Chapter 2

Structural Dynamics

2.1 Solution of the Euler-Bernoulli free beam bending vibrations

Consider an homogenous beam with constant properties along the span: E elastic modulus, J_2 the area moment of inertia of the section about the y axis, and m the mass per unit span. The indefinite equilibrium equation is

$$EJ_2 w'''' + m\ddot{w} = 0 \tag{2.1}$$

To solve the problem it is necessary to add 4 boundary conditions and two initial conditions

$$\begin{aligned} w(x,0) &= f_1(x), \\ \dot{w}(x,0) &= f_2(x). \end{aligned}$$

Assume that it is possible to apply the variable separation i.e.,

$$w(x,t) = f(x)q(t) \tag{2.2}$$

Substitutin in eq. (2.1)

$$EJ_2 f''''q + mf\ddot{q} = 0 \tag{2.3}$$

This equation could be rearranged as

$$-\frac{\ddot{q}}{q} = \frac{EJ_2}{m}\frac{f''''}{f} = \omega^2 = \text{const.} \tag{2.4}$$

Since the left hand side is only function of time and the right hand side is only function of space it results that the only valid condition is that both are equal to constant ω^2. So

$$\begin{cases} \ddot{q} + \omega^2 q = 0 \\ f'''' - \dfrac{m\omega^2}{EJ_2} f = 0 \end{cases} \tag{2.5}$$

Calling $\beta^4 = m\omega^2/EJ_2$ it is possible to see that the second homogenous equation of (2.5)

$$f'''' - \beta^4 f = 0 \tag{2.6}$$

Admits an infinite number of pairs $(\beta^4, f(x))$, that are the generalized EIGENVALUES and EIGENFUNCTIONS, that satisfy the equation. To find these values it is necessary to start from the general solution of this fourth order linear differential equation

$$f(x) = f_0 \mathrm{e}^{\lambda x} \tag{2.7}$$

that results in

$$\lambda^4 = \beta^4 \tag{2.8}$$

and so

$$\begin{cases} \lambda_1 = \beta \\ \lambda_2 = -\beta \\ \lambda_3 = \mathrm{j}\beta \\ \lambda_4 = -\mathrm{j}\beta \end{cases} \tag{2.9}$$

The general solution could be written as

$$f(x) = a\mathrm{e}^{\beta x} + b\mathrm{e}^{-\beta x} + c\mathrm{e}^{\mathrm{j}\beta x} + d\mathrm{e}^{-j\beta x} \tag{2.10}$$

Remembering the Euler formulas, it is possible to re-write the general solution in terms of trigonometric and hyperbolic functions as

$$f(x) = a\sinh\beta x + b\cosh\beta x + c\sin\beta x + d\cos\beta x \tag{2.11}$$

To identify the valid values of β and the amplitude constants it is necessary to impose the boundary conditions. Consider the case where a cantilever is analysed. At the constrained end it is imposed that the translation and the rotation are null i.e.,

$$f(0) = 0,\ f'(0) = 0. \tag{2.12}$$

At the free end it is instead imposed that the shear force and the bending moment are null i.e.,

$$M(L) = 0, \rightarrow \qquad f''(L) = 0 \tag{2.13}$$

$$S(L) = 0, S = M' \ \rightarrow \quad f'''(L) = 0 \tag{2.14}$$

Imposing the boundary conditions leads to the following system of equations

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ \sinh\beta L & \cosh\beta L & -\sin\beta L & -\cos\beta L \\ \cosh\beta L & \sinh\beta L & -\cos\beta L & \sin\beta L \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \\ d \end{Bmatrix} = \mathbf{0} \tag{2.15}$$

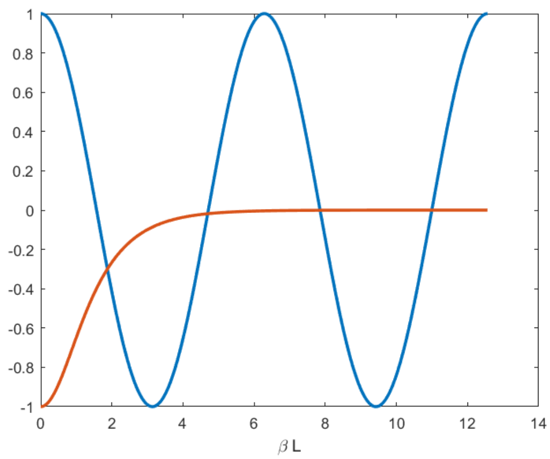

Figure 2.1: Plot of the functions $\cos\beta L$ and $-1/\cosh\beta L$ to identify the βL values that are solution of $\det(\mathbf{H})$.

This is an homogeneous system with a matrix $\mathbf{H}$. To find non trivial solutions it is necessary to identify the values of β for which $\det(\mathbf{H}) = 0$ this leads to

$$\cos\beta L \cosh\beta L = -1. \tag{2.16}$$

There are infinite values $\beta_i L = p_i$, $i = 1, \dots, \infty$ that solve this nonlinear equation. The different solutions can be identified graphically as shown in Figure 2.1. The first few values of the solution are

$$\beta L = 0.597\pi, 1.49\pi, \frac{5}{2}\pi, \frac{7}{2}\pi, \frac{9}{2}\pi, \dots \tag{2.17}$$

With the known values of β it is possible to identify exactly the constants (a, b, c, d), and so the eigensolutions, and also the valid values of the frequency ω^2

$$\omega_i = \sqrt{\frac{EJ_2}{m}\beta_i^4} = \frac{p_i^2}{L^2}\sqrt{\frac{EJ_2}{m}}. \tag{2.18}$$

With these values of ω it is possible to solve the first of the two homogeneous equations (2.5) to compute also the solution in time

$$q(t) = g\sin\omega t + h\cos\omega t \tag{2.19}$$