vartheta$

$$V_G(x,k) = e^{-jk\tilde{x}}\, W\, e^{-jk\tilde{x}_0}$$

$$V_G(\vartheta,k) = e^{-jk\cos\vartheta}\, W\, e^{-jk\tilde{x}_0}$$

$$e^{-jk\cos\vartheta} = J_0(k) + 2\sum_{n=1}^{\infty} j^n J_n(k)\cos n\vartheta$$

$J_n(k)$ BESSEL FUNCTIONS OF THE 1° KIND $\quad J_n(-k) = (-1)^n J_n(k)$

Gust: Sears function

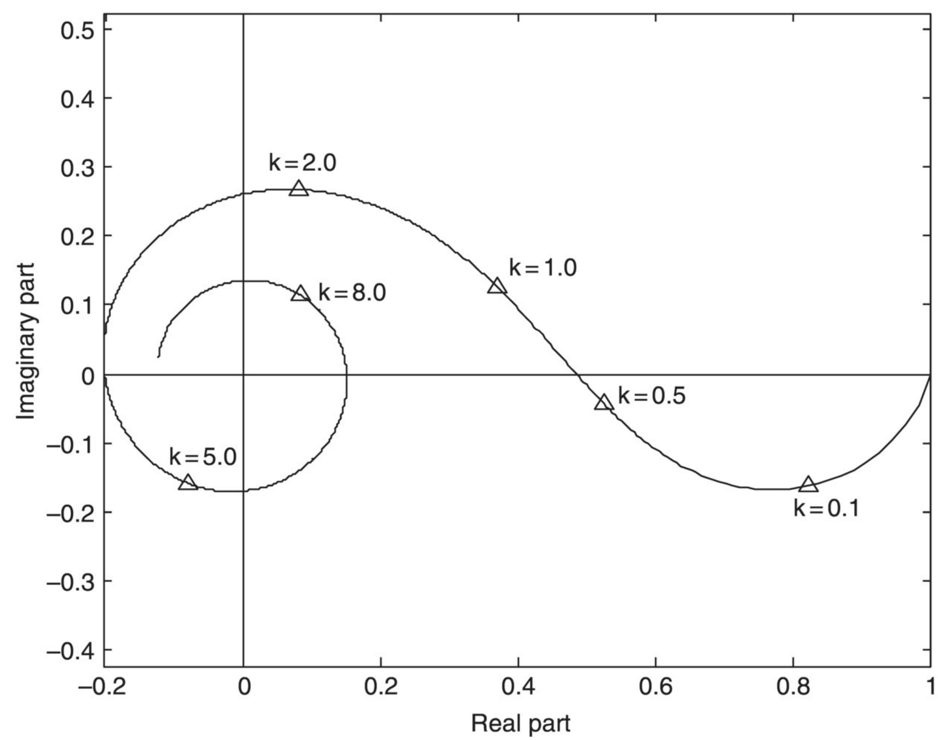
$$v_G(\theta,k) = w\, e^{-jk\tilde{x}_0}\left(\bar{J}_0(k) + 2\sum_{n=1}^{\infty} j^n \bar{J}_n(k)\cos n\theta\right)$$

$$P_0 = \frac{w\, e^{-jk\tilde{x}_0}}{U_\infty} \qquad P_n = \frac{w}{U_\infty}(-j)^n \bar{J}_n(k)$$

$$L = \pi\rho 2b\, U_\infty w\, \phi(k)$$

$$M_{1/2} = L\cdot\frac{b}{2}$$

$$\phi(k) = \left[\bar{J}_0(k) - i\,\bar{J}_1(k)\right]C(k) + j\,\bar{J}_1(k)$$



The final results is…

The frequency response function for

$$\begin{bmatrix} L \\ M \\ M_h \end{bmatrix} = \mathbf{H}_{am}(k) \begin{bmatrix} h \\ \alpha \\ \beta \end{bmatrix} + \mathbf{H}_{ag}(k) v_g$$