POLITECNICO DI MILANO

Aerospace Control Systems
Systems theory – stability

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Outline

Lyapunov theory: autonomous systems

- Equilibria and stability
- Lyapunov's stability theorem
- LaSalle invariance principle
- Stability of linear time-invariant systems

Autonomous systems

Equilibria and their stability

For the autonomous system

$$\dot{x} = f(x)$$

for which an equilibrium is known

$$\bar{x}: \qquad 0 = f(\bar{x})$$

we first want to recall the definition of stability of the equilibrium and then characterise and study such property.

Equilibria and their stability

Assume for simplicity that $\bar{x} = 0$

Then the equilibrium is:

- Stable, if for all $\varepsilon > 0$ there exists $\delta=\delta(\varepsilon)>0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t > 0$$

- Unstable, if it is not stable

- Asymptotically stable (AS), if it is stable and if δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

Stability and linearisation

Let x=0 an equilibrium for the nonlinear system

$$\dot{x} = f(x)$$

and consider

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$$

We then have that:

- x=0 is an AS equilibrium if Re $\lambda_i < 0$ for all the eigenvalues of A
- x=0 is an unstable equilibrium if Re $\lambda_i > 0$ for at least one of the eigenvalues of A.

Example: the pendulum

State equations for a pendulum (mass *m*, length *l*) with friction

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2
\end{aligned}
$$

Linearisation in x=0

$$
\begin{aligned}
\dot{\delta}x_1 &= \delta x_2 \\
\dot{\delta}x_2 &= -\frac{g}{l}\delta x_1 - \frac{k}{m}\delta x_2
\end{aligned}
$$

In this case the eigenvalues have negative real parts so x=0 is AS.

Example: the pendulum

State equations for a pendulum without friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1)$$

Linearisation in x=0

$$\dot{\delta x}_1 = \delta x_2$$
$$\dot{\delta x}_2 = -\frac{g}{l}\delta x_1$$

The eigenvalues are imaginary, so the linearisation criterion does not allow us to conclude about the stability of the equilibrium.

Example: the pendulum

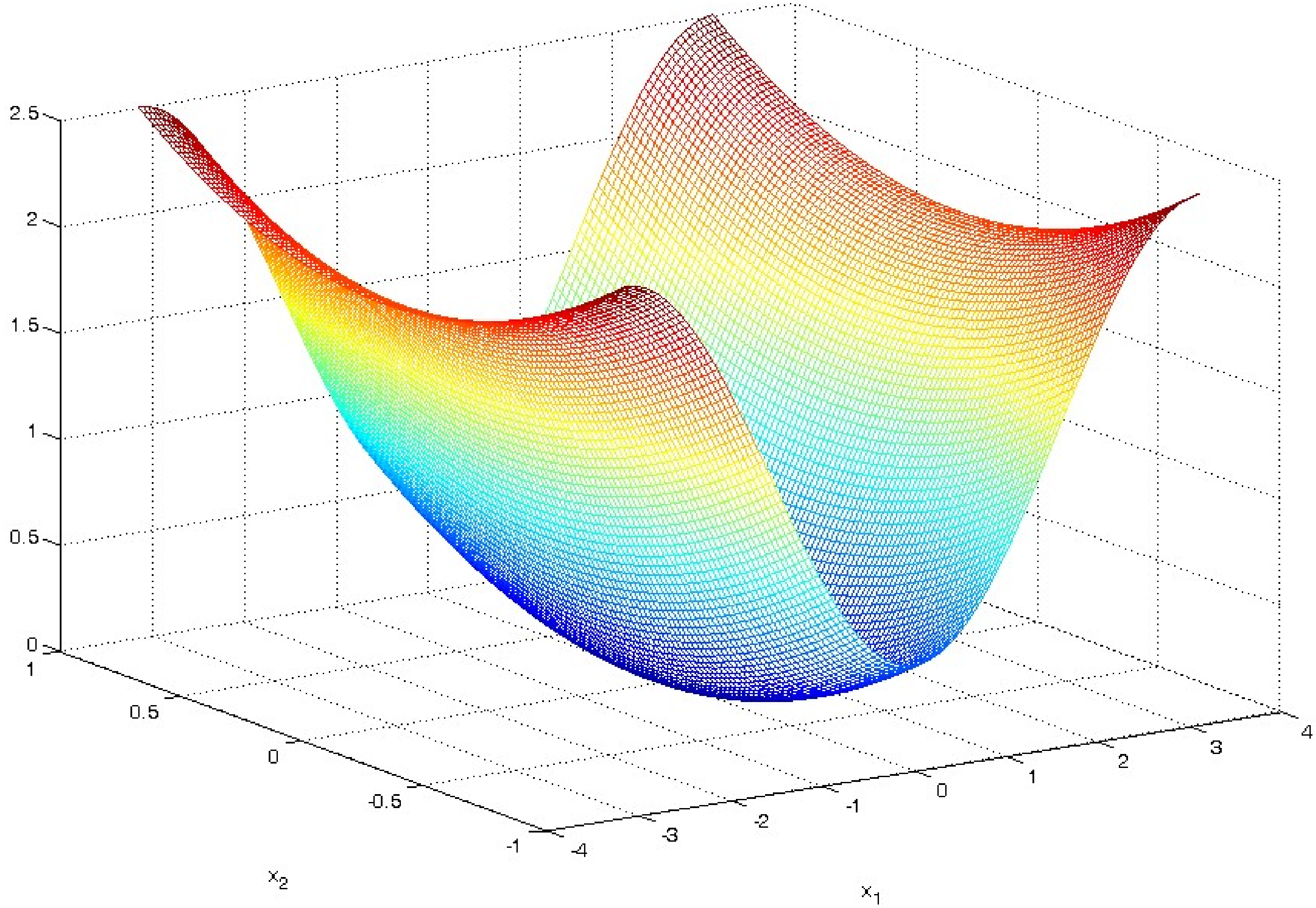
Energy-based analysis of the system:

the total mechanical energy of the pendulum is given by

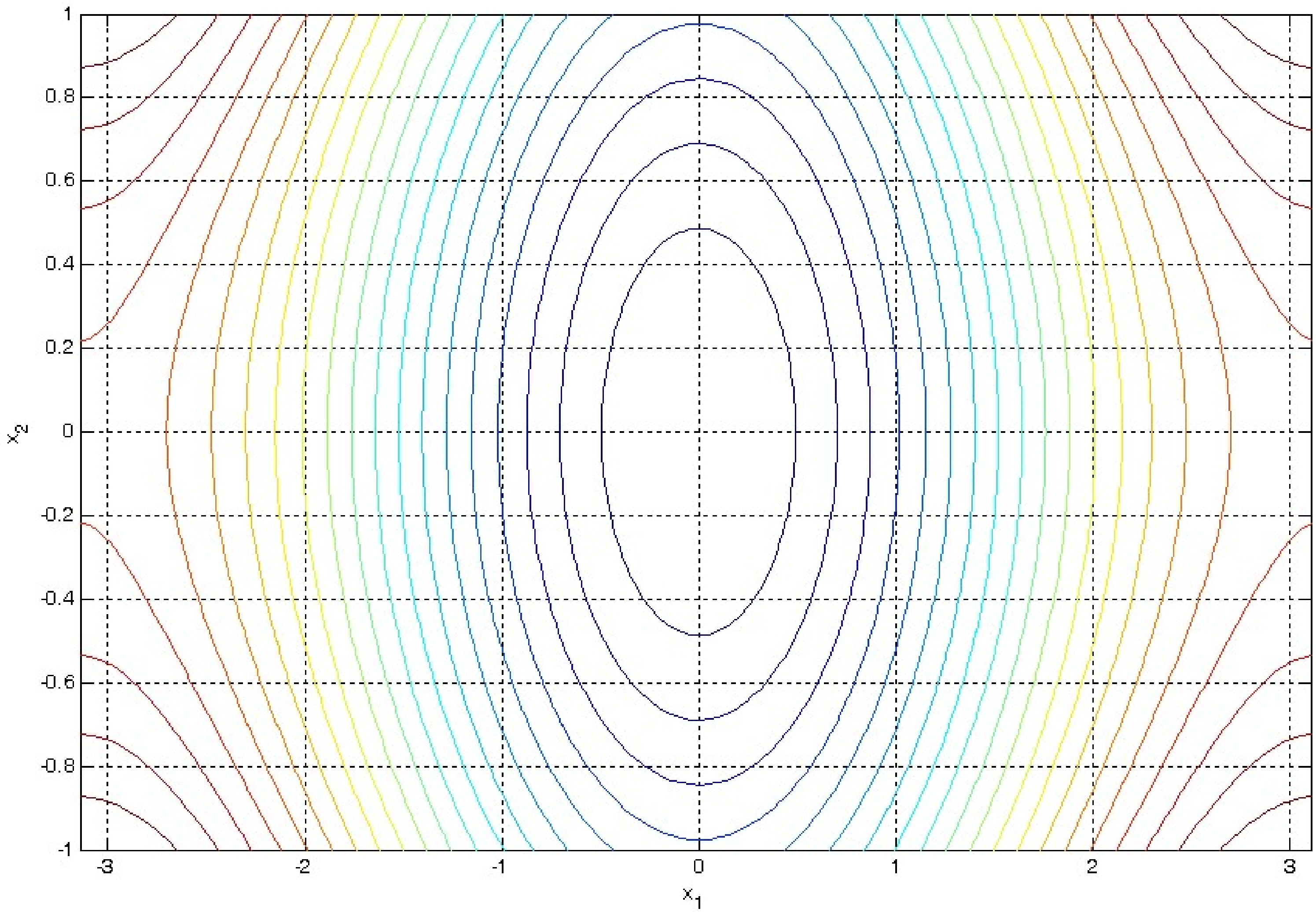
$$E(x) = \frac{g}{l}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

(reference for the potential energy chosen such that E(0)=0).

Total mechanical energy



Total mechanical energy



Example: the pendulum

Without friction:

the system is conservative, so $E(x)=c$, c function of the initial condition (and $dE/dt=0$)

$E(x)=c$ is a closed curve enclosing $x=0$

Therefore, for sufficiently small c x remains arbitrarily close to zero.

Hence, $x=0$ is a stable equilibrium.

Example: the pendulum

With friction:

the system dissipates energy, so E decreases and $dE(t)/dt<0$.

E keeps decreasing as long as the pendulum is moving.

So x tends to zero.

Hence $x=0$ is an AS equilibrium.

Lyapunov functions

- The stability of the pendulum's x=0 equilibrium can be studied using an energy-based approach.

- Lyapunov proved that more general functions can be used to this purpose.

- Consider an energy-like continuously differentiable function V(x) and compute its derivative along the trajectories of the system:

$$\dot{V}(x) = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\dot{x}_i = \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x)$$

Lyapunov functions

Key observation:

if $\dot{V}(x)$ is negative along the solution of the state equation, then V(x) is decreasing.

Therefore, if we can construct a function V(x) with this property, we have a tool to carry out a stability analysis.

Lyapunov's stability theorem

Theorem: let x=0 an equilibrium of the sistem and D a domain which includes x=0. Then given a smooth function V: D $\rightarrow$ R such that

$$V(0) = 0, \quad V(x) > 0 \quad \text{in D}, \quad x \neq 0$$
$$\dot{V}(x) \leq 0 \quad \text{in D}$$

then the equilibrium is stable.

If in addition

$$\dot{V}(x) < 0 \quad \text{in D}, \quad x \neq 0$$

then the equilibrium is asymptotically stable.

Proof: stability

Given $\varepsilon > 0$, choose $r \in (0, \varepsilon]$:

$$B_r = \{x \in \mathcal{R}^n : \|x\| \leq r\} \in D$$

Let

$$\alpha = \min_{\|x\|=r} V(x), \qquad (\alpha > 0)$$

choose $\beta \in (0, \alpha)$ and define

$$\Omega_\beta = \{x \in B_r : V(x) \leq \beta\}$$

NOTE: Ω_β is a closed and bounded set, hence it is a compact set.

Proof: stability

Note that since

$$\dot{V}(x(t)) \leq 0$$

then

$$V(x(t)) \leq V(x(0)) \quad \forall t \geq 0$$

and if x(0) $\in \Omega_\beta$ then we have x(t) $\in \Omega_\beta$ $\forall$ t $\geq$ 0.

Furthermore, V(x) is continuous and V(0)=0 so

$$\exists \delta : \qquad \|x\| \leq \delta \Rightarrow V(x) < \beta$$

and therefore

$$B_\delta \subset \Omega_\beta \subset B_r$$

Proof: stability

Then we have

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

and this implies that

$$\|x(0)\| \leq \delta \Rightarrow \|x(t)\| < r \leq \epsilon \quad \forall t \geq 0$$

and so the equilibrium is stable.

Proof: asymptotic stability

We need to show that x(t) $\to$ 0, t $\to$ ∞, or equivalently

$$\forall a > 0 \quad \exists T > 0 : \|x(t)\| < a \quad \forall t > T$$

To this purpose it is sufficient to show that

V(x(t)) $\to$ 0, t $\to$ ∞.

V(x(t)) monotonically decreasing, bounded from below, so

V(x(t)) $\to$ c, c $\geq$ 0, t $\to$ ∞.

Proof: asymptotic stability

Assume that c $>$ 0.

Then there exists d>0 such that $B_d \subset \Omega_c$

and so x(t) remains outside B_d for all t $\geq$ 0.

Note now that

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t$$

where
$$-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$$

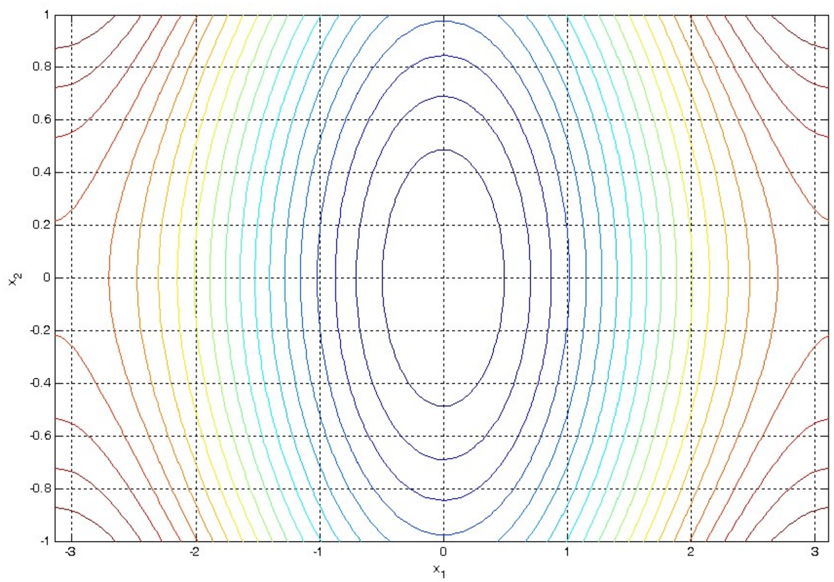
Since the right hand side becomes negative for suff. large t, then c cannot be positive.

Lyapunov function

A smooth function such that

$$V(0) = 0, \quad V(x) > 0 \quad \text{in D}, \quad x \neq 0$$
$$\dot{V}(x) \leq 0 \quad \text{in D}$$

Geometric interpretation via level curves:



Quadratic Lyapunov functions

Quadratic functions are often chosen as candidate Lyapunov functions:

$$V(x) = x^T P x = \sum_{i=1}^{n} \sum_{j=1}^{n} p_{ij} x_i x_j, \qquad P = P^T$$

The function is positive (semi)definite iff matrix P is, and this condition is easy to check.

Comments

Pros and cons of Lyapunov Theorem:

- Enables stability analysis without the need to solve the state equation
- Does not provide criteria for the choice of V(x) (though physics usually help);
- The stability condition is *only* sufficient.

Example: the pendulum

Without friction:

$$V(x) = \frac{g}{l}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

Having checked that V(0)=0 and V(x)>0 in $-\pi < x_1 < \pi$ we compute the derivative along the trajectories of the system

$$\dot{V}(x) = \frac{g}{l}\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 = \frac{g}{l}\sin(x_1)x_2 - \frac{g}{l}\sin(x_1)x_2 = 0$$

Therefore we can prove that the origin is stable.

Example: the pendulum

With friction:

$$V(x) = \frac{g}{l}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

Having checked that V(0)=0 and V(x)>0 in $-\pi < x_1 < \pi$ we compute the derivative along the trajectories of the system

$$\dot{V}(x) = \frac{g}{l}\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 = -\frac{k}{m}x_2^2 \leq 0$$

Therefore we can prove that the origin is stable.

To show asymptotic stability we can either resort to linearisation or use other tools.

Region of attraction

Assume that x=0 is an AS equibrium; then for which initial x the solution of the state equation converges to 0?

We define the *region of attraction* as the set of points x such that

$$\lim_{t \to \infty} \phi(t, x) = 0$$

$\phi(t,x)$ being the solution of the state equation.

Estimation of the region of attraction

Can we compute an estimate of the region of attraction?

In principle we can use the set Ω_c defined in the proof of Lyapunov Theorem.

Indeed, if V(x) satisfies the conditions for AS, then all trajectories starting in Ω_c will be confined to the set and will converge to x=0.

Global asymptotic stability

Are there equilibria for which the region of attraction coincides with the entire state space?

I.e., equilibria for which

$$\lim_{t\to\infty} \phi(t,x) = 0, \forall x$$

Such equilibria are called globally asymptotically stable (GAS).

Which conditions are needed for an equilibrium to be GAS?

Barbashin-Krasovski Theorem

Theorem: let x=0 an equilibrium of the system and choose a smooth function V(x) such that

$$V(0) = 0, \qquad V(x) > 0, \forall x \neq 0$$
$$\|x\| \to \infty \Rightarrow V(x) \to \infty$$
$$\dot{V}(x) < 0, \qquad \forall x \neq 0$$

then the equilibrium is GAS.

NOTE: if x=0 is GAS, then it is the sole equilibrium of the system!

Examples

Consider the first order system given by

$$\dot{x} = -x^2$$

verify that x=0 is an equilibrium and study its stability.

The equilibrium condition is

$$0 = -x^2$$

which has the unique solution x=0, so this point is the only equilibrium of the system.

Examples

To study its stability we can try using the linearisation approach.

$$\dot{x} = -x^2$$

To this purpose we compute

$$\frac{\partial f(x)}{\partial x}|_{x=0} = -2x|_{x=0} = 0$$

which tells us that we cannot conclude anything about AS using linearisation alone.

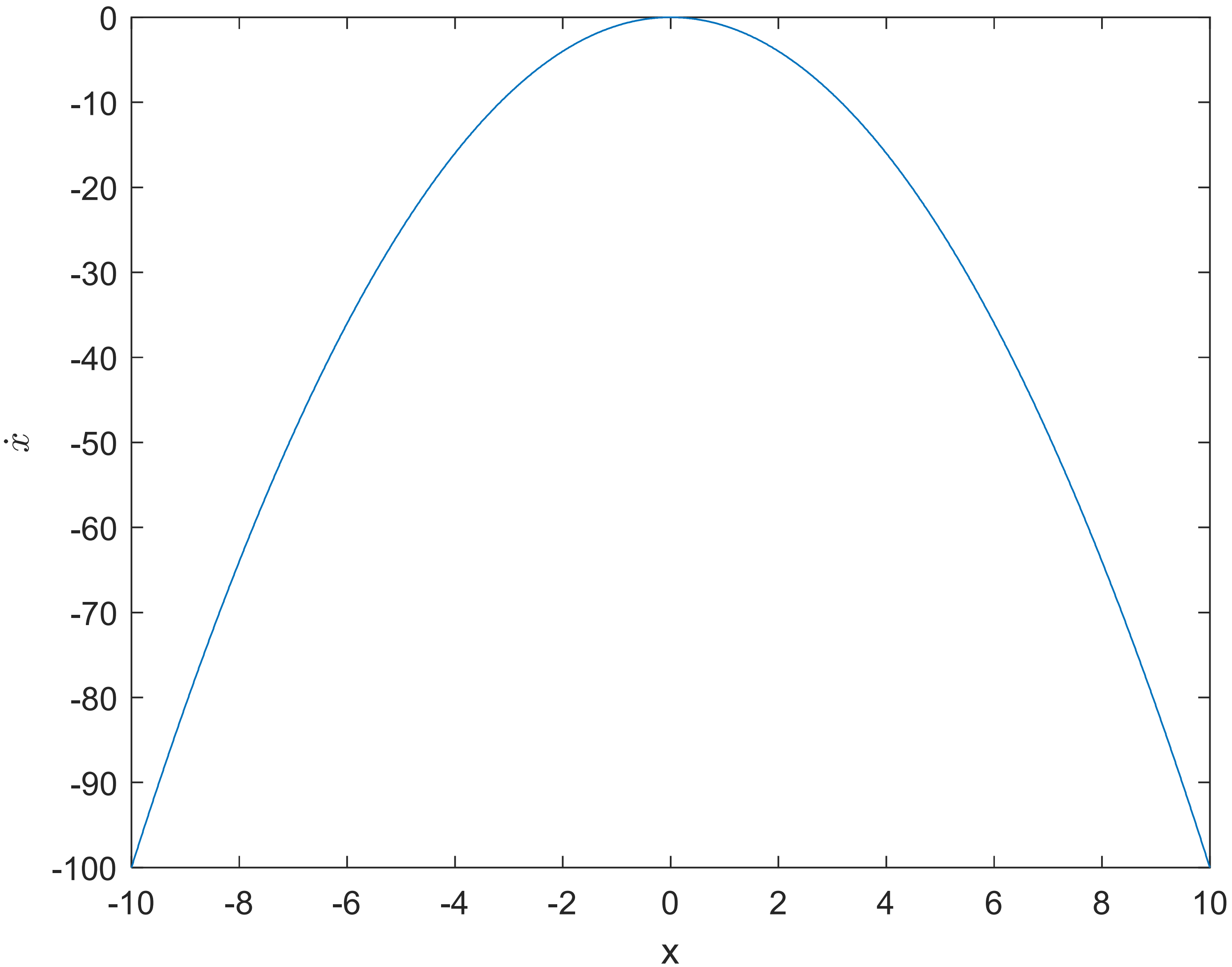
Examples

We then try a graphical approach to the stability analysis, using the definition of stability.

A plot of $\dot{x} = -x^2$ allows us to study the sign of the derivative as a function of the value of the state.

Hence we can study if at a given value of *x* the corresponding derivative is positive (increasing state) or negative (decreasing state).

Examples



Examples

From the plot we see that:

- For $x(0) < 0$ we have a negative derivative, so the state will further decrease, moving away from the equilibrium.
- For $x(0) > 0$ we have a negative derivative, so the state will decrease, returning to the equilibrium.

Since we have found (arbitrarily small) perturbations leading to unbounded motion of the state away from equilibrium we can conclude that the equilibrium is unstable.

Examples

Consider the first order system given by

$$\dot{x} = -x^3$$

verify that x=0 is an equilibrium and study its stability.

The equilibrium condition is

$$0 = -x^3$$

which has the unique solution x=0, so this point is the only equilibrium of the system.

Examples

To study its stability we can try using the linearisation approach.

$$\dot{x} = -x^3$$

To this purpose we compute

$$\frac{\partial f(x)}{\partial x}|_{x=0} = -3x^2|_{x=0} = 0$$

which tells us that we cannot conclude anything about AS using linearisation alone.

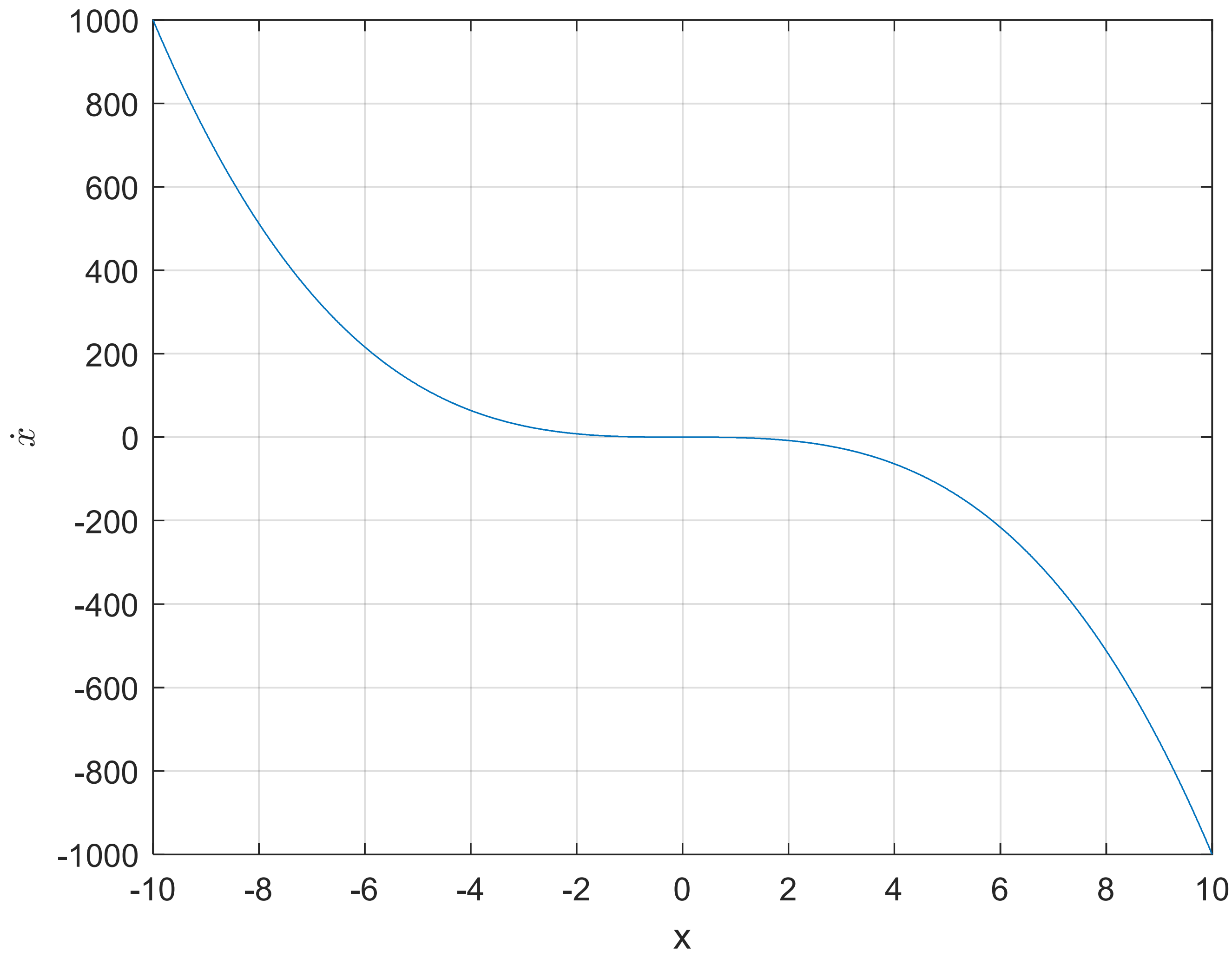
Examples

We then try a graphical approach to the stability analysis, using the definition of stability.

A plot of $\dot{x} = -x^3$ allows us to study the sign of the derivative as a function of the value of the state.

Hence we can study if at a given value of *x* the corresponding derivative is positive (increasing state) or negative (decreasing state).

Examples



1000
800
600
400
200
0
-200
-400
-600
-800
-1000
$\dot{x}$
-10
-8
-6
-4
-2
0
2
4
6
8
10
x

From the plot we see that:

- For $x(0) < 0$ we have a positive derivative, so the state will increase, moving toward the equilibrium.
- For $x(0) > 0$ we have a negative derivative, so the state will decrease, returning to the equilibrium.

Therefore, the equilibrium is AS.

Note that it is also GAS, as the sign of the derivative over each half axis never changes.

Examples

For this example we can also consider the application of Lyapunov's theorem.

Considering the dynamics $\dot{x} = -x^3$ we consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x^2$$

for which we note that

$$V(0) = 0$$
$$V(x) > 0 \quad \forall x \neq 0$$
$$lim_{\|x\| \to \infty} V(x) = +\infty$$

Computing the derivative along the trajectories we get

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x}\dot{x} = x(-x^3) = -x^4 < 0$$

So the equilibrium is AS (Lyapunov Theorem) and also GAS (Barbashin-Krasovski Theorem).

Instability theorems

In some cases it is useful to prove *instability* of an equilibrium rather than its stability or asymptotic stability.

There exist a number of instability theorems, we will look at one in detail.

Chetaev Theorem

Theorem: let x=0 an equilibrium of the system and D a domain including x=0.

Let V: D $\rightarrow$ R a smooth function such that

$$V(0) = 0$$

$$V(x_0) > 0$$

with x_0 of arbitrarily small norm. Then, chosen r>0 such that $B_r \subset$ D define the set

$$U = \{x \in B_r : V(x) > 0\}$$

So, if

$$\dot{V}(x) > 0 \quad \text{in U}$$

then the equilibrium x=0 is unstable.

LaSalle invariance principle

Again, the pendulum (with friction):

We saw that with

$$V(x) = \frac{g}{l}(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

we have

$$\dot{V}(x) = \frac{g}{l}\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 = -\frac{k}{m}x_2^2 \leq 0$$

so we cannot conclude about the AS of the equilibrium.

BUT: is it possible for the pendulum to have trajectories different from the zero equilibrium along which V(x) is constant?

LaSalle invariance principle

Note however that:

$\dot{V}(x) = 0$ only for $x_2=0$;

$x_2 \equiv 0$ implies $x_1 \equiv 0$;

So the only trajectory of the system for which $\dot{V}(x) = 0$ is the origin and therefore $V(x(t))$ and $x(t)$ go to zero for $t \to \infty$.

LaSalle invariance principle

In other words, if we can find a domain D enclosing the origin such that :

$\dot{V}(x) \leq 0$ in D;

No trajectory of the system can coincide with points such that $\dot{V}(x) = 0$
except the origin. Then the origin is AS.

This result is called LaSalle invariance principle.
Let's now look at the rigorous formulation of the principle.

Some definitions

Given a solution x(t) of $\dot{x} = f(x)$ we define:

p: positive limit point of x(t) if there exists a sequence $\{t_n\}$,

$t_n \to \infty$ n $\to \infty$ such that

$x(t_n) \to p$, n $\to \infty$.

L^+: positive limit set, i.e., the set of the positive limit points.

Invariant set M:

$x(0) \in M \Rightarrow x(t) \in M, \ \forall t \in \mathbb{R}$

Positive invariant set M:

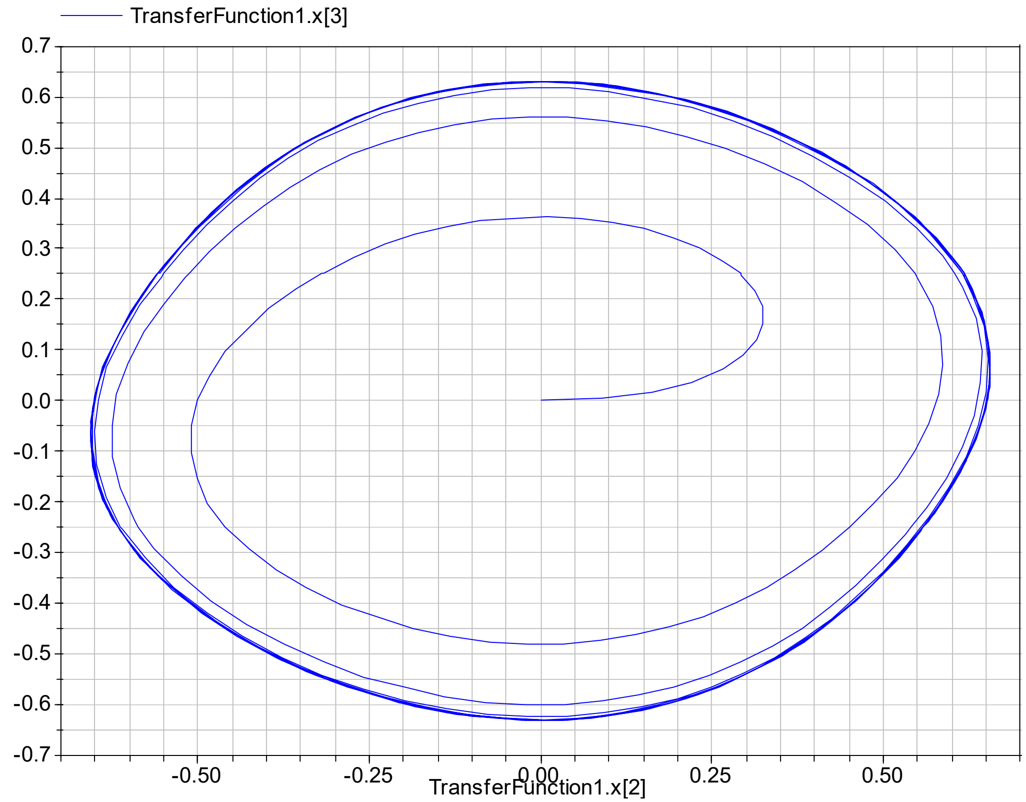
$x(0) \in M \Rightarrow x(t) \in M, t \geq 0$

What is the meaning of $x(t) \to M$?

Note:

$x(t) \to M$ does not imply the existence of the limit of $x(t)$ for $t \to \infty$!

Example: M limit cycle



A Lemma

If a solution $x(t)$ of the system is bounded and belongs to D for $t \geq 0$, then L^+ is a non empty, compact and invariant set.

In addition we have that

$x(t) \to L^+$ for $t \to \infty$

Proof: see Khalil, Appendix A.2.

LaSalle Theorem

Let:

$\Omega \subset$ D a compact set which is positively invariant for the system;

V: D $\rightarrow \mathbb{R}$ a smooth function such that $\dot{V}(x) \leq 0$ in Ω

E the set of points such that $\dot{V}(x) = 0$

M the largest invariant set contained in E

Then every solution of the system starting in Ω tends to M for t $\rightarrow \infty$

Proof

Let x(t) a solution starting in Ω.

Then V(x(t)) admits a limit *a* for $t \to \infty$.

Furthermore $L^+ \subset \Omega$, as Ω is a closed set.

By definition of L^+, for all $p \in L^+$ there exists a sequence $\{t_n\}$, $t_n \to \infty$ $n \to \infty$ such that

$x(t_n) \to p$, $n \to \infty$.

By continuity of V(x)

$$V(p) = \lim_{n \to \infty} V(x(t_n)) = a$$

Proof

Then V(x)=*a* in L^+.

L^+ is invariant, so $\dot{V}(x) = 0$ in L^+.

Therefore

$$L^+ \subset M \subset E \subset \Omega$$

and since $x(t) \to L^+$ for the Lemma, we also have that $x(t) \to M$.

Comments

- The set Ω provides a more flexible tool to characterise the region of attraction;
- LaSalle Theorem is applicable also to study an equilibrium set which is not necessarily given by a single point;
- Function V does not have to be positive definite.

Reference:

- J.P. LaSalle, Some extensions of Lyapunov's second method, IRE Transactions on Circuit Theory, Dicembre 1960, pp. 520-527.

Example: adaptive control

Consider the linear system

$$\dot{y} = ay + u$$

with **unknown** *a* and let's verify that the controller

$$u = -ky$$
$$\dot{k} = \gamma y^2, \qquad \gamma > 0$$

guarantees *globally* that y(t) $\to$ 0, t $\to \infty$.

Example: adaptive control

State space form for the feedback system: letting x_1=y and x_2=k we get

$$\dot{x}_1 = ax_1 - x_1x_2 = -(x_2 - a)x_1$$
$$\dot{x}_2 = \gamma x_1^2$$

From the equations we see that the line x_1=0 is an equilibrium set for the feedback system.

Example: adaptive control

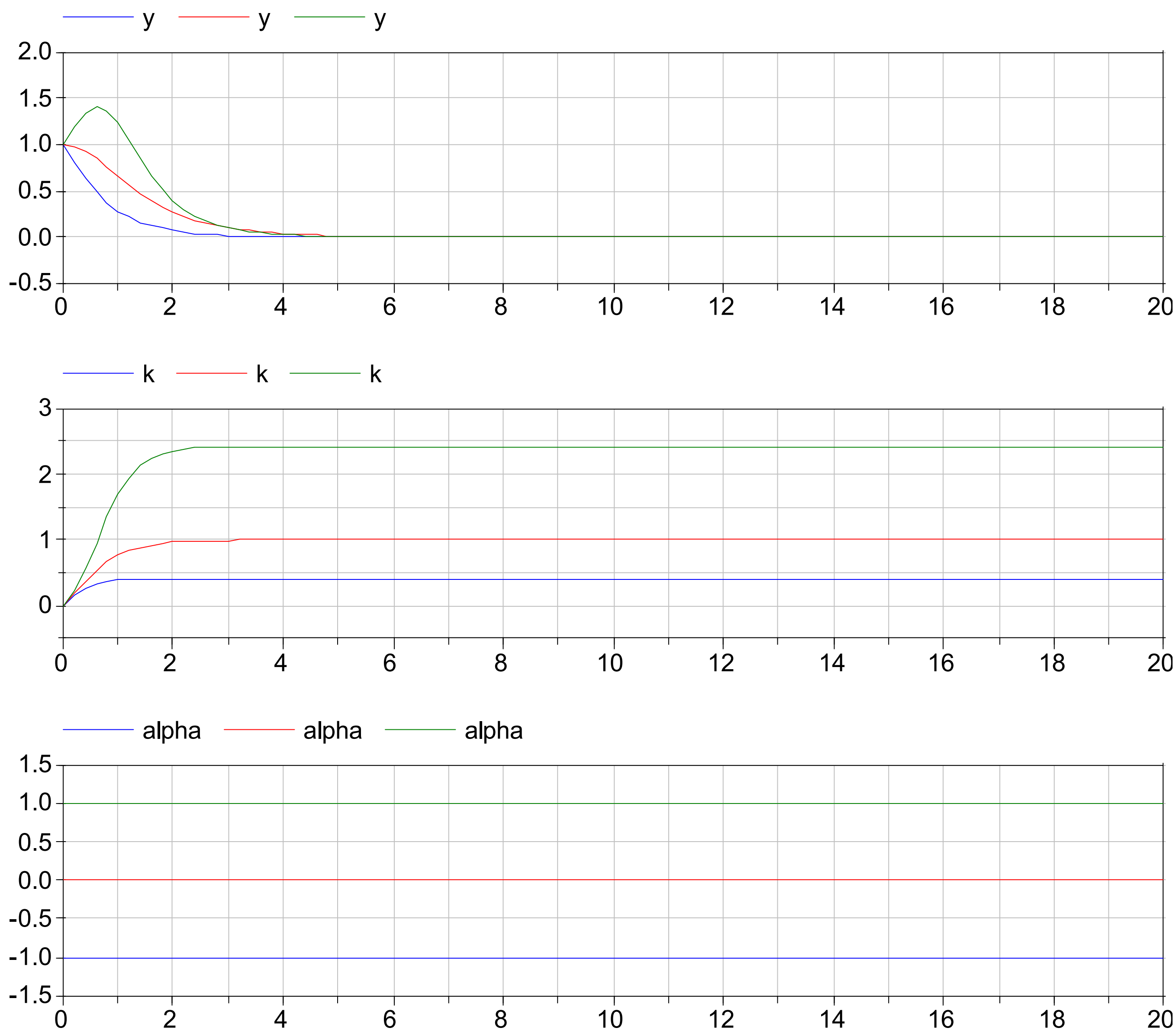
We now choose the function

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2, \quad b > a$$

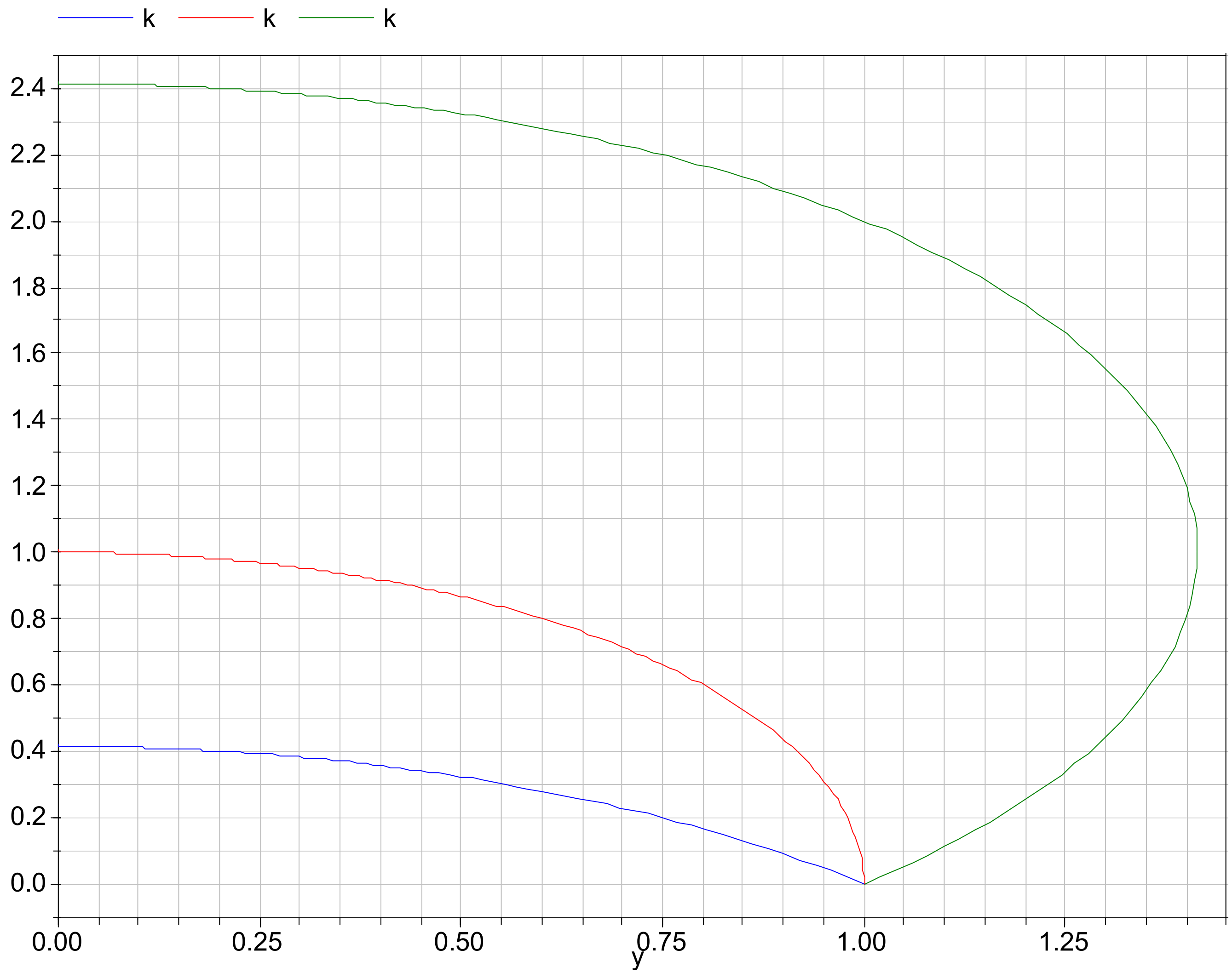
and compute its derivative along the trajectories of the system

$$\dot{V}(x) = x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 =$$
$$= -x_1^2(x_2 - a) + x_1^2(x_2 - b) = -x_1^2(b - a) \leq 0$$

Simulations (γ=1): response of y and k



Simulations (γ=1): L^+ in state space



Further comments

- In the example the Lyapunov function depends on a parameter *b* which has to satisfy *b*>*a*. The value of the parameter is not known explicitly, but we know that it always exists.
- Hence another feature of Lyapunov methods: we must ensure the existence of V(x) but its explicit determination is not required by the theorem.

Linear time invariant systems

Given $A \in \mathbb{R}^{n \times n}$ and $C \in \mathbb{R}^{n \times n}$, the Lyapunov equation in the unknown $P \in \mathbb{R}^{n \times n}$ is given by

$$PA + A^T P = C$$

This equation is tightly related to the stability of the LTI system

$$\dot{x} = Ax$$

Stability of LTI systems

Lyapunov stability theorem for LTI systems: the system

$$\dot{x} = Ax$$

is AS iff for all Q=Q^T>0 there exists a unique P=P^T>0 such that

$$PA + A^T P = -Q$$

Sufficient condition. For the system

$$\dot{x} = Ax$$

the function $V(x)=x^T P x$, $P=P^T>0$ is a Lyapunov function if

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} < 0$$

But this is equivalent to

$$x^T \left(PA + A^T P\right) x = -x^T Q x < 0$$

Therefore V(x) is a Lyapunov function if and only if Q>0.

Let

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

and note that if the system is AS then P is the unique solution of the Lyapunov equation and $P=P^T>0$. Indeed

$$PA + A^T P = \int_0^\infty e^{A^T t} Q e^{At} A dt + \int_0^\infty A^T e^{A^T t} Q e^{At} dt =$$

$$= \int_0^\infty \frac{d}{dt}\left(e^{A^T t} Q e^{At}\right) dt = \left[e^{A^T t} Q e^{At}\right]_0^\infty = -Q$$

So if the system is AS, then P satisfies the equation.

Stability of LTI systems

P is positive definite since $u^T Pu$ can be written as

$$u^T P u = \int_0^\infty u^T e^{A^T t} Q e^{At} u dt > 0, \qquad u \neq 0$$

Uniqueness of P. Assume that two solutions P_1 e P_2 exist. Then

$$(P_1 - P_2) A + A^T (P_1 - P_2) = 0$$

but this in turn implies

$$e^{A^T t} (P_1 - P_2) A + A^T (P_1 - P_2) e^{At} = 0$$

$$\frac{d}{dt}\left(e^{A^T t} (P_1 - P_2) e^{At}\right) = 0$$

and so $P_1 = P_2$.

Necessary condition.

Given an eigenvalue/eigenvector pair (x,λ) of A compute

$$x^H(PA + A^T P)x = -x^H Qx$$

$$\lambda x^H Px + \bar{\lambda} x^H Px = -x^H Qx$$

but as P>0 e Q>0 we get

$$(\lambda + \bar{\lambda}) < 0$$

And so the system is AS.

A computational approach

Given A, how do I check if there exists $P=P^T>0$ such that

$$PA + A^T P < 0$$

This condition is a Linear Matrix Inequality (LMI)

It can be checked in a computationally efficient way using SemiDefinite Programming techniques.

In Matlab, using the SeDuMi and Yalmip tools.

A computational approach

Example:

```
A=[-1, 2, 3; 0, -2, 1; 0, 0, -5];

P = sdpvar(3,3);

F = [P >=0];
F = [F, P*A+A'*P<0]
solvesdp(F)
double(P)
```

State feedback stabilisation

Given the continuous LTI system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

and the state feedback control law

$$u = -Kx + v$$

the closed loop system is given by

$$\dot{x} = (A - BK)x + Bv$$
$$y = (C - DK)x + Dv$$

State feedback stabilisation: find K: A-BK is stable.

When is the problem solvable? Following lectures…

Theorem. Assume (A,B) controllable and let β a scalar such that $\beta > |\lambda_{max}(A)|$; let

$$K = B^T Z^{-1}$$

where $Z=Z^T>0$ is such that

$$-(A+\beta I)Z - Z(A+\beta I)^T = -2BB^T$$

then A-BK is stable.

A Lyapunov stabilisation method (cont.d)

Proof.

Under the considered assumptions, $-(A+\beta I)$ is stable.

Therefore, equation

$$-(A+\beta I)Z - Z(A+\beta I)^T = -2BB^T$$

has a unique symmetric positive definite solution Z.

But

$$-(A+\beta I)Z - Z(A+\beta I)^T = -2BB^T$$

can be written as

$$(A-BK)Z + Z(A-BK)^T = -2\beta Z$$

and therefore A-BK is stable.

Recap

- Lyapunov theory: a general approch for equilibrium stability analysis

- In general only sufficient condition; for LTI systems it is necessary and sufficient

- Lyapunov equations can be used for stability analysis and for stabilisation

- Computational methods (LMI solvers) exist to check Lyapunov inequalities

- Elementary case of a broad class of methods.