POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

05 Dynamic Aeroelasticity: Introduction to Flutter

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References

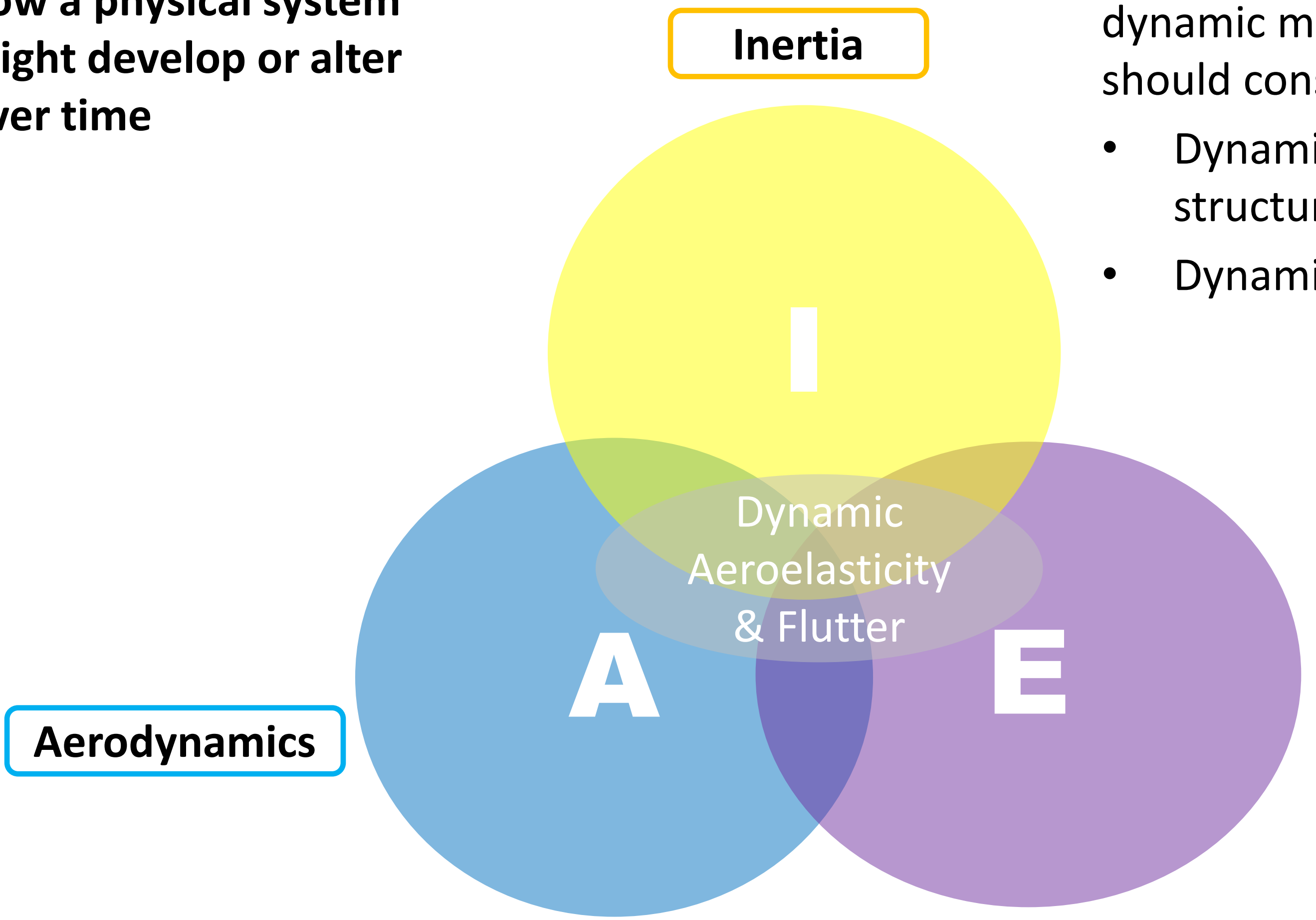
Dowell Section 3.2, 3.3.0 (not the rest)

Dowell Section 3.3.5 3.3.6

Cooper Section 10 up to 10.5.2

Dynamics

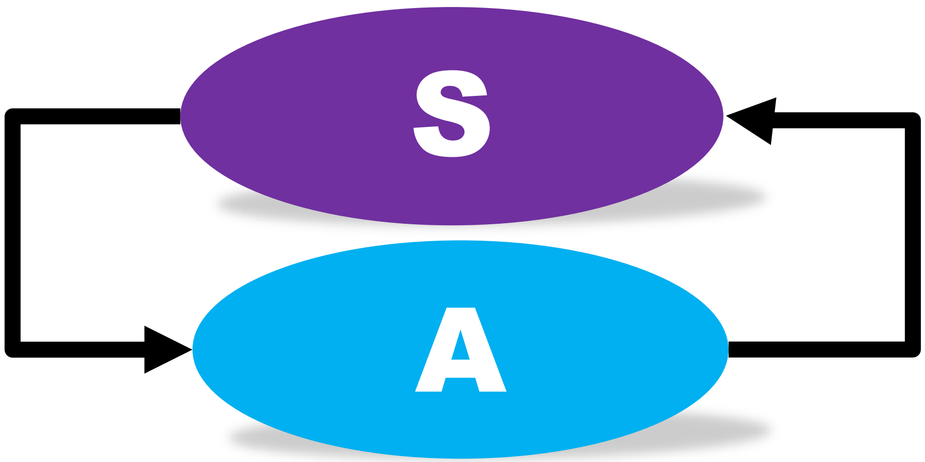
how a physical system might develop or alter over time



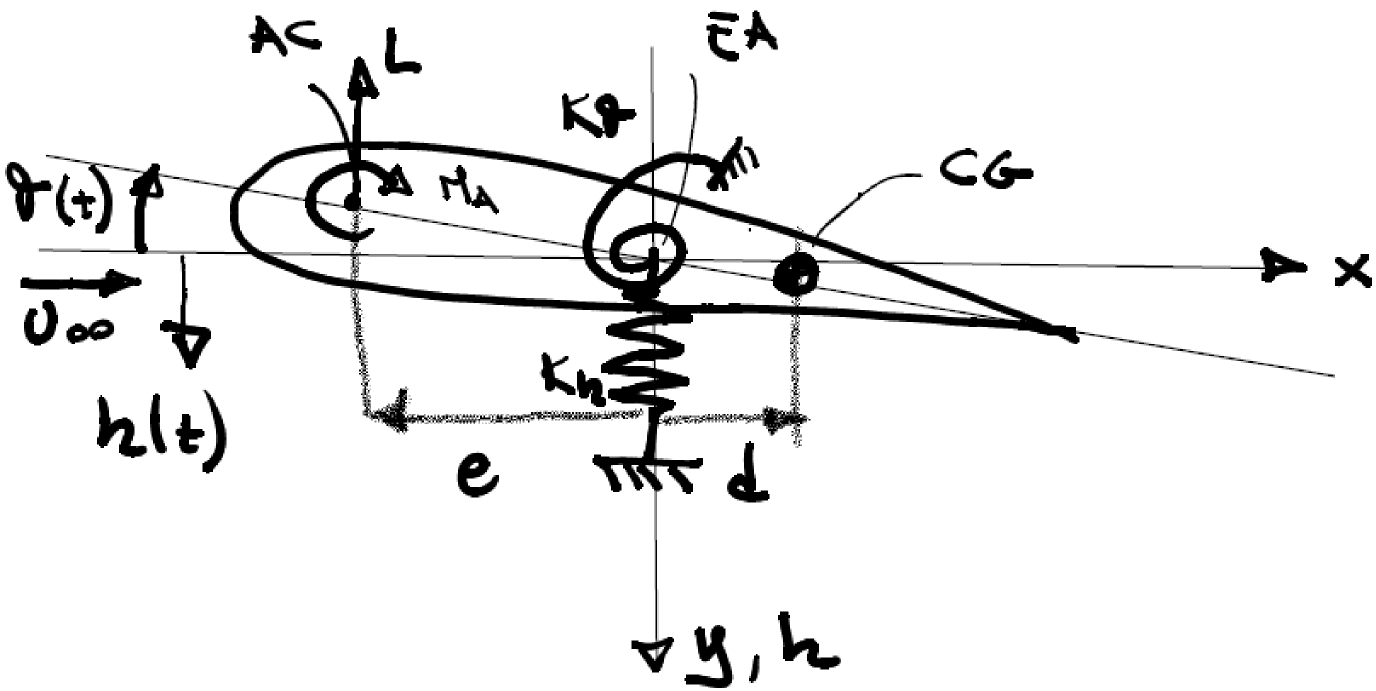
We want to consider the full dynamic model. In principle we should consider:

- Dynamics of the deformable structures
- Dynamics of aerodynamics

Dynamics



Typical section



mass m

inertia moment with respect to CG I_0

$h(t)$	PLUNGE positive downward
$\theta(t)$	PITCH positive clockwise

Write the equation of motion using Lagrange approach

$$\begin{bmatrix} m & md \\ md & I_o + md^2 \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} -L \\ eL + M_{CA} \end{Bmatrix}$$

$S_\theta = md$ is the static moment of inertia
$I_\theta = I_0 + md^2$ is the moment of inertia with respect to the elastic axis

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} -L \\ eL + M_{CA} \end{Bmatrix}$$

Write the equation of motion using different approaches

- ✓ VWP directly
- ✓ D'Alambert approach

Compute the solution for the homogeneous problem q = 0

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

with

$$\mathbf{M} = \begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \qquad \mathbf{K} = \begin{bmatrix} k_h & 0 \\ 0 & k_h \end{bmatrix} \quad \mathbf{q} = \begin{Bmatrix} h \\ \theta \end{Bmatrix}$$

The general solution to a linear homegeneous differential equation is

$$\mathbf{q} = \mathbf{q}_0 \mathrm{e}^{\lambda t}$$

so

Always $\neq 0$

$$\left(\lambda^2 \mathbf{M} + \mathbf{K}\right) \mathbf{q}_0 \mathrm{e}^{\lambda t} = \mathbf{0}$$

$$\left(\lambda^2 \mathbf{M} + \mathbf{K}\right) \mathbf{q}_0 = \mathbf{0}$$

To have a nontrivial solution i.e., one where $\mathbf{q}_0 \neq 0$

$$\det\left(\lambda^2 \mathbf{M} + \mathbf{K}\right) = 0$$

Compute the solution for the homogeneous problem q = 0

$$\det \begin{bmatrix} m\lambda^2 + kh & S\lambda^2 \\ S\lambda^2 & I_\theta \lambda^2 + k_\theta \end{bmatrix} = 0$$

$$\begin{aligned} (m\lambda^2 + k_h)(I_\theta\lambda^2 + k_\theta) - S^2\lambda^4 &= 0 \\ \left(\lambda^2 + \frac{k_h}{m}\right)\left(\lambda^2 + \frac{k_\theta}{I_\theta}\right) - \frac{S^2}{mI_\theta}\lambda^4 &= 0 \\ \left(\lambda^2 + \omega_h^2\right)\left(\lambda^2 + \omega_\theta^2\right) - \frac{md^2}{I_\theta}\lambda^4 &= 0 \end{aligned}$$

Natural frequency of the isolated plunge movement

$$\omega_h = \sqrt{\frac{k_h}{m}}$$

Natural frequency of the isolated pitch movement

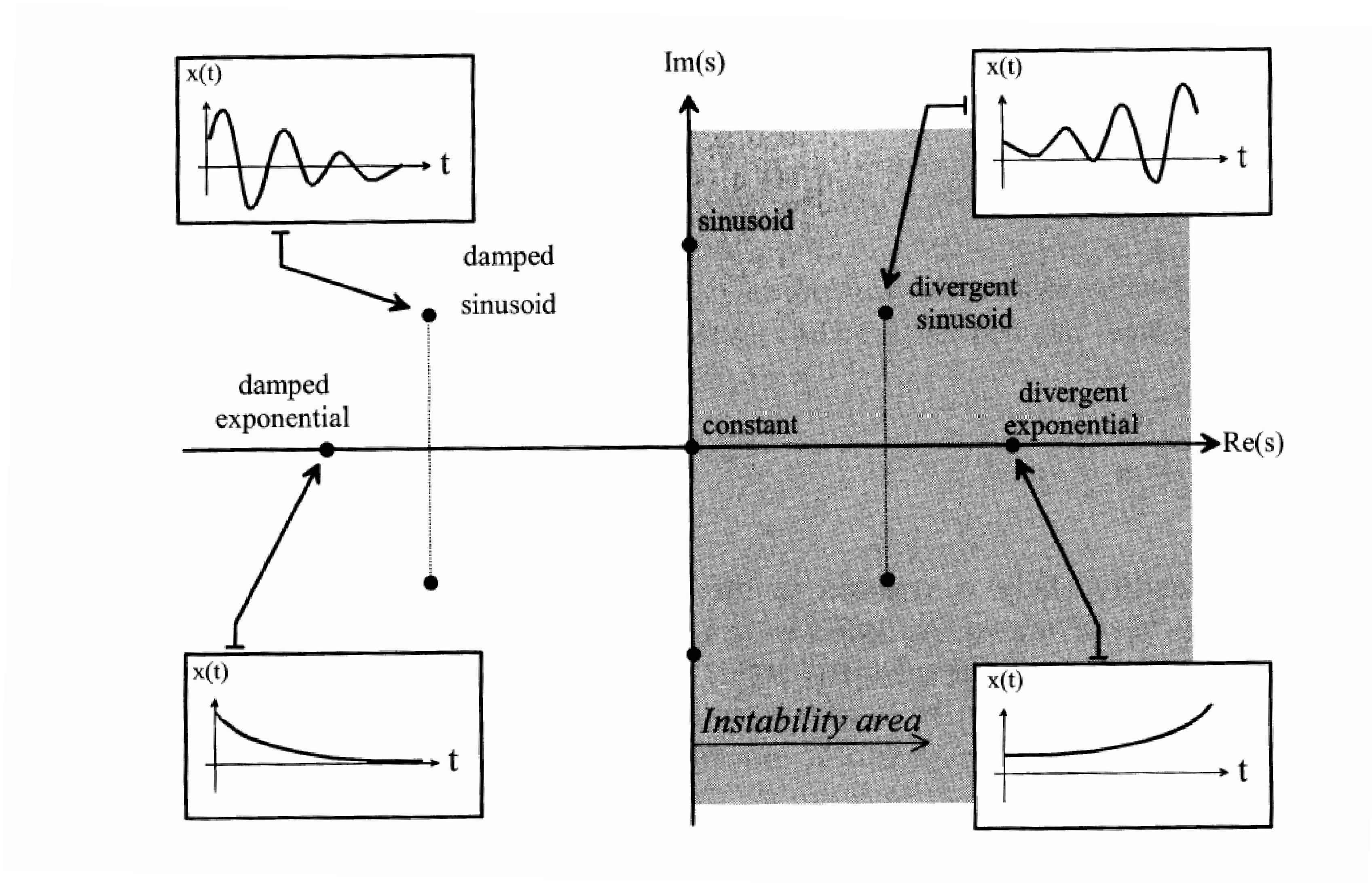
$$\omega_\theta = \sqrt{\frac{k_\theta}{I_\theta}}$$

$$\left(1 - \frac{md^2}{I_\theta}\right)\lambda^4 + (\omega_h^2 + \omega_\theta^2)\lambda^2 + \omega_h^2\omega_\theta^2 = 0$$

$$1 - \frac{md^2}{I_\theta} = 1 - \frac{I_\theta - I_0}{I_\theta} = \frac{I_0}{I_\theta}$$

$$\lambda^4 + \frac{(\omega_h^2 + \omega_\theta^2)}{I_0/I_\theta}\lambda^2 + \frac{\omega_h^2\omega_\theta^2}{I_0/I_\theta} = 0$$

Eigenvalues and stability



Compute the solution for the homogeneous problem q = 0

$$\lambda_{\pm}^2 = -\frac{1}{2}\frac{I_\theta}{I_0}\left(\omega_h^2+\omega_\theta^2\right) \pm \sqrt{\frac{1}{4}\frac{I_\theta^2}{I_0^2}\left(\omega_h^2+\omega_\theta^2\right)^2 - \frac{I_\theta}{I_0}\omega_h^2\omega_\theta^2}$$

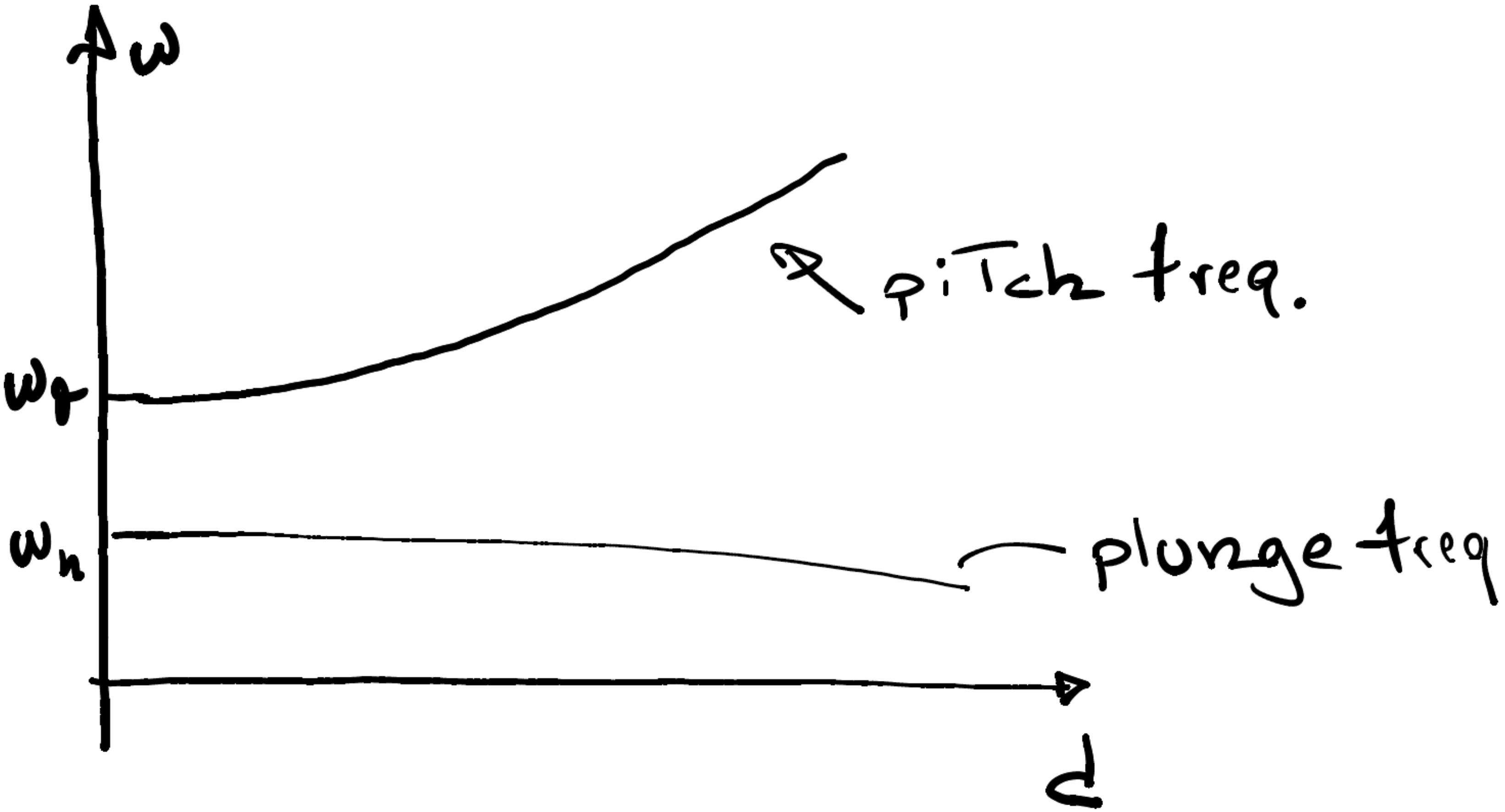
$$\lambda_{\pm}^2 < 0 \Rightarrow \begin{cases} \lambda_{1/2} = \pm \mathrm{j}\sqrt{|\lambda_+^2|} = \pm \mathrm{j}\omega_1 \in \mathbb{C} \\ \lambda_{3/4} = \pm \mathrm{j}\sqrt{|\lambda_-^2|} = \pm \mathrm{j}\omega_2 \in \mathbb{C} \end{cases}$$

<u>THE SYSTEM IS STABLE BUT NOT ASYMPTOTICALLY STABLE</u>
because the real part is null (but not positive).

Determination of the associated eigenvectors

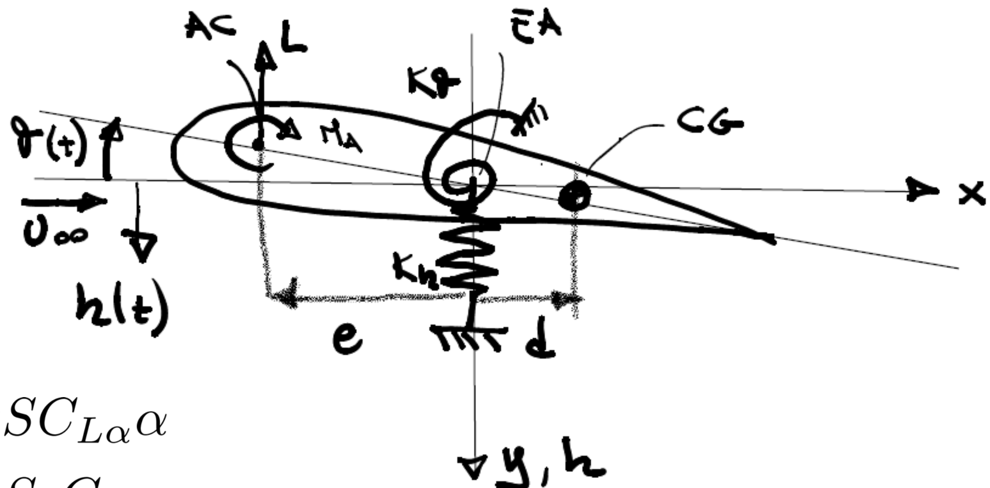
$$\begin{bmatrix} -m\omega_i^2 + kh & -md\omega_i^2 \\ -md\omega_i^2 & -I_\theta\omega_i^2 + k_\theta \end{bmatrix} \begin{Bmatrix} h_0 \\ \theta_0 \end{Bmatrix}_i = 0 \qquad \left(\frac{h_0}{\theta_0}\right)_i = \frac{\omega_i^2 md}{-\omega_i^2 m + k_h} = \frac{-\omega_i^2 I_\theta + k_\theta}{\omega_i^2 md}$$

Behavior of system eigenvalues (frequencies)



$$\left(\frac{h_0}{\theta_0}\right)_i = \frac{\omega_i^2 d}{-\omega_i^2 + \omega_h^2} = \frac{d}{1 - \frac{\omega_h^2}{\omega_i^2}}$$

Inclusion of steady aerodynamics



$$
\begin{aligned}
L &= qSC_{L\alpha}\alpha \\
M_{AC} &= qScC_{m_{AC}}
\end{aligned}
$$

<u>STEADY APPROXIMATION</u> $\alpha = \theta$

$$
\begin{Bmatrix} -L \\ eL + M_{AC} \end{Bmatrix} = \underbrace{qSC_{L\alpha}\begin{bmatrix} 0 & -1 \\ 0 & e \end{bmatrix}}_{\mathbf{K}_A} \begin{Bmatrix} h \\ \theta \end{Bmatrix} + qS \begin{Bmatrix} 0 \\ cC_{m_{CA}} \end{Bmatrix}
$$

$$
\mathbf{M}\ddot{\mathbf{q}} + (\mathbf{K}_s - q\mathbf{K}_A)\mathbf{q} = \mathbf{F}_0
$$

Compute the eigenvalues of the new homogeneous solution (with q > 0)

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \left(\begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} - qSC_{L\alpha} \begin{bmatrix} 0 & -1 \\ 0 & e \end{bmatrix} \right) \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \mathbf{0}$$

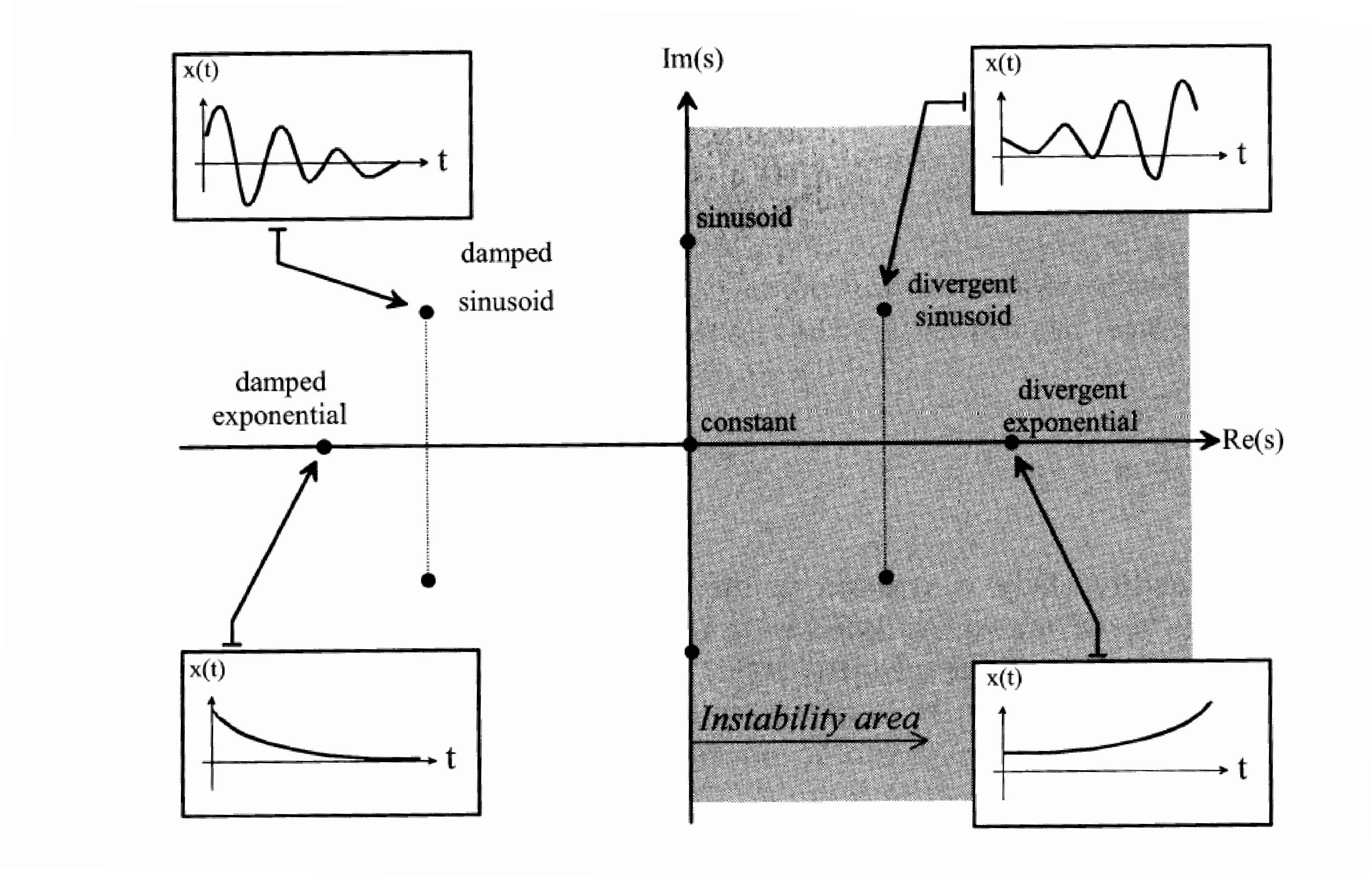
$$\mathbf{q} = \mathbf{q}_0 \mathrm{e}^{\lambda t} \qquad \left(\mathbf{M}\lambda^2 + \mathbf{K}_s - q\mathbf{K}_A \right) \mathbf{q}_0 \mathrm{e}^{\lambda t} = 0$$

$$\underbrace{\begin{bmatrix} m\lambda^2 + k_h & S_\theta\lambda^2 + qSC_{L\alpha} \\ S_\theta\lambda^2 & I_\theta\lambda^2 + k_\theta - qSeC_{L\alpha} \end{bmatrix}}_{\mathbf{A}} \begin{Bmatrix} h_0 \\ \theta_0 \end{Bmatrix} = \mathbf{0}$$

$$\det(\mathbf{A}(\lambda)) = 0 \;\Rightarrow\; a\lambda^4 + b\lambda^2 + c = 0$$

Nonlinear eigenvalue problem or
generalized eigenvalue problem

Eigenvalues and stability



How we characterize an eigenvalue

$$\det(\mathbf{A}(\lambda)) = 0 \;\Rightarrow\; a\lambda^4 + b\lambda^2 + c = 0$$

Characteristic polynomial or
Characteristic equation

The solutions of this algebraic equation are four complex <u>EIGENVALUES</u> $\lambda_i \in \mathbb{C}$

$$\lambda = \sigma + \mathrm{j}\omega \begin{cases} \sigma = \mathrm{Re}(\lambda) \\ \omega = \mathrm{Im}(\lambda) \end{cases}$$

$\omega = \mathrm{Im}(\lambda)$ <u>DAMPED NATURAL FREQUENCY</u>

$\xi = -\dfrac{\mathrm{Re}(\lambda)}{|\lambda|}$ <u>DAMPING RATIO</u>

$$\begin{cases} \xi > 0 & \text{Asymphthotically Stable} \\ \xi = 0 & \text{Stable} \\ \xi < 0 & \text{Unstable} \end{cases}$$

How we characterize an eigenvalue

Damped harmonic oscillator

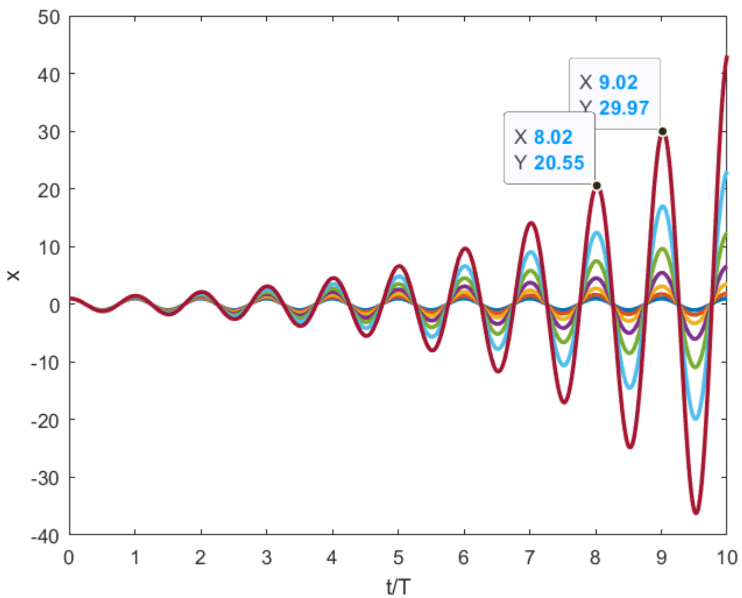
ξ = 0 : 0.1 : 1.0

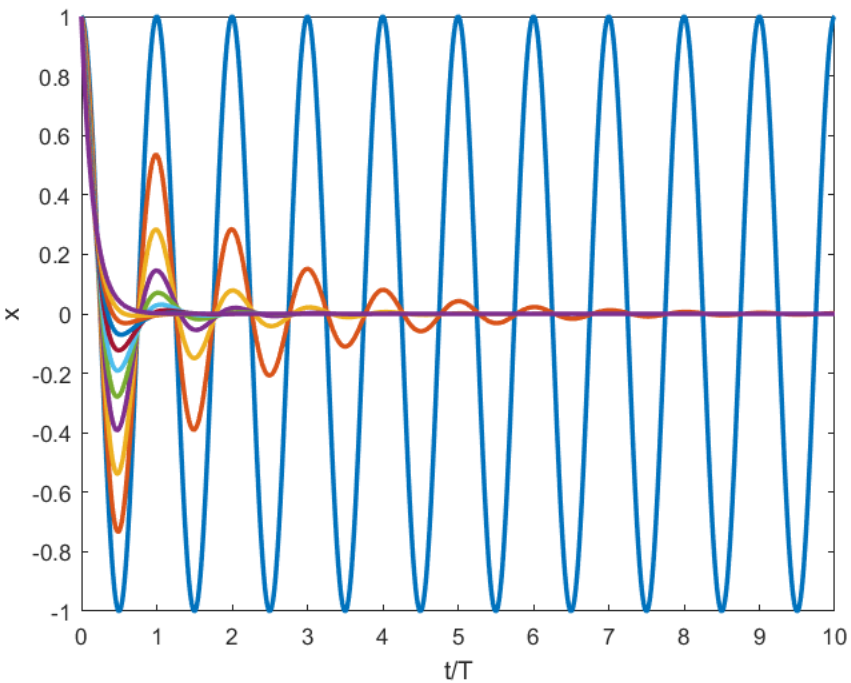
$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\det(\lambda^2 + 2\xi\omega_n\lambda + \omega_n^2) = 0$$

$$\lambda_{1/2} = -\xi\omega_n \pm \mathrm{j}\omega_n\sqrt{1-\xi^2}$$

ξ = 0 : -0.01 : -0.06





$$-1 < \xi < 1$$

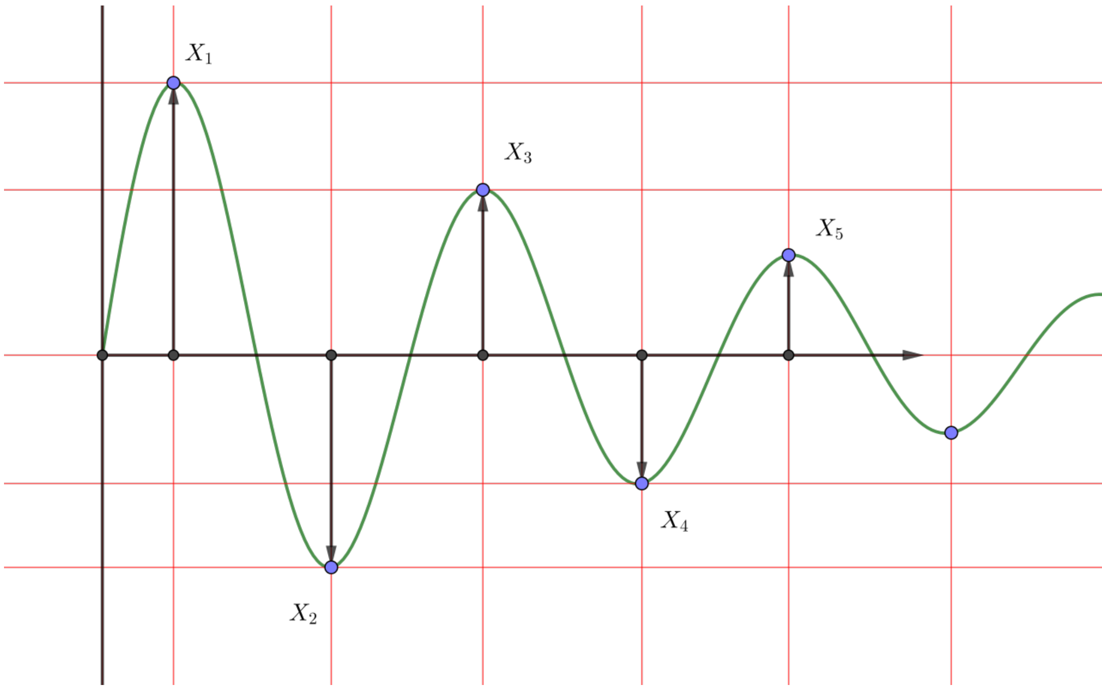
$|\xi| = 1$ means that the eigenvalue is real

$$\omega_n = \frac{\mathrm{Im}(\lambda)}{\sqrt{1-\xi^2}} \quad \underline{\text{NATURAL FREQUENCY}}$$

Logarithmic decrement

Are you able to compute the damping ratio looking at those diagrams?

Use the logarithmic decrement



$$\delta = \ln \frac{x(t)}{x(t+T)}$$

$$\xi = \frac{1}{\sqrt{1+\left(\frac{2\pi}{\delta}\right)^2}}$$

$$T = \frac{2\pi}{\omega_d}$$

$$x(t+T) = x(t)\mathrm{e}^{-\xi\omega_n T}$$

$$\delta = \xi\omega_n T = \xi\frac{2\pi}{\sqrt{1-\xi^2}}$$

$$|\xi| \ll 1 \quad \xi \approx \frac{\delta}{2\pi}$$

ξ and g damping

Often in aeroelasticity to characterize damping it is used the parameter g

$$g = \frac{\mathrm{Re}(\lambda)}{\mathrm{Im}(\lambda)}$$

<u>WARNING</u> For real eigenvalues $g \to \infty$ so it not good to characterize systems Real eigenvalues

If $\xi \ll 1$ then $g \approx -\xi$

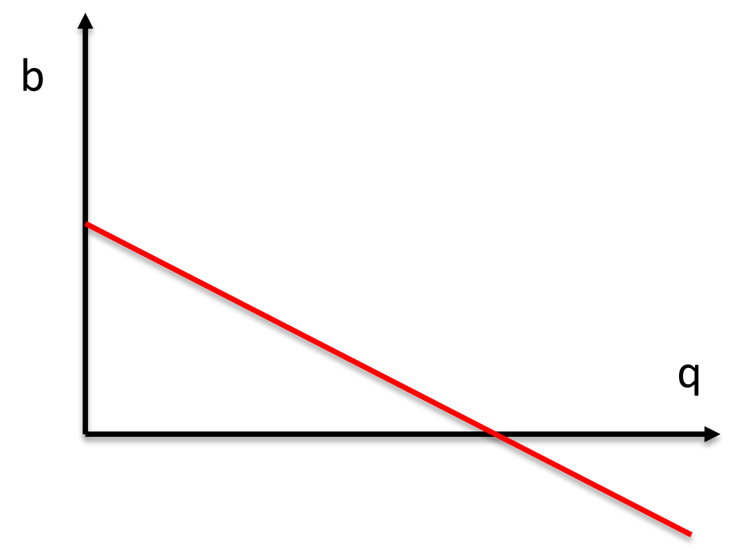
Stability of the typical section with steady aero

$$\begin{bmatrix} m\lambda^2 + k_h & S_\theta\lambda^2 + qSC_{L\alpha} \\ S_\theta\lambda^2 & I_\theta\lambda^2 + k_\theta - qSeC_{L\alpha} \end{bmatrix} \begin{Bmatrix} h_0 \\ \theta_0 \end{Bmatrix} = \mathbf{0}$$

$$\det(\mathbf{A}(\lambda)) = 0 \;\Rightarrow\; a\lambda^4 + b\lambda^2 + c = 0 \;\Rightarrow\; \lambda^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\begin{aligned} a &= mI_\theta - S_\theta^2 = \det(\mathbf{M}) > 0 \text{ always} \\ b &= m\left(k_\theta - qSeC_{L\alpha}\right) + I_\theta k_h - S_\theta qSC_{L\alpha} \\ c &= k_h(k_\theta - qSeC_{L\alpha}) > 0 \quad \text{if } q < q_D \end{aligned}$$

$$b = \underbrace{mk_\theta + I_\theta k_h}_{> 0} - qSC_{L\alpha}(em + S_\theta)$$



Stability of the typical section with steady aero

- $b > 0$ valid for $q = 0$ and for the first range of values of q
- $b < 0$ only for very large q values (not important in practice)

Case 1) $(b^2 - 4ac) > 0$ and $b > 0$

$$\lambda^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda_i^2 \in \mathbb{R}, \quad \lambda_i^2 < 0 \, \forall i$$

$$\Rightarrow \lambda_j = \pm \mathrm{j}\omega_j \in \mathbb{C}$$

In this case the 4 eigenvalues are complex numbers with the real part equal to zero

$\Rightarrow$ the system is <u>STABLE</u>

Stability of the typical section with steady aero

$$\lambda^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Case 2) $(b^2 - 4ac) < 0 \Rightarrow \sqrt{b^2 - 4ac} \in \mathbb{C}$, and so

$$\lambda_j^2 \in \mathbb{C} \quad \text{with} \; \mathrm{Re}(\lambda_j^2) < 0$$

$$\begin{aligned} \lambda_j^2 &= |\lambda_j^2| \mathrm{e}^{\mathrm{j}\varphi_j} \\ \varphi_j &= \tan^{-1}\left(\frac{\mathrm{Im}(\lambda_j^2)}{\mathrm{Re}(\lambda_j^2)}\right) \\ \Rightarrow \qquad & \lambda_{j1} = |\lambda_j| \mathrm{e}^{\mathrm{j}\frac{\varphi_j}{2}}, \lambda_{j2} = |\lambda_j| \mathrm{e}^{\mathrm{j}\frac{\varphi_j}{2} + \mathrm{j}\pi} \end{aligned}$$

Root square of $z^2 \in \mathbb{C}$

$$z^2 = r\mathrm{e}^{\mathrm{j}\varphi}$$

$$\begin{cases} z_1 = \sqrt{r}\mathrm{e}^{\mathrm{j}\frac{\varphi}{2}} \\ z_2 = \sqrt{r}\mathrm{e}^{\mathrm{j}\frac{\varphi}{2} + \mathrm{j}\pi} \end{cases}$$

At least one of the eigenvalues λ will have the real part positive!
$\Rightarrow$ the system is <u>UNSTABLE</u>

Stability of the typical section with steady aero

The limit of stability represents the condition where the system is transformed from being STABLE into an UNSTABLE one.This condition is also identified as FLUTTER.

To identify it, it is necessary to find the dynamic pressure at which the quantity $(b^2 - 4ac)$ changes sign (from postive to negative).

$$\text{FLUTTER} : \text{find}\, q_F \;:\; b^2 - 4ac = 0$$

$$b^2 - 4ac = 0 \;\rightarrow\; f(q_F) = 0$$

q_F FLUTTER DYNAMIC PRESSURE.

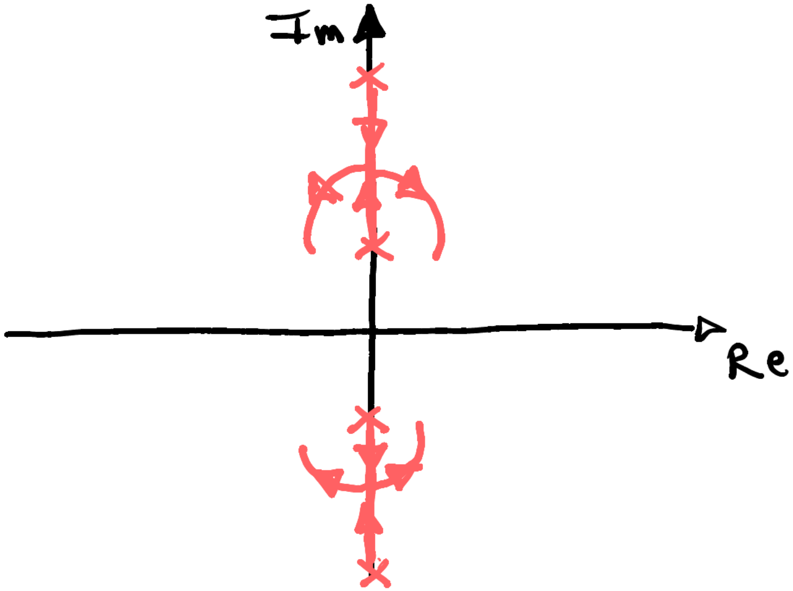
It is possible to verify that $f(q_F)$ is a second order equation in q_F i.e.,

$$c_1 q_F^2 + c_2 q_F + c_3 = 0$$

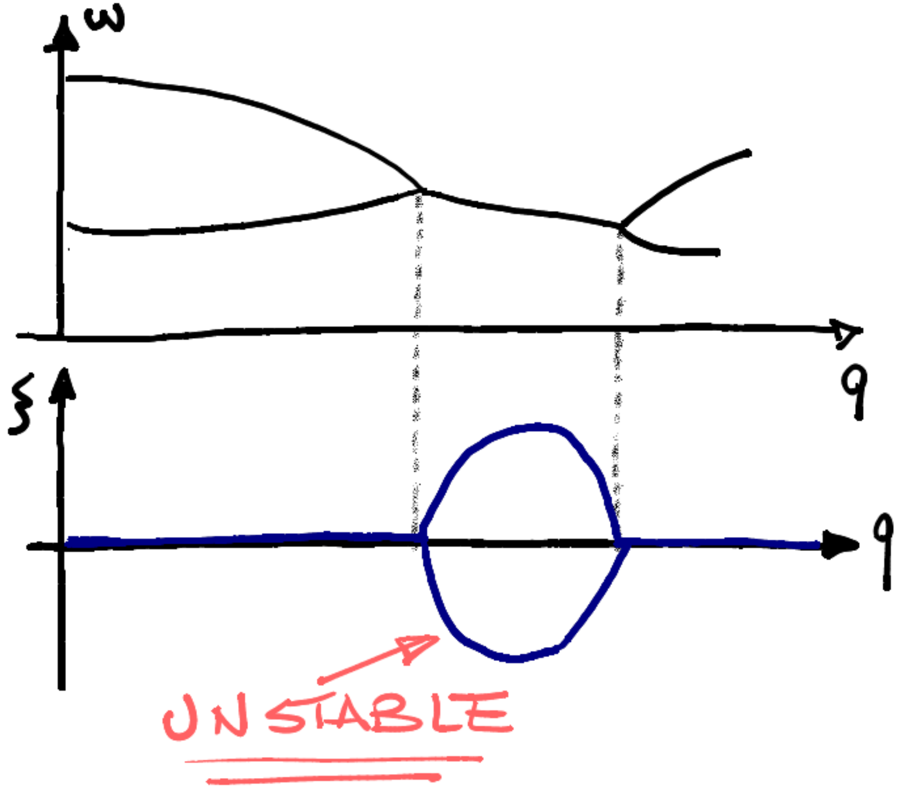
If $q_{Fi} < 0 \forall i$ or $q_{Fi} \in \mathbb{C}$ there is no physically meaningful flutter dynamic pressure.

It is possible to verify that this happens if $S_\theta \leq 0$ (for the TS with this aerodynamic approximation).

Stability of the typical section with steady aero



Evolution of the eigenvalues for increasing values of dynamic pressure



Stability of the typical section with steady aero

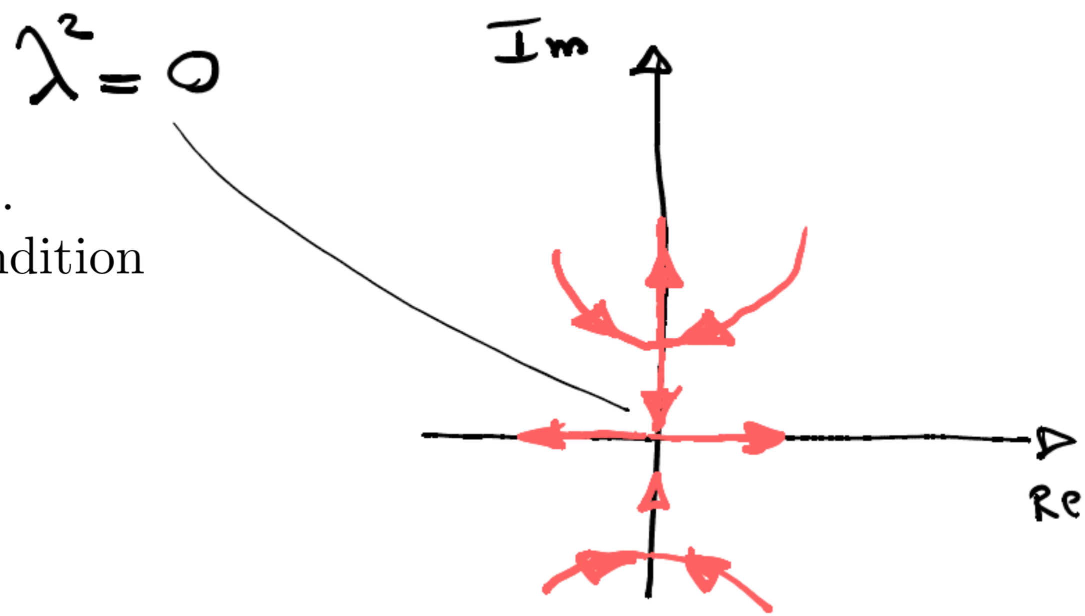
If the dynamic pressure is incremented further it will be possible to reach the point where $c = 0$

$$k_\theta - qSeC_{L\alpha} = 0, \text{ so } q = q_D$$

The characteristic equation becomes

$$a\lambda^4 + b\lambda^2 = 0$$
$$\lambda^2(a\lambda^2 + b) = 0$$

$\lambda^2 = 0$ two eigenvalues are null.
This is the DIVERGENCE condition already studied.



Stability of the typical section with steady aero

$$\left(\Omega^2 \begin{bmatrix} 1 & \bar{d} \\ \bar{d} & \bar{r}_\theta^2 \end{bmatrix} + \begin{bmatrix} R^2 & 0 \\ 0 & \bar{r}_\theta^2 \end{bmatrix} - \bar{U}^2 \frac{C_{L\alpha}}{\pi\mu} \begin{bmatrix} 0 & -1 \\ 0 & \bar{e} \end{bmatrix}\right) \begin{Bmatrix} \bar{h} \\ \theta \end{Bmatrix} = \mathbf{0}$$

$$r_\theta^2 = \frac{I_\theta}{m} \text{ Gyration radius}$$

$$R^2 = \frac{\omega_h^2}{\omega_\theta^2} \text{ Frequency ratio}$$

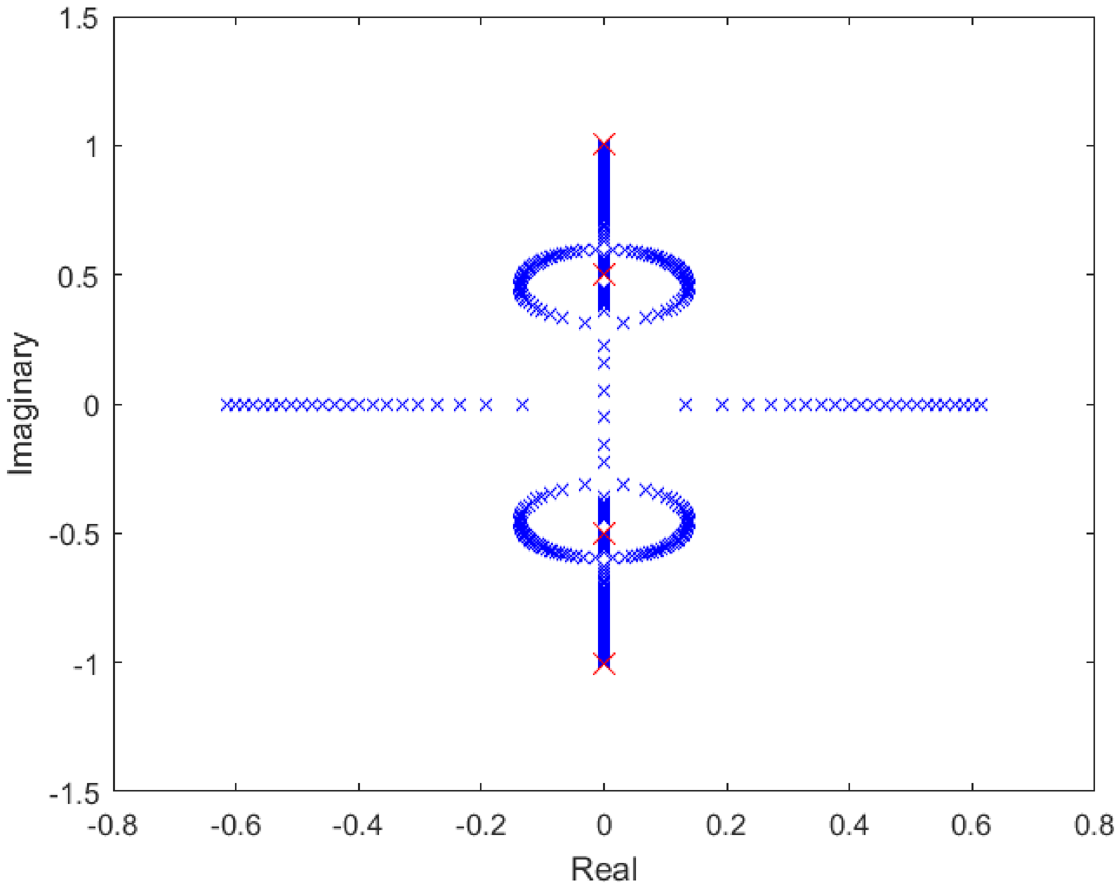
$$\bar{U} = \frac{2U}{c\omega_\theta}$$

$$\begin{aligned} \mu &= 10 \\ \bar{d} &= 0.05 \\ \bar{r}_\theta &= 0.5 \\ \bar{e} &= 0.5 \\ C_{L\alpha} &= 2\pi \\ R &= 0.5 \\ \bar{U} &= 0 - 1.8 \end{aligned}$$

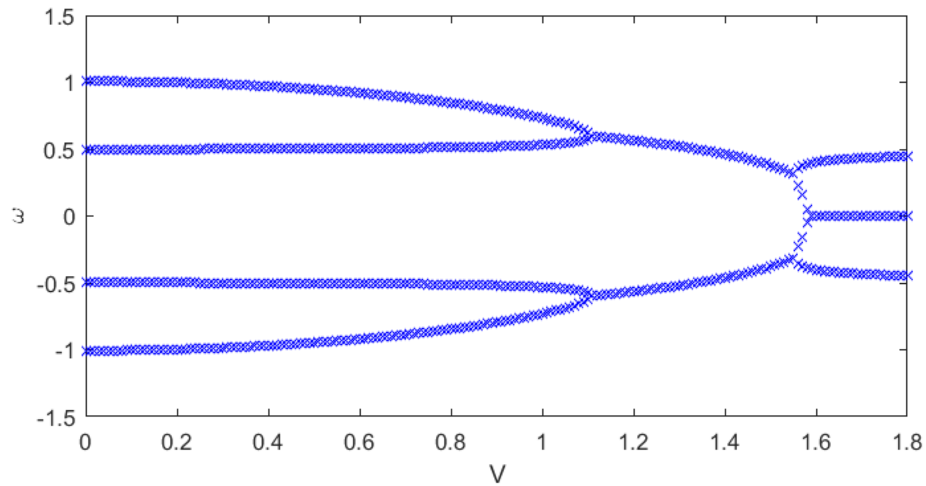
$$\mu = \frac{2m}{\rho\pi\frac{c}{2}S} = \frac{\text{airfoil mass}}{\text{air volume mass}}$$

Compute the eigenvalues in this range of non-dimensional speeds

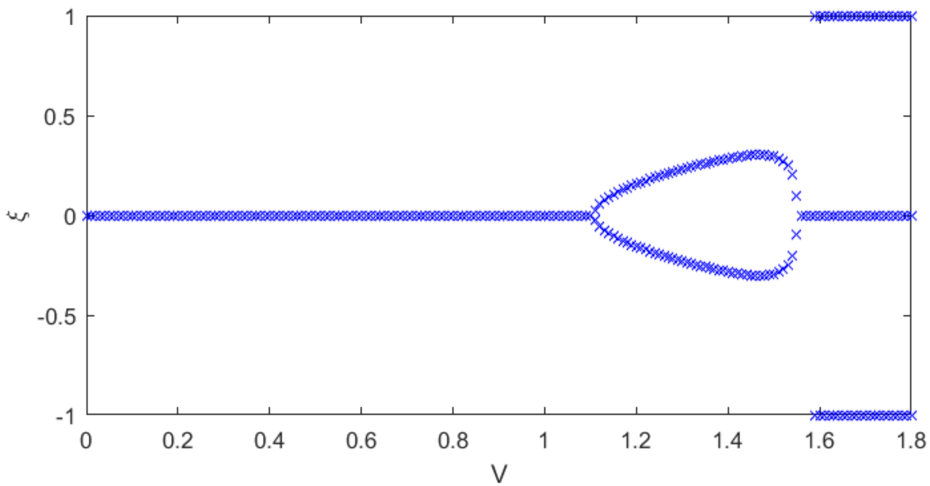
Eigenvalue plot (root locus)



V-ω V-ξ

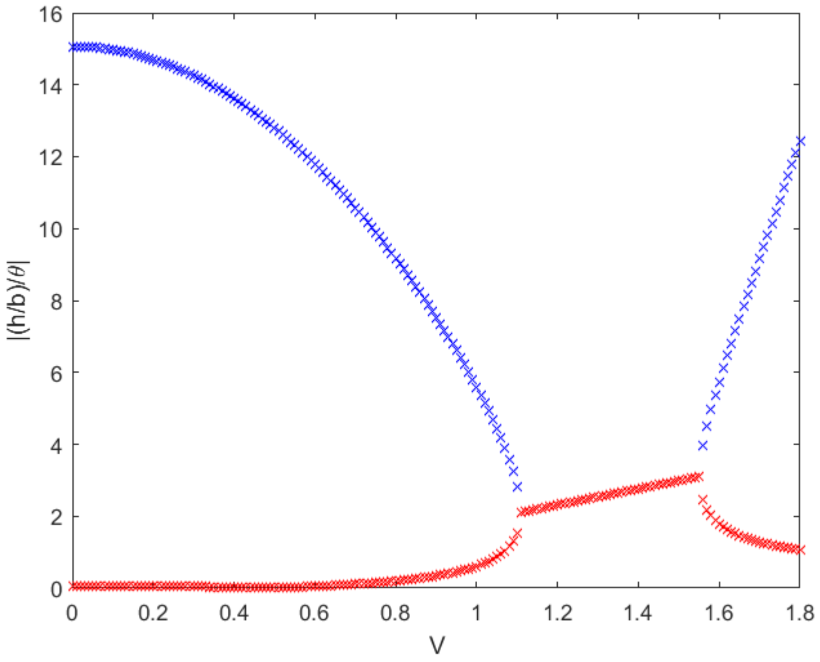


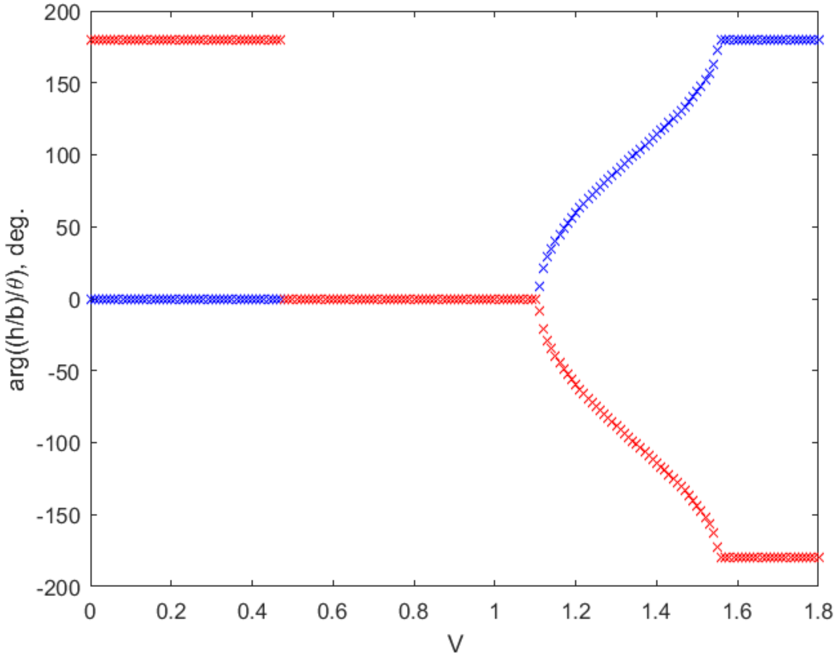
1.5
1
0.5
0
-0.5
-1
-1.5
ω
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
V



1
0.5
0
-0.5
-1
ξ
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
V

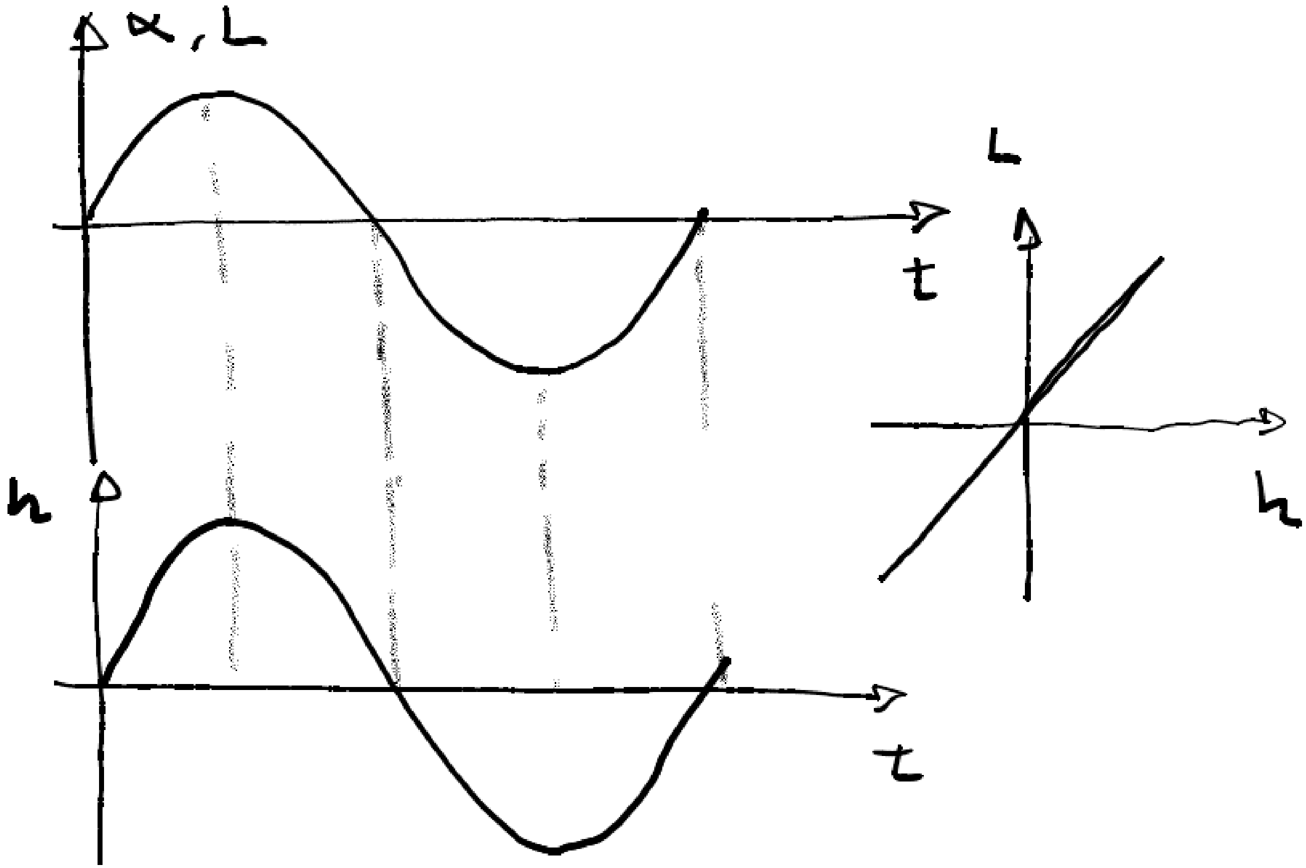
Eigenvectors



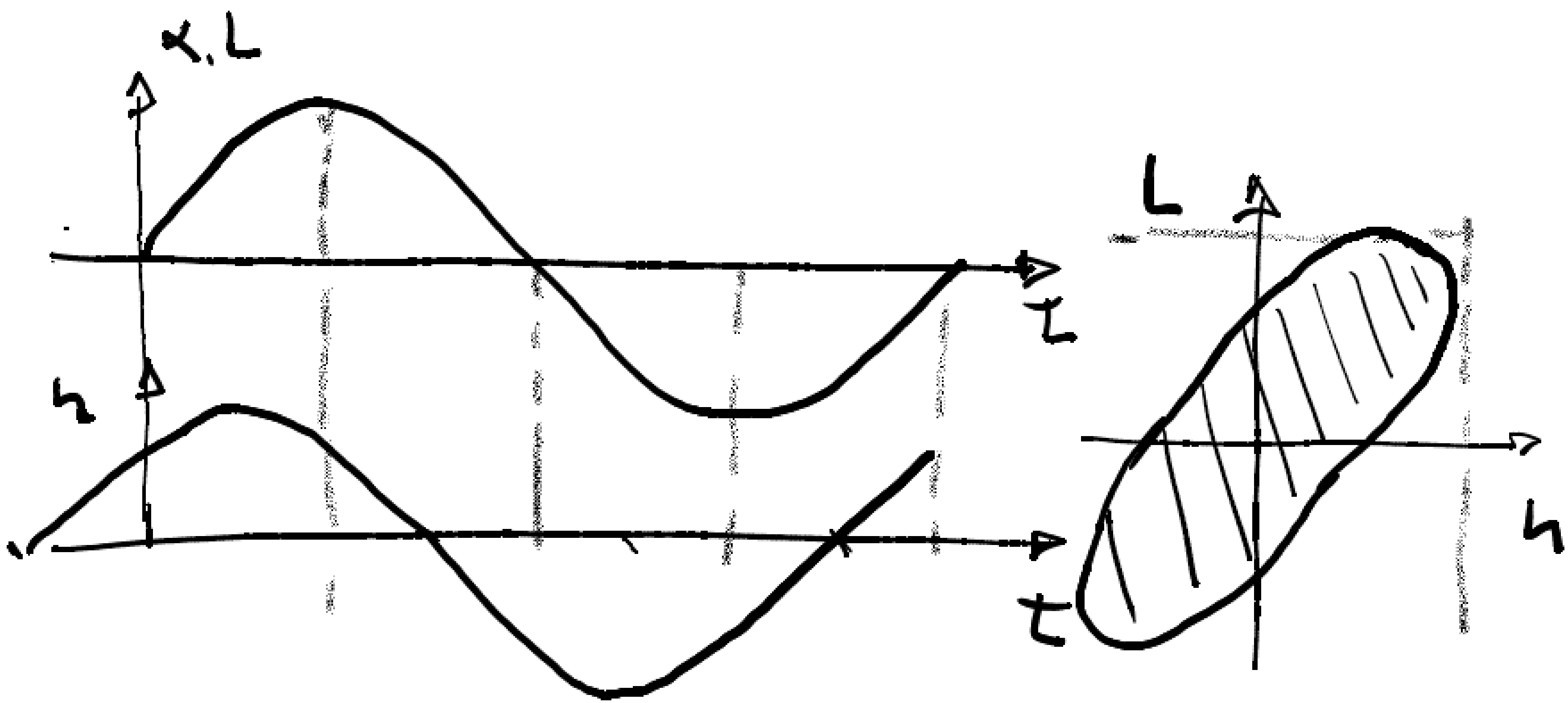


It is the rise of a phase shift between plunge and pitch that is at the base of flutter phenomenon

Effect of phase shift



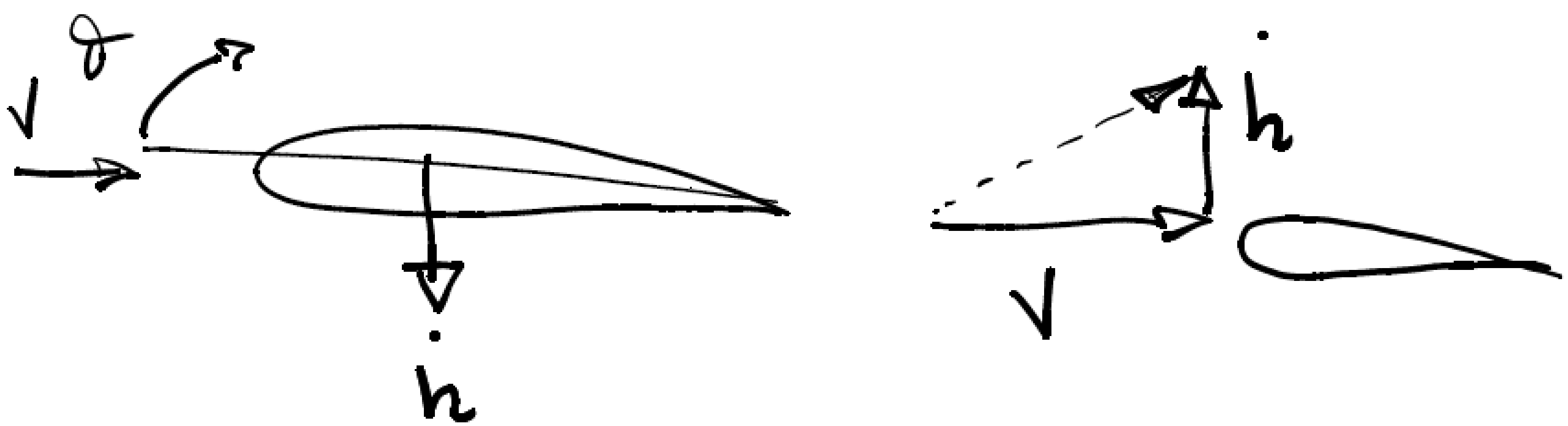
Effect of phase shift



Above flutter conditions the airflow pumps energy into the structure that necessarily amplifies its oscillations

Quasi-steady aerodynamic model

$$L = qSC_{L\alpha}\left(\theta + \frac{\dot{h}}{U}\right)$$



Quasi-steady model

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \frac{qSC_{L\alpha}}{U} \begin{bmatrix} 1 & 0 \\ -e & 0 \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\theta} \end{Bmatrix} + \left(\begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} - qSC_{L\alpha} \begin{bmatrix} 0 & -1 \\ 0 & e \end{bmatrix} \right) \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \mathbf{0}$$

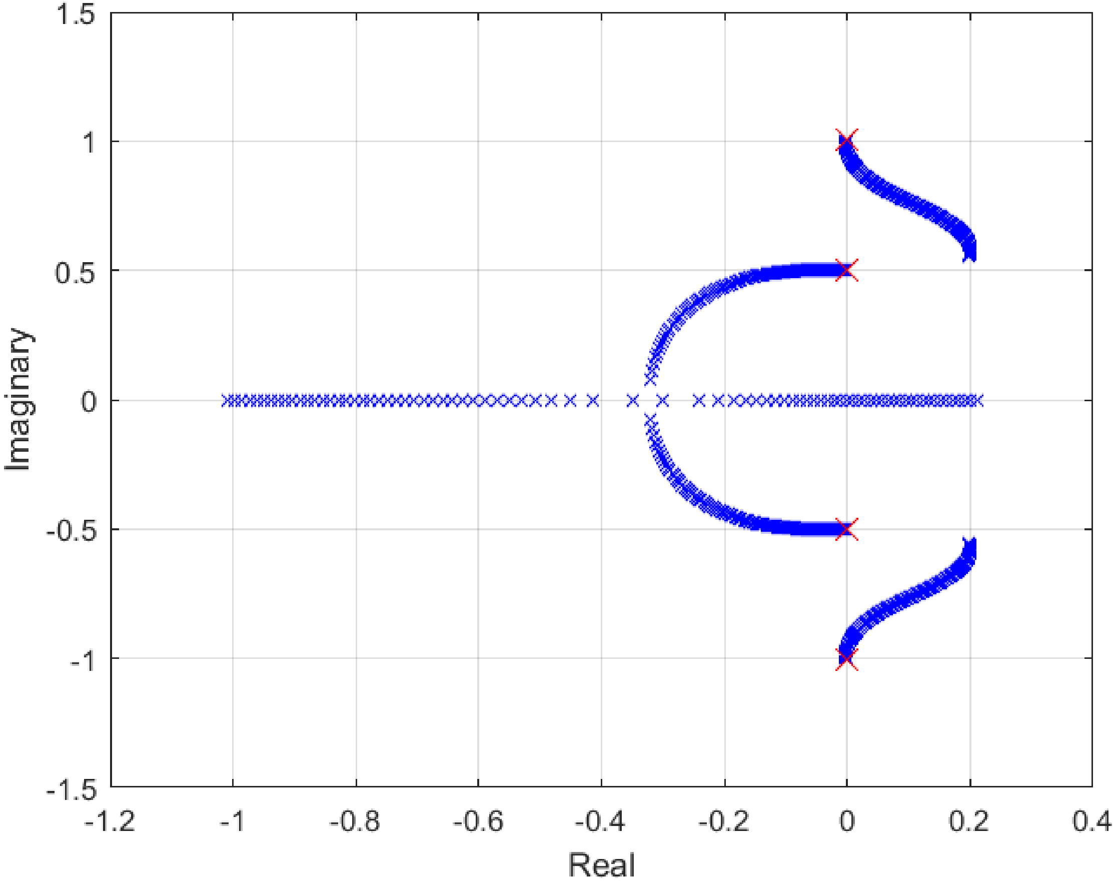
$$\mathbf{M}\ddot{\mathbf{q}} + q\mathbf{C}_A\dot{\mathbf{q}} + (\mathbf{K}_s - q\mathbf{K}_A)\,\mathbf{q} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -(\mathbf{K}_s - q\mathbf{K}_A) & -q\mathbf{C}_A \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix}$$

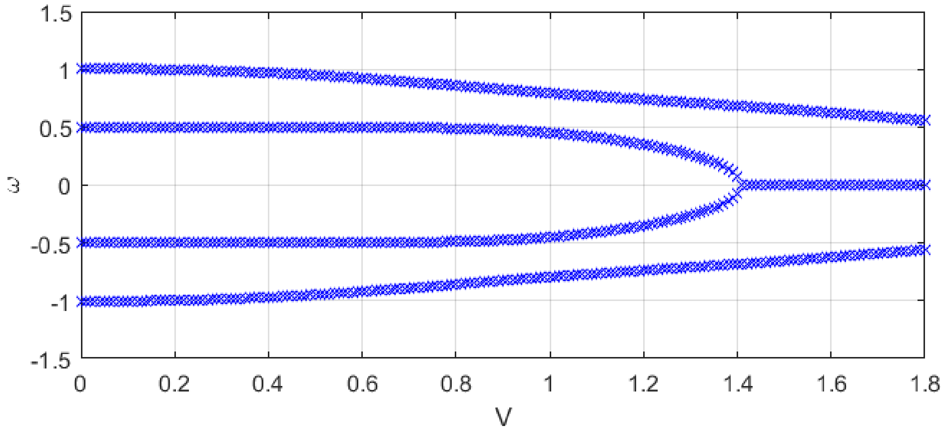
The second order system could be transformed in first order state-space form

$$\begin{Bmatrix} \dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{M}^{-1}(\mathbf{K}_s - q\mathbf{K}_A) & -q\mathbf{M}^{-1}\mathbf{C}_A \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \dot{\mathbf{q}} \end{Bmatrix}$$

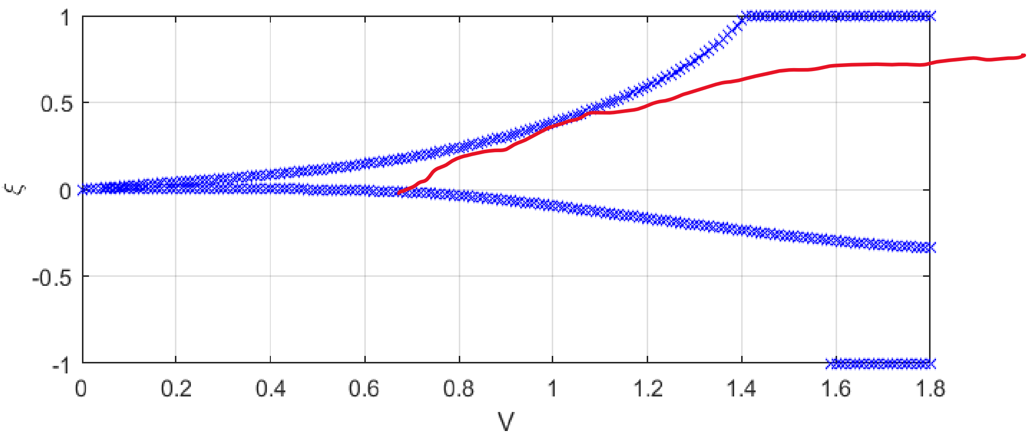
Quasi-steady model



Quasi-steady model

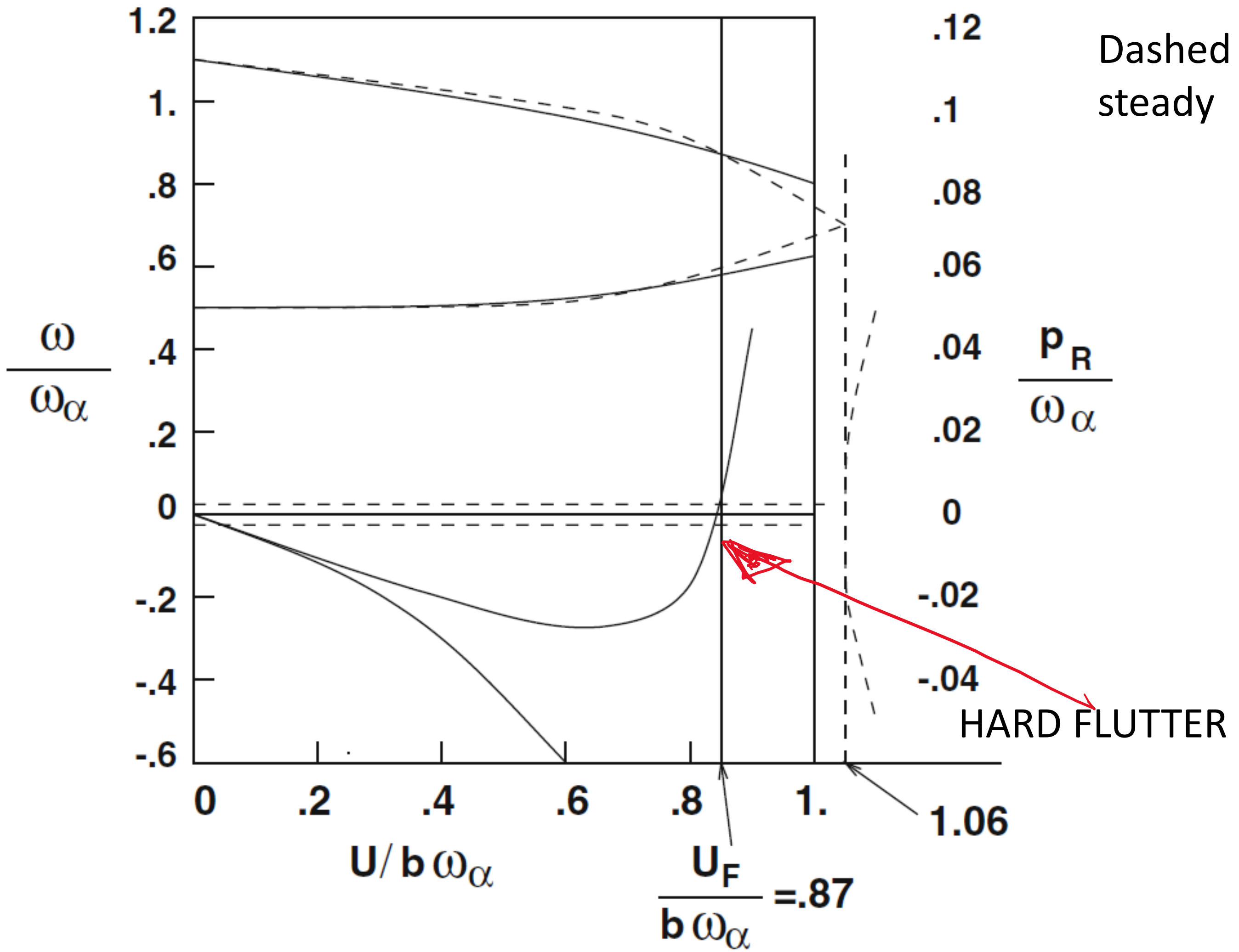


There is no need to get to coincident frequencies to see a flutter



Flutter @0.47

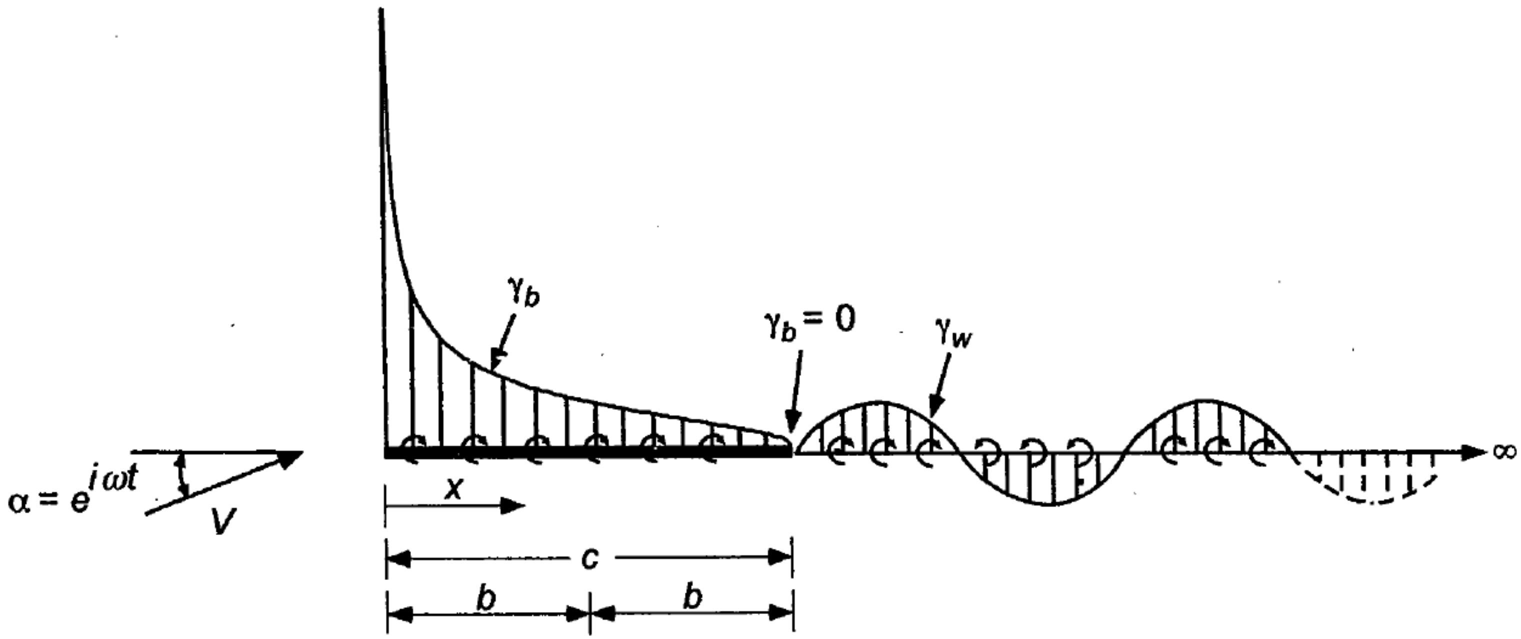
Quasi-steady model



Dashed line is steady

$$
\begin{aligned}
\mu &= 10 \\
\bar{d} &= 0.2 \\
\bar{r}_\theta &= 0.5 \\
\bar{e} &= 0.5 \\
C_{L\alpha} &= 2\pi \\
R &= 0.5 \\
\bar{U} &= 0 - 1.8
\end{aligned}
$$

Unsteady aerodynamics



The variation of lift on the airfoil cause a variation of vorticity released in the wake. The wake will be composed by vortices of variable intensity that are convected in the flow field. Those vortex induce effects on the airfoil that cause a change of lift distribution on it. There is an internal feedback loop in the aerodynamics too.

Reduced frequency

The structure oscillates at the frequency ω

$\Rightarrow T = \dfrac{2\pi}{\omega}$ interval of time necessary to complete a cycle

A fluid particle interacts with the airfoil for a time that is approximatively

$\Rightarrow \quad \tau = \frac{c}{2U}$

If $\tau \ll T$ the interaction could be considered static, otherwise dynamic effects coud be expected

$$k = \frac{\tau}{T} 2\pi = \frac{\omega c}{2V} \text{ REDUCED FREQUENCY}$$

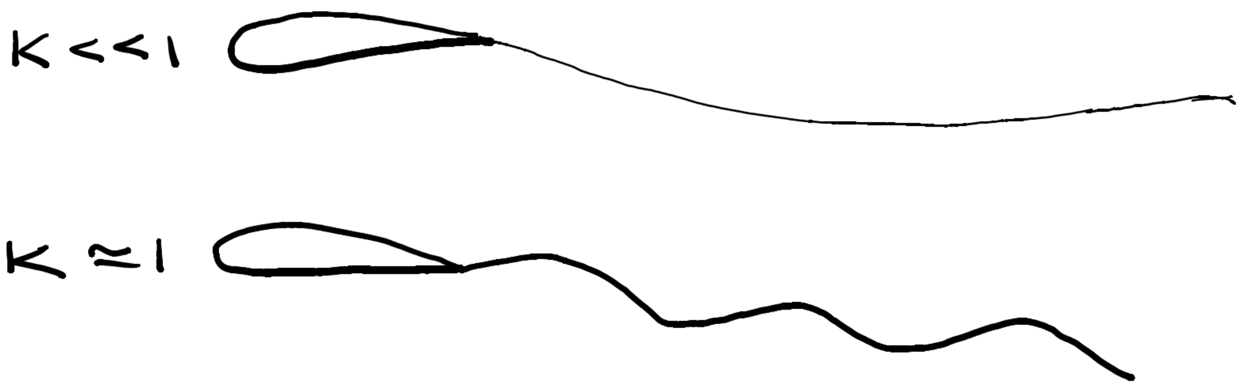
The reduced frequency allows to discriminate bewteeen cases where a static interaction should be exprected ($k \ll 1$) and others where a dynamic interaction should be considered.

Reduced frequency

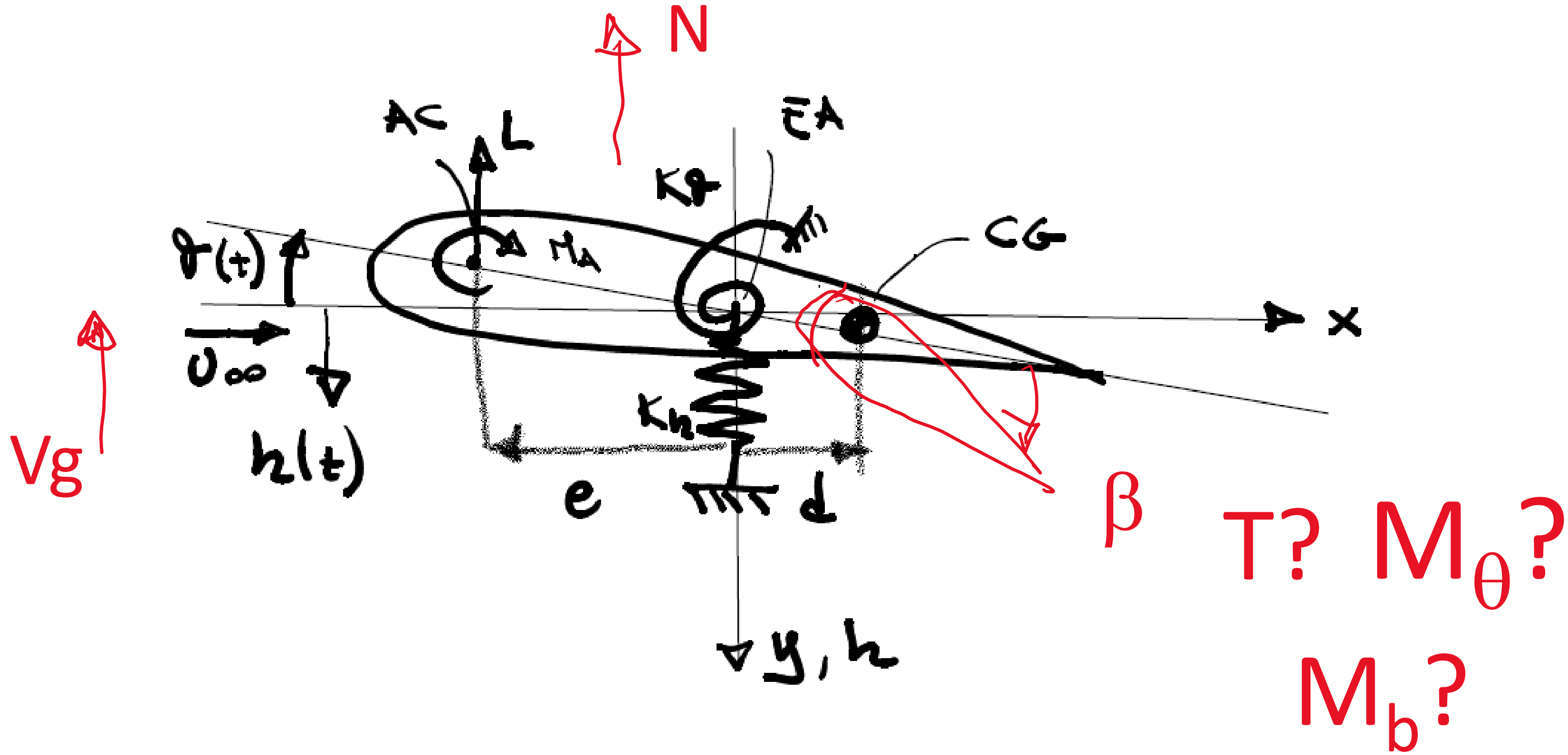
Wave lenght of the wake η

$$\eta = UT = U\frac{2\pi}{\omega} = U2\pi\frac{c}{2Uk}$$
$$\eta = \frac{\pi c}{k}$$

When $k = 0.1 \rightarrow \eta \approx 31.5c$



Response



h(t) PLUNGE positive downward

θ(t) PITCH positive clockwise

Appendix

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \left(\begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} - qSC_{L\alpha} \begin{bmatrix} 0 & -1 \\ 0 & e \end{bmatrix} \right) \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \mathbf{0}$$

Divide all terms by $m\omega_\theta^2$

$$\frac{1}{\omega_\theta^2} \begin{bmatrix} 1 & d \\ d & r_\theta^2 \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \left(\begin{bmatrix} \frac{k_h}{m\omega_\theta^2} & 0 \\ 0 & \frac{k_\theta}{m\omega_\theta^2} \end{bmatrix} - \frac{qSC_{L\alpha}}{m\omega_\theta^2} \begin{bmatrix} 0 & -1 \\ 0 & e \end{bmatrix} \right) \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \mathbf{0}$$

$$r_\theta^2 = \frac{I_\theta}{m} \text{ Gyration radius}$$

$$R^2 = \frac{\omega_h^2}{\omega_\theta^2} \text{ Frequency ratio}$$

$$\begin{bmatrix} R^2 & 0 \\ 0 & r_\theta^2 \end{bmatrix}$$

$$\bar{(\cdot)} = \frac{(\cdot)}{c/2}$$

$$\mu = \frac{m}{\rho\pi\left(\frac{c}{2}\right)^2 L} = \frac{2m}{\rho\pi\frac{c}{2}S} = \frac{\text{airfoil mass}}{\text{air volume mass}}$$

with c the chord of the aifoil

Appendix

$$\frac{\rho U^2 S C_{L\alpha}}{2m\omega_\theta^2} = \frac{2}{c\pi\mu}\frac{U^2 C_{L\alpha}}{\omega_\theta^2} = \left(\frac{2U}{c\omega_\theta}\right)^2 \frac{cC_{L\alpha}}{2\pi\mu} = \bar{U}^2 \frac{cC_{L\alpha}}{2\pi\mu}$$

$$\bar{U} = \frac{2U}{c\omega_\theta} \ \mathrm{non-dimensional\ speed}$$

$$\Omega = \frac{\lambda}{\Omega_\theta} \ \mathrm{non-dimensional\ eigenvalue}$$

$$\left(\Omega^2 \begin{bmatrix} 1 & \bar{d} \\ \bar{d} & \bar{r}_\theta^2 \end{bmatrix} + \begin{bmatrix} R^2 & 0 \\ 0 & \bar{r}_\theta^2 \end{bmatrix} - \bar{U}^2 \frac{C_{L\alpha}}{\pi\mu} \begin{bmatrix} 0 & -1 \\ 0 & \bar{e} \end{bmatrix}\right) \begin{Bmatrix} \bar{h} \\ \theta \end{Bmatrix} = \mathbf{0}$$

All equations have been non-dimensionalized dividing them by $(c/2)^2$

Appendix

If a QUASI-STEADY approximation for aerodynamic forces is used it is necessary to add this extra term

$$\frac{qSC_{L\alpha}}{Um\omega_\theta^2}\begin{bmatrix}1 & 0\\ -e & 0\end{bmatrix}\begin{Bmatrix}\dot{h}\\ \dot{\theta}\end{Bmatrix} \rightarrow \lambda\frac{1}{2}\rho U\frac{SC_{L\alpha}}{m\omega_\theta^2}\begin{bmatrix}1 & 0\\ -e & 0\end{bmatrix}\begin{Bmatrix}h_0\\ \theta_0\end{Bmatrix}$$

$$\Omega\bar{U}\frac{C_{L\alpha}}{\pi\mu}\begin{bmatrix}1 & 0\\ -\bar{e} & 0\end{bmatrix}\begin{Bmatrix}\bar{h}_0\\ \theta_0\end{Bmatrix}$$