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055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

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15 The aeroelastic problem: Formulations of the aeroelastic stability problem

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Material

Masarati's notes Section Aeroelasticity 8.2 8.3

Assembly of the aeroelastic system

Consider a structural model where a set of generalized coordinates $\mathbf{q}$ together with the associated shape functions are used to describe the diplacements of any structural node $\boldsymbol{\eta}$

$$\boldsymbol{\eta}_s = \mathbf{N}\mathbf{q}$$

The structural model will be written as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = q\mathbf{F}_a$$

The vector $q\mathbf{F}_a$ represents the unsteady aerodynamic forces. We need to undertand how to represent this vector.

If we use the PLV

$$\delta W_a = q \int_{A_s} \delta\boldsymbol{\eta}_a^T \cdot \mathbf{n}_a C_p(\mathbf{x}_a) \mathrm{d}s_a$$

where A_s is the aerodynamic surface where the aerodynamic grid $\mathbf{x}_a$ is defined. The vector $\boldsymbol{\eta}_a$ is the vector of the displacements of the aerodynamic nodes.

Assembly of the aeroelastic system

Using the defined interface scheme it will be possible to express

$$\boldsymbol{\eta}_a = \mathbf{H}\boldsymbol{\eta}_s$$

So, it will result

$$\delta W_a = q \int_{A_s} \delta\boldsymbol{\eta}_s^T \mathbf{H}^T \cdot \mathbf{n}_a C_p(\mathbf{x}_a) \mathrm{d}s$$

$$\delta W_a = q \delta\mathbf{q}^T \int_{A_s} \mathbf{N}^T \mathbf{H}^T \cdot \mathbf{n}_a C_p(\mathbf{x}_a) \mathrm{d}s$$

$$\delta W_a = q \delta\mathbf{q}^T \underbrace{\sum_i \int_{A_{s_i}} \mathbf{N}^T \mathbf{H}^T \cdot \mathbf{n}_a \mathrm{d}s \, C_p(\mathbf{x}_{a_i})}_{\mathbf{F}_a} = q \delta\mathbf{q}^T \mathbf{D} \mathbf{C}_p$$

$$\mathbf{D} = [n_q \times n_a]$$

Assembly of the aeroelastic system

$$
\begin{aligned}
\mathbf{C}_p &= \mathbf{P}(k)\boldsymbol{\varphi} \\
\boldsymbol{\varphi} &= \mathbf{Y}(k, M)^{-1}\mathbf{Z}(M)\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} \\
\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} &= \frac{\partial \boldsymbol{\eta}_a}{\partial x} + \frac{\mathrm{j}\omega}{U_\infty}\boldsymbol{\eta}_a = \mathbf{H}\left(\frac{\partial \boldsymbol{\eta}_s}{\partial x} + \frac{\mathrm{j}\omega}{U_\infty}\boldsymbol{\eta}_s\right) = \mathbf{H}\left(\mathbf{N}_{/x} + \frac{\mathrm{j}k}{b}\mathbf{N}\right)\mathbf{q} \\
\frac{\partial \boldsymbol{\varphi}}{\partial \mathbf{n}} &= \left(\mathbf{A} + \mathrm{j}k\mathbf{B}\right)\mathbf{q}
\end{aligned}
$$

$$
\delta W_a = q\delta\mathbf{q}^T\mathbf{D}\mathbf{P}(k)\mathbf{Y}^{-1}(k, M)\mathbf{Z}(M)\left(\mathbf{A} + \mathrm{j}k\mathbf{B}\right)\mathbf{q}
$$

$$
\Rightarrow \mathsf{\mathbf{F}}_a(k) = \mathsf{\mathbf{H}}_{am}(k, M)\mathsf{\mathbf{q}}
$$

The aeroelastic equation

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = q \int_0^t \mathbf{h}_{am}(t-\tau)\mathbf{q}(\tau)\mathrm{d}\tau$$

$$\mathcal{L} \rightarrow \left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} - q\mathbf{H}_{am}(p, M)\right)\mathbf{q} = \mathbf{0}$$

$$p = \frac{sb}{U_\infty} = \frac{\sigma b}{U_\infty} + \frac{\mathrm{j}\omega b}{U_\infty}$$

$$\det\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K} - q\mathbf{H}_{am}(p, M)\right) = 0$$

In general we do not have the full transfer function $\mathbf{H}_{am}(p, M)$ but we only have available the frequency response, i.e. $\mathbf{H}_{am}(k, M)$

We need to find methods to compute the correct solution to the eigenvalue problem having this partial information.

Quasi-steady approximation

It is possible to express the transfer function using the McLaurin series

$$\mathbf{H}(s) = \mathbf{H}|_{s=0} + s\,\frac{\partial \mathbf{H}}{\partial s}\bigg|_{s=0} + \frac{s^2}{2}\,\frac{\partial^2 \mathbf{H}}{\partial s^2}\bigg|_{s=0} + \cdots$$

$$\mathbf{F}_a(s) = \left(\mathbf{H}|_{s=0} + s\,\frac{\partial \mathbf{H}}{\partial s}\bigg|_{s=0} + \frac{s^2}{2}\,\frac{\partial^2 \mathbf{H}}{\partial s^2}\bigg|_{s=0}\right)\mathbf{q}(s)$$

Using the inverse Laplace transformation

$$\mathcal{L}^{-1} \rightarrow \mathbf{F}_a(t) = \mathbf{H}(0)\mathcal{L}^{-1}\left[\mathbf{q}(s)\right] + \frac{\partial \mathbf{H}(0)}{\partial s}\mathcal{L}^{-1}\left[s\mathbf{q}(s)\right] + \frac{1}{2}\frac{\partial^2 \mathbf{H}(0)}{\partial s^2}\mathcal{L}^{-1}\left[s^2\mathbf{q}(s)\right]$$

Quasi-steady approximation

$$\mathbf{F}_a(t) = \mathbf{H}(0)\mathbf{q}(t) + \frac{\partial \mathbf{H}(0)}{\partial s}\dot{\mathbf{q}}(t) + \frac{1}{2}\frac{\partial^2 \mathbf{H}(0)}{\partial s^2}\ddot{\mathbf{q}}(t)$$

$$\begin{aligned} \mathbf{H}(0) &= \mathbf{K}_a \\ \frac{\partial \mathbf{H}(0)}{\partial p} &= \mathbf{C}_a \\ \frac{1}{2}\frac{\partial^2 \mathbf{H}(0)}{\partial p^2} &= \mathbf{M}_a \end{aligned}$$

$$\mathbf{H}(p, M) \rightarrow \quad \begin{aligned} \frac{\partial \mathbf{H}}{\partial s} &= \frac{\partial \mathbf{H}}{\partial p}\frac{\partial p}{\partial s} = \frac{\partial \mathbf{H}}{\partial p}\frac{b}{U_\infty} \\ \frac{\partial^2 \mathbf{H}}{\partial s^2} &= \frac{\partial^2 \mathbf{H}}{\partial p^2}\left(\frac{b}{U_\infty}\right)^2 \end{aligned}$$

$$\mathbf{F}_a(t) = \mathbf{K}_a\mathbf{q}(t) + \frac{b}{U_\infty}\mathbf{C}_a\dot{\mathbf{q}}(t) + \frac{1}{2}\left(\frac{b}{U_\infty}\right)^2\mathbf{M}_a\ddot{\mathbf{q}}(t)$$

$$\left(\mathbf{M} - q\frac{b^2}{U_\infty^2}\mathbf{M}_a\right)\ddot{\mathbf{q}} + \left(\mathbf{C} - q\frac{b}{U_\infty}\mathbf{C}_a\right)\dot{\mathbf{q}} + (\mathbf{K} - q\mathbf{K}_a)\,\mathbf{q} = \mathbf{0}$$

Meaning of QS approximation in time domain

$$\begin{cases} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \end{cases}$$

By setting $\mathbf{x}^{(n)} = \mathbf{0}$

$$\mathbf{y} = \left(-\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{u} + \sum_{i=1}^{n} \left(-\mathbf{C}\mathbf{A}^{-(i+1)}\mathbf{B}\right)\mathbf{u}^{(i)}$$

$$\mathbf{H}(0) \qquad\qquad \frac{\partial^i \mathbf{H}(0)}{\partial p^i}$$

Let's impose that $\ddot{\mathbf{x}} = \mathbf{0}$

$$\begin{aligned} \ddot{\mathbf{x}} &= \mathbf{A}\dot{\mathbf{x}} + \mathbf{B}\dot{\mathbf{u}} = \mathbf{0} \\ \ddot{\mathbf{x}} &= \mathbf{A}\left(\mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\right) + \mathbf{B}\dot{\mathbf{u}} = \mathbf{0} \end{aligned}$$

$$\mathbf{A}^2\mathbf{x} + \mathbf{A}\mathbf{B}\mathbf{u} + \mathbf{B}\dot{\mathbf{u}} = \mathbf{0}$$

$$\rightarrow \mathbf{x} = -\mathbf{A}^{-1}\mathbf{B}\mathbf{u} - \mathbf{A}^{-2}\mathbf{B}\dot{\mathbf{u}}$$

$$\mathbf{y} = \left(\mathbf{C}\mathbf{A}^{-1}\mathbf{B} + \mathbf{D}\right)\mathbf{u} - \mathbf{C}\mathbf{A}^{-2}\mathbf{B}\dot{\mathbf{u}}$$

A QS approximation of order n is equivalent to say that the derivative (n+1) in time is negligible

Computation of the QS approximation from the frequency response

$$\frac{\partial \mathbf{H}}{\partial p} = \frac{\partial \mathbf{H}}{\partial \mathrm{j}\omega} = -\mathrm{j}\frac{\partial \Re(\mathbf{H})}{\partial \omega} + \frac{\partial \Im(\mathbf{H})}{\partial \omega}$$

$$\frac{\partial \Re(\mathbf{H})}{\partial \omega} = 0$$

because the real part
of the frequency response is even

$$\frac{\partial^2 \mathbf{H}}{\partial p^2} = -\frac{\partial^2 \mathbf{H}}{\partial \omega^2} = -\frac{\partial^2 \Re(\mathbf{H})}{\partial \omega^2} - \mathrm{j}\frac{\partial^2 \Im(\mathbf{H})}{\partial \omega^2}$$

$$\frac{\partial^2 \Im(\mathbf{H})}{\partial \omega^2} = 0$$

because the imaginary part
of the frequency response is odd

$$\frac{\partial \mathbf{H}}{\partial p} = \frac{\partial \Im(\mathbf{H})}{\partial \omega} \qquad \frac{\partial^2 \mathbf{H}}{\partial p^2} = -\frac{\partial^2 \Re(\mathbf{H})}{\partial \omega^2}$$

Computation of flutter speed

$$q = \frac{1}{2}\rho_h U_\infty^2 = \frac{1}{2}\rho_0 U_{EAS}^2$$

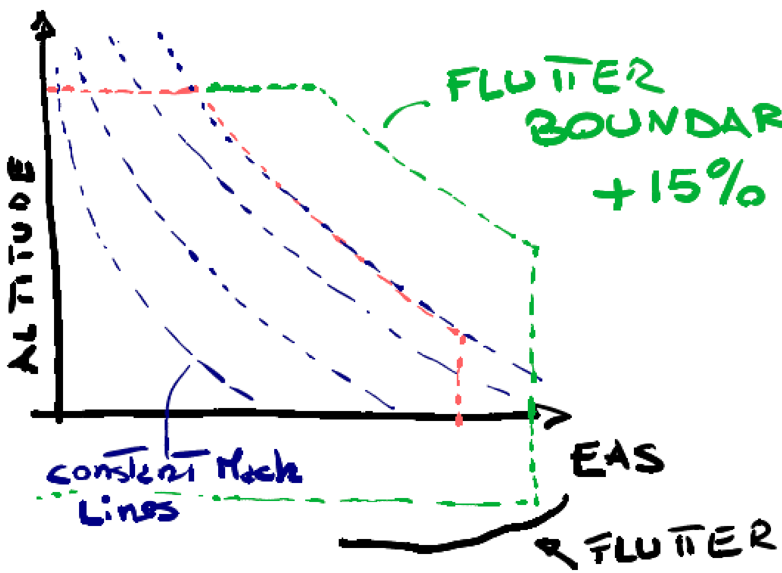
where EAS means Equivalent Air Speed is the speed that will give the same dynamic pressure at zero altitude, so

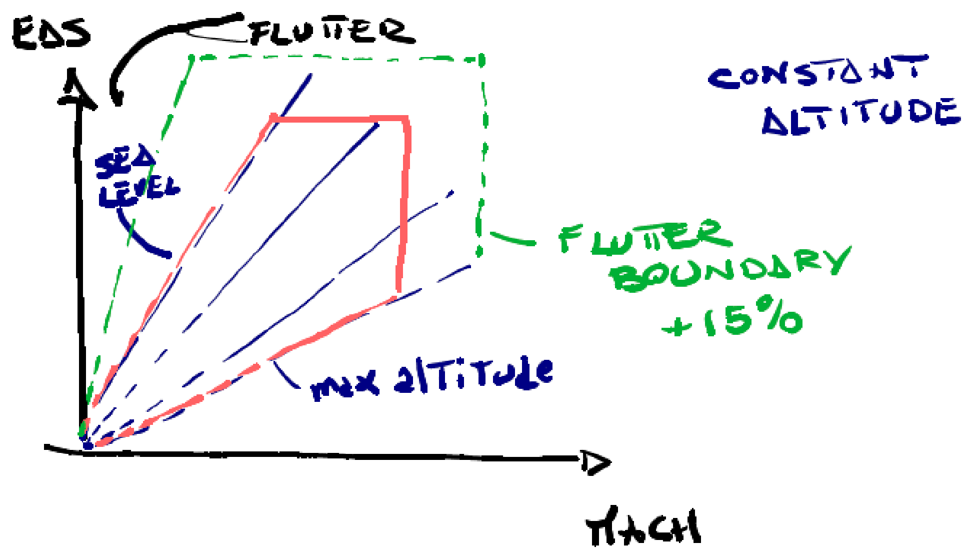
$$U_{EAS} = \sqrt{\frac{\rho_h}{\rho_0}} U_\infty$$

$$M_\infty^2 = \frac{U_\infty^2}{a_\infty^2} = \frac{U_\infty^2 \rho_h}{\gamma p_h} = \frac{2q_\infty}{\gamma p_h} \rightarrow p_h = \frac{2q_\infty}{\gamma M_\infty^2}$$

Assign M_∞ $\rightarrow$ find the $q|U_{EAS}$ for flutter $\rightarrow$ detect at wich p_h (and so altitude) is happening

Computation of flutter speed





Direct solution for flutter speed

At flutter

$$s = 0 + \mathrm{j}\omega_F$$
$$p = 0 + \mathrm{j}k_F$$
$$q_F = \frac{1}{2}\rho_h U_F^2$$

$$\det\left(-\omega_F^2\mathbf{M} + \mathrm{j}\omega_F\mathbf{C} + \mathbf{K} - q_F\mathbf{H}_{am}(k_F, M_F)\right) = 0$$

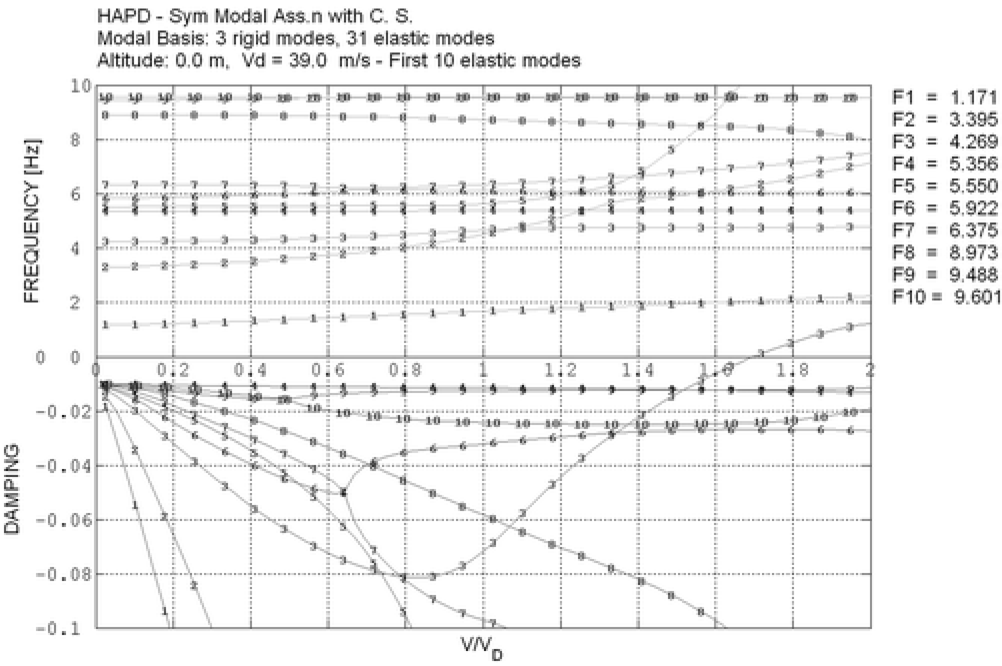
This is a nonlinear imaginary equation that can be solved if we impose M_F and look for ω_F and q_F. $k_F = f(\omega_F)$

When q_F is known the corresponding ρ_h and so the altitude can be computed.

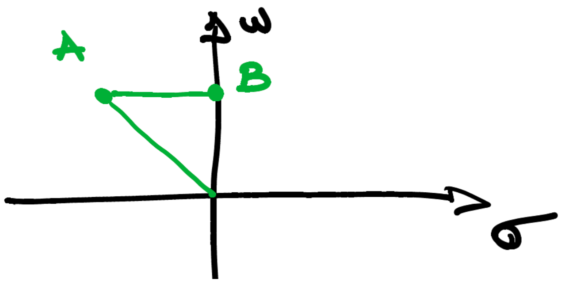
Flutter V-f V-g diagrams

In practice, it is more convenient to compute all the (desired) eigenvalues for a range of EAS velocities (or dynamic pressures) starting from zero and evaluate their real part. Flutter is detected as the lowest airstream speed at which the real part of at least one eigenvalue crosses the zero axis from negative to positive.

The knowledge of all eigenvalues from zero to the desired airstream velocity not only allows the analyst to check for flutter clearance, but also to determine how the eigenvalues evolve with the EAS velocity allowing an experimental verification of the aircraft aeroelastic stability.



Flutter P-K method



$$H_{am}(p) \simeq \overset{0^\circ}{H_{am}(k_i)} + \overset{1^\circ}{\frac{\partial H_{am}}{\partial p}(p-k_i)} + \overset{2^\circ}{\frac{1}{2}\frac{\partial^2 H_{am}}{\partial p^2}(p-k_i)^2} + \dots$$

$$H_{am}(p) \simeq H_{am_R}(k_i) + j k_i \frac{H_{am_I}(k_i)}{k_i}$$

$$\simeq H_{am_R}(k_i) + p \frac{H_{am_I}(k_i)}{k_i}$$

The idea is to use a Taylor approximation using the knowledge of the transfer function on point B to approximate the value at point A

The approximation is exact when A belongs to the imaginary axis, i.e. at flutter speed

Usually, an approximation of order 0 is used but higher order approximations are possible

Flutter P-K Method

For a given set of parameters
U_∞ $(\rho, h, M, \dots)$

1. Compute a preliminary guess for an eigenvalue λ_{0i}
2. Compute $\omega_i = \mathrm{Im}(\lambda_i) \rightarrow K_i = \frac{\omega_i b}{U_\infty}$ and $H_{am}(K_i)$
3. Solve for
$$\det\left(\lambda^2 M + \lambda C + q_\infty \frac{\lambda b}{U_\infty} \frac{H_I(K)}{K} + K - q_\infty H_R(K)\right) = 0$$
4. identify the closest eigenvalue λ_i to λ_{0i}
5. set λ_i as the new initial guess and iterate $|\lambda_{i_{new}} - \lambda_{i_{old}}| < \varepsilon$

The algorithm is repeated for every eigenvalue of interest.

As starting guess it is possible to take the structural frequencies at zero speed

State-space representation of unsteady aerodynamics

Analitic continuation of $H_{am}(k)$ function is exploited to extend the frequency response

$$H_{am}(ik) \longrightarrow H_{am}(p)$$

$$H_{am}(ik) = \frac{N(ik)}{D(ik)} \longrightarrow H_{am}(p) = \frac{N(p)}{D(p)}$$

$$H_{am}(p) = D_0 + D_1 p + D_2 p^2 + \sum_{i=1}^{n} \frac{D_{i+2}\, p}{p + b_i}$$

State-space formulation

$$\varepsilon^2 = \sum_{h=1}^{m} \left[H_{em}(k_h) - \tilde{\hat{H}}_{em}(k_h) \right]^* \left[H_{em_h} - \tilde{\hat{H}}_{em_h} \right]$$

Find $D_i \quad i = 0, \dots, n$ that minimize ε^2

$b_i \quad i = 1, \dots, n$

$$D_0 = H_{em}(0)$$

$$D_1 = \frac{\partial H_{em}(0)}{\partial p} \qquad D_2 = \frac{\partial^2 H_{em}(0)}{\partial p^2}$$

State-space formulation

$$\sum_{i=1}^{n} \frac{D_{i+2}\, p}{p - b_i} = C_A (p I - A)^{-1} B_A p$$

$$\tau = \frac{t U_\infty}{b}$$

$$\frac{b}{U_\infty} \dot{x}_A = A x_A + \frac{b}{U_\infty} B_A \dot{q}$$

$$F_a = C_A x_A + D_0 q + \frac{b}{U_\infty} D_1 \dot{q} + \left(\frac{b}{U_\infty}\right)^2 D_2 \ddot{q}$$

State-space formulation

$$\begin{bmatrix} I & 0 & 0 \\ 0 & \tilde{M} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{Bmatrix} \dot{q} \\ \ddot{q} \\ \dot{x}_A \end{Bmatrix} = \begin{bmatrix} 0 & I & 0 \\ -\tilde{K} & -\tilde{C} & q_\infty C_A \\ 0 & B_A & \frac{U_\infty}{b} A \end{bmatrix} \begin{Bmatrix} q \\ \dot{q} \\ x_A \end{Bmatrix}$$

$$\tilde{K} = K - q_\infty D_0$$

$$\tilde{C} = C - q_\infty \frac{b}{U_\infty} D_1$$

$$\tilde{M} = M - q_\infty \left(\frac{b}{U_\infty}\right)^2 D_2$$

Sta-space formulation

$$\Phi(\tau) = 1 - A_1 e^{-b_1 \tau} - A_2 e^{-b_2 \tau}$$

$$\tau = \frac{t U_\infty}{b}$$

$$\Phi(t) = 1 - A_1 e^{-b_1 U_\infty / b\, t} - A_2 e^{-b_2 \frac{U_\infty}{b} t}$$

$$\phi(t) = \int_0^t h(\tau) \overset{\text{STEP}}{H}(t-\tau)\, d\tau = \int_0^t h(\tau)\, d\tau$$

$$h(t) = \dot{\phi}(t)$$

$$h(t) = A_1 b_1 \frac{U_\infty}{b} e^{-b_1 \frac{U_\infty}{b} t} + A_2 b_2 \frac{U_\infty}{b} e^{-b_2 \frac{U_\infty}{b} t}$$

$$\tilde{C}(k) = \frac{1}{1 + \frac{\pi}{2} j k}$$

$$\tilde{C}(p) = \frac{1}{1 + \frac{\pi}{2} p}$$

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \frac{U_\infty}{b} \begin{bmatrix} -b_1 & 0 \\ 0 & -b_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \frac{U_\infty}{b} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y = \begin{bmatrix} A_1 b_1 & A_2 b_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

Flutter speed: the continuation approach

The eigenvalue problem could be seen as a nonlinear algebraic problem

$$\left(s^2 M + sC + K + q_\infty H(p, M)\right) q = 0$$

$$F(s)\, q = 0 \qquad f(s, q) = 0$$

$s, q \in \mathbb{C}$ UNKNOWNS $(n+1)$

$2(n+1)$ REAL

However, the eigenvector are defined but for a constant, so the unknowns are 2 n + 1 and there are 2n equations

$$q^* q = 1$$

Flutter speed: the continuation approach

$$\begin{bmatrix} \frac{\partial f}{\partial q} & \frac{\partial f}{\partial s} \\ 2q_i^* & 0 \end{bmatrix} \begin{Bmatrix} \Delta q \\ \Delta s \end{Bmatrix} = \begin{Bmatrix} -f(q_i, s_i) \\ 1 - q_i^* q_i \end{Bmatrix}$$

$$q = q_i + \Delta q \qquad s = s_i + \Delta s$$

iterate until $|\Delta q| < \varepsilon \quad |\Delta s| < \varepsilon$

$$\frac{\partial f}{\partial q} = s_i^2 M + s_i C + K - q_\infty H_{am}(p_i, M)$$

$$\frac{\partial f}{\partial s} = \left(2 s_i M + C - q_\infty \frac{\partial H_{am}}{\partial p} \frac{b}{U_\infty}\right) q_i$$

$$\frac{\partial H_{am}}{\partial p} \rightarrow -j \frac{\partial H_{am}}{\partial k}$$

The algorithm could be applied for any eigenvalue/eigenvector of interest starting from a set of initial guesses and at each speed of interest.

Flutter speed: continuation

To obtain ai initial guess at a new speed it is possible to use the value at the previous speed or try to estimate the change of the eigenvalue/eigenvector with speed

$$\frac{\partial q_i}{\partial U_\infty} \qquad \frac{\partial s_i}{\partial U_\infty} \qquad q_i^N = q_i + \frac{\partial q_i}{\partial U_\infty} \Delta U_\infty$$

$$s_i^N = s_i + \frac{\partial s_i}{\partial U_\infty} \Delta U_\infty$$

$$\begin{cases} f(U_\infty, q_i, s_i) = 0 \\ q_i^* q_i = 1 \end{cases}$$

$$\begin{cases} \dfrac{df}{dU_\infty} = \dfrac{\partial f}{\partial q} \dfrac{\partial q_i}{\partial U_\infty} + \dfrac{\partial f}{\partial s} \dfrac{\partial s_i}{\partial U_\infty} + \dfrac{\partial f}{\partial U_\infty} = 0 \\ 2 q_i^* \dfrac{\partial q_i}{\partial U_\infty} = 0 \end{cases}$$

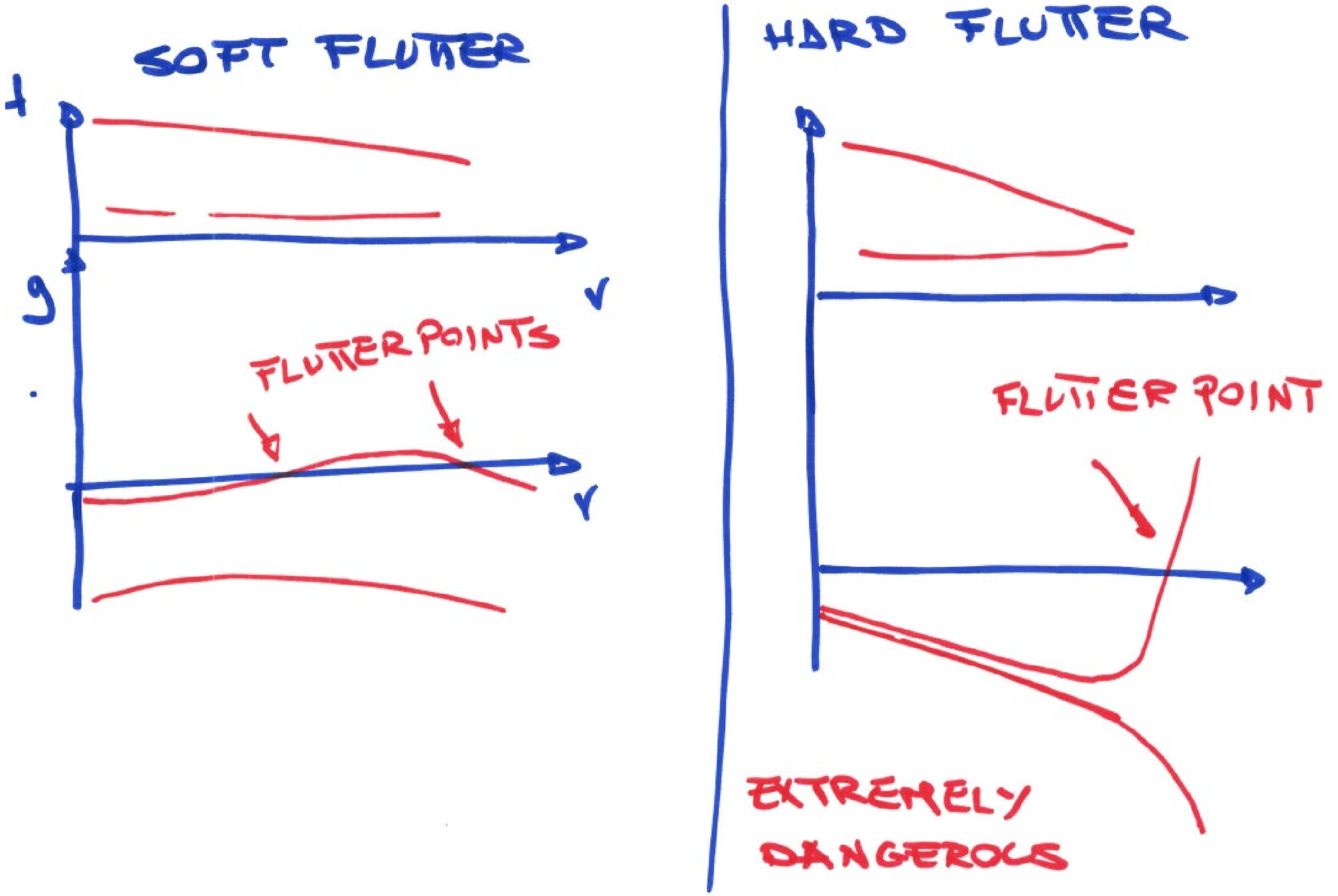
$$\begin{bmatrix} \dfrac{\partial f}{\partial q}(s_i, q_i) & \dfrac{\partial f}{\partial s} \\ 2 q_i^* & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{\partial q_i}{\partial U_\infty} \\ \dfrac{\partial s_i}{\partial U_\infty} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{\partial f}{\partial U_\infty} q_i \\ 0 \end{Bmatrix}$$

Flutter speed: the continuation approach

$$\begin{cases} \left(s^2 M + sC + K - q\, H_{am}(p, M)\right) q = 0 \\ q^* q = 1 \end{cases}$$

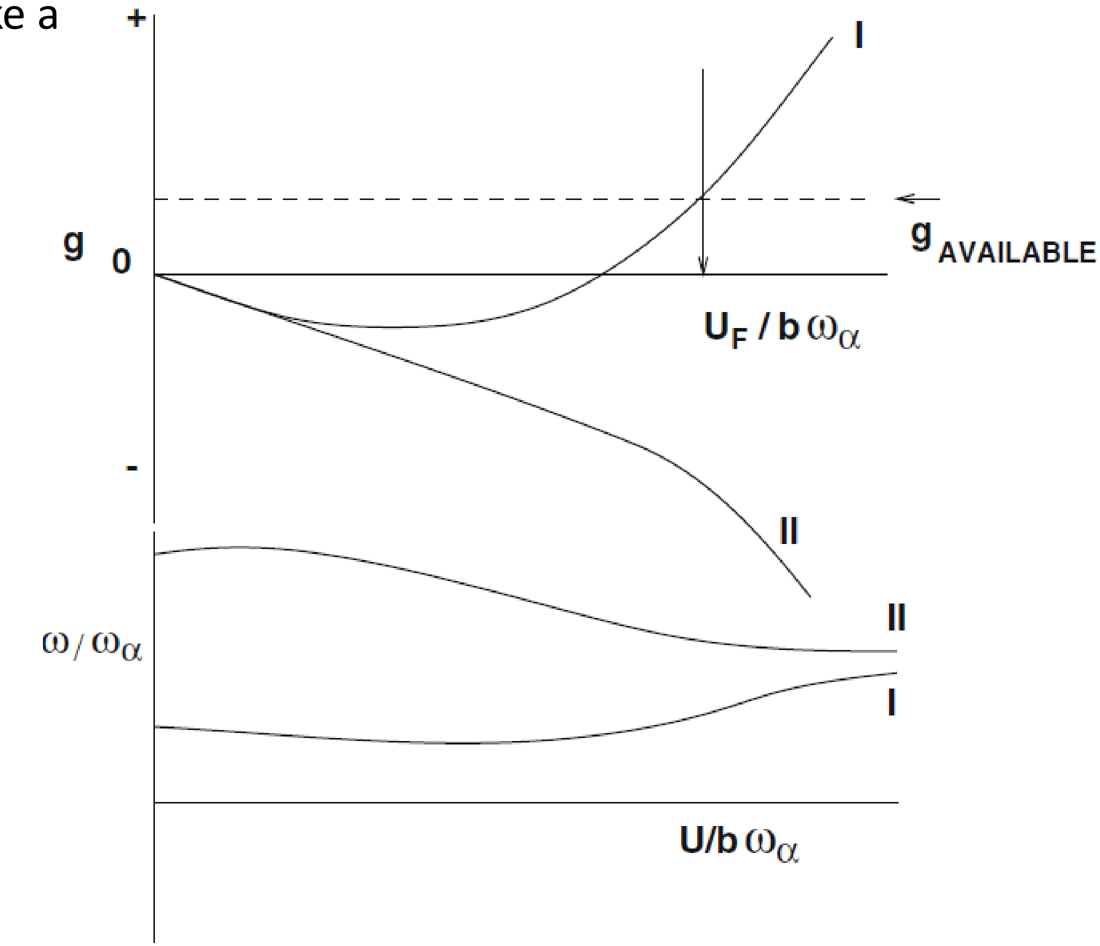
$$\begin{cases} \dfrac{\partial f}{\partial q} \Delta q + \dfrac{\partial f}{\partial s} \Delta s = -f(q_i, s_i) \\ 2 q_i^* \Delta q = 1 - q_i^* q_i \end{cases}$$

Types of flutter

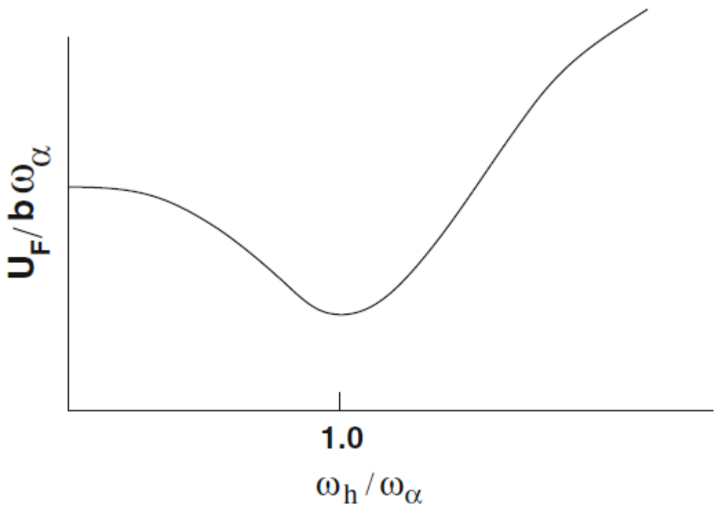


Types of Flutter

Sometimes it is possible to make a hypothesis on the minimum damping available without considering it in the model

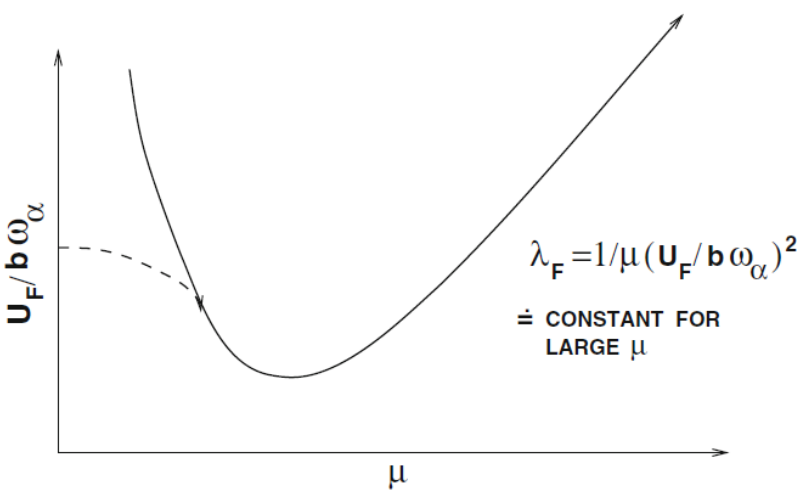


Dependency of flutter on main parameters



Dependency on the ratio between bending and torsional structures

Dependency on the mass ratio



Transonic dip

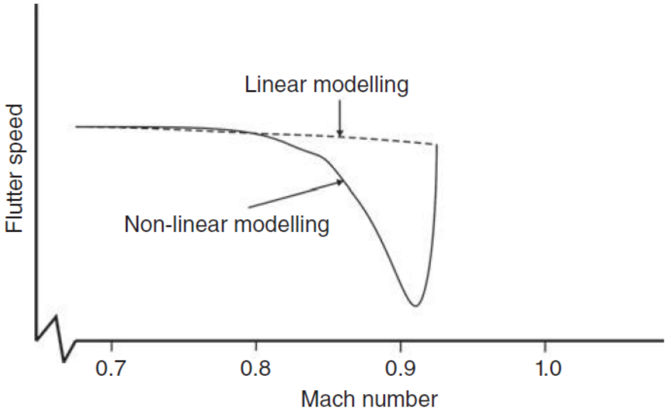
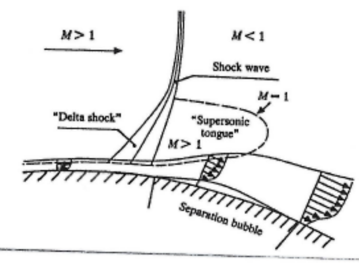


Figure 10.25 Typical flutter speed behaviour in a transonic regime.

The aerodynamic pressure on an airfoil is normally greatest near Mach number equal to one and hence, the flutter speed (or dynamic pressure) tends to be a minimum there.



The formation of shock waves on the wing in the transonic region influences the aerodynamic transfer function leading in the transonic region to lower flutter speeds (transonic dip).

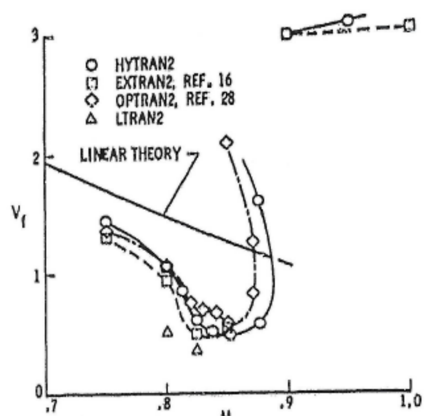


Fig. 7 Comparison of NACA 64A010 calculated flutter boundaries.

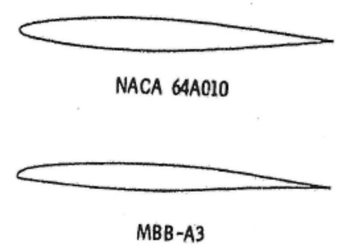


Fig. 3 Airfoil profiles.