Chapter 4

Compressible Unsteady Potential Flows

In this Chapter the differential formulation for compressible potential flow will be presented (Sec. 4.1; in Section 4.2 the nonlinear and linearized equation for the velocity potential is obtained when an ideal gas flow is considered. Finally, in Section 4.3 an integral representation of the velocity potential in the frequency and time domain has been obtained for a flow around a translating lifting body.

4.1 Compressible Potential Flows Model

From conservation of mass, momentum and energy and Gibbs thermodynamics with the additional assumptions that the fluid is *inviscid* and *adiabatic,* we have obtained the continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{4.1}$$

Euler equation

$$\frac{D\mathbf{v}}{Dt} = \mathbf{f} - \frac{1}{\rho} \nabla p \tag{4.2}$$

and the entropy equation

$$\frac{DS}{Dt} = 0 \tag{4.3}$$

In this section we want to show that if the flow is *initially isentropic* and *initally irrotational,* then the flow is irrotational (and hence potential) at all times provided that no shock waves arise in the flow field.

<u>ISENTROPIC FLOWS</u>

Assume that at time $t = 0$ the flow is isentropic

$$S(\mathbf{x}, 0) = S_0 \tag{4.4}$$

and that shock waves do not arise in the field flow. Then

$$S(\mathbf{x}, t) = S_0 \tag{4.5}$$

i.e., the flow is isentropic at all times (Eq. 2.27). Note that this implies that the state-space equation for fluids (Eq. 2.8) is given by

$$e \;=\; e(\tau) \tag{4.6}$$

Then, assuming that the force field is conservative, *i.e.*,

$$\mathbf{f} = -\nabla\Omega \tag{4.7}$$

(where Ω = potential energy) Euler equation may be rewritten in a more convenient form.

In order to accomplish this, introduce the enthalpy h, defined as

$$h := e + \frac{p}{\rho} = e + p\tau \tag{4.8}$$

Note that (see Eqs. 2.11, 2.18, and 2.20)

$$dh = de + d(p\tau) = \theta dS + \tau dp \tag{4.9}$$

Hence, for isentropic flows

$$dh = \frac{1}{\rho} dp \tag{4.10}$$

or

$$\nabla h = \frac{1}{\rho}\nabla p \qquad\qquad \frac{Dh}{Dt} = \frac{1}{\rho}\frac{Dp}{Dt} \tag{4.11}$$

Noting that (Lagrange acceleration formula)

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\zeta}\times\mathbf{v} + \nabla\left(\frac{v^2}{2}\right) \tag{4.12}$$

and using Eq. 4.7 and 4.11 Euler equation may be rewritten as

$$\frac{\partial\mathbf{v}}{\partial t} + \boldsymbol{\zeta}\times\mathbf{v} + \nabla(\frac{v^2}{2} + h + \Omega) = 0 \tag{4.13}$$

<u>KELVIN'S THEOREM</u>

This theorem states that, if C_M is a material contour, (i.e., a closed line composed of material points), then

$$\frac{d\Gamma}{dt} = 0 \tag{4.14}$$

where the circulation Γ is defined as

$$\Gamma = \oint_{C_M} \mathbf{v} \cdot d\mathbf{x} = \iint_\sigma \boldsymbol{\zeta} \cdot \mathbf{n} d\sigma \tag{4.15}$$

with the vorticity $\boldsymbol{\zeta}$ given by

$$\boldsymbol{\zeta} = \nabla \times \mathbf{v} \tag{4.16}$$

The proof of this theorem is given in the following: let the contour C_M be divided into segments having extremes $\mathbf{x}_k, \mathbf{x}_{k+1}$, then (using Eqs. 4.2, 4.7 and 4.11)

$$\begin{aligned} \frac{d}{dt} \oint_{C_M} \mathbf{v} \cdot d\mathbf{x} &= \frac{D}{Dt} \lim_{\Delta x_k \to 0} \sum_k \mathbf{v}_k \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k) \\ &= \lim_{\Delta x_k \to 0} \sum_k \left(\frac{D\mathbf{v}_k}{Dt} \cdot (\mathbf{x}_{k+1} - \mathbf{x}_k) + \mathbf{v}_k \cdot (\mathbf{v}_{k+1} - \mathbf{v}_k) \right) \\ &= \oint_{C_M} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \oint_{C_M} \mathbf{v} \cdot d\mathbf{v} = -\oint_{C_M} d(h + \Omega) + \oint_{C_M} d\frac{v^2}{2} = 0 \end{aligned} \tag{4.17}$$

because the contour integral of the exact differential of a single-valued function is equal to zero.

IRROTATIONAL FLOWS

Assume that at time $t = 0$ the flow is irrotational, that is

$$\boldsymbol{\zeta}(\mathbf{x}, 0) = 0 \tag{4.18}$$

or

$$\Gamma(0) = 0 \quad \text{for any } \mathrm{C_M} \tag{4.19}$$

then, Kelvin theorem yields

$$\Gamma(t) = 0 \quad \text{for almost any } \mathrm{C_M}, \text{ any t} \tag{4.20}$$

i.e.,

$$\boldsymbol{\zeta}(\mathbf{x}, t) = 0 \;\; \text{for almost any } \mathbf{x} \text{ any t} \tag{4.21}$$

that is, the field is irrotational at all times at almost any point: to be excluded are those contours that are intersected by the aircraft, i.e., the points emanating from the trailing edge. These points form a surface called a wake (a more detailed discussion of this issue is given in Ref. 2).

Next, consider a theorem on the equivalence of irrotational and potential fields: if the velocity field is irrotational ($\boldsymbol{\zeta} = 0$), then there exists a function φ (velocity potential) such that

$$\mathbf{v} = \text{grad } \varphi \qquad \text{(potential field)} \tag{4.22}$$

In order to prove this theorem, note that if $\boldsymbol{\zeta}(\mathbf{x}, t) = 0$ (any x, any t), then

$$\oint_{C_M} \mathbf{v} \cdot d\mathbf{x} = 0 \qquad (\text{any } \mathrm{C_M}, \text{ any t}) \tag{4.23}$$

Thus

$$\int_{\mathbf{x}_o}^{\mathbf{x}_1} \mathbf{v} \cdot d\mathbf{x} = \qquad \text{path independent} \tag{4.24}$$

i.e.,

$$\int_0^{\mathbf{x}} \mathbf{v} \cdot d\mathbf{x} = \qquad \text{function of } \mathbf{x} = \varphi(\mathbf{x}) \tag{4.25}$$

Note that

$$d\varphi = \mathbf{v} \cdot d\mathbf{x} \tag{4.26}$$

which implies

$$\mathbf{v} = \nabla \varphi \tag{4.27}$$

The potential function is single valued if the flow region is simply connected. However, if the region is multiply connected (e.g., the flow around an airfoil in two dimensions or that around a doughnut in three dimensions) then the potential may be multivalued.

Using the above theorem, we may conclude that for a perfect fluid under a conservative force field, if the flow is shockfree, initially isentropic and initially irrotational then the flow is isentropic and irrotational (that is, potential) at all times.

4.2 Potential Flows

Starting from the continuity equation, Euler equation and entropy equation, we have obtained that for a perfect fluid subject to a conservative force field, if the flow is shock-free, initially isentropic, initially irrotational and then such flow is isentropic, irrotational and therefore potential at all times. Such flows are goverened by continuity equation and Euler equation. Here we will derive a differential equation for the velocity potential. For the sake of simplicity, we will limit ourselves to the case in which the conservative force field is negligible, that is

$$\Omega = 0 \tag{4.28}$$

Our starting equations will be the continuity equation

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0 \tag{4.29}$$

The Euler equation for isentropic irrotational flows (Eq. 4.13 with $\Omega = 0$ and $\boldsymbol{\zeta} = 0$)

$$\frac{\partial \mathbf{v}}{\partial t} + \nabla \left(\frac{v^2}{2} + h \right) = 0 \tag{4.30}$$

the potential representation for the velocity

$$\mathbf{v} = \text{grad } \varphi \tag{4.31}$$

Note that, in order to close the problem in terms of primitive variables, we need the equation of state. Since the flow is isentropic,

$$S = S_0 \tag{4.32}$$

the pressure is a function only of the density

$$p = p(\rho) \tag{4.33}$$

A fluid for which the equation of state is given by Eq. 4.33 is called barotropic. In addition, Eq. 4.9 implies, for isentropic flows,

$$dh = \frac{1}{\rho} dp \tag{4.34}$$

BERNOULLI'S THEOREM

Combining Eq. 4.30 and 4.31, one obtains

$$\nabla \left(\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + h \right) = 0 \tag{4.35}$$

which yields Bernoulli's theorem:

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + h = \text{ function of t only} \tag{4.36}$$

If $h = h_\infty$ at infinity, then

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + h = h_\infty \tag{4.37}$$

EQUATION OF THE AERODYNAMIC POTENTIAL

Note that combining Eq. 4.34 with 4.33 yields

$$\frac{Dh}{Dt} = \frac{1}{\rho} \frac{Dp}{Dt} = \frac{1}{\rho} \frac{dp}{d\rho} \frac{D\rho}{Dt} \tag{4.38}$$

or combining with Bernoulli's theorem, Eq. 4.37 and continuity equation, Eq. 4.29, one obtains the desired equation for the velocity potential

$$\frac{D}{Dt} \left(\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} \right) = \frac{dp}{d\rho} \nabla^2 \varphi \tag{4.39}$$

In expanded form, Eq. 4.39 may be written in the air frame of reference as

$$\begin{aligned} \frac{dp}{d\rho} \left(\varphi_{xx} + \varphi_{yy} + \varphi_{zz} \right) &= \varphi_{tt} + \varphi_x^2 \varphi_{xx} + \varphi_y^2 \varphi_{yy} + \varphi_z^2 \varphi_{zz} \\ &+ 2 \left(\varphi_x \varphi_y \varphi_{xy} + \varphi_y \varphi_z \varphi_{yz} + \varphi_z \varphi_x \varphi_{zx} \right. \\ &+ \left. \varphi_x \varphi_{xt} + \varphi_y \varphi_{yt} + \varphi_z \varphi_{zt} \right) \end{aligned} \tag{4.40}$$

A more physical form for the equation of the aerodynamic potential may by obtained by noting that

$$\begin{aligned}\frac{D}{Dt}\left(\frac{\partial\varphi}{\partial t}+\frac{1}{2}\mathbf{v}\cdot\mathbf{v}\right) &= \frac{D}{Dt}\left(\frac{\partial\varphi}{\partial t}+\mathbf{v}_c\cdot\mathbf{v}\right) \\ &= \left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right)\left(\frac{\partial}{\partial t}+\mathbf{v}_c\cdot\nabla\right)\varphi=\frac{D}{Dt}\frac{D_c}{Dt}\varphi=\frac{D_c^2\varphi}{Dt^2}\end{aligned} \tag{4.41}$$

where c indicates that $\mathbf{v}_c$ is kept constant during the application of the operator $\frac{D}{Dt}$. Hence the equation of the velocity potential is

$$\nabla^2\varphi=\frac{1}{\frac{dp}{d\rho}}\frac{D_c^2\varphi}{Dt^2} \tag{4.42}$$

In a frame of reference moving with (constant) speed of the fluid at a given point and a given time, Eq. 4.42 reduces (at that point an that time) to the classical (linear) wave equation. Therefore Eq. 4.42 is a nonlinear wave equation and

$$a=\sqrt{\frac{dp}{d\rho}} \tag{4.43}$$

is the local speed of propagation of the waves (speed of sound).

BOUNDARY CONDITIONS

The following boundary conditions are typically considered for the solution of the velocity potential equation. The surface of the body is considered here impermeable. This implies $\mathbf{v}\cdot\mathbf{n}=\mathbf{v}_B\cdot\mathbf{n}$, or, using Eq. 4.31, the condition is given as

$$\frac{\partial\varphi}{\partial n}=\mathbf{v}_B\cdot\mathbf{n} \tag{4.44}$$

At infinity

$$\varphi=0 \tag{4.45}$$

On the wake the conditions are $(\mathbf{v}_1-\mathbf{v}_2)\cdot\mathbf{n}=0$, and $\Delta p=0$. This implies $\Delta h=0$, or using Eq. 4.37

$$\frac{\partial}{\partial t}(\varphi_1-\varphi_2)+\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{v}_1-\mathbf{v}_2)=0 \tag{4.46}$$

which may be rewritten as

$$\frac{D_w}{Dt}\Delta\varphi=\left(\frac{\partial}{\partial t}+\mathbf{v}_w\cdot\nabla\right)(\varphi_1-\varphi_2)=0 \tag{4.47}$$

with

$$\mathbf{v}_w=\frac{1}{2}(\mathbf{v}_1+\mathbf{v}_2) \tag{4.48}$$

This equation may be integrated to yield

$$\Delta\varphi=\varphi_1-\varphi_2=\text{ constant} \tag{4.49}$$

following wake point having velocity $\mathbf{v}_w$. Finally the initial conditions are

$$\begin{aligned} \varphi(\mathbf{x},0) &= \varphi_0(\mathbf{x}) \\ \dot{\varphi}(\mathbf{x},0) &= \varphi_1(\mathbf{x}) \end{aligned} \tag{4.50}$$

One is not aware of a general theorem of existence and uniqueness for the equation of the velocity potential although it is common belief that this equation with the above conditions and the addition of Kutta - Joukowski assumption of smooth flow at the trailing edge has a unique solution. Discussions of these issues with special emphasis on the wake and trailing edge conditions is given in Ref. 2.

INCOMPRESSIBLE FLOWS

If a fluid is incompressibe (or the flow is such that the compressibility is negligible), then $\rho =$ constant and

$$\left(\frac{dp}{d\rho}\right)^{-1} = \frac{1}{a^2} = 0 \tag{4.51}$$

Equation 4.42 reduces to

$$\nabla^2\varphi = 0 \tag{4.52}$$

with the same boundary conditions

$$\frac{\partial\varphi}{\partial n} = \mathbf{v}_B \cdot \mathbf{n} \quad \text{on} \quad \mathcal{S}_{\mathrm{B}} \tag{4.53}$$

$$\varphi = 0 \quad \text{at} \quad \infty \tag{4.54}$$

$$\frac{D_w}{Dt}\Delta\varphi = 0 \quad \text{on} \quad \mathcal{S}_{\mathrm{w}} \tag{4.55}$$

IDEAL GASES

The assumption of ideal gases with constant specific heat coefficients is next introduced in order to obtain an explicit expression for the speed of sound. The state equations for an ideal gas are

$$\frac{p}{\rho} = R\theta \tag{4.56}$$

$$e = c_V\theta \tag{4.57}$$

where R is the constant for the *termically perfect* gases and c_V is the constant-volume specific heat, which is constant with respect to θ for *calorically perfect* gases and which represents the heat quantity necessary to increase the temperature of a mass kilogram of gas of one degree keeping constant the gas volume. Then

$$h = e + \left(\frac{p}{\rho}\right) = (c_V + R)\theta = c_P\theta = \frac{c_P}{R}\frac{p}{\rho} = \frac{\gamma}{\gamma - 1}p\tau \tag{4.58}$$

where

$$
\begin{aligned}
c_P &= c_V + R \\
\gamma &:= \frac{c_P}{c_V}
\end{aligned}
\tag{4.59}
$$

Note that for isentropic flow

$$dh = \tau dp \tag{4.60}$$

Combining Eqs. 4.58 and 4.60, one obtains

$$\frac{dp}{p} + \gamma \frac{d\tau}{\tau} = 0 \tag{4.61}$$

which may be integrated to yield

$$p\tau^{\gamma} = \text{ const } = \mathrm{A} \tag{4.62}$$

or

$$p = A\rho^{\gamma} \tag{4.63}$$

which is the well known law for isentropic transformations for ideal gases. Using Eq. 4.63, one obtains

$$a^2 = \frac{dp}{d\rho} = \gamma \frac{p}{\rho} = (\gamma - 1)h \tag{4.64}$$

Substituting into Bernoulli's theorem, one obtains

$$\frac{\partial \varphi}{\partial t} + \frac{v^2}{2} + \frac{a^2}{\gamma - 1} = \frac{a_\infty^2}{\gamma - 1} \tag{4.65}$$

i.e., the desired expression for the speed of sound

$$a^2 = a_\infty^2 + (\gamma - 1)\left[-\frac{\partial \varphi}{\partial t} - \frac{v^2}{2}\right] \tag{4.66}$$

Note that combining Eqs. 4.40, 4.43 and 4.66 yields an equation which contains φ as the only unknown.

LINEARIZED EQUATIONS

Consider the velocity potential and assume

$$\left|\frac{\nabla \varphi}{U_\infty}\right| = O(\epsilon) \tag{4.67}$$

Isolating terms of order ϵ^2 and considering the time derivative for a frame of reference in a uniform translation with velocity U_∞ yields

$$\frac{D_c^2 \varphi}{Dt^2} = \left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x}\right)^2 \varphi + O(\epsilon^2) \tag{4.68}$$

and

$$a^2\nabla^2\varphi = a_\infty^2\nabla^2\varphi + O(\epsilon^2) \tag{4.69}$$

which yields, see Eq. 4.42,

$$\nabla^2\varphi = \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + U_\infty\frac{\partial}{\partial x}\right)^2\varphi + \sigma \tag{4.70}$$

where $\sigma = O(\epsilon^2)$ comprises all the nonlinear terms.

Neglecting σ one obtains the linearized equation for the potential in the body frame of reference

$$\nabla^2\varphi = \frac{1}{a_\infty^2}\left(\frac{\partial}{\partial t} + U_\infty\frac{\partial}{\partial x}\right)^2\varphi \tag{4.71}$$

For *steady state* ($\frac{\partial}{\partial t} = 0$), Eq. 4.71 reduces to

$$(1 - M_\infty^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \tag{4.72}$$

where

$$M_\infty = \frac{U_\infty}{a_\infty} \tag{4.73}$$

For *subsonic flows* ($M_\infty < 1$), the governing equation is elliptic and may be written as

$$\varphi_{XX} + \varphi_{YY} + \varphi_{ZZ} = 0 \tag{4.74}$$

where X, Y, Z are the well known Prandtl-Glauert variables

$$X = \frac{x}{\ell\sqrt{1-M^2}} \quad Y = y/\ell \quad Z = z/\ell \tag{4.75}$$

For *transonic flows* ($M_\infty \sim 1$), the nonlinear terms cannot be neglected and one obtains an equation of mixed (elliptic/hyperbolic) type. A *Transonic Small Perturbation* (TSP) equation is often used in which only the dominant nonlinear terms are used. As shown for instance by Murman and Cole (Ref. [3]), the higher order terms to be included in this case are

$$\sigma = \frac{U_\infty}{a_\infty^2}(\gamma + 1)\varphi_x\varphi_{xx} \tag{4.76}$$

4.3 Airplane Potential Aerodynamics Integral formulation

In this section we present the integral formulation for the linear velocity-potential equation, for a frame of reference in uniform translation. The present approach is in frequency domain.

4.3.1 Integral Equation

The linear velocity-potential equation (Eq. 4.70 with $\sigma = 0$) written in a body frame of reference that moves in the direction of the negative x-axis is given by:

$$\nabla^2 \varphi = \frac{1}{a_\infty^2} \left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x} \right)^2 \varphi \tag{4.77}$$

Introducing the dimensionless Prandtl-Glauert variables, $X = x/\beta\ell$, $Y = y/\ell$, $Z = z/\ell$, and $T = U_\infty t/\ell$, with

$$\beta = \sqrt{1 - M_\infty^2} \tag{4.78}$$

and taking the Laplace transform with zero initial conditions ($p := s\ell/U_\infty$, s Laplace variable), one obtains

$$\tilde{\varphi}_{XX} + \tilde{\varphi}_{YY} + \tilde{\varphi}_{ZZ} = M_\infty^2 2p \frac{1}{\beta} \tilde{\varphi}_X + M_\infty^2 p^2 \tilde{\varphi} \tag{4.79}$$

where $\tilde{\varphi}$ denotes the Laplace transform of φ.

Next, setting

$$\tilde{\varphi} = \hat{\varphi} e^{M_\infty \bar{p} X} \tag{4.80}$$

with $\bar{p} = M_\infty p/\beta$, yields

$$\nabla_{\mathbf{X}}^2 \hat{\varphi} - \bar{p}^2 \hat{\varphi} = 0 \tag{4.81}$$

where

$$\nabla_{\mathbf{X}}^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2} + \frac{\partial^2}{\partial Z^2} \tag{4.82}$$

is the Laplacian operator in the Prandtl-Glauert space, $X,\ Y,\ Z$. The corresponding fundamental solution satisfies the equation

$$\nabla_{\mathbf{X}}^2 \hat{G} - \bar{p}^2 \hat{G} = \delta(\mathbf{X} - \mathbf{X}_*) \tag{4.83}$$

The solution to this equation is different for subsonic and supersonic flows. Here, for simplicity, we limit ourselves to subsonic flows *i.e.*, $M_\infty < 1$.[1] In this case the solution to Eq. 4.83 is given by[2]

$$\hat{G} = G_0 e^{-\bar{p} R} \tag{4.88}$$

[1] For supersonic flows see, *e.g.*, Ref. [7], Morino (1974).

[2] If one consider that the Laplacian operator in spheric co-ordinates (R, θ, ϕ) is given by

$$\nabla_{sf}^2 \bullet := \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial \bullet}{\partial R} \right) + \frac{1}{R^2 \sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial \bullet}{\partial \theta} \right) + \frac{1}{R^2 \sin^2\theta} \frac{\partial^2 \bullet}{\partial \phi^2} \tag{4.84}$$

where

$$G_0 = -\frac{1}{4\pi R} \tag{4.89}$$

Using the Green's identity, Eq. 3.18, yields

$$E_* \hat{\varphi}_* = \oint\!\!\oint_{\mathcal{S}_0} \left(\frac{\partial \hat{\varphi}}{\partial N} \, \hat{G} - \hat{\varphi} \frac{\partial \hat{G}}{\partial N} \right) d\mathcal{S}_0 \tag{4.90}$$

where $\mathcal{S}_0$ is the image of $\mathcal{S}$ in the X, Y, Z space, and $\partial/\partial N$ is the directional derivative in the direction of the normal $\mathbf{N}$ to $\mathcal{S}_0$.

Recalling that $\tilde{\varphi} = \hat{\varphi} e^{M_\infty \bar{p} X}$, Eq. 4.80, we have

$$\begin{aligned} E_* \tilde{\varphi}_* e^{-M_\infty \bar{p} X_*} &= \oint\!\!\oint_{\mathcal{S}_0} \left(\frac{\partial \tilde{\varphi}}{\partial N} - M_\infty \bar{p} \frac{\partial X}{\partial N} \tilde{\varphi} \right) e^{-\bar{p} M_\infty X} \; G_0 \; e^{-\bar{p} R} \; d\mathcal{S}_0 \\ &- \oint\!\!\oint_{\mathcal{S}_0} \tilde{\varphi} e^{-\bar{p} M_\infty X} \left[\frac{\partial G_0}{\partial N} e^{-\bar{p} R} - \bar{p} \; G_0 \; e^{-\bar{p} R} \; \frac{\partial R}{\partial N} \right] d\mathcal{S}_0 \end{aligned} \tag{4.91}$$

or, recalling that $\bar{p} = M_\infty p / \beta$,

$$E_* \tilde{\varphi}_* = \oint\!\!\oint_{\mathcal{S}_0} \left[\frac{\partial \tilde{\varphi}}{\partial N} e^{-p\Theta} G_0 - \tilde{\varphi} e^{-p\Theta} \frac{\partial G_0}{\partial N} + p \tilde{\varphi} e^{-p\Theta} G_0 \frac{\partial \hat{\Theta}}{\partial N} \right] d\mathcal{S}_0 \tag{4.92}$$

where

$$\Theta = \frac{M_\infty}{\beta} \left[M_\infty (X - X_*) + R \right] \tag{4.93}$$

$$\hat{\Theta} = \frac{M_\infty}{\beta} \left[M_\infty (X_* - X) + R \right] \tag{4.94}$$

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then, assuming as particular solution of the Eq. 4.83

$$\hat{G} = \frac{A}{R} e^{-\bar{p} R} \tag{4.85}$$

with A constant to be determined, one can obtain that

$$\nabla^2_{sf} \hat{G} \equiv \bar{p}^2 \hat{G} \tag{4.86}$$

i.e., this solution satisfies the Helmholtz Eq. 4.83 in all the points $\mathbf{X}$ but the point $\mathbf{X}_*$.

Finally, the constant A can be determined integrating Eq. 4.83 in a spherical volume with center in the point $\mathbf{X}_*$ and taking the limit for $R \to 0$; then, one has (considering the Gauss theorem)

$$\begin{aligned} 1 &= \lim_{R \to 0} \oint\!\!\oint_{S_{sf}} \frac{\partial \hat{G}}{\partial R} d\mathcal{S} - \bar{p}^2 \lim_{R \to 0} \iiint_{V_{sf}} \frac{A}{R} e^{-\bar{p} R} dV = \lim_{R \to 0} \oint\!\!\oint_{\Omega_{sf}} \frac{\partial \hat{G}}{\partial R} R^2 d\Omega - \bar{p}^2 \lim_{R \to 0} \iiint_{V_{sf}} \frac{A}{R} e^{-\bar{p} R} \; R^2 \sin\theta \; dR d\theta d\phi \\ &= \lim_{R \to 0} \oint\!\!\oint_{\Omega_{sf}} \left(-\frac{A}{R^2} - \bar{p} \frac{A}{R} \right) e^{-\bar{p} R} R^2 d\Omega \; = \; -4\pi A \end{aligned} \tag{4.87}$$

where changes of co-ordinates in spherical coo-ordinates (for the volume integral) and in solid-angle Ω (for the surface integral) has been used ($dS = R^2 d\Omega$). Then, the final result for the function $\hat{G}$ is given by the Eq. 4.88.

The treatment of the wake is similar to that for incompressible flows (Subsection 3.3.6) and yields

$$
\begin{aligned}
E_*\tilde{\varphi}_* &= \oiint_{\mathcal{S}_{0B}} \left[\frac{\partial \tilde{\varphi}}{\partial N} e^{-p\Theta} G_0 - \tilde{\varphi} e^{-p\Theta} \frac{\partial G_0}{\partial N} + p\tilde{\varphi} e^{-p\Theta} G_0 \frac{\partial \hat{\Theta}}{\partial N} \right] d\mathcal{S}_{0B} \\
&- \iint_{\mathcal{S}_{0W}} \left[\Delta\tilde{\varphi} e^{-p\Theta} \frac{\partial G_0}{\partial N} - p\Delta\tilde{\varphi} e^{-p\Theta} G_0 \frac{\partial \hat{\Theta}}{\partial N} \right] d\mathcal{S}_{0W} \qquad (4.95)
\end{aligned}
$$

Going back to the time domain one obtains

$$
\begin{aligned}
E(\mathbf{X}_*)\varphi(\mathbf{X}_*, T_*) &= \oiint_{\mathcal{S}_{0B}} \left[\frac{\partial \varphi}{\partial N} \frac{-1}{4\pi R} - \varphi \frac{\partial}{\partial N} \left(\frac{-1}{4\pi R} \right) + \frac{\partial \varphi}{\partial T} \frac{-1}{4\pi R} \frac{\partial \hat{\Theta}}{\partial N} \right]_{T_*-\Theta} d\mathcal{S}_0 \\
&- \oiint_{\mathcal{S}_{0W}} \left[\Delta\varphi \frac{\partial}{\partial N} \left(\frac{-1}{4\pi R} \right) - \frac{\partial \Delta\varphi}{\partial T} \frac{-1}{4\pi R} \frac{\partial \hat{\Theta}}{\partial N} \right]_{T_*-\Theta} d\mathcal{S}_0 \qquad (4.96)
\end{aligned}
$$

Physical interpretation

In order to obtain a physical interpretation of the time delay θ, consider Fig. 4.1. The point $\mathbf{x}_0$ is the point from which the disturbance started at time t. After a time θ, the disturbance moving at speed a_∞ reaches the point $\mathbf{x}_*$, whereas the point of the aircraft causing the disturbance travels at a speed U_∞ and moves to the point $\mathbf{x}$. Hence, the distance between $\mathbf{x}_*$ and $\mathbf{x}_0$ is $\mathrm{a}_\infty\theta$, whereas that between $\mathbf{x}$ and $\mathbf{x}_0$ is $\mathrm{U}_\infty\theta$. Introduce a frame of reference ξ, η, ζ (connected with the undisturbed air), in which

$$
\begin{aligned}
\xi &= x - U_\infty t \\
\eta &= y \\
\zeta &= z
\end{aligned}
$$

or

$$
\begin{aligned}
x &= \xi + U_\infty \tau \\
y &= \eta \\
z &= \zeta
\end{aligned}
$$

Note also that (Fig. 4.1)

$$
\begin{aligned}
\Delta x &= x_* - x = \xi_* - \xi + U_\infty \theta \qquad (4.97) \\
\Delta y &= y_* - y = \eta_* - \eta \\
\Delta z &= z_* - z = \zeta_* - \zeta
\end{aligned}
$$

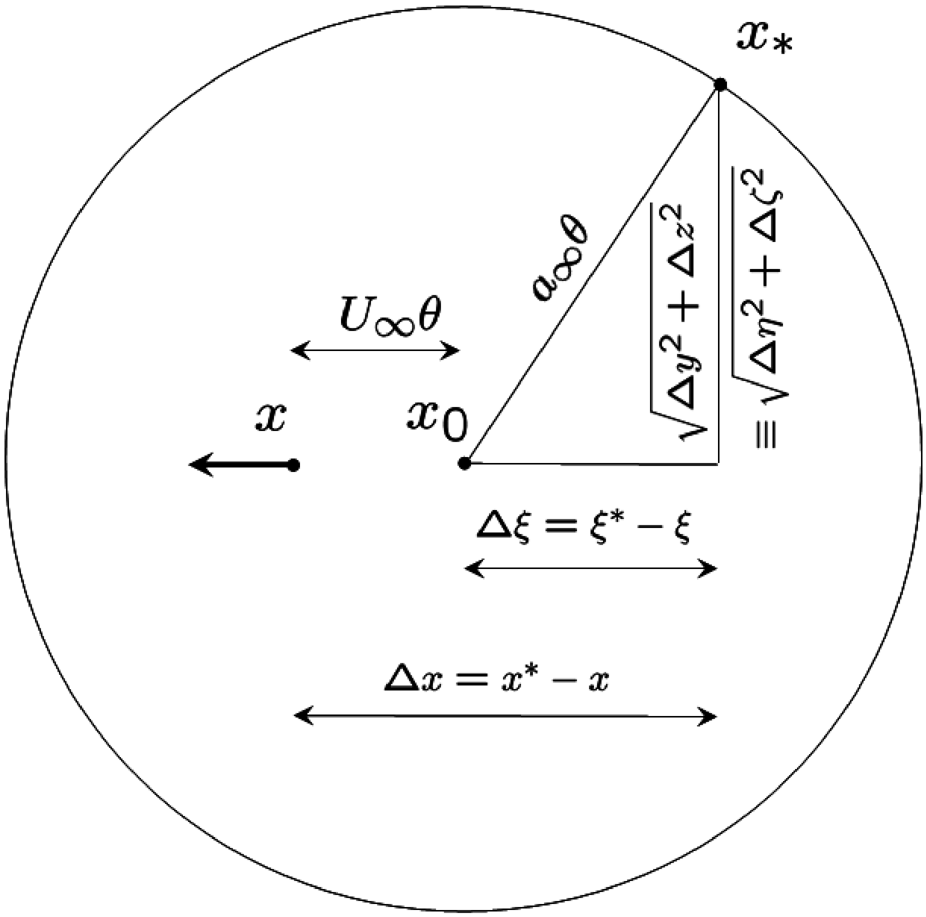

Figure 4.1: Acoustic delay evaluation

Therefore, using geometrical considerations, one obtains

$$\begin{aligned} a_\infty^2\theta^2 &= \Delta\xi^2+\Delta\eta^2+\Delta\zeta^2 \\ &= (\Delta x-U_\infty\theta)^2+\Delta y^2+\Delta z^2 \\ &= \Delta x^2+\Delta y^2+\Delta z^2-2U_\infty\theta\Delta x+U_\infty^2\theta^2 \end{aligned} \tag{4.98}$$

or

$$(a_\infty^2-U_\infty^2)\theta^2+2U_\infty\Delta x\theta-r^2=0 \tag{4.99}$$

i.e., solving for θ, and using the plus sign to satisfy the condition $\theta>0$

$$\theta=\frac{-M\Delta x+r_\beta}{\beta^2 a_\infty} \tag{4.100}$$

with $r_\beta:=\sqrt{\Delta x^2+\beta^2(\Delta y^2+\Delta z^2)}$.

Thus, the compressibility delay is expressed in terms of the x,y,z variables in Eq. 4.100: once this formula is made dimensionless multiplying it by U_∞/ℓ and considering the definition $\Delta x:=x_*-x$, one can easily verify that the evaluated delay is that defined in Eq. 4.93.

4.3.2 Discretization

The discretization is also similar to that for incompressible flows (Subsection 3.3.6): for the sake of simplicity consider a zero-th order discretization subdividing the body (closed) surface in M panels and the wake (opened) surface in N panels. Rewriting $\Delta\varphi_{TE} = \varphi_{Upper} - \varphi_{Lower}$ (see Eq. 3.64) as

$$\Delta\varphi_{TE_n} = \sum_m^M S_{nm}\varphi_m \tag{4.101}$$

(where $S_{nm} = 1$ at the upper trailing-edge node, $S_{nm} = -1$ at the lower trailing-edge node, and $S_{nm} = 0$ otherwise), the resulting discretized equations in the time domain are

$$\begin{aligned} \frac{1}{2}\varphi_k(T) &= \sum_h^M B_{kh}\chi_h(T-\Theta_{kh}) + \sum_h^M C_{kh}\varphi_h(T-\Theta_{kh}) + \sum_h^M D_{kh}\dot{\varphi}_h(T-\Theta_{kh}) \\ &+ \sum_n^N F_{kn}\Delta\varphi_n(T-\Theta_{kn}) + \sum_n^N G_{kn}\Delta\dot{\varphi}_n(T-\Theta_{kn}) \end{aligned} \tag{4.102}$$

with (similar form can be given to $\Delta\dot{\varphi}_n(T-\Theta_{kn})$)

$$\Delta\varphi_n(T-\Theta_{kn}) = \Delta\varphi_{TE_n}(T-\Theta_{kn}-\Pi_n) = \sum_m^M S_{nm}\varphi_m(T-\Theta_{kn}-\Pi_n) \tag{4.103}$$

where Π_n is the dimensionless time ($\Pi = \tau U_\infty/\ell$) required for a wake point to be convected from the trailing edge to the wake located at the point $\mathbf{X}_n$ and the other coefficients are defined as in the following

$$B_{km} = \iint_{\mathcal{S}_{0_m}} \frac{-1}{4\pi\|\mathbf{X}-\mathbf{X}_k\|} d\mathcal{S}_{0_m} \tag{4.104}$$

$$C_{km} = \iint_{\mathcal{S}_{0_m}} \frac{\partial}{\partial N}\left(\frac{1}{4\pi\|\mathbf{X}-\mathbf{X}_k\|}\right) d\mathcal{S}_{0_m} \tag{4.105}$$

$$D_{km} = \iint_{\mathcal{S}_{0_m}} \frac{\partial\hat{\Theta}_{km}}{\partial N}\left(\frac{-1}{4\pi\|\mathbf{X}-\mathbf{X}_k\|}\right) d\mathcal{S}_{0_m} \tag{4.106}$$

$$F_{kn} = \iint_{\mathcal{S}_{0_n}} \frac{\partial}{\partial N}\left(\frac{1}{4\pi\|\mathbf{X}-\mathbf{X}_k\|}\right) d\mathcal{S}_{0_n} \tag{4.107}$$

$$G_{kn} = \iint_{\mathcal{S}_{0_n}} \frac{\partial\hat{\Theta}_{kn}}{\partial N}\left(\frac{-1}{4\pi\|\mathbf{X}-\mathbf{X}_k\|}\right) d\mathcal{S}_{0_n} \tag{4.108}$$

If the hypothesis of prescribed wake is taken into account, the corresponding equations in the frequency domain are

$$\sum_h^M Y_{kh}\tilde{\varphi}_h = \sum_h^M Z_{kh}\tilde{\chi}_h \tag{4.109}$$

where

$$
\begin{aligned}
Y(p)_{kh} &:= \frac{1}{2}\delta_{kh} - (C_{kh} + pD_{kh})e^{-p\Theta_{kh}} - \sum_{n}^{N}(F_{kn} + pG_{kn})e^{-p(\Theta_{kn}+\Pi_n)}S_{nh} &\quad (4.110)\\
Z(p)_{kh} &:= B_{kh}e^{-p\Theta_{kh}} &\quad (4.111)
\end{aligned}
$$

It is important to point out that all the coefficients defined above depend on the compressibility condition of the flow only through the Mach number M_∞.

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