POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

14 Unsteady Aerodynamics: Panel method for unsteady compressible flows (Morino's method)

Giuseppe Quaranta
Dipartimento di Scienze e Tecnologie Aerospaziali

Material

Dowell Section 4.3.2

Morino Mastroddi notes

Introduction

- ✓ The methodology stems from a smart usage of **Green's theorem** (or divergence theorem).
- ✓ The advantage is to lead to a very simple integral expression that can be easily solved using a numerical approach
- ✓ Several authors worked on this formulation. However, the most complete work was developed Prof. Morino who applied it to subsonic and supersonic flows. This is why here it will be nominated as "Morino's method"
- ✓ This method represents a prototype for all possible panel methods in 2D/3D compressible/incompressible subsonic/supersonic flows

Morino L, Chen LT, Suciu EO (1975) Steady and oscillatory subsonic and supersonic aerodynamics around complex configurations. AIAA J 13(3):368–374

Green's theorem

The divergence theorem reads

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{b} \, \mathrm{d}v = \int_{\partial V} \mathbf{n} \cdot \mathbf{b} \, \mathrm{d}s$$

The generic vector $\mathbf{b}$ could be seen as the product of a scalar φ times the gradient of another scalar function G, i.e.

$$\mathbf{b} = \varphi \boldsymbol{\nabla} G$$

Applying the divergence theorem to this function it results

$$\int_V \boldsymbol{\nabla} \cdot (\varphi \boldsymbol{\nabla} G) \, \mathrm{d}v = \int_{\partial V} \varphi \, \mathbf{n} \cdot \boldsymbol{\nabla} G \, \mathrm{d}s$$

It is possible to verify that for any scalar a and vector $\mathbf{b}$

$$\boldsymbol{\nabla} \cdot (a\mathbf{b}) = a \boldsymbol{\nabla} \cdot \mathbf{b} + \boldsymbol{\nabla} a \cdot \mathbf{b}$$

Green's identity

So

$$\boldsymbol{\nabla} \cdot (\varphi \boldsymbol{\nabla} G) = \varphi \boldsymbol{\nabla}^2 G + \boldsymbol{\nabla} G \cdot \boldsymbol{\nabla} \varphi$$

and at the same time

$$\boldsymbol{\nabla} \cdot (G \boldsymbol{\nabla} \varphi) = G \boldsymbol{\nabla}^2 \varphi + \boldsymbol{\nabla} G \cdot \boldsymbol{\nabla} \varphi$$

Using the divergence theorem it is possible to say that

$$\int_V (\boldsymbol{\nabla} \cdot (\varphi \boldsymbol{\nabla} G) - \boldsymbol{\nabla} \cdot (G \boldsymbol{\nabla} \varphi)) \, \mathrm{d}v = \int_{\partial V} \mathbf{n} \cdot (\varphi \boldsymbol{\nabla} G - G \boldsymbol{\nabla} \varphi) \, \mathrm{d}s$$

Using the vectorial identities above

$$\int_V (\varphi \boldsymbol{\nabla}^2 G - G \boldsymbol{\nabla}^2 \varphi) \, \mathrm{d}v = \int_{\partial V} \left(\varphi \frac{\partial G}{\partial \mathbf{n}} - G \frac{\partial \varphi}{\partial \mathbf{n}} \right) \mathrm{d}s$$

Green's Identity

Fundamental solution of Laplace equation

Consider to take as function G the fundamental solution to Laplace equation, i.e. the computation of the potential that results from having a singularity in the point $\mathbf{x}_0$

$$\boldsymbol{\nabla}^2 G = \delta(|\mathbf{x} - \mathbf{x}_0|) = \prod_{i=1}^{n} \delta(x_i - x_{0_i})$$

The solution to this problem is known and equal to

$$G = \begin{cases} -\frac{1}{2\pi} \log r & n = 2 \\ -\frac{1}{4\pi r} & n = 3 \end{cases}$$

with $r = |\mathbf{x} - \mathbf{x}_0|$.

Basic expression of Morino's method

Using as G the fundamental solution to the LAplace problem it is easy to verify that

$$\int_V \varphi \boldsymbol{\nabla}^2 G \, \mathrm{d}v = \int_V \varphi \delta(|\mathbf{x} - \mathbf{x}_0|) \, \mathrm{d}v = E(\mathbf{x}_0)\varphi(\mathbf{x}_0)$$

where

$$E(\mathbf{x}_0) = \begin{cases} 0 & \text{if } \mathbf{x}_0 \notin V, \text{ External} \\ 1 & \text{if } \mathbf{x}_0 \in V \text{ and } \mathbf{x}_0 \notin \partial V, \text{ Internal} \\ \dfrac{1}{2} & \text{if } \mathbf{x}_0 \in \partial V \end{cases}$$

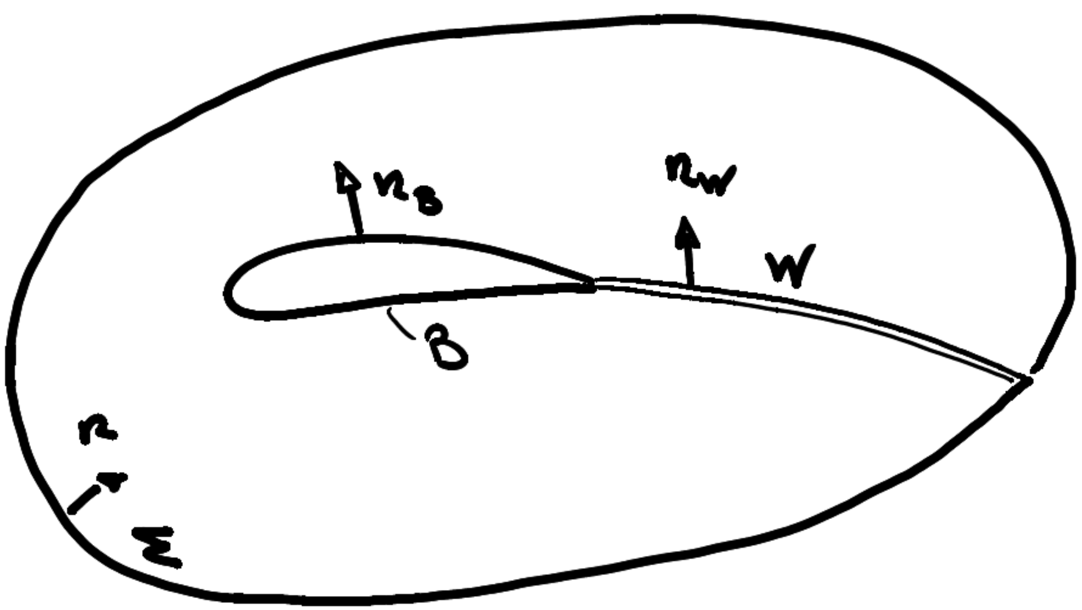
Using this expression in the Green's identity, it result that

$$\boxed{E(\mathbf{x}_0)\varphi(\mathbf{x}_0) = \int_{\partial V} \left(\varphi \frac{\partial G}{\partial \mathbf{n}} - G \frac{\partial \varphi}{\partial \mathbf{n}} \right) \mathrm{d}s + \int_V G \boldsymbol{\nabla}^2 \varphi \, \mathrm{d}v}$$

This expression could be used to compute the potential φ when the solution to the fundamental problem G is known.

Incompressible flow

$$\nabla^2 \varphi = 0 \qquad \lim_{R \to \infty} \varphi = \varphi|_\Sigma = 0$$



$$\partial V = \Sigma \cup B \cup W$$

The boundary of the domain is the union of:

1. The body B
2. The wake W
3. The infinite Σ

The volume integral disappears, as the boundary integral on Σ.

Basic expression and BC for incompressible case

$$E(\mathbf{x}_0)\varphi(\mathbf{x}_0) = \int_B \left(\varphi\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial \varphi}{\partial \mathbf{n}}\right) \mathrm{d}s + \int_W \left(\varphi\frac{\partial G}{\partial \mathbf{n}} - G\frac{\partial \varphi}{\partial \mathbf{n}}\right) \mathrm{d}s$$

Remembering the boundary conditions on the body

$$\left.\frac{\partial \varphi}{\partial \mathbf{n}}\right|_B = \boldsymbol{\nabla}\varphi \cdot \mathbf{n}_B = \mathbf{u} \cdot \mathbf{n}_B = \mathbf{v}_B \cdot \mathbf{n}_B = \frac{\partial \eta}{\partial t} + U_\infty \frac{\partial \eta}{\partial x}$$

For the wake we have to remember that the wake is aligned with the local flow field, so

$$\mathbf{u}|_W \cdot \mathbf{n}_W = 0 \rightarrow \left.\frac{\partial \varphi}{\partial \mathbf{n}}\right|_W = 0$$

$$E(\mathbf{x}_0)\varphi(\mathbf{x}_0) = \int_B \varphi\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}s - \int_B G\frac{\partial \varphi}{\partial \mathbf{n}}\mathrm{d}s + \int_W \varphi\frac{\partial G}{\partial \mathbf{n}}\mathrm{d}s$$

Steady case: analysis of the wake

The wake is unloaded, so the jump of pressure across it must be equal to zero. Using the unsteady Bernoulli theorem

$$\Delta C_p|_W = -\frac{2}{U_\infty^2}\left(U_\infty \frac{\partial \Delta\varphi}{\partial x} + \frac{\partial \Delta\varphi}{\partial t}\right)_W = 0$$

Since the flow is steady, and so all derivatives with respect to time are null

$$\frac{2}{U_\infty}\frac{\partial}{\partial x}\left(\varphi_U - \varphi_L\right)_W = 0$$

The jump of potential on the wake along the direction of the flow x is null. So it is only function of span y, i.e.

$$\left(\varphi_U - \varphi_L\right)_W(x, y) = \left(\varphi_U - \varphi_L\right)_W(x = TE, y) = \Delta\varphi_{TE}(y)$$

The jump of potential on the wake at the Traling Edge (TE) is equal to the circulation at that span station y, as prescibed by Kutta condition

$$\Delta\varphi_{TE}(y) = \Gamma(y) \propto L(y)$$

where L is the lift per unit span of the wing.

Incompressible, steady; basic formulation

The obtained integral expression can be used to compute the potential on every point in the flow field when we know the potential on B, or to compute the potential on B itself

For $\mathbf{x}_0 \in B$ and $\mathbf{x} \subset \mathbb{R}^3$

$$\frac{1}{2}\varphi(\mathbf{x}_0) = -\frac{1}{4\pi}\underbrace{\int_B \varphi \frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{r}\right)}_{\text{SOURCES}} \mathrm{d}s + \frac{1}{4\pi}\underbrace{\int_B \frac{1}{r}\frac{\partial \varphi}{\partial \mathbf{n}}}_{\text{DOUBLETS}}\mathrm{d}s - \frac{1}{4\pi}\int_W \Delta\varphi_{TE}\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{r}\right)\mathrm{d}s$$

$$2\pi\varphi(\mathbf{x}_0) = -\int_B \varphi \frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{r}\right)\mathrm{d}s + \int_B \frac{1}{r}\frac{\partial \varphi}{\partial \mathbf{n}}\mathrm{d}s - \int_W \Delta\varphi_{TE}\frac{\partial}{\partial \mathbf{n}}\left(\frac{1}{r}\right)\mathrm{d}s$$

Discretization: panel method

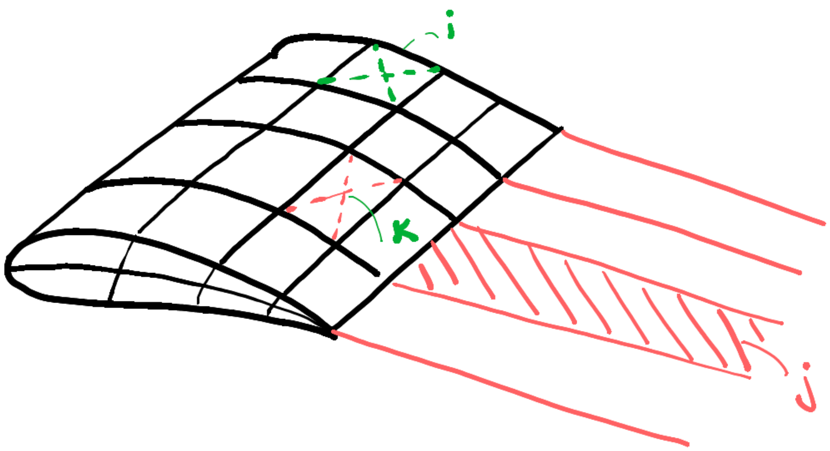
1. Divide the surface into $N = N_x \times N_y$ panels and the wake into N_y semi-infinite strips

2. Define the type of approximation for the potential φ over each panel

   - 0-th order $\rightarrow$ φ constant on each panel
   - 1-st order $\rightarrow$ φ linear on each panel, i.e. differnt values of φ on each node

3. decide how and where the BC must be imposed on each panel

For 0-th order panel the best collocation point (point where the Boundary Condition is applied) is at the center of the panel

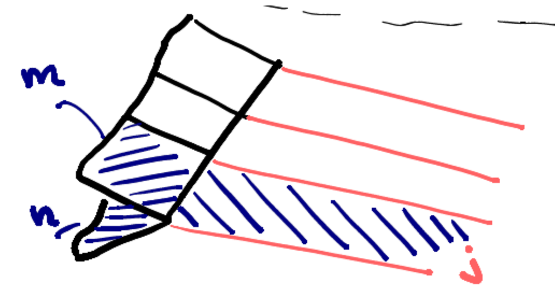


Discretization

$$2\pi\varphi_i = -\sum_{k=1}^{N}\varphi_k \int_{B_k} \frac{\partial}{\partial \mathbf{n}_k}\left(\frac{1}{r_{ik}}\right)\mathrm{d}s + \sum_{k=1}^{N}\frac{\partial \varphi_k}{\partial \mathbf{n}_k}\int_{B_k}\frac{1}{r_{ik}}\mathrm{d}s - \sum_{j=1}^{N_y}\Delta\varphi_{TE_j}\int_{W_j}\frac{\partial}{\partial \mathbf{n}_j}\left(\frac{1}{r_{ij}}\right)\mathrm{d}s$$

$$\sum_{j=1}^{N_y}\Delta\varphi_{TE_j}\int_{W_j}\frac{\partial}{\partial \mathbf{n}_j}\left(\frac{1}{r_{ij}}\right)\mathrm{d}s = \sum_{k=1}^{N} h_k\varphi_k \int_{W_j}\frac{\partial}{\partial \mathbf{n}_k}\left(\frac{1}{r_{ik}}\right)\mathrm{d}s$$

$$h_k = \begin{cases} 0 & k \text{ is not a TE panel} \\ 1 & k \text{ is a TE upper panel} \\ -1 & k \text{ is a TE lower panel} \end{cases}$$



It is not necessary to have potential values associated with the wake panels. The jump of potential of the wake pane is equal to the difference between the potential of the corresponding TE panels (imposition of the Kutta condition).

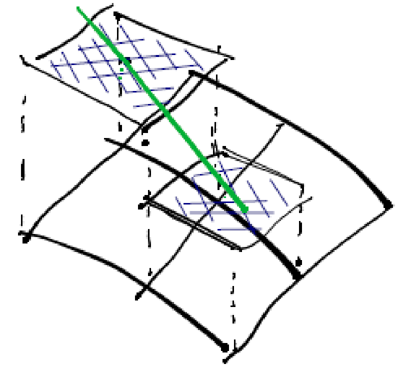
Discretization

$$\mathbf{Y}\varphi = \mathbf{Z}\frac{\partial \varphi}{\partial \mathbf{n}}$$

Kroneker delta

$$\begin{aligned} Y_{ik} &= 2\pi\delta_{ik} + \int_{B_k} \frac{\partial}{\partial \mathbf{n}_k}\left(\frac{1}{r_{ik}}\right) \mathrm{d}s + h_k \int_{W_k} \frac{\partial}{\partial \mathbf{n}_k}\left(\frac{1}{r_{ik}}\right) \mathrm{d}s \\ Z_{ik} &= \int_{B_k} \frac{1}{r_{ik}} \mathrm{d}s \end{aligned}$$

Computation of loads



$$C_p = -\frac{2}{U_\infty}\frac{\partial \varphi}{\partial x}$$

If we approximate the continuous potential function with

$$\tilde{\varphi}(x,y) = \sum_i N_i(x,y)\varphi_i = \mathbf{N}_\varphi \boldsymbol{\varphi}$$

$$\frac{\partial \tilde{\varphi}}{\partial x} = \frac{\partial \mathbf{N}_\varphi}{\partial x}\boldsymbol{\varphi} = \mathbf{P}\boldsymbol{\varphi}$$

If we call $\mathbf{A} = \mathbf{Y}\mathbf{Z}^{-1}$ then

$$\mathbf{A}\boldsymbol{\varphi} = U_\infty \boldsymbol{\alpha} \quad \boldsymbol{\alpha} = \frac{\partial \eta}{\partial x}$$

The pressure distribution will be

$$\mathbf{C}_p = -2\mathbf{P}\mathbf{A}^{-1}\boldsymbol{\alpha}$$

If 0-th order panel are used it is necessary to define a methodology to reconstruct the slope of the geometry at the panel center. So, an interpolated potential $\tilde{\varphi}$ must be defined using shape functions $N\varphi$

Incompressible, unsteady case: wake

$$\Delta C_p\big|_W = 0 \qquad U_\infty \frac{\partial \Delta\varphi}{\partial x} + \frac{\partial \Delta\varphi}{\partial x} = 0$$

$$-\frac{2}{U_\infty^2}\left(U_\infty \frac{\partial \Delta\varphi}{\partial x} + \frac{\partial \Delta\varphi}{\partial t}\right) = 0$$

with the following BC at the TE

$$\Delta\varphi(b, y, t) = \Delta\varphi_{TE}(y, t) = \Gamma(y, t)$$

Transforming in frequency domain

$$\frac{\partial \Delta\varphi}{\partial x} + \frac{\mathrm{j}\omega}{U_\infty}\Delta\varphi = 0$$

The solution of this linear differential equation in space is

$$\Delta\varphi = c\mathrm{e}^{-\mathrm{j}\frac{\omega}{U_\infty}x} = c\mathrm{e}^{-\mathrm{j}k\tilde{x}}, \qquad \tilde{x} = \frac{x}{b}$$

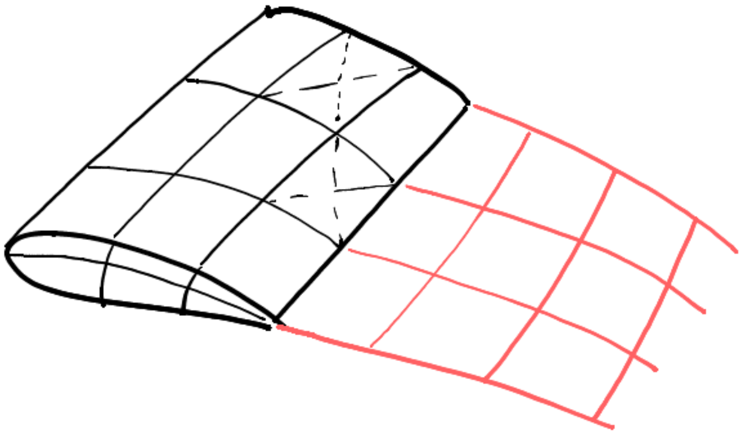
To find the constant c it is necessary to use the BC $c = \Gamma(y)\mathrm{e}^{\mathrm{j}k}$ so

$$\Delta\varphi = \Gamma(y)\mathrm{e}^{-\mathrm{j}k(\tilde{x}-1)}$$

The jump of potential along the wake panels is not constant anymore: it is equal to the product of the jump of potential at the trailing edge multiplied by a delay operator (the exponential term) that express the fact that the vorticity travels in the wake at the asymptotic speed

Discretization

Now it is necessary to divide the wake into individual panels also along the length x direction due to the delay term



$$
\begin{aligned}
Y_{ij} &= 2\pi\delta_{ij} + \int_{B_j} \frac{\partial}{\partial \mathbf{n}_j}\left(\frac{1}{r_{ij}}\right) \mathrm{d}s \\
&+ h_j \mathrm{e}^{-\mathrm{j}k(\tilde{x}_j - 1)} \int_{W_j} \frac{\partial}{\partial \mathbf{n}_j}\left(\frac{1}{r_{ij}}\right) \mathrm{d}s \\
Z_{ij} &= \int_{B_j} \frac{1}{r_{ij}} \mathrm{d}s
\end{aligned}
$$

$$
\mathbf{Y}\boldsymbol{\varphi} = \mathbf{Z}\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} = \mathbf{Z}U_\infty\left(\frac{\partial \boldsymbol{\eta}}{\partial x} + \frac{\mathrm{j}\omega}{U_\infty}\boldsymbol{\eta}\right)
$$

$$
\mathbf{C}_p = -\frac{2}{U_\infty}\left(\mathbf{P}\boldsymbol{\varphi} + \frac{\mathrm{j}\omega}{U_\infty}\boldsymbol{\varphi}\right)
$$

Assigning a reduced frequency k it is possible to compute the loads generated by the different body displacements and so a tabulated version of the aerodynamic transfer matrix

If the distance between panels is larger than the size of the panel the delay could be considered constant and moved out of the integrals

Compressible cases: steady flow

$$\nabla^2\varphi = M_\infty^2 \frac{\partial^2\varphi}{\partial x^2}$$

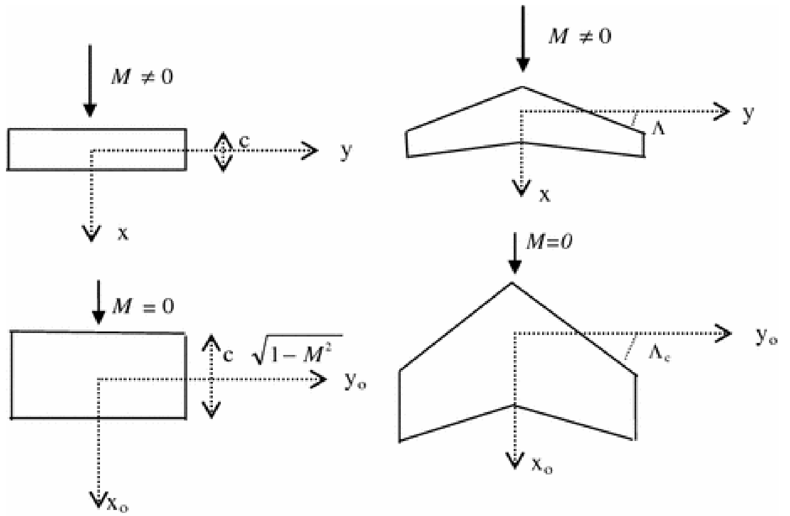
In these cases, it is possible to apply the PRANDTL-GLAUERT transformation to transform the equation into the original (incompressible) format

$$\begin{cases} \hat{x} &= \frac{x}{\beta} \\ \hat{y} &= y \\ \hat{z} &= z \end{cases}$$

$$\Rightarrow \nabla^2\hat{\varphi} = 0$$

with $\hat{\varphi} = \varphi(\hat{x}.\hat{y}, \hat{z})$.

with $\beta^2 = 1 - M_\infty^2$.



Problem formulation

$$\hat{r}^2 = (\hat{x}-\hat{x}_0)^2 + (\hat{y}-\hat{y}_0)^2 + (\hat{z}-\hat{z}_0)^2$$
$$\hat{r}^2 = \frac{1}{\beta^2}\left((x-x_0)^2 + \left[(y-y_0)^2 + (z-z_0)^2\right]\beta^2\right) = \frac{r_\beta^2}{\beta^2}$$

r_β^2 is always positive in subsonic flow while it can be negative in supersonic flows

$$2\pi\hat{\varphi}(\hat{\mathbf{x}}_0) = \int_{\hat{B}} \hat{\varphi}\frac{\partial}{\partial\hat{\mathbf{n}}}\left(\frac{1}{\hat{r}}\right)\mathrm{d}\hat{s} - \int_{\hat{B}} \frac{1}{\hat{r}}\frac{\partial\hat{\varphi}}{\partial\hat{\mathbf{n}}}\mathrm{d}\hat{s} + \int_{\hat{W}} \Delta\hat{\varphi}_{TE}\frac{\partial}{\partial\hat{\mathbf{n}}}\left(\frac{1}{\hat{r}}\right)\mathrm{d}\hat{s}$$

Since $\hat{r} = r_\beta/\beta$ and $\mathrm{d}\hat{s} = \mathrm{d}s/\beta$ then

$$2\pi\varphi(\mathbf{x}_0) = \int_{B} \varphi\frac{\partial}{\partial\mathbf{n}}\left(\frac{1}{r_\beta}\right)\mathrm{d}s - \int_{B} \frac{1}{r_\beta}\frac{\partial\varphi}{\partial\mathbf{n}}\mathrm{d}s + \int_{W} \Delta\varphi_{TE}\frac{\partial}{\partial\mathbf{n}}\left(\frac{1}{r_\beta}\right)\mathrm{d}s$$

$$Y_{ik} = 2\pi\delta_{ik} + \int_{B_k} \frac{\partial}{\partial\mathbf{n}_k}\left(\frac{1}{r_{\beta_{ik}}}\right)\mathrm{d}s + h_k\int_{W_k} \frac{\partial}{\partial\mathbf{n}_k}\left(\frac{1}{r_{\beta_{ik}}}\right)\mathrm{d}s \quad Z_{ik} = \int_{B_k} \frac{1}{r_{\beta_{ik}}}\mathrm{d}s$$

Unsteady supersonic flow: fundamental solution

$$\nabla^2\varphi - \frac{1}{a_\infty^2}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2} = 0$$

We need to look for the fundamental solution of this modified Laplace equation, i.e.

$$\nabla^2 G - \frac{1}{a_\infty^2}\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} = \delta(|\mathbf{x} - \mathbf{x}_0|)\delta(t)$$

Transforming in frequency domain

$$\nabla^2 G + \frac{\omega^2}{a_\infty^2} G = \delta(|\mathbf{x} - \mathbf{x}_0|)$$

The general solution is

$$G = -\frac{1}{4\pi r_\beta}\mathrm{e}^{-\mathrm{j}\omega\tau}$$

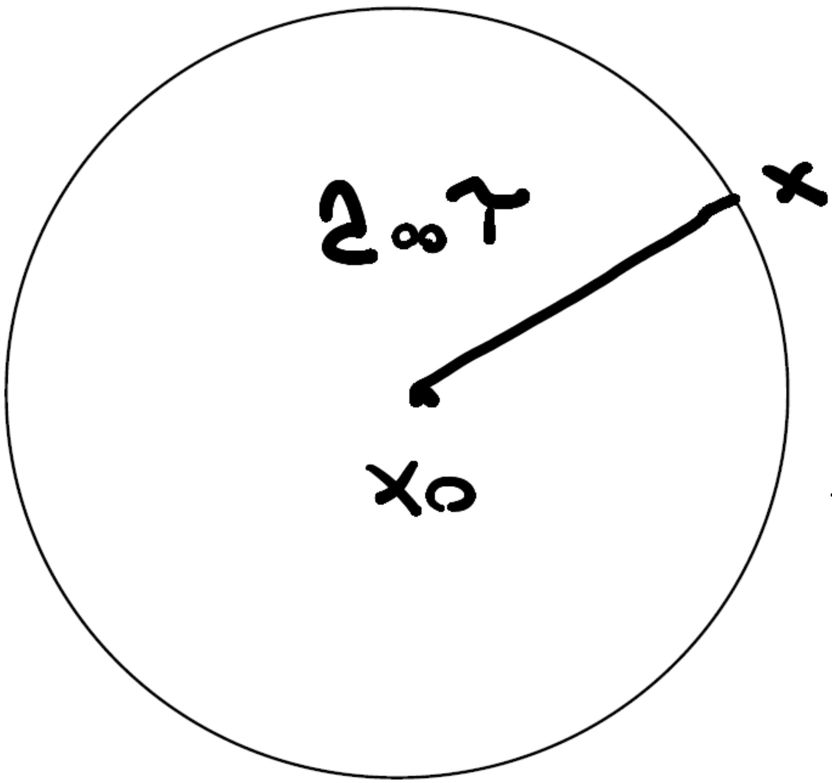
with τ the time required by the perturbation to travel from $\mathbf{x}_0$, the point where it started, to $\mathbf{x}$.

Explanation of τ (for the fluid at rest)

Given $\mathbf{x}$ and $\mathbf{x}_0$, τ is the amount of time we have to go back in $\mathbf{x}_0$ to understand the intensity of the perturbation that reached $\mathbf{x}$ in the current instant of time.

$$a_\infty^2 \tau^2 = r^2 = (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$

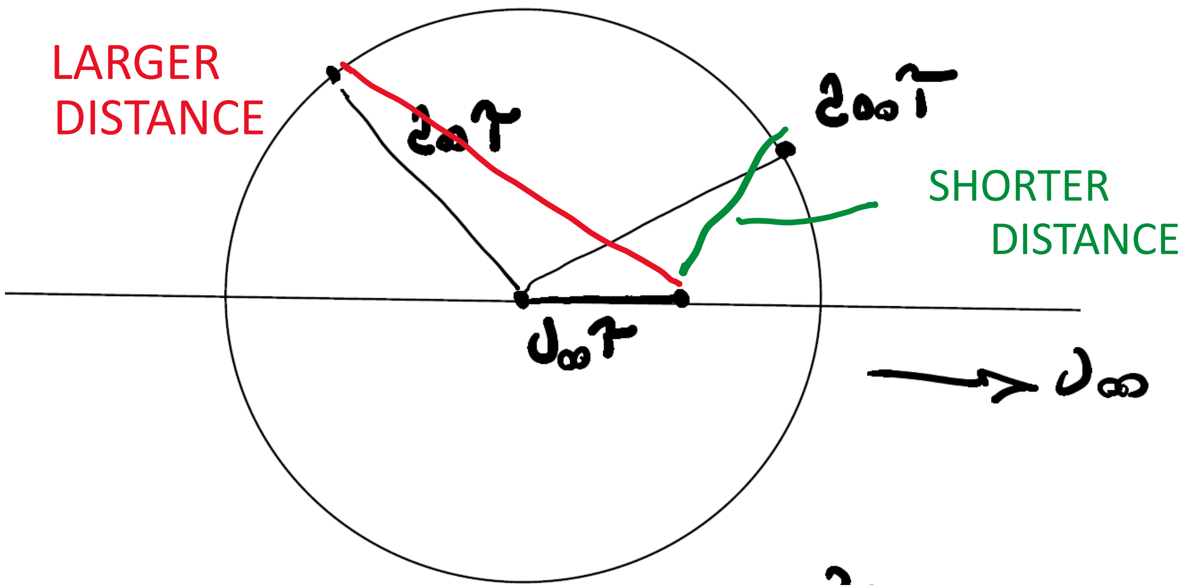
$$\tau = \sqrt{\frac{r^2}{a_\infty^2}}$$



If the fluid is at rest independently from the relative position between x and x_0, the time delay is the ratio between the distance and the speed of sound

Explanation of τ

If the perturbation source is moving forward with speed U_∞ in the direction x



$$a_\infty^2 \tau^2 = (x - x_0 - U_\infty \tau)^2 + (y - y_0)^2 + (z - z_0)^2$$

If the fluid is moving at constant speed, after the time τ x_0 will move to a different place so the delay will be different in the different directions.

Explanation of τ

$$a_\infty^2 \tau^2 = U_\infty^2 \tau^2 - 2U_\infty (x - x_0)\tau + (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2$$
$$\left(U_\infty^2 - a_\infty^2\right)\tau^2 - 2U_\infty (x - x_0)\tau + r^2 = 0$$

Solving for τ we obtain

$$\tau = \frac{U_\infty \Delta x \pm \sqrt{U_\infty^2 \Delta x^2 - r^2 (U_\infty^2 - a_\infty^2)}}{U_\infty^2 - a_\infty^2}$$
$$\tau = \frac{1}{a_\infty} \frac{M_\infty \Delta x \pm \sqrt{M_\infty^2 \Delta x^2 - r^2 (M_\infty^2 - 1)}}{M_\infty^2 - 1}$$

In general there are two solutions

$$\tau = \frac{1}{a_\infty} \frac{-M_\infty \Delta x \mp r_\beta}{\beta^2}$$

Subsonic flow

If $M_\infty < 1$ the there are two real solutions τ_1, τ_2 with $\tau_1 > 0$ and $\tau_2 < 0$. The second solution is physically meaningless, so only the first one is taken, i.e.

$$\tau = \tau_1 = \frac{1}{a_\infty}\frac{r_\beta - M_\infty \Delta x}{\beta^2} = \frac{D^+}{a_\infty} = \frac{D^+ M_\infty}{U_\infty}$$

So, the fundamental solution to the compressible equation of motion for the potential is (in frequency domain)

$$G = -\frac{1}{4\pi r_\beta}\mathrm{e}^{-\mathrm{j}\frac{\omega}{a_\infty}D^+}$$

$$\begin{aligned}
Y_{ij} &= 2\pi\delta_{ij} + \mathrm{e}^{-\mathrm{j}\frac{\omega}{a_\infty}D^+}\int_{B_j}\frac{\partial}{\partial \mathbf{n}_j}\left(\frac{1}{r_{\beta_{ij}}}\right)\mathrm{d}s \\
&+ h_j \mathrm{e}^{-\mathrm{j}k(\tilde{x}_j - 1)}\mathrm{e}^{-\mathrm{j}\frac{\omega}{a_\infty}D^+}\int_{W_j}\frac{\partial}{\partial \mathbf{n}_j}\left(\frac{1}{r_{\beta_{ij}}}\right)\mathrm{d}s \\
Z_{ij} &= \mathrm{e}^{-\mathrm{j}\frac{\omega}{a_\infty}D^+}\int_{B_j}\frac{1}{r_{\beta_{ij}}}\mathrm{d}s
\end{aligned}$$

Supersonic flow

If $M_\infty > 1$ there are different possibilities.

a) If

$$r_\beta = \Delta x^2 + \beta^2 \left(\Delta y^2 + \Delta z^2\right) \geq 0$$
$$\rightarrow \Delta x^2 \geq -\beta^2 \left(\Delta y^2 + \Delta z^2\right)$$

The point is within the Mach cone, and it is reached by two waves that started at different times from different positions

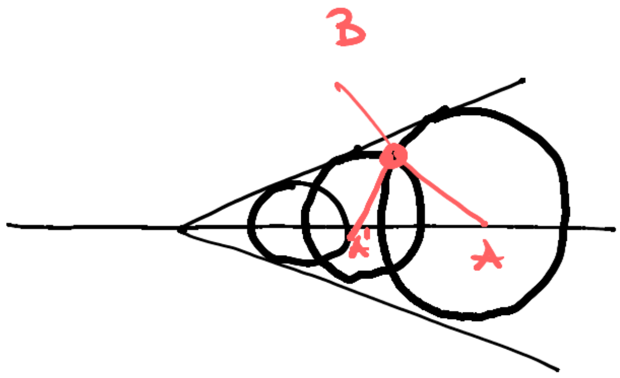
It follows that $\tau_1, \tau_2 > 0$

b) If

$$r_\beta = \Delta x^2 + \beta^2 \left(\Delta y^2 + \Delta z^2\right) \leq 0$$
$$\rightarrow \Delta x^2 \leq -\beta^2 \left(\Delta y^2 + \Delta z^2\right)$$

The point is outside the Mach cone, and it is not reached by any perturbation wave

It follows that $\tau_1, \tau_2 < 0$



Supersonic flow panelization

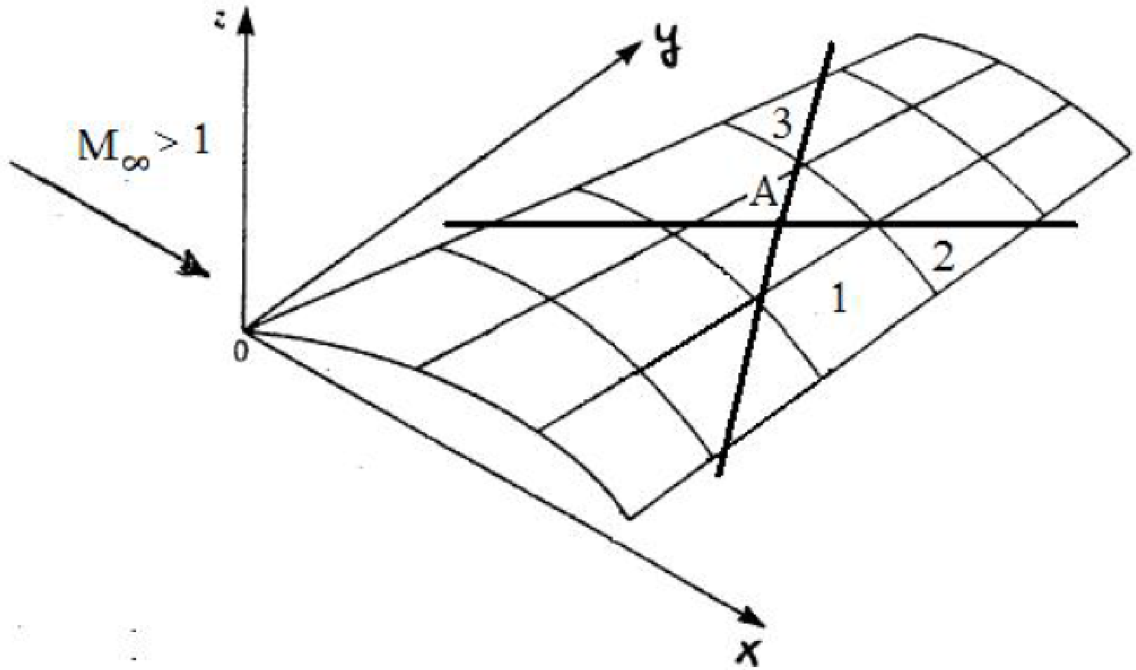
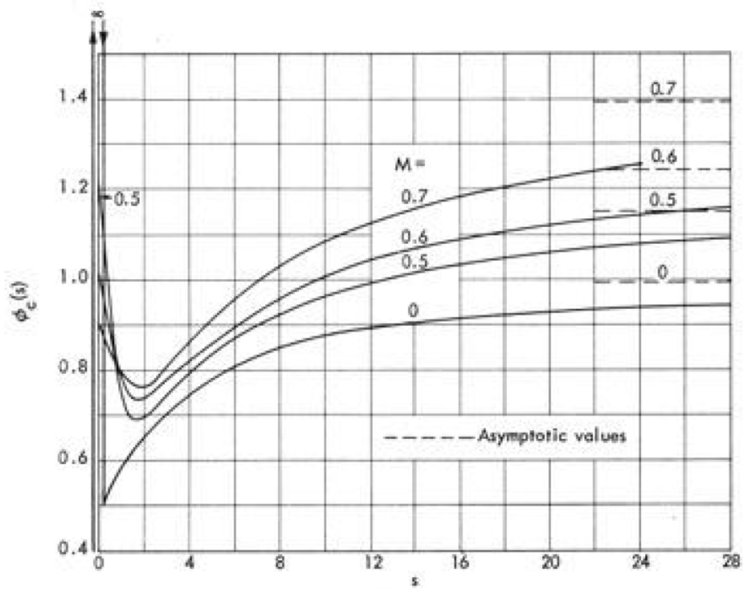


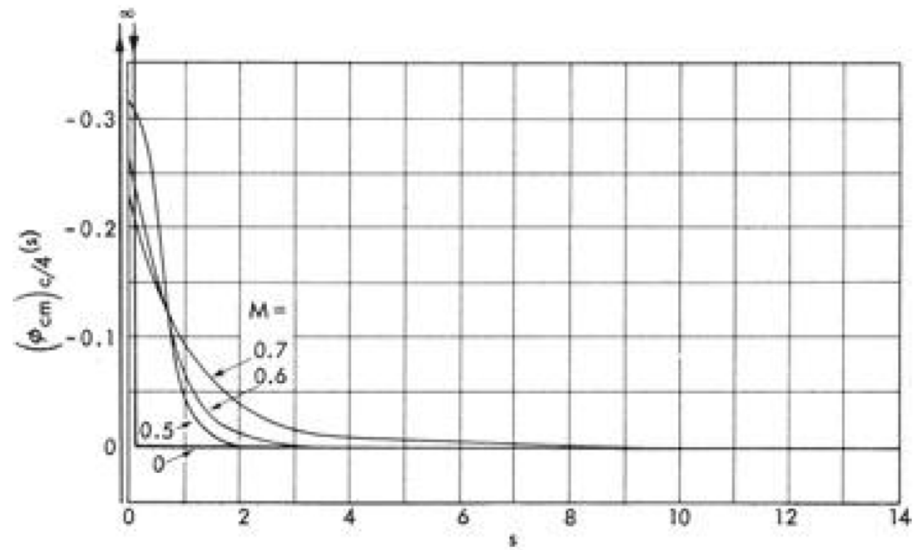
Figure 1.11: Cone of influence over the panelization

Domain of dependance and domain of influence. Panel-A Influences the panels within the Mach cone directed downward and is influenced only by the panels included in the Mach cone directed upward.

Indicial response: compressible vs incompressible flow



(a)

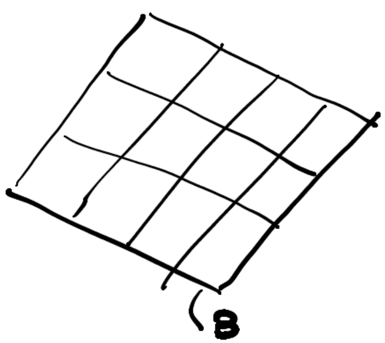


(b)

Lifting surface

The idea of the lifting surface is to obtain a formulation where given the aerodynamic loads developed in a certain lifting surface it is possible to derive the induced inflow (and so the change of angle of attack) on the surface

$$\frac{w(x,y,k)}{U_\infty} = \int_S \underbrace{K}_{\text{KERNEL}}(x-\xi, y-\eta, k, M)\,\Delta C_p(\xi,\eta)\,d\xi\,d\eta$$



$\tau \to 0$

$r_B \to z$

$\int_B \frac{\partial \varphi}{\partial n} \to 0$

The advantage is that it is not required to discretize the wake

Different kernels have been defined in the literature for the different flow conditions

Lifting surface from Morino

$$E\,4\pi\varphi = \int_B \Delta\varphi \, \frac{\partial}{\partial z}\left(\frac{1}{r}\right) dS_B$$

$$E\,4\pi\frac{\partial \varphi}{\partial z} = \int_\infty \Delta\varphi \, \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) dS_B$$

$$\Delta C_p = -\frac{2}{U_\infty^2}\left(U_\infty \frac{\partial \Delta\varphi}{\partial x} + j\omega\,\Delta\varphi\right)$$

$$\Delta C_p = -\frac{2}{U_\infty b}\, e^{-jk\tilde{x}} \frac{\partial}{\partial x}\left(e^{+jk\tilde{x}}\,\Delta\varphi\right)$$

$$\int_{-\infty}^{\tilde{x}} \Delta C_p(\xi)\, \frac{U_\infty b}{2}\, e^{jk\xi}\, d\xi = -\,e^{-jk\tilde{x}}\,\Delta\varphi(\tilde{x})$$

$$\Delta\varphi(\tilde{x}) = -\frac{U_\infty b}{2}\, e^{+jk\tilde{x}} \int_{-\infty}^{\tilde{x}} \Delta C_p(\xi)\, e^{jk\xi}\, d\xi$$

$$E\,4\pi\frac{\partial \varphi}{\partial z} = -\frac{U_\infty b}{2}\int_S \int_{-\infty}^{\tilde{x}} e^{jk\xi}\, \Delta C_p(\xi, y)\, d\xi \cdot e^{jk\tilde{x}}\, \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) dS$$

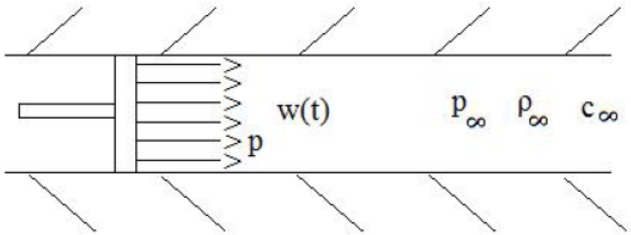
The problem is solved using a Ritz approximation for loads and then imposing the BC

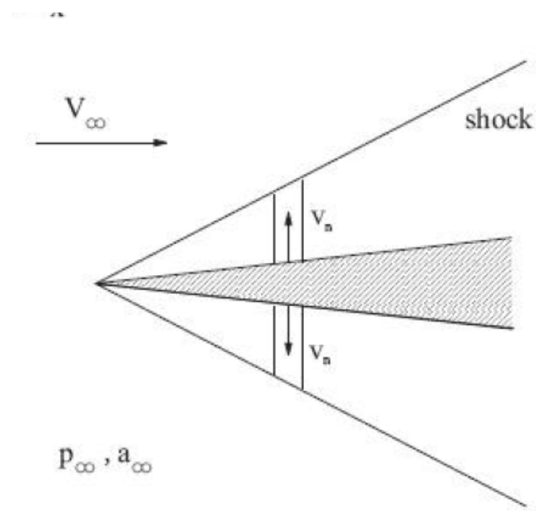
$$\Delta C_p(x,y) = \left[N(x,y)\right]\left\{q(k)\right\}$$

Piston Theory

When Mach number is very large the region of influence and dependance are very small, so it is reasonable to consider the variation of pressure on every point only function of the movement of that point (not influenced by others).

Every local points function as a “piston”





If $M_\infty \gg 1$ (like $M_\infty > 2$ or higher)

$$\frac{p}{p_\infty} = \left(1 + \frac{\gamma - 1}{2}\frac{w}{a_\infty}\right)^{\frac{2\gamma}{\gamma-1}}$$

Piston Theory

If $\frac{w}{a_\infty} \ll 1$ then

1° ORDER EXPANSION

$$p - p_\infty = \rho_\infty a_\infty w = \frac{1}{2}\rho_\infty U_\infty^2 \frac{2}{M_\infty}\frac{w}{U_\infty}$$

$$C_p = \frac{2}{M_\infty}\alpha$$

2° ORDER EXPANSION

$$p - p_\infty = \rho_\infty a_\infty^2 \left(\frac{w}{a_\infty} + \frac{\gamma+1}{4}\left(\frac{w}{a_\infty}\right)^2\right)$$

$$C_p = \frac{2}{M_\infty^2}\left(\frac{w}{a_\infty} + \frac{\gamma+1}{4}\left(\frac{w}{a_\infty}\right)^2\right)$$

Recap

2D

$M=0$ $\begin{cases} \text{THEODORSEN} & h, \alpha, \beta \\ \text{CICALA} & \text{any change of } \eta \end{cases}$

$\hookrightarrow H_{am}(k)$

$\begin{cases} \text{WAGNER} \\ + \\ \text{DUHAMEL integral} \end{cases}$ $\rightarrow$ Time dom. formulation

Recap

$M \geq 0$ SUBSONIC OR SUPERSONIC $\begin{cases} \text{PANEL METHOD} \\ \text{LIFTING SURFACE} \\ \text{VORTEX METHOD} \end{cases}$

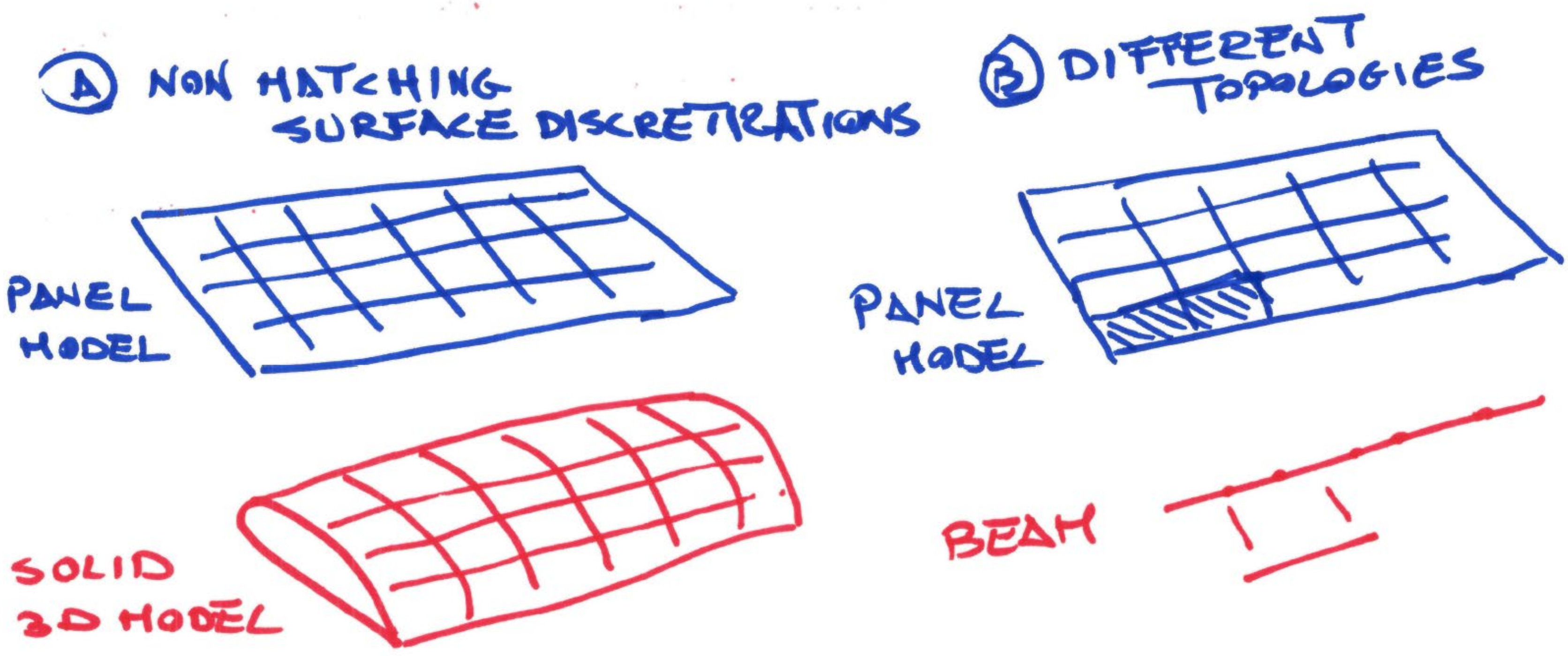
$\hookrightarrow H_{zm}(k)$ $\hookrightarrow$ compute indicial response

$M \gg 1$ PISTON THEORY

$M \simeq 1$ CFD ?

3D $\begin{cases} \text{STRIP THEORY} & \text{combine 2D models} \\ \text{PANEL METHOD} \end{cases}$

Aeroelastic Interface



$$\mathbf{x}_F = \mathbf{x}_{EA} + \boldsymbol{\omega} \times \mathbf{d}$$

Aeroelastic interface

1) Ability to interface matching and not-matching grids
2) Ability to deal with irregular grids and local refinements
3) Correct representation of rigid movements
4) Computational efficiency
5) Independence from the numerical formulation used in the structure or aerodynamic field
6) Ability to manage extrapolation
7) Conservation of exchanged energy
8) Smothness

Aeroelastic Interface

$$\mathbf{x}_{F_i} = \sum_{j=1}^{N} h_{ij}\mathbf{x}_{S_i} \rightarrow \mathbf{x}_F = \mathbf{H}\mathbf{x}_S$$

$$\mathbf{F}_{S_i} = \sum_{j=1}^{N} h_{ij}^F \mathbf{F}_{F_i} \rightarrow \mathbf{F}_S = \mathbf{H}^F \mathbf{F}_F$$

Wrting the virtual work of the two interfaced sides

$$\delta\mathbf{x}_F^T\mathbf{F}_F = \delta\mathbf{x}_S^T\mathbf{F}_S$$

This identity must be ensured to avoid any leackage of energy during the transfer of information between the two grids

$$\delta\mathbf{x}_S^T\mathbf{H}^T\mathbf{F}_F - \delta\mathbf{x}_S^T\mathbf{F}_S = 0 \Rightarrow \mathbf{F}_S = \mathbf{H}^T\mathbf{F}_F$$

or otherwise

$$\mathbf{H}^F = \mathbf{H}^T$$