POLITECNICO DI MILANO





Aerospace Control Systems
Systems theory – performance 1

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Control systems performance

- Aim of control: once stability is guaranteed, make the control error *e* "small"

- The performance of the control system is expressed in terms of the "closeness" of *e* to zero

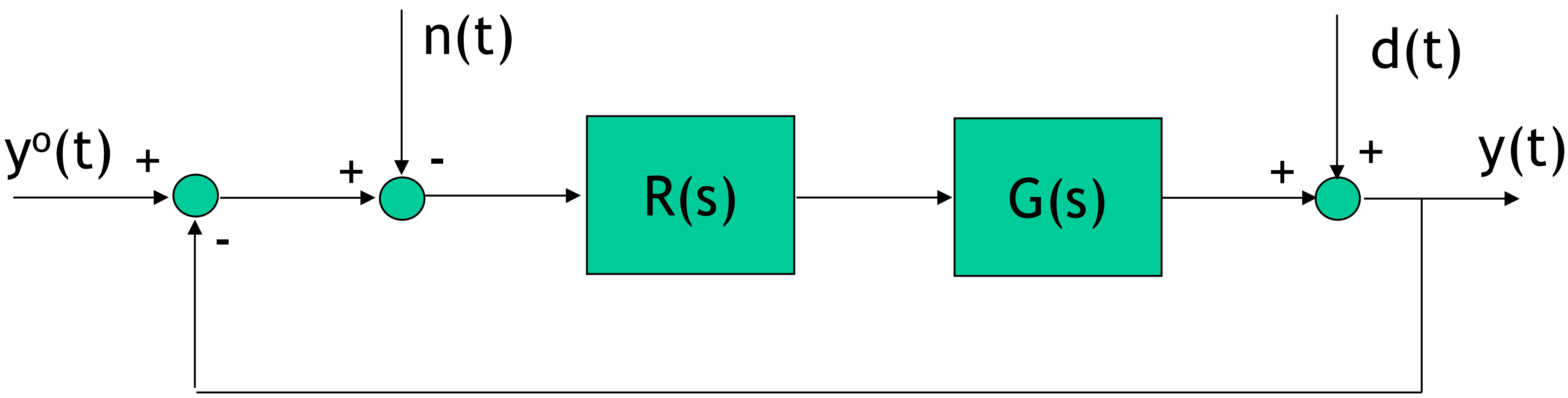
- How can performance be measured?

- Let's first review how this is done for SISO LTI system and then we will try to generalize as much as possible.

- In SISO LTI system we usually focus on two different aspects (static and dynamic performance).

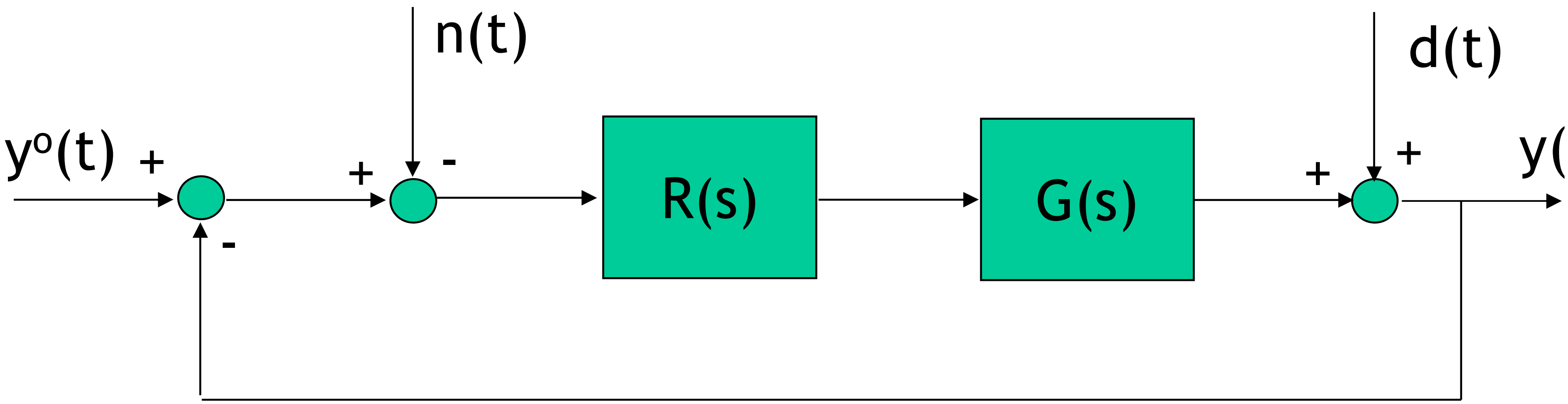
The SISO control loop – stability analysis tools

Consider the SISO control loop described by the block diagram



How do we check closed-loop stability and/or design R(s) for closed-loop stability?

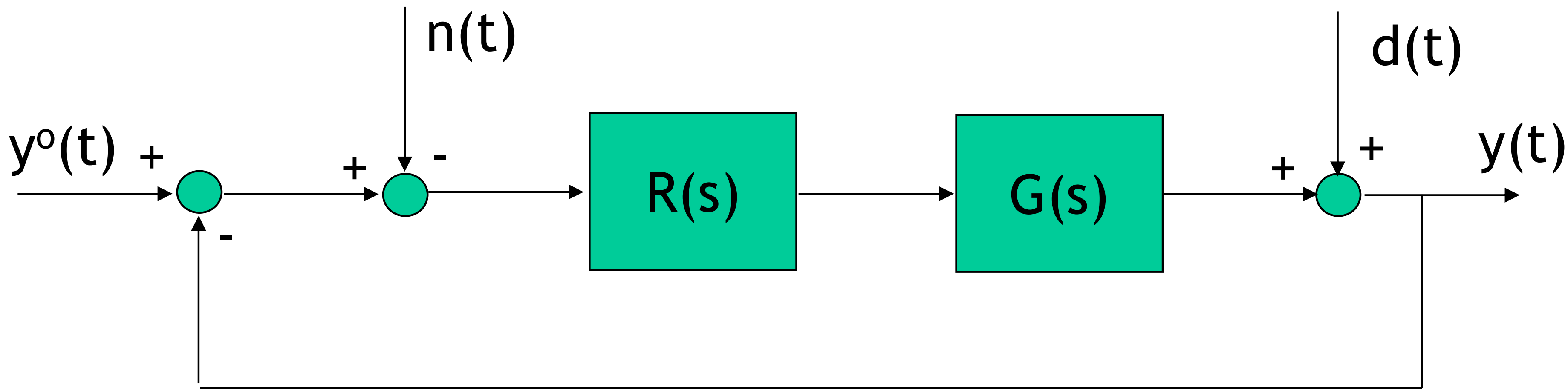
The SISO control loop – stability analysis tools



Classical tools:

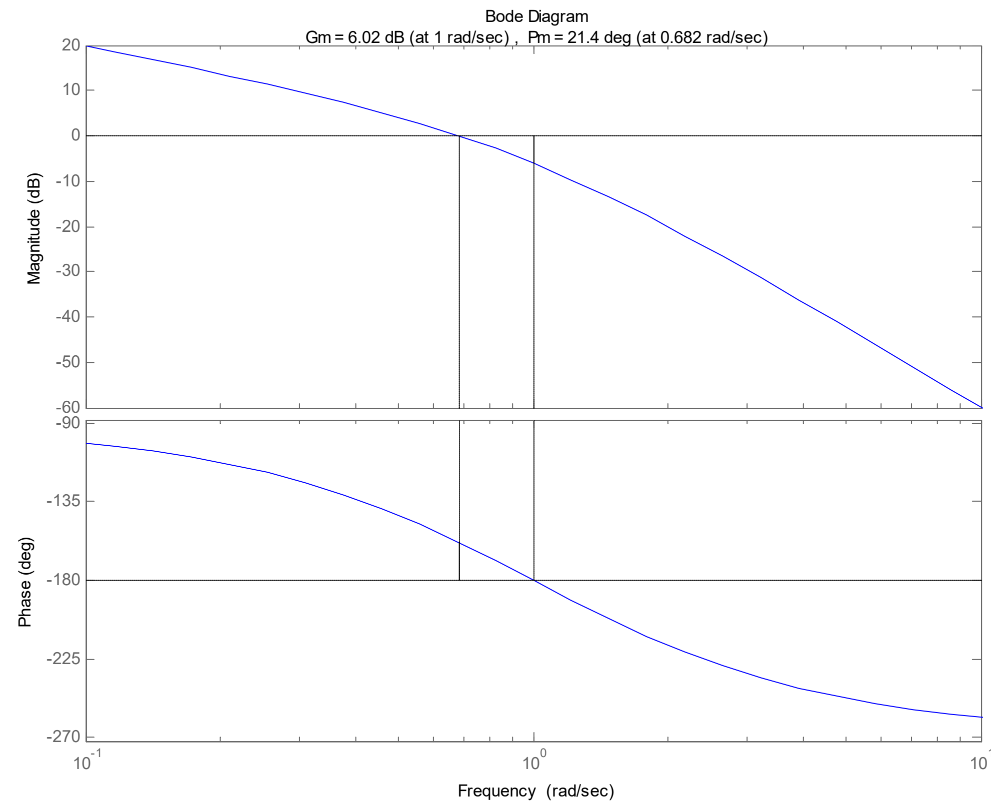
- Nyquist criterion: wide validity, not practical
- Bode criterion:
  - restricted to R(s)G(s) without Right Half Plane (RHP) poles
  - not suitable for applications such as rotorcraft – most helicopters are open-loop unstable
- Root-locus analysis.

The SISO control loop – stability robustness



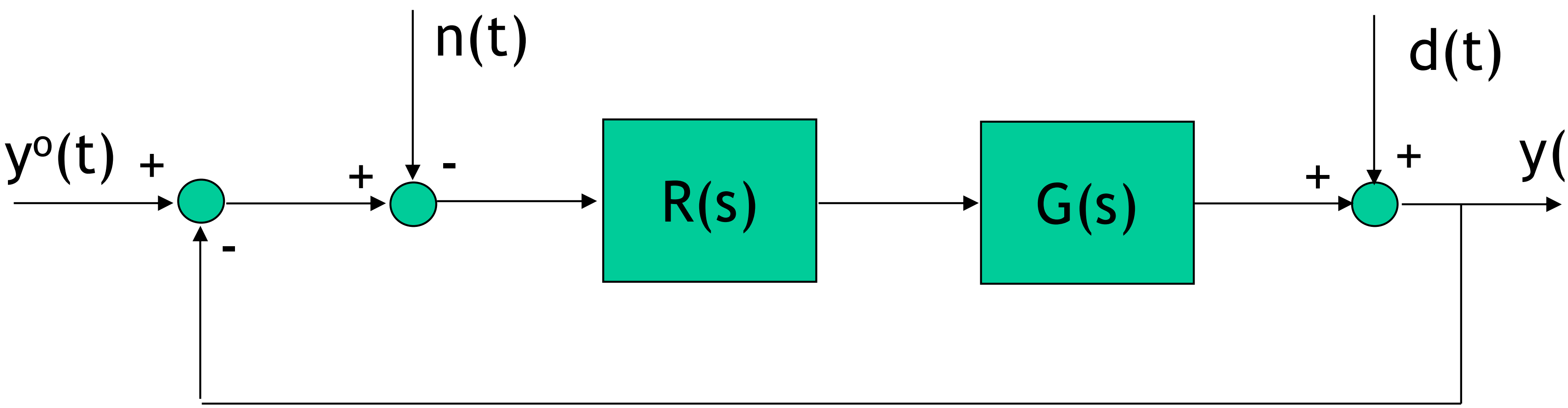
We recall the classical robustness indicators:

- phase margin
- gain margin.



The SISO control loop – defining performance

Consider the SISO control loop described by the block diagram



And assume that n(t)=d(t)=0 and $y^o(t)$=step(t).

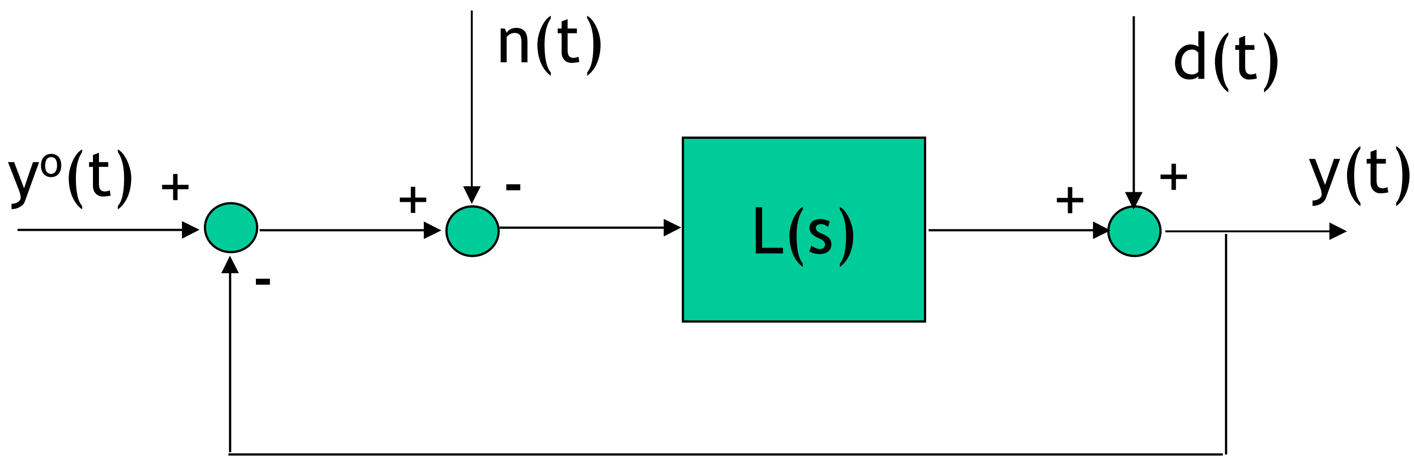
What will y(t) look like?

The SISO control loop – defining performance

Assumption: the closed-loop system is asymptotically stable.

We then have

$$Y(s) = \frac{R(s)G(s)}{1+R(s)G(s)} Y^o(s) = \frac{L(s)}{1+L(s)} \frac{1}{s}$$

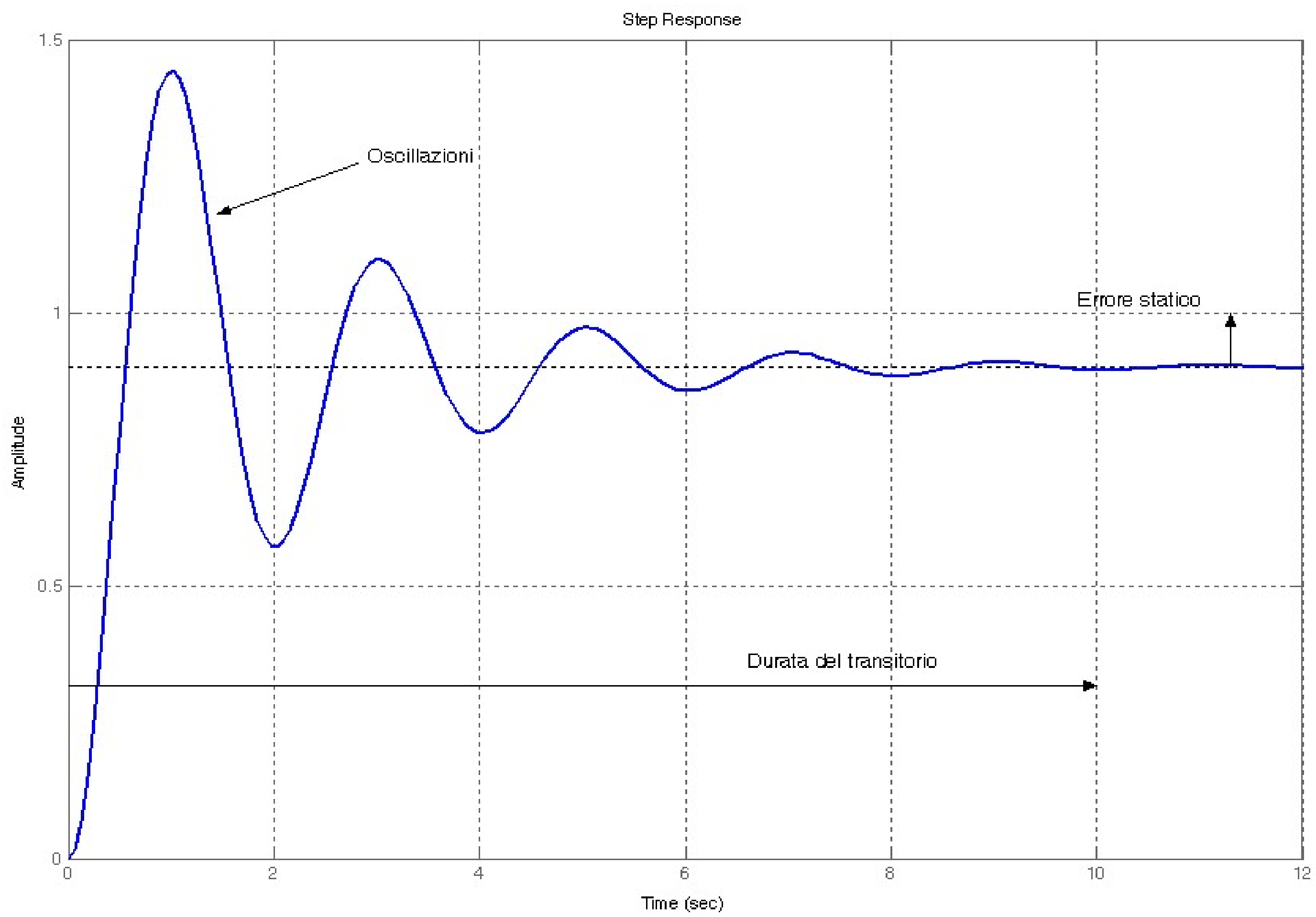


The SISO control loop – defining performance

y(t) will:

- Tend to a constant value
- Have a transient depending on the shape of the transfer function from y^o to y, which will affect
  - the shape of the transients (*e.g.*, delay, rise time, presence or absence of oscillations);
  - the duration of transients (settling time).

The SISO control loop – defining performance



Static and dynamic performance

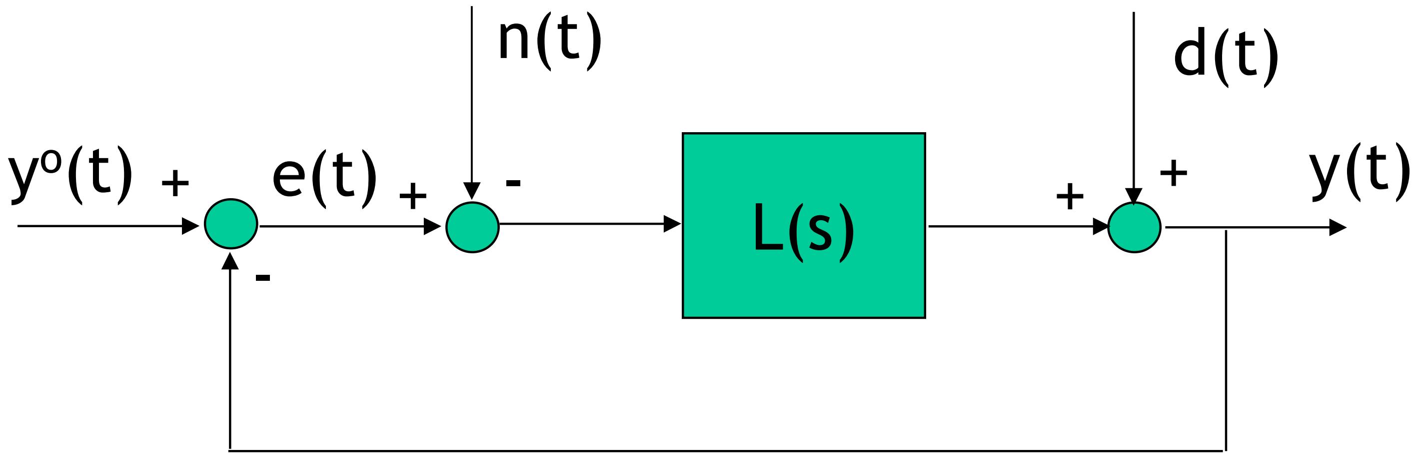
- Static performance: the behaviour of the control system *at steady state i.e.*, for $t \to \infty$.

- Dynamic performance: the behaviour of the control system *during transients*, defined in terms of
  - shape and
  - duration of transients.

- Goal: understanding how performance can be characterised in terms of $L(s)$.

Sensitivity functions

Note that the loop is completely described by the following relations



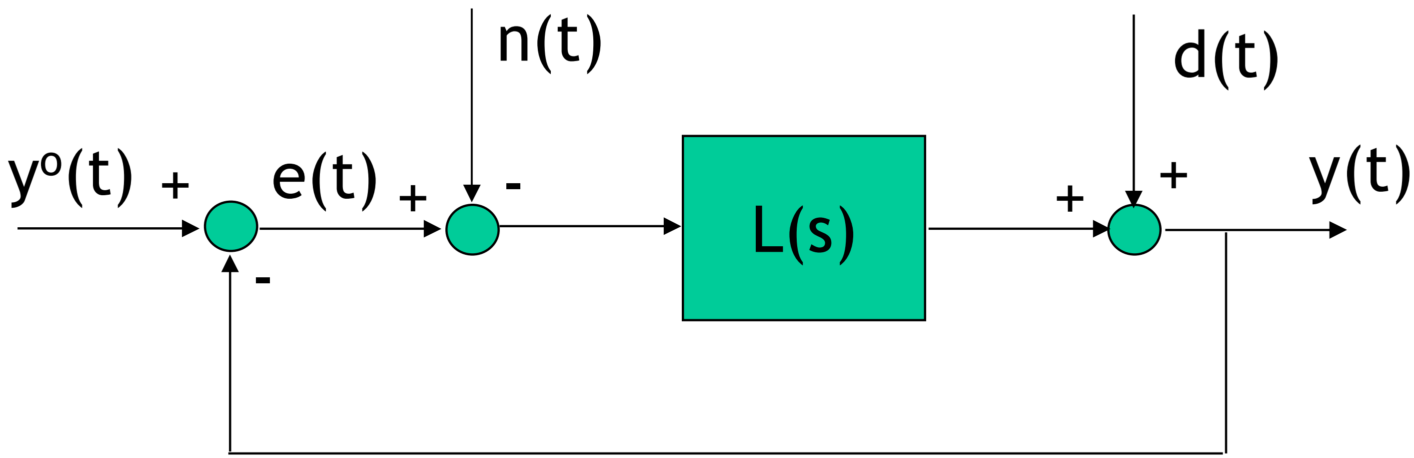
$$E(s) = \frac{1}{1+L(s)}Y^o(s) - \frac{1}{1+L(s)}D(s) + \frac{L(s)}{1+L(s)}N(s)$$

so the loop performance is completely described by two transfer functions only.

Let

$$S(s) = \frac{1}{1+L(s)} \quad \text{sensitivity function}$$

$$F(s) = \frac{L(s)}{1+L(s)} \quad \text{complementary sensitivity function}$$



Sensitivity functions

S(s) describes:

- The effect of y^o on e (ideally: 0)
- The effect of d on e (ideally: 0)

F(s) describes:

- The effect of n on e (ideally: 0)… but also
- The effect of y^o on y (ideally: 1)!

NOTE THAT: S(s) and F(s) are not indipendent but

$$\boxed{S(s) + F(s) = 1 \quad \forall s}$$

For analysis purposes, assumptions on the classes of inputs to be considered are needed.

Let's consider the two following cases:

- Canonical inputs (step, ramp, parabola…)
- Sinusoidal inputs.

As mentioned previously, it will be assumed that the closed-loop system is asymptotically stable.

We study S(s) (effect of y^o and d on e).

Assume, *e.g.*, that y^o has a Laplace transform of the type

$$Y^o(s) = \frac{A}{s^r}, \qquad r > 0$$

(canonical input) then the Laplace transform of e will be given by

$$E(s) = S(s)\frac{A}{s^r} = \frac{1}{1+L(s)}\frac{A}{s^r}$$

Static performance: canonical inputs (2)

As the closed-loop system is asymptotically stable we can study the limit

$$\lim_{t \to \infty} e(t)$$

using the final value theorem, so

$$\lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{1}{1 + L(s)} \frac{A}{s^r}$$

How can the above limit be computed?

Static performance: canonical inputs (3)

Let's consider the most general possible form for $L(s)$

$$L(s) = \frac{\mu}{s^g} \frac{\prod_i (T_i s + 1)}{\prod_i (\tau_i s + 1)} \frac{\prod_i \left(\frac{s^2}{\alpha_{ni}^2} + \frac{2\zeta_i s}{\alpha_{ni}} + 1 \right)}{\prod_i \left(\frac{s^2}{\omega_{ni}^2} + \frac{2\xi_i s}{\omega_{ni}} + 1 \right)}$$

and note that for $s \to 0$

$$L(s) \to \frac{\mu}{s^g}$$

so the static error is given by

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+L(s)}\frac{A}{s^r} = \lim_{s\to 0} \frac{s^g}{s^g + \mu}\frac{A}{s^{r-1}}$$

Static performance: canonical inputs (4)

Let's analyse the result in detail

r=1 (step)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s^g}{s^g+\mu} A = \begin{cases} g=0 & \frac{1}{1+\mu}A \\ g\geq 1 & 0 \end{cases}$$

r=2 (ramp)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s^{g-1}}{s^g+\mu} A = \begin{cases} g=0 & \infty \\ g=1 & \frac{1}{1+\mu}A \\ g\geq 2 & 0 \end{cases}$$

Static performance: canonical inputs (5)

Comments:

- In order to achieve zero static error the type of $L(s)$ must be at least equal to the type of the considered canonical input ($g=r$).

- If the type of $L(s)$ is strictly lower ($g=r-1$) finite static error is obtained, which can be reduced by acting on μ (as long as this is possible).

Static performance: canonical inputs (6)

Let's now study $F(s)$ (effect of n on e).

Assume that the Laplace transform of n is given by

$$N(s) = \frac{A}{s^r}, \qquad r > 0$$

(canonical input) then the Laplace transform of e will be given by

$$E(s) = F(s)\frac{A}{s^r} = \frac{L(s)}{1+L(s)}\frac{A}{s^r}$$

Static performance: canonical inputs (7)

So the static error is given by

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{L(s)}{1+L(s)}\frac{A}{s^r} = \lim_{s\to 0} \frac{\mu}{s^g+\mu}\frac{A}{s^{r-1}}$$

r=1 (step)

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{\mu}{s^g+\mu}A = \begin{cases} g=0 & \frac{\mu}{1+\mu}A \\ g\geq 1 & A \end{cases}$$

r=2 (ramp)

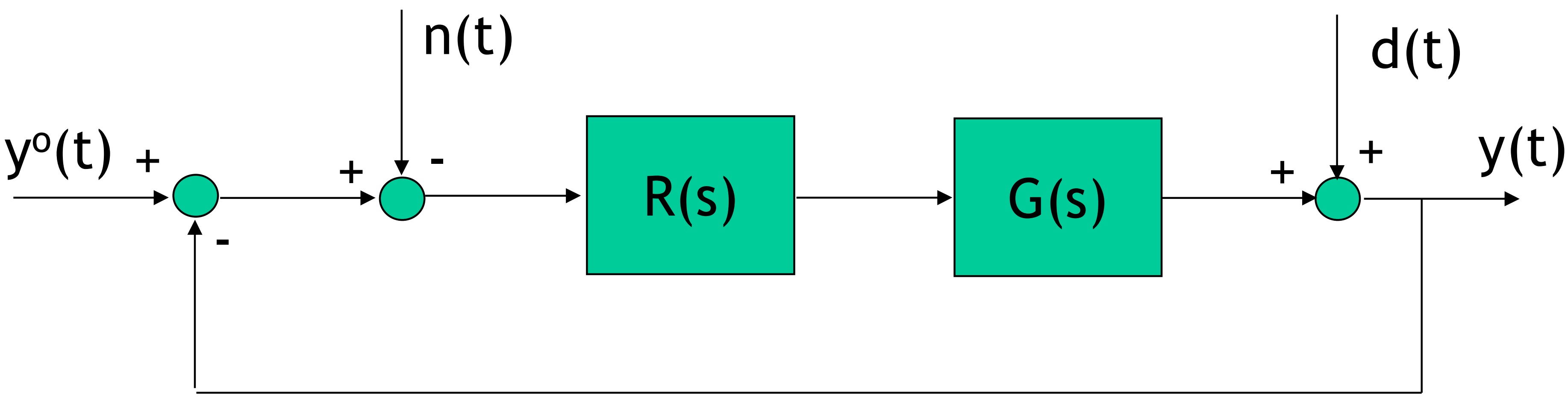
$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{\mu}{s^g+\mu}\frac{A}{s} = \infty \quad \forall g \geq 0$$

Comments:

- If $n \neq 0$ it is not possible to achieve zero static error.
- If $n(t)$=step(t) the static error is finite and can be reduced by acting on μ in the case g=0.
- If n is of type greater than zero then the static error is not finite.

Static performance: sinusoidal inputs

Consider again the SISO control loop



and assume that $n(t)=d(t)=0$ and $y^o(t)=\sin(\Omega t)$.

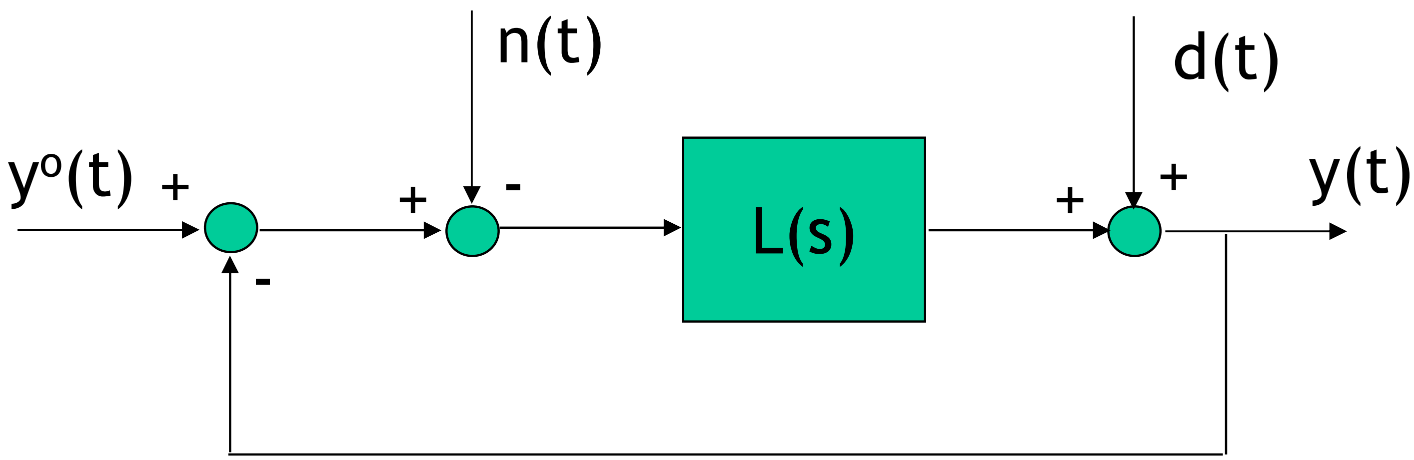
What will the time history of y(t) look like?

Static performance: sinusoidal inputs (2)

Assumption: the closed-loop system is asymptotically stable.

We then have

$$Y(s) = \frac{R(s)G(s)}{1+R(s)G(s)} Y^o(s) = \frac{L(s)}{1+L(s)} \frac{\Omega}{s^2+\Omega^2}$$



Static performance: sinusoidal inputs (3)

Let's study the transfer function $S(s)$ (effect of y^o and d on e).

Assume that the Laplace transform of y^o is given by

$$Y^o(s) = \frac{as+b}{s^2+\Omega^2}$$

(sum of a sine and a cosine) then the Laplace transform of e will be given by

$$E(s) = S(s)\frac{as+b}{s^2+\Omega^2} = \frac{1}{1+L(s)}\frac{as+b}{s^2+\Omega^2}$$

Static performance: sinusoidal inputs (4)

How can we study the behaviour of $e(t)$ for $t \to \infty$?

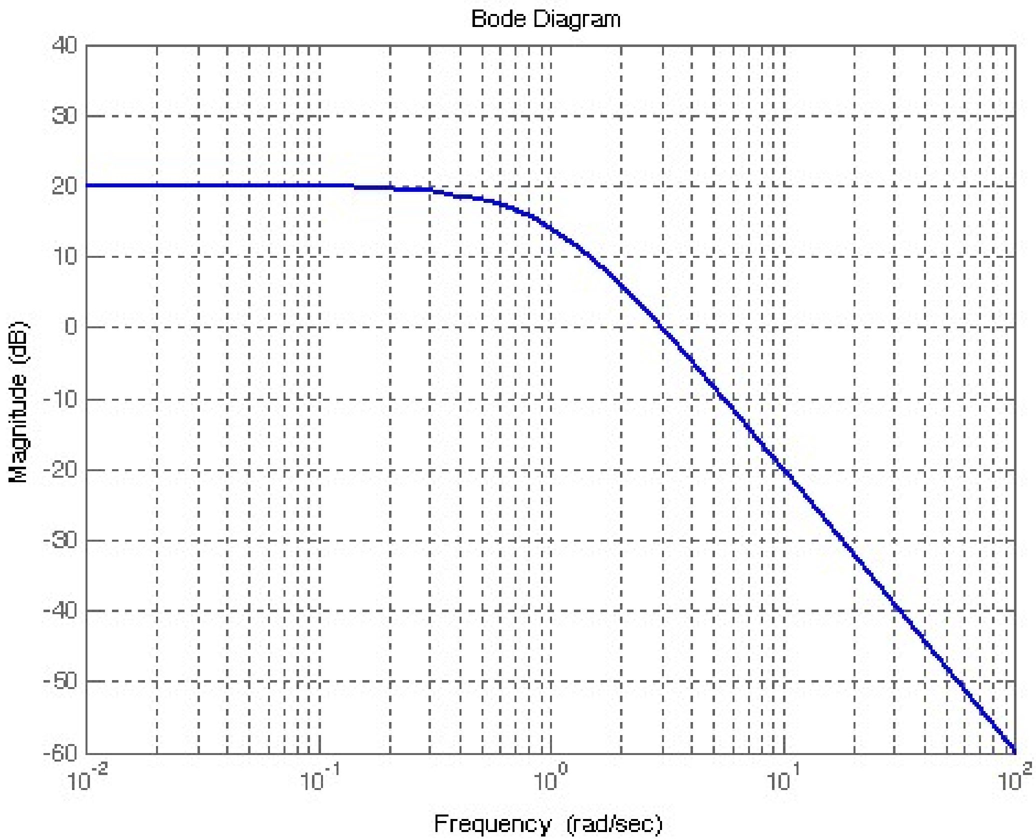
$E(s)$ has two poles on the imaginary axis, so the final value theorem cannot be applied.

However, the frequency response theorem can be used:

$$e(t) \to |S(j\Omega)| \sin(\Omega t + \angle(S(j\Omega)))$$

Frequency response of S(s)

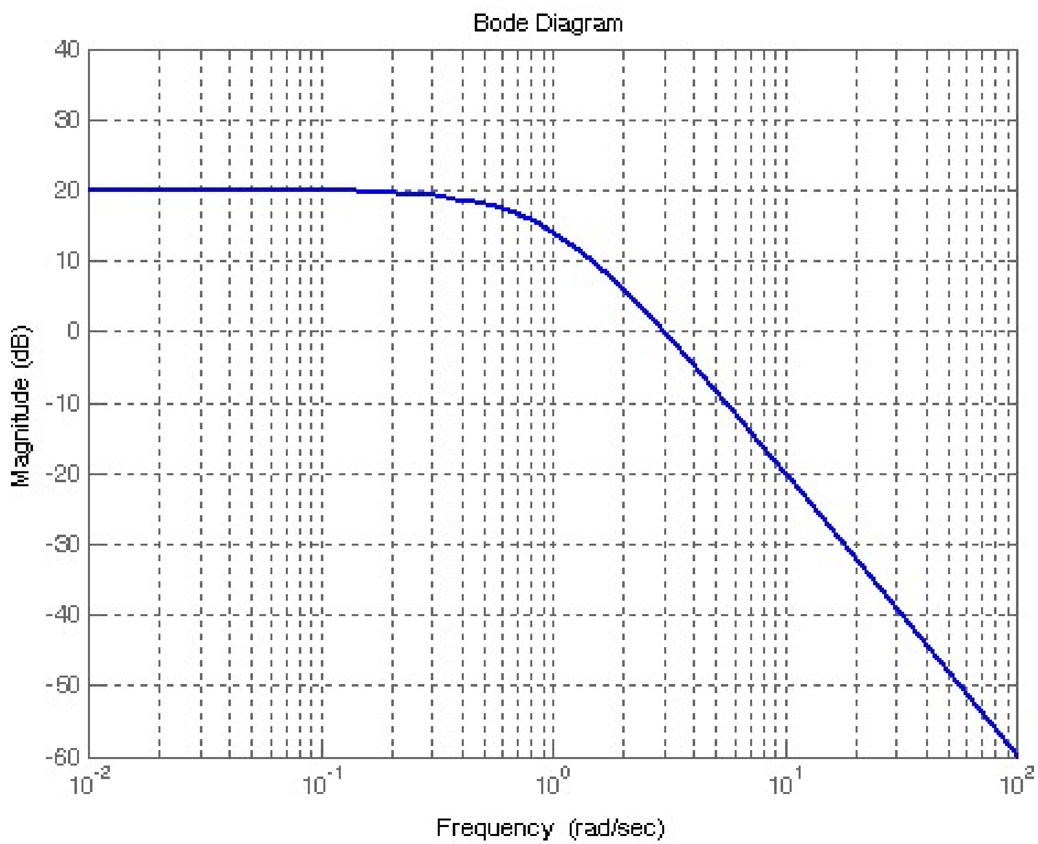
Assume that the magnitude of the frequency response of $L(s)$ has a Bode plot of the form



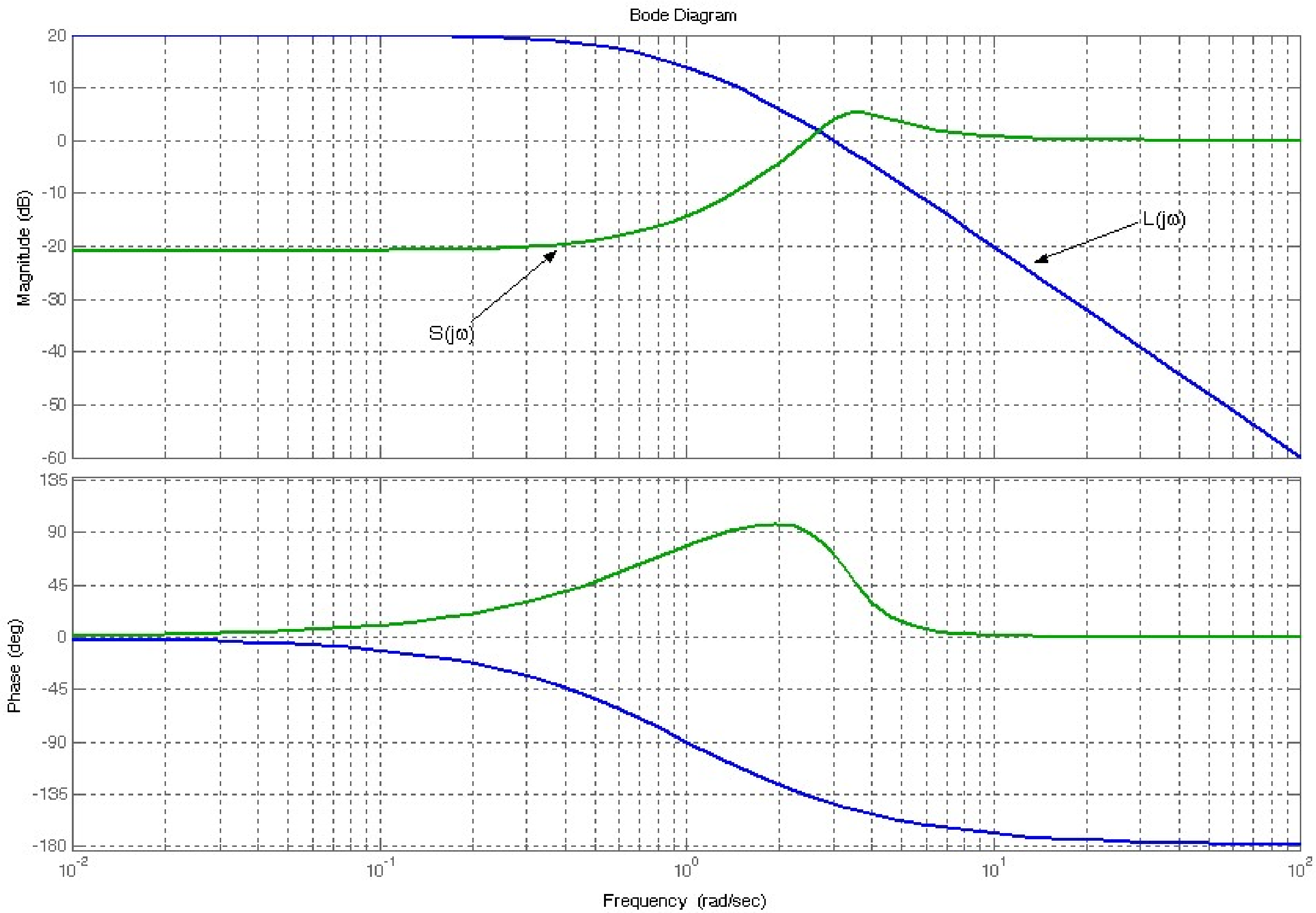
and denote with ω_c the crossover frequency, *i.e.*, $\omega_c : \quad |L(j\omega_c)| = 1$

Frequency response of S(s) (2)

$$|S(j\omega)| = \frac{1}{|1+L(j\omega)|} \simeq \begin{cases} \omega \ll \omega_c & \dfrac{1}{|L(j\omega)|} \\ \omega \gg \omega_c & 1 \end{cases}$$



Frequency response of S(s) (3)



Static performance: sinusoidal inputs (5)

Let's now study *F*(*s*) (effect of *n* on *e*).

Assume again that the Laplace transform of *n* is given by

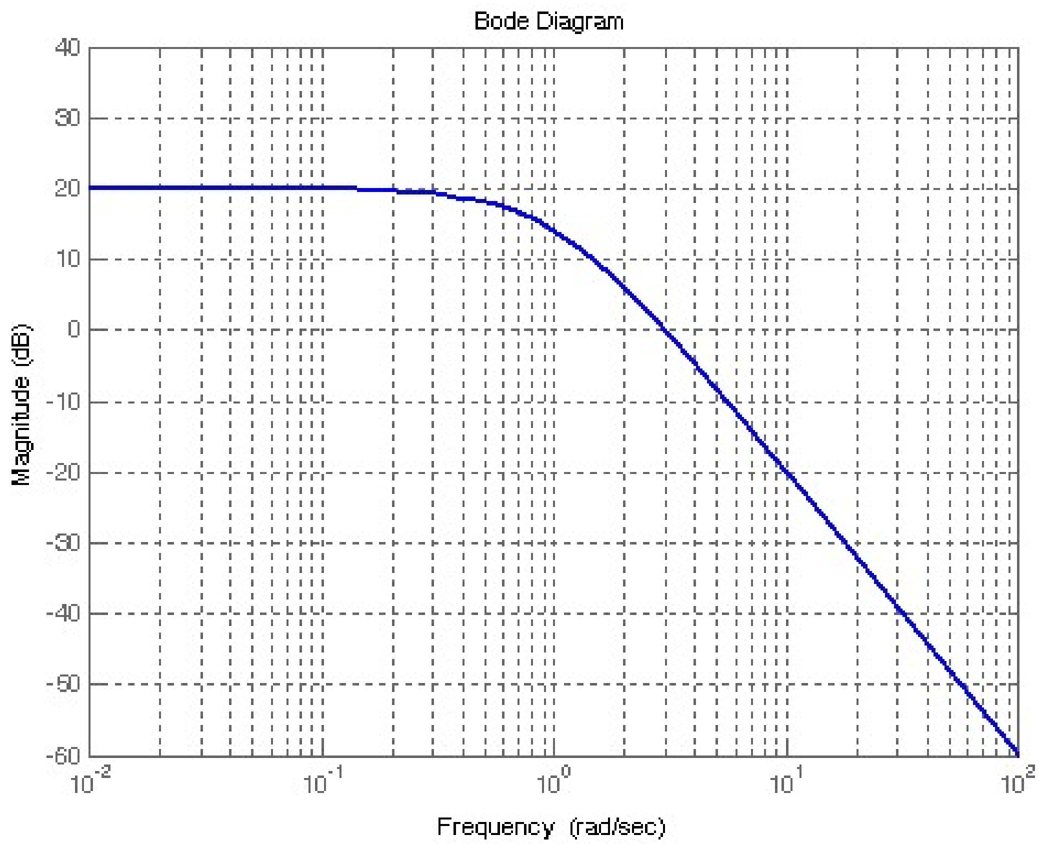
$$N(s) = \frac{as+b}{s^2+\Omega^2}$$

then the Laplace transform of *e* will be given by

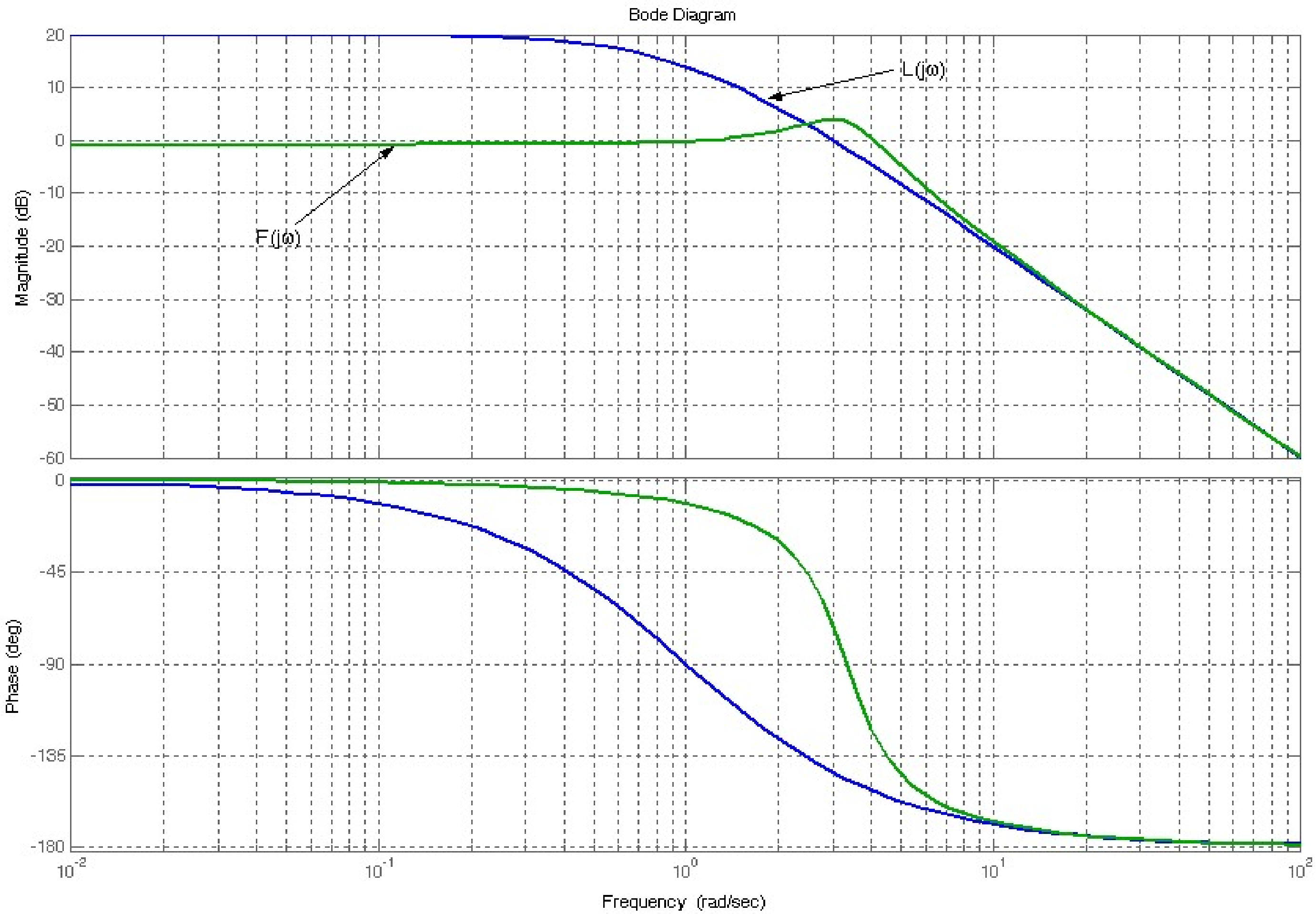
$$E(s) = F(s)\frac{as+b}{s^2+\Omega^2} = \frac{L(s)}{1+L(s)}\frac{as+b}{s^2+\Omega^2}$$

Frequency response of F(s)

$$|F(j\omega)| = \frac{|L(j\omega)|}{|1+L(j\omega)|} \simeq \begin{cases} \omega \ll \omega_c & 1 \\ \omega \gg \omega_c & |L(j\omega)| \end{cases}$$



Frequency response of F(s) (2)



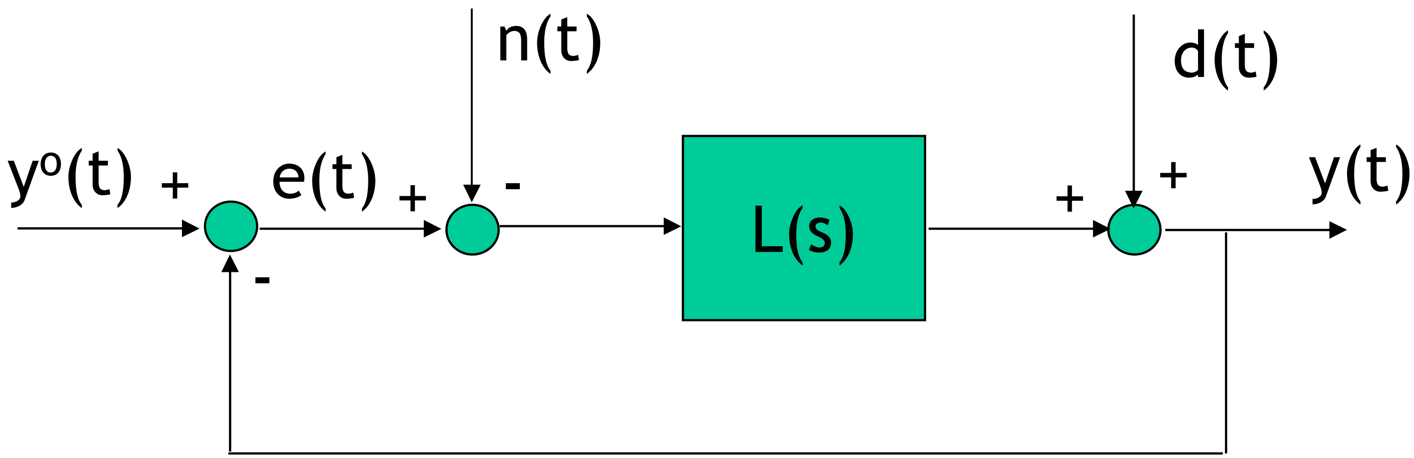
Comments

- The effect of a sinusoidal input on the control error can be analysed directly from the Bode plots of the frequency response of $L(s)$.
- The crossover frequency ω_c provides important information about the performance of the control system.
- Need for accurate tradeoffs between disturbance attenuation and tracking performance.

Dynamic performance: problem statement

We focus on $F(s)$, *i.e.*, on the shape and duration of transients due to variations of y^o.

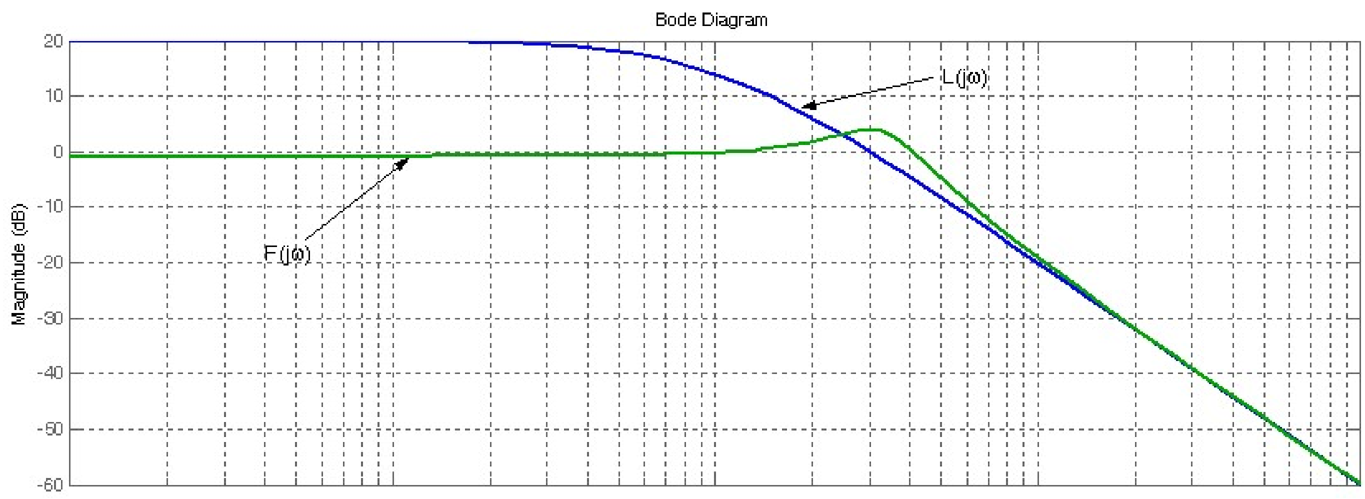
Goal: to relate transient characteristics to suitable parameters of the frequency response of $L(s)$.



We will seek a second order approximation for *F*(*s*):

$$F_2(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

Recall that $|F(j\omega)|$ looks like



how can we choose ω_n and ξ to model F(s) in an accurate way?

In order to get a slope change at $\omega=\omega_c$ we choose $\omega_n=\omega_c$

We then choose ξ such that

$$|F(j\omega_c)| = |F_2(j\omega_c)|$$

i.e., in order to have that $F_2(s)$ has the same (possible) resonant peak as F(s).

We get

$$|F(j\omega_c)| = \frac{|L(j\omega_c)|}{|1 + L(j\omega_c)|} = \frac{1}{|1 + e^{j\phi_c}|}$$

where the crossover phase φ_c is defined as $\phi_c = \angle L(j\omega_c)$

Recall now that

$$|L(j\omega_c)| = 1$$
$$\measuredangle L(j\omega_c) = \phi_c$$

so

$$|F(j\omega_c)| = \frac{|L(j\omega_c)|}{|1 + L(j\omega_c)|} = \frac{1}{|1 + e^{j\phi_c}|} = \frac{1}{2\sin(\phi_m/2)}$$

$$|F_2(j\omega_c)| = \frac{\omega_c^2}{|-\omega_c^2 + 2\xi\omega_c^2 j + \omega_c^2|} = \frac{1}{2\xi}$$

from which

$$\xi = \sin(\phi_m/2) \simeq \frac{\phi_m}{2}\text{rad} \simeq \left(\frac{\phi_m}{100}\right)^o$$

Therefore:

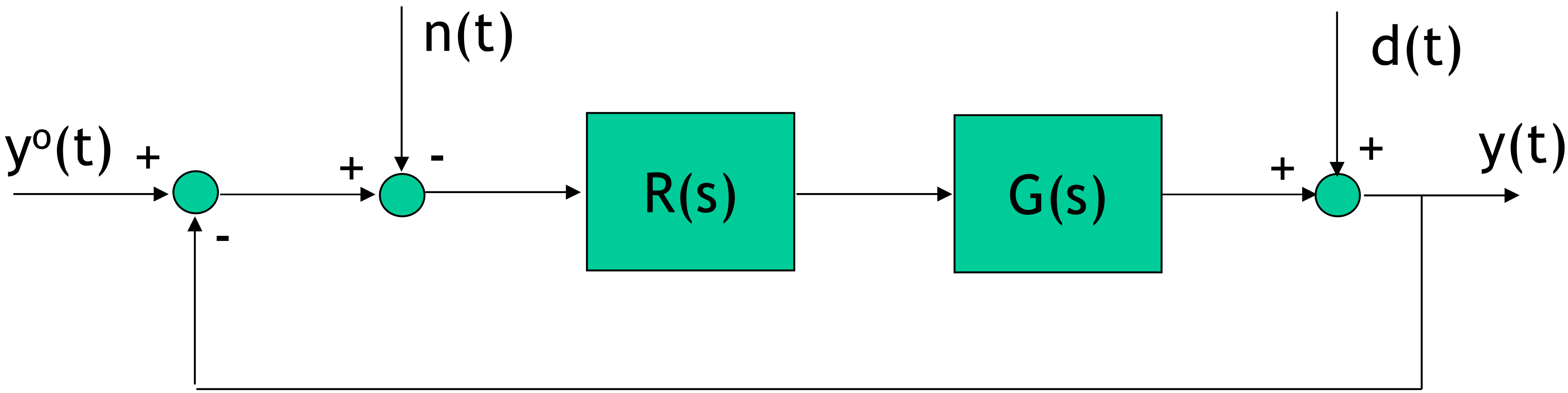
- the settling time of the approximate second order model will be given by

$$t_A = 4 \div 5 \frac{1}{\xi \omega_n} \simeq 4 \div 5 \frac{100}{\phi_m \omega_c}$$

- The second order model also makes it possible to predict the shape of the transient and the overshoot of oscillations (if any):

$$S_{\%} = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100$$

An example



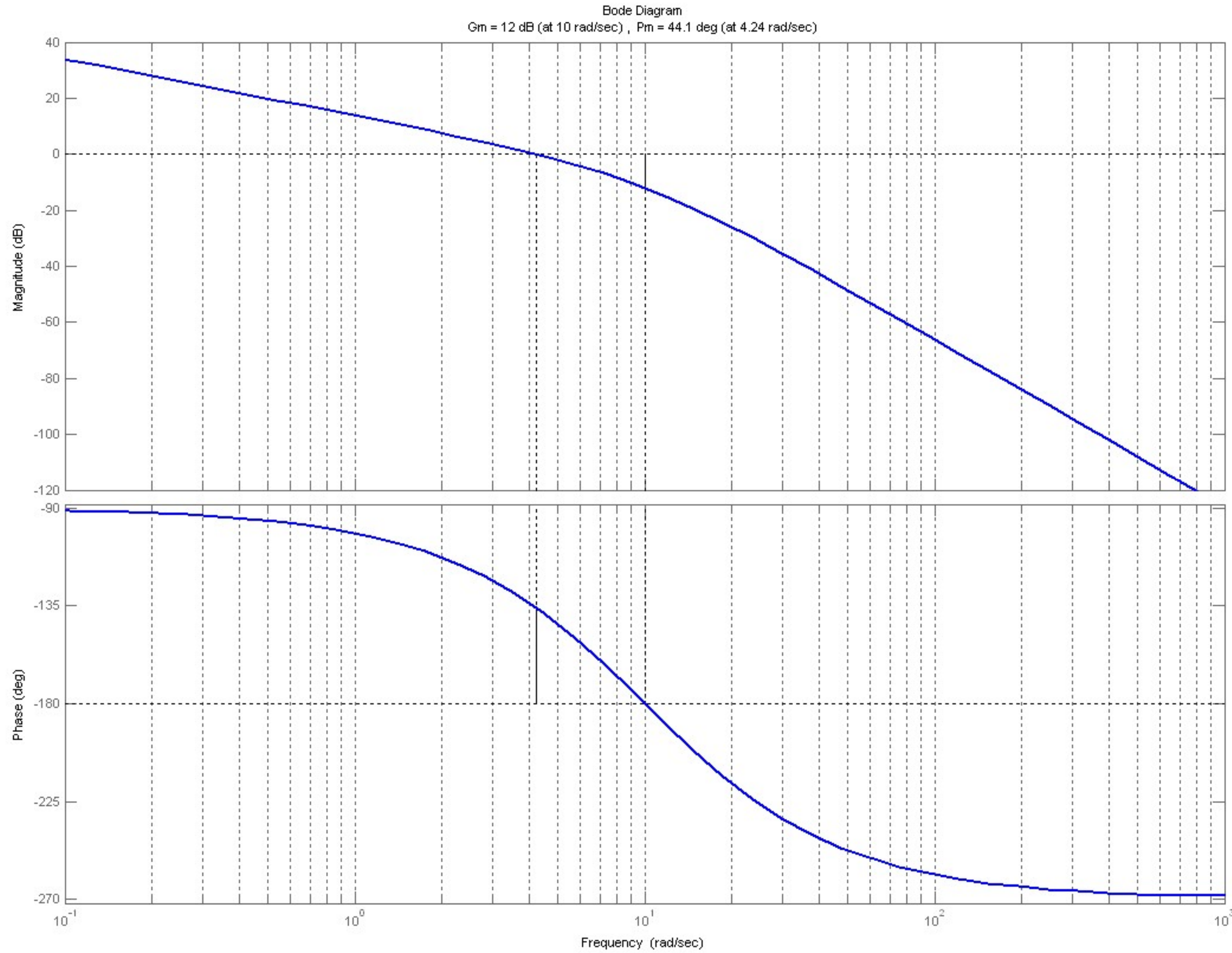
$$R(s) = \frac{5(s+1)}{s}, \qquad G(s) = \frac{1}{(s+1)(0.1s+1)^2}$$

$$L(s) = \frac{5}{s(0.1s+1)^2}$$

An example (2)

ω_c=4 rad/s

ϕ_m=44°



An example (3)

The approximate analysis leads to $F_2(s)$ given by

$$F_2(s) = \frac{\omega_c^2}{s^2 + 2\frac{\phi_m}{100}\omega_c s + \omega_c^2} = \frac{16}{s^2 + 3.52s + 16}$$

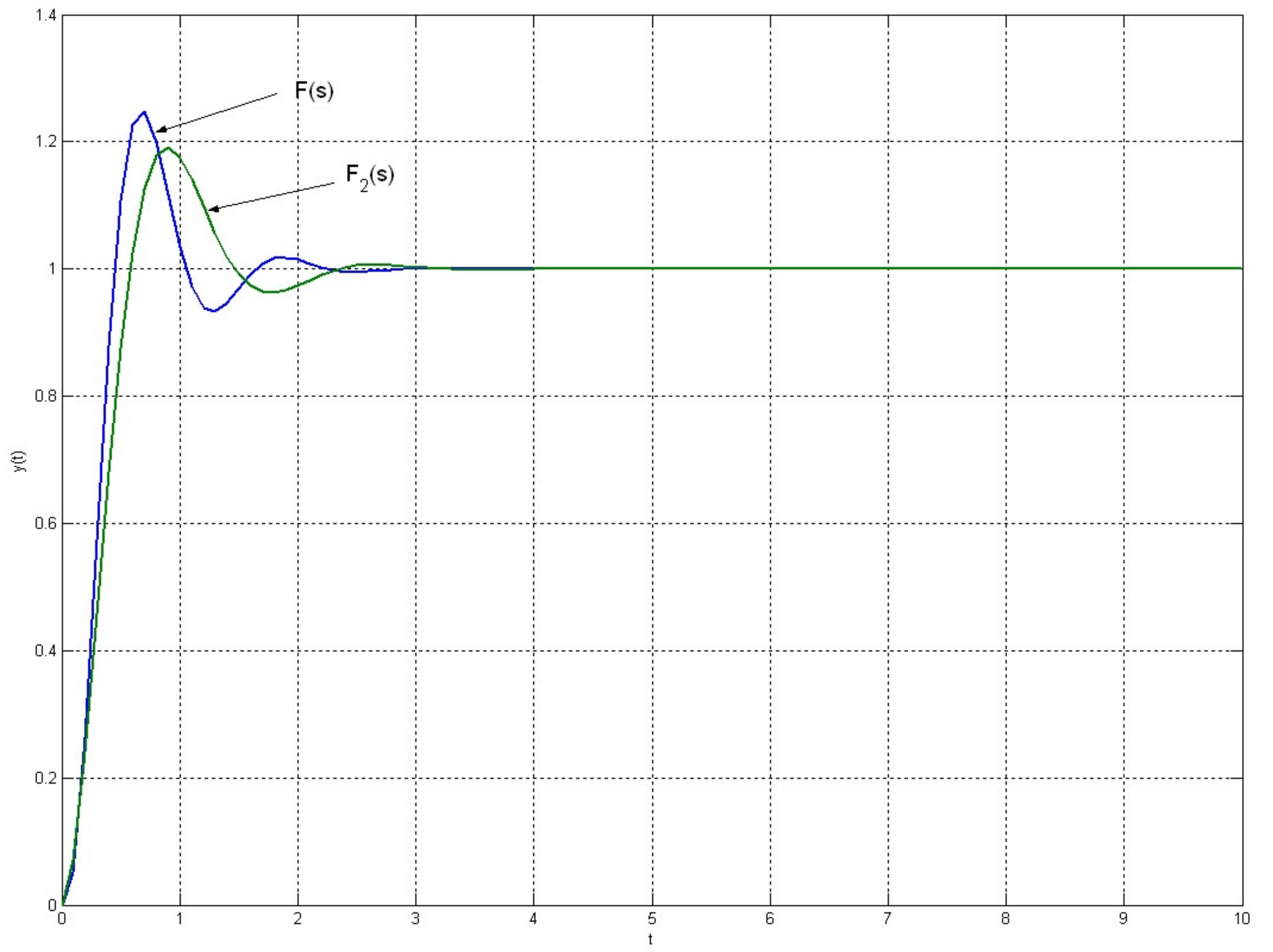
and so to the estimated settling time and overshoot

$$t_A \simeq 5\frac{100}{\phi_m \omega_c} = 2.84s$$

$$S_{\%} = e^{-\frac{\phi_m}{100}\pi/\sqrt{1-(\frac{\phi_m}{100})^2}} \times 100 = 21\%$$

An example (4)

Comparison between the step responses of $F(s)$ and $F_2(s)$:

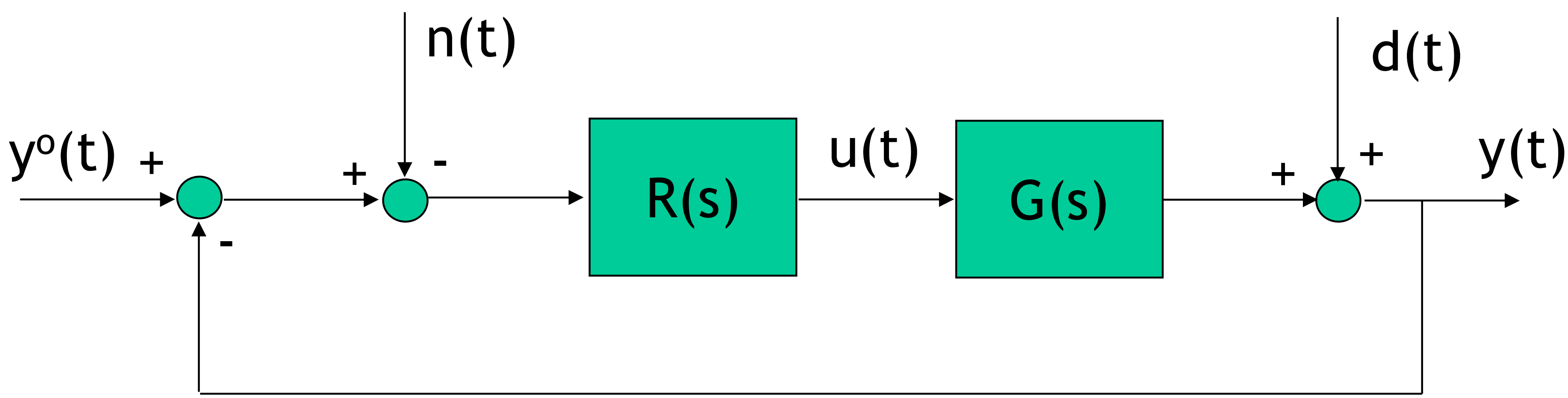


An example (5)

Comments:

- The step response of $F_2(s)$ is not identical to the $F(s)$ one but…
- …the relevant parameters are estimated in a fairly accurate way.
- Similar conclusions can be reached by analysing the resonance peak of S(s).

So far only the effect of inputs on e and y has been studied;
Undertainding how u behaves is also important, particularly during transients (risk of saturation).
To this purpose, the control sensitivity function $Q(s)$ is introduced.



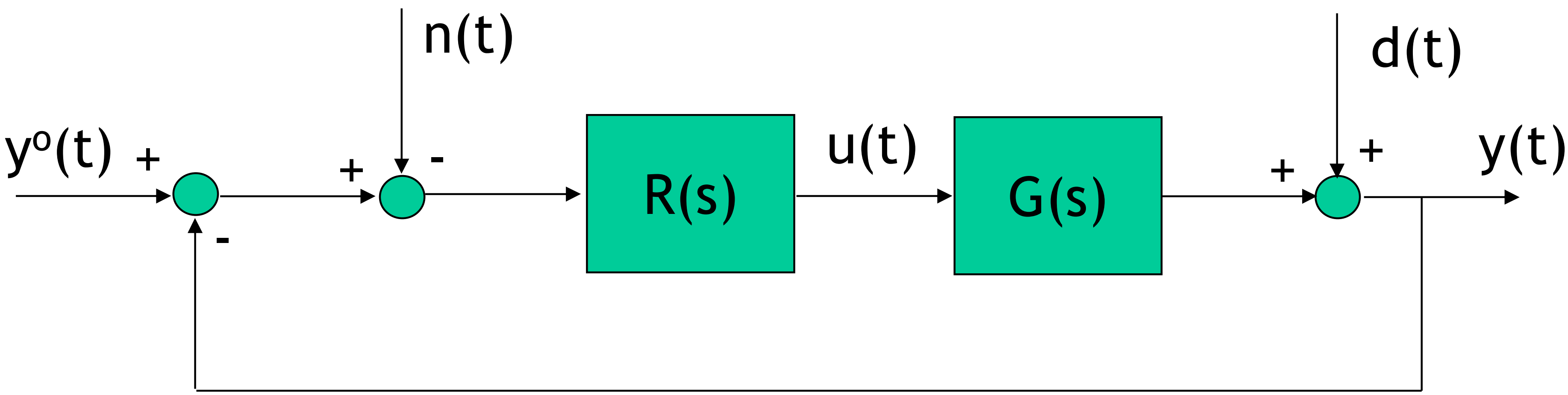
Control sensitivity (2)

Q(*s*) is defined as

$$Q(s) = \frac{R(s)}{1 + L(s)}$$

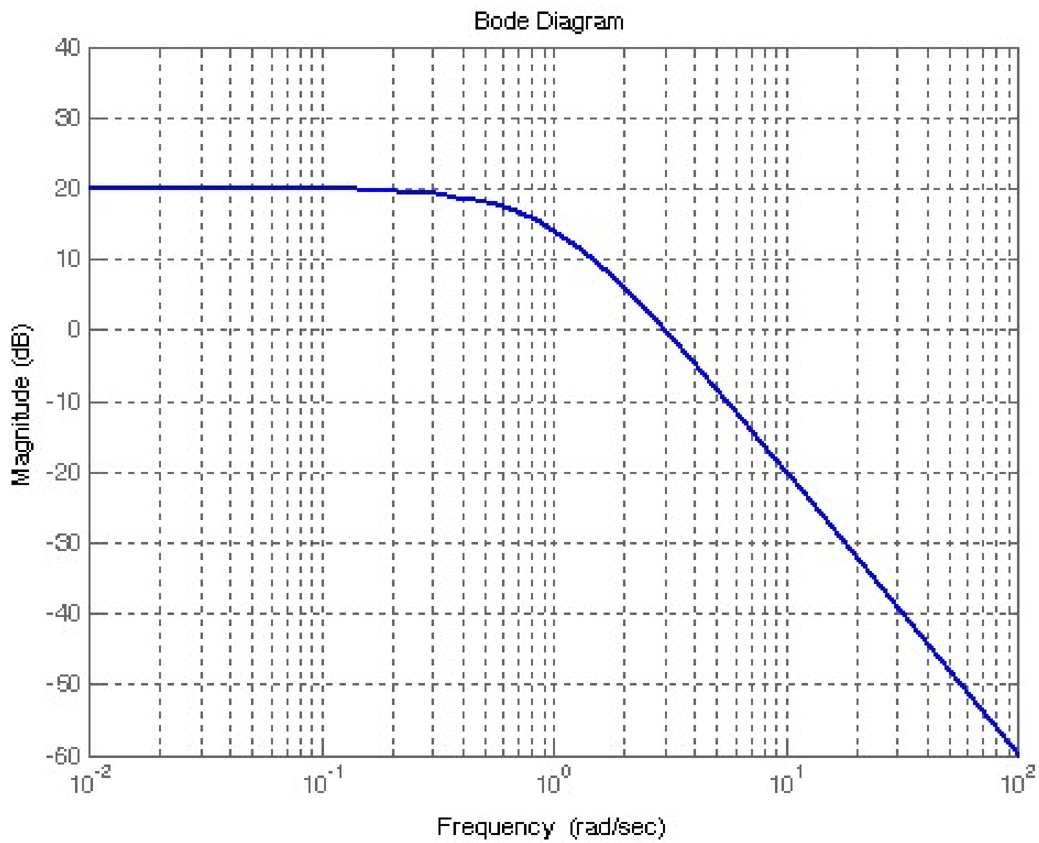
and so represents:

- the effect of y^o on u
- the effect of d on u



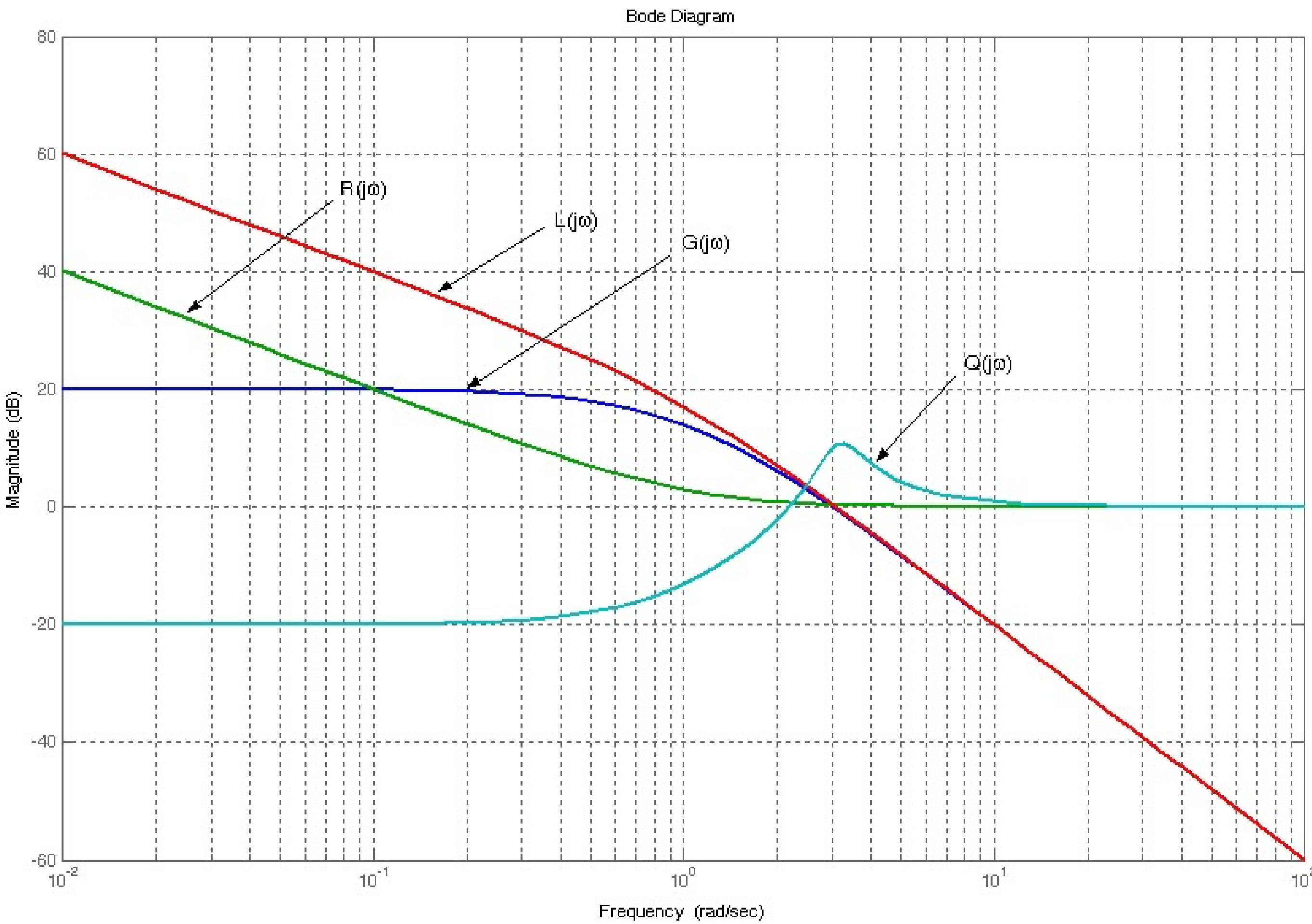
Frequency response of Q(s)

$$|Q(j\omega)| = \frac{|R(j\omega)|}{|1+L(j\omega)|} \simeq \begin{cases} \omega \ll \omega_c & \dfrac{1}{|G(j\omega)|} \\ \omega \gg \omega_c & |R(j\omega)| \end{cases}$$



An example:

$$R(s) = \frac{s+1}{s}, \qquad G(s) = \frac{10}{(s+1)^2}$$



Performance requirements as frequency weights

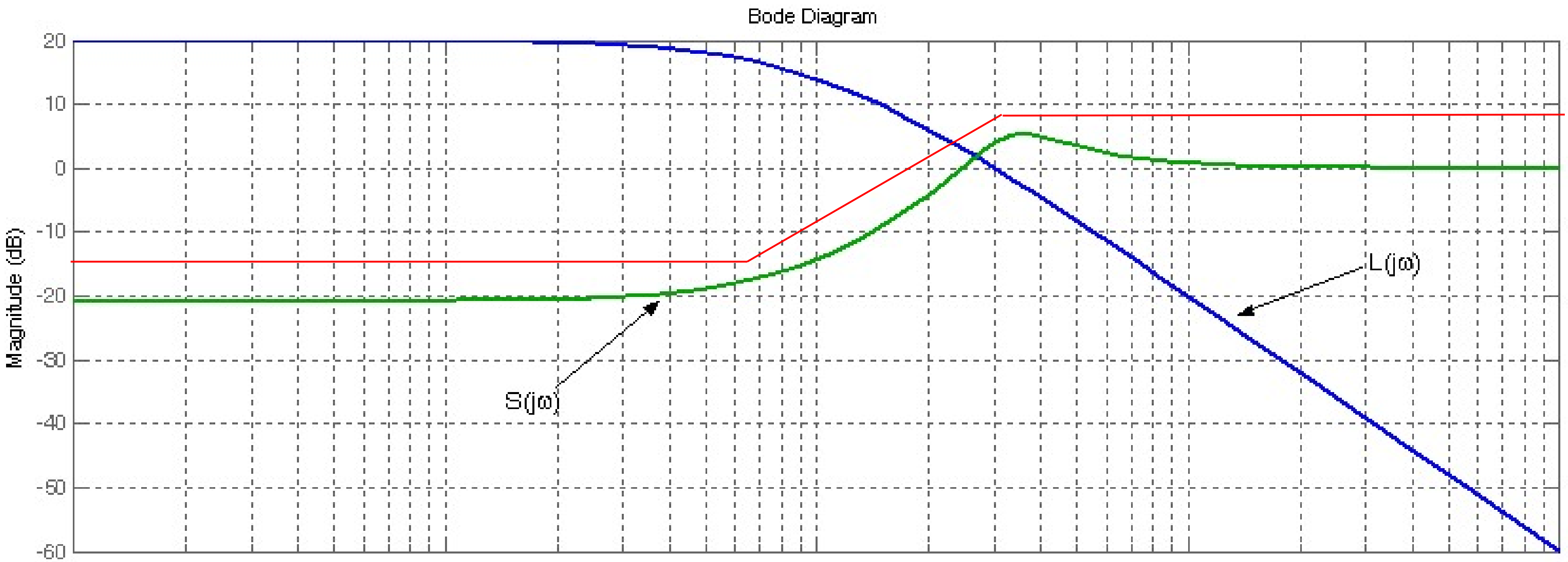
- In summary, one can say that the shape of the frequency response of the sensitivity function defines the actual closed-loop performance of the feedback system.

- Different aspects of performance relate to different properties of the frequency response, but it should be possible to represent requirements concisely as *frequency-dependent* weights on the response.

- Consider the sensitivity function as an example.

Performance requirements as frequency weights

- Consider the sensitivity function as an example



- And assume a transfer function $W_p(s)$ can be found with the property that:

$$|S(j\omega)| \leq \frac{1}{|W_p(j\omega)|} \quad \forall \omega$$

Performance requirements as frequency weights

- Transfer function $W_p(s)$ can be chosen to have
  - The desired slope or value at low frequency (which defines the steady-state error for canonical inputs)
  - The desired magnitude over the control system bandwidth (which defines the steady-state error for sinusoidal/periodic/finite-energy inputs)
  - The desired crossover frequency and peak amplitude (which define settling time and maximum overshoot of the step response)
- The set of inequalities $|S(j\omega)| \leq \dfrac{1}{|W_p(j\omega)|} \quad \forall\omega$ however can be only verified qualitatively if a graphical approach is used.