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MODES OF VIBRATION OF N-DOF STRUCTURES

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STRUCTURAL DYNAMICS AND AEROELASTICITY
Exercises

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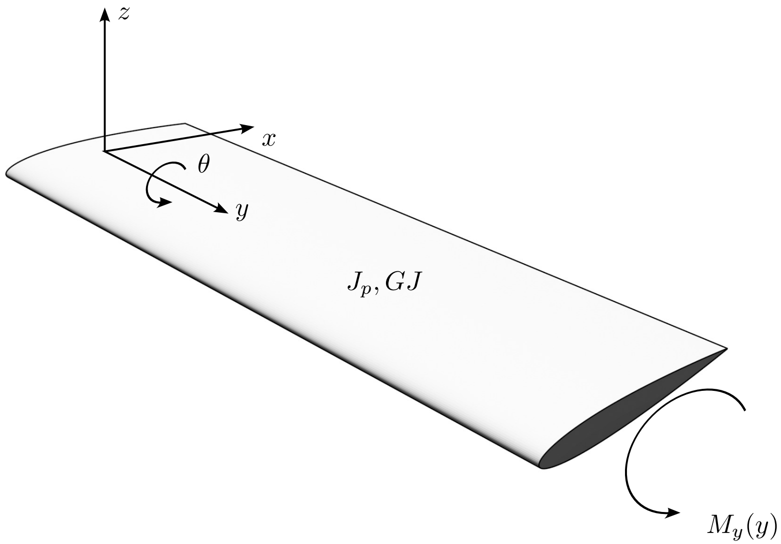
Let's consider a cantilever, **straight wing** subjected to **pure torsion** (bar)



Continuum Mechanics approach:
Infinitesimal chunk equilibrium

$$\cancel{-M_y} + \cancel{M_y} + dM_y - J_p\ddot{\theta}dy = 0$$
$$M_{y/y}\cancel{dy} - J_p\ddot{\theta}\cancel{dy} = 0$$

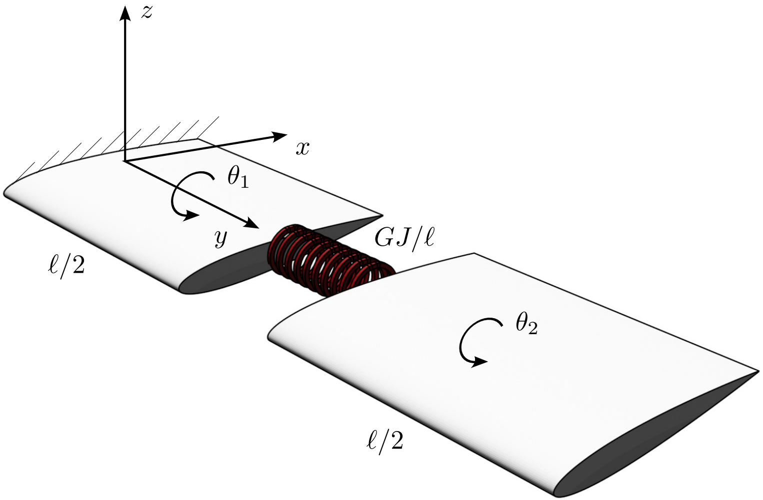
the internal torsion moment linear with the torsion deformation.
Indefinite equilibrium equation (we'll learn to solve it) $M_y = GJ\theta_{/y}$

$$GJ\theta_{/yy} - J_p\ddot{\theta} = 0$$

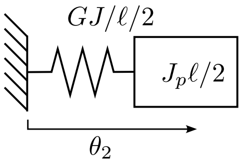
Natural frequencies and mode shapes:

$$\omega_n = \sqrt{\frac{GJ}{J_p\ell^2}}\frac{\pi}{2}(1+2n)$$
$$a_n(y) = A_n \sin\left((1+2n)\frac{y}{\ell}\frac{\pi}{2}\right)$$

Let's consider a simplified, **rigid** model



Equivalent model



Virtual Work:

$$\delta\mathcal{W} = \delta\theta_2^T \left(-J_p \frac{\ell}{2} \ddot{\theta}_2 - \frac{2GJ}{\ell} \theta_2 \right) = 0$$

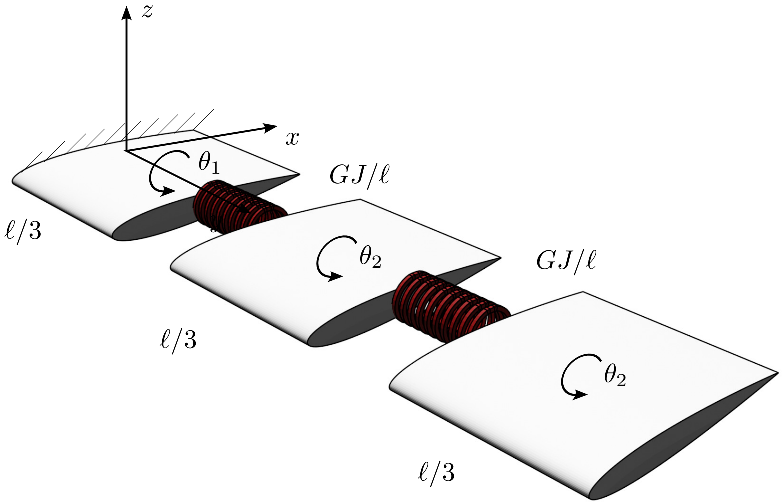
Equation of (free) motion:

$$J_p \frac{\ell}{2} \ddot{\theta}_2 + \frac{2GJ}{\ell} \theta_2 = 0$$

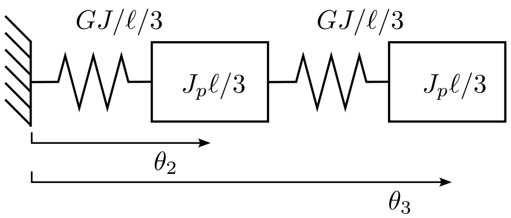
Natural frequency estimate:

$$\omega = 2\sqrt{\frac{GJ}{J_p \ell^2}} \tag{1}$$

Let's consider a simplified, **rigid** model



Equivalent model

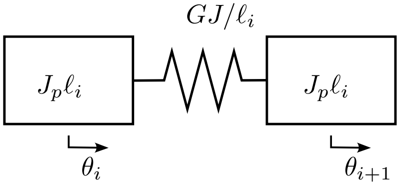


Equations of motion:

$$\frac{1}{3}J_p\ell\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+3\frac{GJ}{\ell}\begin{bmatrix}2 & -1\\ -1 & 1\end{bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix} \quad (2)$$

Let's be more **general**...

Generic **building block**

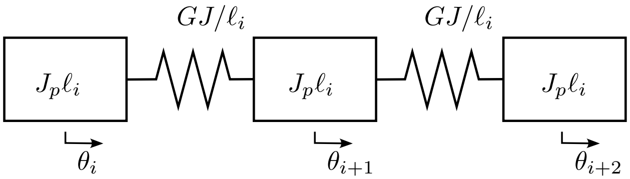


Virtual Work contributions:

$$\delta\mathcal{W}_{\text{ext}} = -\begin{Bmatrix}\delta\theta_i\\ \delta\theta_{i+1}\end{Bmatrix}^T J_p\ell_i \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_i\\ \ddot{\theta}_{i+1}\end{Bmatrix}$$

$$\delta\mathcal{W}_{\text{int}} = -\begin{Bmatrix}\delta\theta_i\\ \delta\theta_{i+1}\end{Bmatrix}^T \frac{GJ}{\ell_i}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_i\\ \theta_{i+1}\end{Bmatrix}$$

Assembly of 2 building blocks



Virtual Work contributions:

$$\delta\mathcal{W}_{\text{ext}} = -\begin{Bmatrix}\delta\theta_i\\ \delta\theta_{i+1}\\ \delta\theta_{i+2}\end{Bmatrix}^T J_p\ell_i \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_i\\ \ddot{\theta}_{i+1}\\ \ddot{\theta}_{i+2}\end{Bmatrix}$$

$$\delta\mathcal{W}_{\text{int}} = -\begin{Bmatrix}\delta\theta_i\\ \delta\theta_{i+1}\\ \delta\theta_{i+2}\end{Bmatrix}^T \frac{GJ}{\ell_i}\begin{bmatrix}1 & -1 & 0\\ -1 & 1{+}1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}\theta_i\\ \theta_{i+1}\\ \theta_{i+2}\end{Bmatrix}$$

The resulting matrices are:

$$\mathbf{M} = J_p \ell_i \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \dots & 1 & \end{bmatrix}$$

$$\mathbf{M} \in \mathcal{R}^{N \times N}$$

$$\mathbf{K} = \frac{GJ}{\ell_i} \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{K} \in \mathcal{R}^{N \times N}$$

The **clamp** constraint at the wing root can be added by **eliminating** the first section rotation θ_0 from the dofs, thus eliminating the first row and column of both $\mathbf{M}$ and $\mathbf{K}$.

$$\mathbf{M} = J_p \ell_i \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 \dots & 1 & \end{bmatrix}$$

$$\mathbf{M} \in \mathcal{R}^{(N-1) \times (N-1)}$$

$$\mathbf{K} = \frac{GJ}{\ell_i} \begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$$

$$\mathbf{K} \in \mathcal{R}^{(N-1) \times (N-1)}$$

The equations of (free) motion are

$$\mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\theta} = \mathbf{0}$$

the free motion solution is (undamped case)

$$\boldsymbol{\theta}_G(t) = \boldsymbol{\Theta}\mathrm{e}^{j\omega t}$$

therefore

$$\left(-\omega^2\mathbf{M} + \mathbf{K}\right)\boldsymbol{\Theta}\cancel{\mathrm{e}^{j\omega t}} = \mathbf{0}$$

which leads to the (generalized) eigenvalue problem

$$\lambda\mathbf{M}\boldsymbol{\Theta} = \mathbf{K}\boldsymbol{\Theta}$$

with $\lambda = -\omega^2$

- ω_n is the n-th **natural frequency**
- $\boldsymbol{\Theta}_n$ the n-th **mode shape**

Since the matrices are *symmetric*, the modal matrix

$$\boldsymbol{\psi} = \begin{bmatrix}\boldsymbol{\Theta}_1 & \boldsymbol{\Theta}_2 & \dots & \boldsymbol{\Theta}_n\end{bmatrix}$$

is *orthogonal*, and as such

$$\mathbf{m} = \boldsymbol{\psi}^T\mathbf{M}\boldsymbol{\psi}$$
$$\mathbf{k} = \boldsymbol{\psi}^T\mathbf{K}\boldsymbol{\psi}$$

are **diagonal** matrices.

When damping is present

$$\mathbf{M}\ddot{\boldsymbol{\theta}}\mathbf{C}\dot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\theta} = \mathbf{0}$$

If damping is **proportional** to **M** and **K**:

$$\mathbf{C} = \alpha\mathbf{M} + \beta\mathbf{K}$$

then,
the mode shapes remain the same of the undamped case,
and the eigenvalues are

$$\lambda = -\xi_i\omega_i \pm \sqrt{1 - \xi_i^2}\omega_i$$

with

$$\omega_i = \sqrt{k_i/m_i}$$

$$\xi_i = \frac{\alpha}{2\omega_i} + \frac{\beta\omega_i}{2}$$

when proportional damping is **not** a good model for the system's damping, then defining the state vector

$$\mathbf{x} = \begin{Bmatrix} \dot{\boldsymbol{\theta}} \\ \boldsymbol{\theta} \end{Bmatrix} \tag{3}$$

we can write the equations of motion in state-space form

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{4}$$

with

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{5}$$

substituting the general solution $\mathbf{x}(t) = \mathbf{X}\mathrm{e}^{\lambda t}$ the eigenproblem is recovered

$$(\lambda\mathbf{I} - \mathbf{A})\,\mathbf{X} = \mathbf{0}$$

- compare the numerical results of the lumped parameter model with the analytical results from continuum mechanics (see slide 3)
- plot the Frequency Response Function (FRF) to an harmonic torsion moment applied to the wing tip, using as output the rotation at the root (first dof)

$$\mathbf{M}\ddot{\boldsymbol{\theta}} + \mathbf{K}\boldsymbol{\theta} = \mathbf{T}_0 \mathrm{e}^{i\Omega t}$$

- plot the FRF between the first modal coordinate q_1 and the same torsion moment, after projecting the system's equation in the modal space, with

$$\boldsymbol{\psi}^T \mathbf{M}\boldsymbol{\psi}\ddot{\mathbf{q}} + \boldsymbol{\psi}^T \mathbf{K}\boldsymbol{\psi}\mathbf{q} = \boldsymbol{\psi}^T \mathbf{T}_0 \mathrm{e}^{i\Omega t}$$

- evaluate the system time response – you can use MATLAB's `lsim` – to a pseudo-random input generated from a Power Spectral Density that is non-zero in a bandwith narrower with respect to the sytem's bandwith

$$T(t) = \sum_{i=1}^{n} \sqrt{\frac{S(\Omega_n) d\Omega}{2}} \sin\left(\Omega_n t + \phi\right)$$

  with ϕ random and $\Omega_n \in \begin{bmatrix} 0 & \Omega_M \end{bmatrix}$, $\Omega_M < \max(\omega_n)$
- repeat the same analysis using only the first 2 vibration modes, i.e. considering the reduced system

$$\boldsymbol{\psi}_r^T \mathbf{M} \boldsymbol{\psi}_r \ddot{\mathbf{q}} + \boldsymbol{\psi}_r^T \mathbf{K} \boldsymbol{\psi}_r \mathbf{q} = \boldsymbol{\psi}_r^T \mathbf{T}_0 \mathrm{e}^{i\Omega t}$$

  with $\boldsymbol{\psi}_r = \begin{bmatrix} \boldsymbol{\Theta}_1 & \boldsymbol{\Theta}_2 \end{bmatrix}$