POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

Unsteady Typical Section Model

Giuseppe Quaranta

Dipartimento di Scienze e Tecnologie Aerospaziali

Basic Model

Equilibrium equation with respect to the elastic axis

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\theta \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\theta} \end{Bmatrix} \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} -L \\ M_\theta \end{Bmatrix}$$

$$S_\theta = md$$

$$\begin{bmatrix} m & S_\theta \\ S_\theta & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_h & 0 \\ 0 & c_\theta \end{bmatrix} \begin{Bmatrix} \dot{h} \\ \dot{\theta} \end{Bmatrix} \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = qS \begin{Bmatrix} -C_L \\ bC_{M_\theta} \end{Bmatrix}$$

Non-dimensionalization

Let’s divide the equation by $m\omega_\theta^2$

Remembering that

$$r_\theta^2 = \frac{I_\theta}{m},\ k_h = m\omega_h^2,\ k_\theta = m\omega_\theta^2,\ c_h = 2\xi_h m\omega_h,\ c_\theta = 2\xi_\theta I_\theta \omega_\theta$$

$$\frac{1}{\omega_\theta^2}\begin{bmatrix}1 & d\\ d & r_\theta^2\end{bmatrix}\begin{Bmatrix}\ddot{h}\\ \ddot{\theta}\end{Bmatrix} + \frac{1}{\omega_\theta}\begin{bmatrix}2\xi_h\frac{\omega_h}{\omega_\theta} & 0\\ 0 & 2\xi_\theta r_\theta^2\end{bmatrix}\begin{Bmatrix}\dot{h}\\ \dot{\theta}\end{Bmatrix} +$$

$$\begin{bmatrix}\left(\frac{\omega_h}{\omega_\theta}\right)^2 & 0\\ 0 & r_\theta^2\end{bmatrix}\begin{Bmatrix}h\\ \theta\end{Bmatrix} = \frac{qS}{m\omega_\theta^2}\begin{Bmatrix}-C_L\\ bC_{M_\theta}\end{Bmatrix}$$

Non-dimensionalization

$$t = \omega_\theta \tau,\ R = \frac{w_h}{w_\theta},\ \bar{(\cdot)} = \frac{(\cdot)}{b}$$

$$\begin{bmatrix} 1 & \bar{d} \\ \bar{d} & \bar{r}_\theta^2 \end{bmatrix} \begin{Bmatrix} \bar{h}'' \\ \theta'' \end{Bmatrix} + \begin{bmatrix} 2\xi_h R & 0 \\ 0 & 2\xi_\theta \bar{r}_\theta^2 \end{bmatrix} \begin{Bmatrix} \bar{h}' \\ \theta' \end{Bmatrix} + \begin{bmatrix} R^2 & 0 \\ 0 & \bar{r}_\theta^2 \end{bmatrix} \begin{Bmatrix} \bar{h} \\ \theta \end{Bmatrix} = \frac{qS}{bm\omega_\theta^2} \begin{Bmatrix} -C_L \\ C_{M_\theta} \end{Bmatrix}$$

Non-dimensionalization

$$\bar{U} = \frac{U}{b\omega_\theta}, \ \mu = \frac{2m}{\rho b S} = \frac{m}{\rho b^2 s}$$

s is the span of the wing so that the surface is
S = 2 b s

$$\begin{bmatrix} 1 & \bar{d} \\ \bar{d} & \bar{r}_\theta^2 \end{bmatrix} \begin{Bmatrix} \bar{h}'' \\ \theta'' \end{Bmatrix} + \begin{bmatrix} 2\xi_h R & 0 \\ 0 & 2\xi_\theta \bar{r}_\theta^2 \end{bmatrix} \begin{Bmatrix} \bar{h}' \\ \theta' \end{Bmatrix} + \begin{bmatrix} R^2 & 0 \\ 0 & \bar{r}_\theta^2 \end{bmatrix} \begin{Bmatrix} \bar{h} \\ \theta \end{Bmatrix} = \frac{\bar{U}^2}{\pi\mu} \begin{Bmatrix} -C_L \\ C_{M_\theta} \end{Bmatrix}$$

Theodorsen model

$$L = \rho\pi b^2 \left(\ddot{h} + U\dot{\theta} - ab\ddot{\theta}\right) + \rho b U 2\pi C(k) w$$

$$M = -\rho\pi b^3 \left(\frac{1}{2}\ddot{h} + U\dot{\theta} - \left(\frac{b}{8} - \frac{ab}{2}\right)\ddot{\theta}\right) + \left(\frac{b}{2} + ab\right) L$$

$$w = \dot{h} + U\alpha + \left(\frac{b}{2} - ab\right)\dot{\alpha}$$

$$L = \rho\pi b^2 \left(\ddot{h} + U\dot{\theta} - ab\ddot{\theta}\right) + \rho b U 2\pi C(k) w$$

$$M = \rho\pi b^3 \left(a\ddot{h} - U\left(\frac{1}{2} - a\right)\dot{\theta} - \left(\frac{b}{8} + a^2 b\right)\ddot{\theta}\right) + \left(\frac{b}{2} + ab\right)\rho b U 2\pi C(k) w$$

Theodorsen model

$$C_L = \frac{Ls}{1/2\rho U^2 2bs} = \frac{L}{\rho b U^2}$$

$$C_{M_\theta} = \frac{M_\theta s}{1/2\rho U^2 2bsb} = \frac{M_\theta}{\rho b^2 U^2}$$

$$C_L = \pi\left(\frac{b}{U^2}\ddot{h} + \frac{b}{U}\dot{\theta} - \frac{ab^2}{U^2}\ddot{\theta}\right) + 2\pi C(k)\frac{w}{U}$$

$$C_{M_\theta} = \pi\left(\frac{ab}{U^2}\ddot{h} - \frac{b}{U}\left(\frac{1}{2} - a\right)\dot{\theta} - \frac{b}{U^2}\left(\frac{b}{8} + a^2 b\right)\ddot{\theta}\right) + \left(\frac{1}{2} + a\right)2\pi C(k)\frac{w}{U}$$

Theodorsen model

$$C_L = \pi\left(\frac{1}{\bar{U}^2}\bar{h}'' + \frac{1}{\bar{U}}\theta' - \frac{a}{\bar{U}^2}\theta''\right) + 2\pi C(k)\frac{w}{U}$$

$$C_{M_\theta} = \pi\left(\frac{a}{\bar{U}^2}\bar{h}'' - \frac{1}{\bar{U}}\left(\frac{1}{2} - a\right)\theta' - \frac{1}{\bar{U}^2}\left(\frac{1}{8} + a^2\right)\theta''\right) + \left(\frac{1}{2} + a\right)2\pi C(k)\frac{w}{U}$$

$$\frac{w}{U} = \frac{1}{\bar{U}}\bar{h}' + \theta + \left(\frac{1}{2} - a\right)\frac{1}{\bar{U}}\theta'$$

$$\begin{aligned}\frac{\bar{U}^2}{\pi\mu}\begin{Bmatrix}-C_L \\ C_{M_\theta}\end{Bmatrix} = {} & \frac{1}{\mu}\begin{bmatrix}-1 & a \\ a & -\left(\frac{1}{8} + a^2\right)\end{bmatrix}\begin{Bmatrix}\bar{h}'' \\ \theta''\end{Bmatrix} + \frac{\bar{U}}{\mu}\begin{bmatrix}0 & 1 \\ 0 & -\left(\frac{1}{2} - a\right)\end{bmatrix}\begin{Bmatrix}\bar{h}' \\ \theta'\end{Bmatrix} + \\ & \frac{2\bar{U}}{\mu}\begin{bmatrix}-1 & -\left(\frac{1}{2} - a\right) \\ \left(\frac{1}{2} + a\right) & \left(\frac{1}{4} - a^2\right)\end{bmatrix}C(k)\begin{Bmatrix}\bar{h}' \\ \theta'\end{Bmatrix} + \frac{2\bar{U}^2}{\mu}\begin{bmatrix}0 & -1 \\ 0 & \left(\frac{1}{2} + a\right)\end{bmatrix}C(k)\begin{Bmatrix}\bar{h} \\ \theta\end{Bmatrix}\end{aligned}$$

Theodorsen model

$$\left(\begin{bmatrix}1 & \bar{d} \\ \bar{d} & \bar{r}_\theta^2\end{bmatrix} + \frac{1}{\mu}\begin{bmatrix}1 & -a \\ -a & \left(\frac{1}{8}+a^2\right)\end{bmatrix}\right)\begin{Bmatrix}\bar{h}'' \\ \theta''\end{Bmatrix} +$$
$$\left(\begin{bmatrix}2\xi_h R & 0 \\ 0 & 2\xi_\theta \bar{r}_\theta^2\end{bmatrix} + \frac{\bar{U}}{\mu}\left(\begin{bmatrix}0 & -1 \\ 0 & \left(\frac{1}{2}-a\right)\end{bmatrix} + \begin{bmatrix}2 & (1-2a) \\ -(1+2a) & -\left(\frac{1}{2}-2a^2\right)\end{bmatrix} C(k)\right)\right)\begin{Bmatrix}\bar{h}' \\ \theta'\end{Bmatrix} +$$
$$\left(\begin{bmatrix}R^2 & 0 \\ 0 & \bar{r}_\theta^2\end{bmatrix} + \frac{2\bar{U}^2}{\mu}\begin{bmatrix}0 & 1 \\ 0 & -\left(\frac{1}{2}+a\right)\end{bmatrix} C(k)\right)\begin{Bmatrix}\bar{h} \\ \theta\end{Bmatrix} = \mathbf{0}$$

P-K Method

$$C(k) = \mathrm{Re}(C(k)) + s\frac{\mathrm{Im}(C(k))}{k}$$

$$\det\left(s^2\left(\mathbf{M}_s + \frac{1}{\mu}\mathbf{M}_a + \frac{\bar{U}}{\mu}\mathbf{C}_{aC}\frac{\mathrm{Im}(C(k))}{k}\right) + \right.$$
$$+ s\left(\mathbf{C}_s + \frac{\bar{U}}{\mu}\left(\mathbf{C}_{aNC} + \mathbf{C}_{aC}\mathrm{Re}(C(k)) + 2\bar{U}\mathbf{K}_{aC}\frac{\mathrm{Im}((C(k)))}{k}\right)\right) +$$
$$\left. + \left(\mathbf{K}_s + \frac{2\bar{U}^2}{\mu}\mathbf{K}_{aC}\mathrm{Re}(C(k))\right)\right) = 0$$

Sta-space Approach

$$\frac{\bar{U}^2}{\pi\mu}\begin{Bmatrix}-C_L\\ C_{M_\theta}\end{Bmatrix}=\frac{1}{\mu}\begin{bmatrix}-1 & a\\ a & -\left(\frac{1}{8}+a^2\right)\end{bmatrix}\begin{Bmatrix}\bar{h}''\\ \theta''\end{Bmatrix}+\frac{\bar{U}}{\mu}\begin{bmatrix}0 & 1\\ 0 & -\left(\frac{1}{2}-a\right)\end{bmatrix}\begin{Bmatrix}\bar{h}'\\ \theta'\end{Bmatrix}+$$

$$\frac{2\bar{U}}{\mu}\begin{bmatrix}-1\\ \left(\frac{1}{2}+a\right)\end{bmatrix}C(k)\begin{bmatrix}0 & \bar{U} & 1 & \left(\frac{1}{2}-a\right)\end{bmatrix}\begin{Bmatrix}\bar{h}\\ \theta\\ \bar{h}'\\ \theta'\end{Bmatrix}$$

$$\frac{\bar{U}^2}{\pi\mu}\begin{Bmatrix}-C_L\\ C_{M_\theta}\end{Bmatrix}=-\frac{1}{\mu}\mathbf{M}_a\begin{Bmatrix}\bar{h}''\\ \theta''\end{Bmatrix}-\frac{\bar{U}}{\mu}\mathbf{C}_{aNC}\begin{Bmatrix}\bar{h}'\\ \theta'\end{Bmatrix}+\frac{2\bar{U}}{\mu}\mathbf{C}_w C(k)\begin{bmatrix}\bar{U}\mathbf{B}_{w1} & \mathbf{B}_{w2}\end{bmatrix}\begin{Bmatrix}\bar{h}\\ \theta\\ \bar{h}'\\ \theta'\end{Bmatrix}$$

$$\frac{b}{U}\dot{\mathbf{x}}_a=\mathbf{A}\mathbf{x}_a+\mathbf{B}u$$

$$y=\mathbf{C}\mathbf{x}_a+\mathbf{D}u$$

$$\mathbf{x}_a'=\bar{U}\mathbf{A}\mathbf{x}_a+\bar{U}\mathbf{B}u$$

$$y=\mathbf{C}\mathbf{x}_a+\mathbf{D}u$$

Stata-space approach

$$
\begin{bmatrix}
\mathbf{I} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{M}_s + \frac{1}{\mu}\mathbf{M}_a & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{I}
\end{bmatrix}
\begin{Bmatrix}
\dot{\mathbf{q}} \\ \ddot{\mathbf{q}} \\ \dot{\mathbf{x}}_a
\end{Bmatrix} =
$$

$$
\begin{bmatrix}
\mathbf{0} & \mathbf{I} & \mathbf{0} \\
-\mathbf{K}_s + \frac{2\bar{U}^2}{\mu}\mathbf{C}_w\mathbf{D}\mathbf{B}_{w1} & -\mathbf{C}_s - \frac{\bar{U}}{\mu}\mathbf{C}_{aNC} + \frac{2\bar{U}}{\mu}\mathbf{C}_w\mathbf{D}\mathbf{B}_{w2} & \frac{2\bar{U}}{\mu}\mathbf{C}_w\mathbf{C} \\
\bar{U}^2\mathbf{B}\mathbf{B}_{w1} & \bar{U}\mathbf{B}\mathbf{B}_{w2} & \bar{U}\mathbf{A}
\end{bmatrix}
\begin{Bmatrix}
\mathbf{q} \\ \dot{\mathbf{q}} \\ \mathbf{x}_a
\end{Bmatrix}
$$

Continuation algorithm

$$\left(\lambda \mathbf{V} - \mathbf{A}_t(\bar{U})\right) \mathbf{z} = 0$$

$$\mathbf{z}^* \mathbf{z} = 1$$

$$\begin{bmatrix} (\lambda_i \mathbf{V} - \mathbf{A}_t) & \mathbf{V} \mathbf{z}_i \\ 2\mathbf{z}_i^* & 0 \end{bmatrix} \begin{Bmatrix} \delta \mathbf{z}_i \\ \delta \lambda_i \end{Bmatrix} = \begin{Bmatrix} -\left(\lambda_i \mathbf{V} - \mathbf{A}_t\right) \mathbf{z}_i \\ 1 - \mathbf{z}_i^* \mathbf{z}_i \end{Bmatrix}$$

$$\begin{bmatrix} (\lambda_i \mathbf{V} - \mathbf{A}_t) & \mathbf{V} \mathbf{z}_i \\ 2\mathbf{z}_i^* & 0 \end{bmatrix} \begin{Bmatrix} \frac{\partial \mathbf{z}_i}{\partial \bar{U}} \\ \frac{\partial \lambda_i}{\partial \bar{U}} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial \mathbf{A}_t}{\partial \bar{U}} \mathbf{z}_i \\ 0 \end{Bmatrix}$$

Continuation algorithm

$$\frac{\partial \mathbf{A}_t}{\partial \bar{U}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \frac{4\bar{U}}{\mu}\mathbf{C}_w\mathbf{D}\mathbf{B}_{w1} & -\frac{1}{\mu}\mathbf{C}_{aNC} + \frac{2}{\mu}\mathbf{C}_w\mathbf{D}\mathbf{B}_{w2} & \frac{2}{\mu}\mathbf{C}_w\mathbf{C} \\ 2\bar{U}\mathbf{B}\mathbf{B}_{w1} & \mathbf{B}\mathbf{B}_{w2} & \mathbf{A} \end{bmatrix}$$