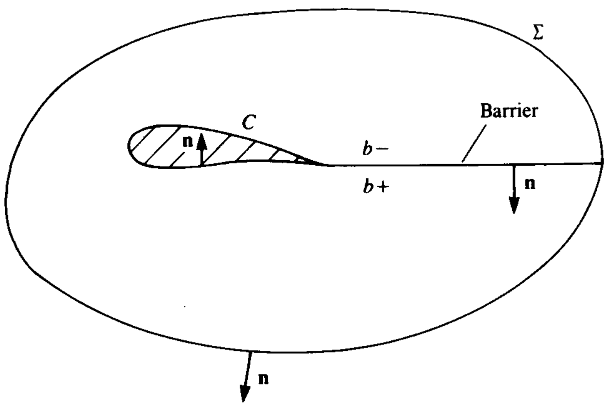

FIGURE 2.8
Doubly connected region exterior to an airfoil.

Insert a barrier b joining C and Σ and denote the two sides of the barrier as $b-$ and $b+$ as shown in the figure. Note that $\mathbf{n}$ is the outward normal to $b-$ and $-\mathbf{n}$ is the outward normal to $b+$. Equation (2.38) then becomes

$$\int_\sigma \nabla\Phi_D \cdot \nabla\Phi_D \, dV = \int_C \Phi_D \frac{\partial \Phi_D}{\partial n} dS + \int_\Sigma \Phi_D \frac{\partial \Phi_D}{\partial n} dS + \int_{b-} \Phi_D \frac{\partial \Phi_D}{\partial n} dS - \int_{b+} \Phi_D \frac{\partial \Phi_D}{\partial n} dS \quad (2.41)$$

The integral around C is zero from the boundary condition and if we let Σ go to infinity the integral around Σ is zero also. Let Φ_D^- be Φ_D on $b-$ and Φ_D^+ be Φ_D on $b+$. Then Eq. (2.41) is

$$\int_\sigma \nabla\Phi_D \cdot \nabla\Phi_D \, dV = \int_{b-} \Phi_D^- \frac{\partial \Phi_D^-}{\partial n} dS - \int_{b+} \Phi_D^+ \frac{\partial \Phi_D^+}{\partial n} dS \quad (2.42)$$

The normal derivative of Φ_D is continuous across the barrier and Eq. (2.42) can be written in terms of an integral over the barrier:

$$\int_\sigma \nabla\Phi_D \cdot \nabla\Phi_D \, dV = \int_{\text{barrier}} (\Phi_D^- - \Phi_D^+) \frac{\partial \Phi_D^-}{\partial n} dS \quad (2.43)$$

If we reintroduce the quantities Φ_1 and Φ_2 and rearrange the integrand we get

$$\int_\sigma \nabla\Phi_D \cdot \nabla\Phi_D \, dV = \int_{\text{barrier}} (\Phi_1^- - \Phi_1^+ + \Phi_2^+ - \Phi_2^-) \frac{\partial \Phi_D^-}{\partial n} dS \quad (2.44)$$

Note that the circulations associated with flows 1 and 2 are given by

$$\Gamma_1 = \Phi_1^+ - \Phi_1^-$$
$$\Gamma_2 = \Phi_2^+ - \Phi_2^-$$

and are constant, and finally

$$\int_\sigma \nabla\Phi_D \cdot \nabla\Phi_D \, dV = (\Gamma_2 - \Gamma_1) \int_{\text{barrier}} \frac{\partial \Phi_D^-}{\partial n} ds \quad (2.45)$$

Since in general we cannot require that the integral along the barrier be zero, the solution to the Neumann exterior problem is only uniquely determined to within a constant when $\Gamma_1 = \Gamma_2$ (when the circulation is specified as part of the problem statement). This result can be generalized for multiply connected regions in a similar manner. The value of the circulation cannot be specified on purely mathematical grounds but will be determined later on the basis of *physical* considerations.

2.9 VORTEX QUANTITIES

In conjunction with the velocity vector, we can define various quantities such as streamlines, stream tubes, and stream surfaces. Corresponding quantities can be defined for the vorticity vector that will prove to be useful later on in the modeling of lifting flows.

The field lines (e.g., in Fig. 2.2) that are parallel to the vorticity vector are called *vortex lines* and these lines are described by

$$\boldsymbol{\zeta} \times d\mathbf{l} = 0 \quad (2.46)$$

where $d\mathbf{l}$ is a segment along the vortex line (as shown in Fig. 2.9). In cartesian coordinates, this equation yields the differential equations for the vortex lines:

$$\frac{dx}{\zeta_x} = \frac{dy}{\zeta_y} = \frac{dz}{\zeta_z} \quad (2.47)$$

The vortex lines passing through an open curve in space form a vortex surface and the vortex lines passing through a closed curve in space form a vortex tube. A vortex filament is defined as a vortex tube of infinitesimal cross-sectional area.

The divergence of the vorticity is zero since the divergence of the curl of

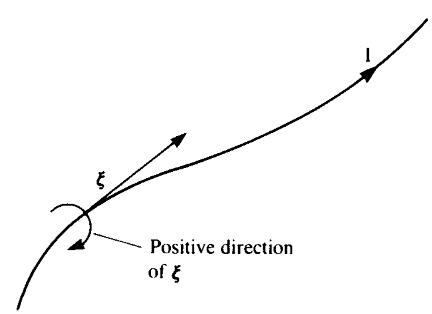

FIGURE 2.9
Vortex line.

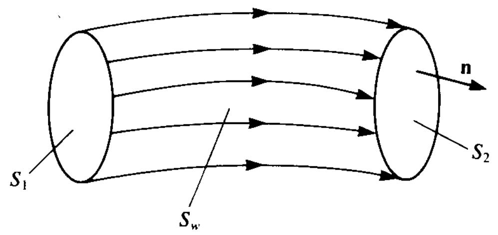

FIGURE 2.10
Vortex tube.

any vector is identically zero:

$$\nabla \cdot \boldsymbol{\zeta} = \nabla \cdot \nabla \times \mathbf{q} = 0 \tag{2.48}$$

Consider, at any instant, a region of space R enclosed by a surface S. An application of the divergence theorem yields

$$\int_S \boldsymbol{\zeta} \cdot \mathbf{n}\, dS = \int_R \nabla \cdot \boldsymbol{\zeta}\, dV = 0 \tag{2.49}$$

At some instant in time draw a vortex tube in the flow as shown in Fig. 2.10. Apply Eq. (2.49) to the region enclosed by the wall of the tube S_w and the surfaces S_1 and S_2 that cap the tube. Since on S_w the vorticity is parallel to the surface, the contribution of S_w vanishes and we are left with

$$\int_S \boldsymbol{\zeta} \cdot \mathbf{n}\, dS = \int_{S_1} \boldsymbol{\zeta} \cdot \mathbf{n}\, dS + \int_{S_2} \boldsymbol{\zeta} \cdot \mathbf{n}\, dS = 0 \tag{2.50}$$

Note that $\mathbf{n}$ is the outward normal and its direction is shown in the figure. If we denote $\mathbf{n}_v$ as being positive in the direction of the vorticity, then Eq. (2.50) becomes

$$\int_{S_1} \boldsymbol{\zeta} \cdot \mathbf{n}_v\, dS = \int_{S_2} \boldsymbol{\zeta} \cdot \mathbf{n}_v\, dS = \text{const.} \tag{2.51}$$

At each instant of time, the quantity in Eq. (2.51) is the same for any cross-sectional surface of the tube. Let C be any closed curve that surrounds the tube and lies on its wall. The circulation around C is given from Eq. (2.4) as

$$\Gamma_C = \int_S \boldsymbol{\zeta} \cdot \mathbf{n}_v\, dS = \text{const.} \tag{2.52}$$

and is seen to be constant along the tube. The results in Eqs. (2.51) and (2.52) express the spatial conservation of vorticity and are purely kinematical.

If Eq. (2.52) is applied to a vortex filament and $\mathbf{n}_v$ is chosen parallel to the vorticity vector, then

$$\Gamma_C = \zeta\, dS = \text{const.} \tag{2.53}$$

and the vorticity at any section of a vortex filament is seen to be inversely proportional to its cross-sectional area. A consequence of this result is that a

vortex filament cannot end in the fluid since zero area would lead to an infinite value for the vorticity. This limiting case, however, is useful for the purposes of modeling and so it is convenient to define a vortex filament with a fixed circulation, zero cross-sectional area, and infinite vorticity as a vortex filament with concentrated vorticity.

Based on results similar to those of Section 2.3 and this section, the German scientist Hermann von Helmholtz (1821–1894) developed his vortex theorems for inviscid flows, which can be summarized as:

1. The strength of a vortex filament is constant along its length.
2. A vortex filament cannot start or end in a fluid (it must form a closed path or extend to infinity).
3. The fluid that forms a vortex tube continues to form a vortex tube and the strength of the vortex tube remains constant as the tube moves about (hence vortex elements, such as vortex lines, vortex tubes, vortex surfaces, etc., will remain vortex elements with time).

The first theorem is based on Eq. (2.53), while the second theorem follows from this. The third theorem is actually a combination of Helmholtz's third and fourth theorems and is a consequence of the inviscid flow assumption (Eq. (2.9)).

2.10 TWO-DIMENSIONAL VORTEX

To illustrate a flowfield frequently called a two-dimensional vortex, consider a two-dimensional rigid cylinder of radius R rotating in a viscous fluid at a constant angular velocity of ω_y, as shown in Fig. 2.11a. This motion results in a flow with circular streamlines and therefore the radial velocity component is zero. Consequently the continuity equation (Eq. (1.35)) in the r–θ plane becomes

$$\frac{\partial q_\theta}{\partial \theta} = 0 \tag{2.54}$$

Integrating this equation results in

$$q_\theta = q_\theta(r) \tag{2.55}$$

The Navier–Stokes equation in the r direction (Eq. (1.36)), after neglecting the body force terms, becomes

$$-\rho \frac{q_\theta^2}{r} = -\frac{\partial p}{\partial r} \tag{2.56}$$

Since q_θ is a function of r only, and owing to the radial symmetry of the problem the pressure must be either a function of r or a constant. Therefore, its derivative will not appear in the momentum equation in the θ direction

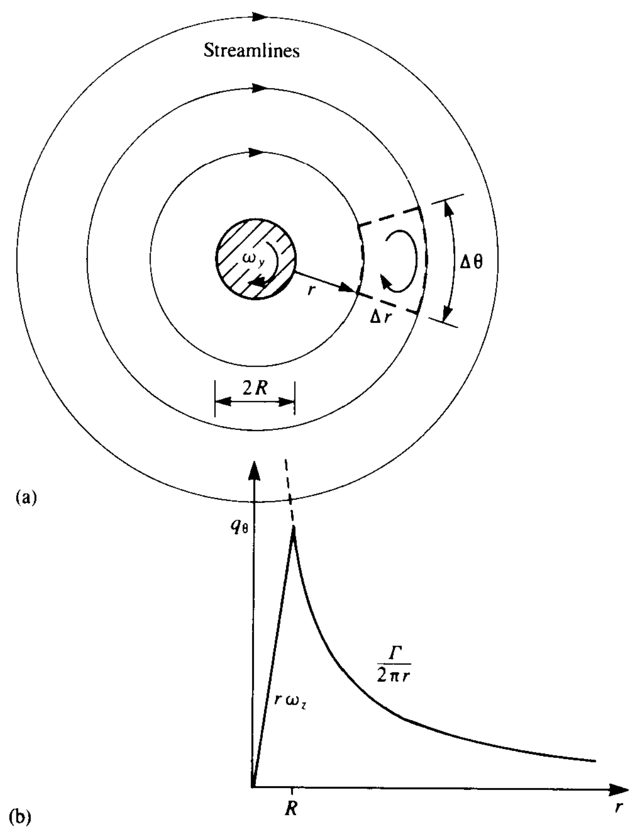

FIGURE 2.11
Two-dimensional flowfield around a cylindrical core rotating as a rigid body.

(Eq. 1.37),

$$0 = \mu\left(\frac{\partial^2 q_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r}\right) \tag{2.57}$$

and since q_θ is a function of r only,

$$0 = \frac{d^2 q_\theta}{dr^2} + \frac{d}{dr}\left(\frac{q_\theta}{r}\right) \tag{2.58}$$

Integrating with respect to r yields

$$\frac{dq_\theta}{dr} + \frac{q_\theta}{r} = C_1$$

where C_1 is the constant of integration. Rearranging this yields

$$\frac{1}{r}\frac{d}{dr}(rq_\theta) = C_1$$

and after an additional integration

$$q_\theta = \frac{C_1}{2}r + \frac{C_2}{r} \tag{2.59}$$

The boundary conditions are

$$q_\theta = -R\omega_y \qquad \text{at } r = R \tag{2.60a}$$

$$q_\theta = 0 \qquad \text{at } r = \infty \tag{2.60b}$$

The second boundary condition is satisfied only if $C_1 = 0$, and by using the first boundary condition, the velocity becomes

$$q_\theta = -\frac{R^2\omega_y}{r} \tag{2.61}$$

From the vortex filament results (Eq. (2.53)), the circulation has the same sign as the vorticity, and is therefore positive in the clockwise direction. The circulation around the circle of radius r, concentric with the cylinder, is found by using Eq. (2.3)

$$\Gamma = \int_{2\pi}^{0} q_\theta r\, d\theta = 2\omega_y \pi R^2 \tag{2.62}$$

and is constant. The tangential velocity can be rewritten as

$$q_\theta = -\frac{\Gamma}{2\pi r} \tag{2.63}$$

This velocity distribution is shown in Fig. 2.11b and is called vortex flow. If $r \to 0$ then the velocity becomes very large near the core, as shown by the dashed lines.

It has been demonstrated that Γ is the circulation generated by the rotating cylinder. However, to estimate the vorticity in the fluid, the integration line shown by the dashed lines in Fig. 2.11a is suggested. Integrating the velocity in a clockwise direction, and recalling that $q_r = 0$, results in

$$\oint \mathbf{q}\cdot d\mathbf{l} = 0\cdot\Delta r + \frac{\Gamma}{2\pi(r+\Delta r)}(r+\Delta r)\,\Delta\theta - 0\cdot\Delta r - \frac{\Gamma}{2\pi r}r\,\Delta\theta = 0$$

This indicates that this vortex flow is irrotational everywhere, except at the core where *all* the vorticity is generated. When the core size approaches zero $(R \to 0)$ then this flow is called an *irrotational vortex* (excluding the core point, where the velocity approaches infinity).

The three-dimensional velocity field induced by such an element is derived in the next section.

2.11 THE BIOT–SAVART LAW

At this point we have an incompressible fluid for which the continuity equation is

$$\nabla \cdot \mathbf{q} = 0 \tag{1.23}$$

and where vorticity ζ can exist and the problem is to determine the velocity field as a result of a known vorticity distribution. We may express the velocity field as the curl of a vector field $\mathbf{B}$, such that

$$\mathbf{q} = \nabla \times \mathbf{B} \tag{2.64}$$

Since the curl of a gradient vector is zero, $\mathbf{B}$ is indeterminate to within the gradient of a scalar function of position and time, and $\mathbf{B}$ can be selected such that

$$\nabla \cdot \mathbf{B} = 0 \tag{2.65}$$

The vorticity then becomes

$$\zeta = \nabla \times \mathbf{q} = \nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B}$$

By applying Eq. (2.65) this reduces to Poisson's equation for the vector potential $\mathbf{B}$:

$$\zeta = -\nabla^2 \mathbf{B} \tag{2.66}$$

The solution of this equation, using Green's theorem (see Karamcheti,[1.5] p. 533) is

$$\mathbf{B} = \frac{1}{4\pi} \int_V \frac{\zeta}{|r_0 - r_1|} dV$$

Here $\mathbf{B}$ is evaluated at point P (which is a distance $\mathbf{r}_0$ from the origin, shown in Fig. 2.12) and is a result of integrating the vorticity ζ (at point $\mathbf{r}_1$) within the volume V. The velocity field is then the curl of $\mathbf{B}$

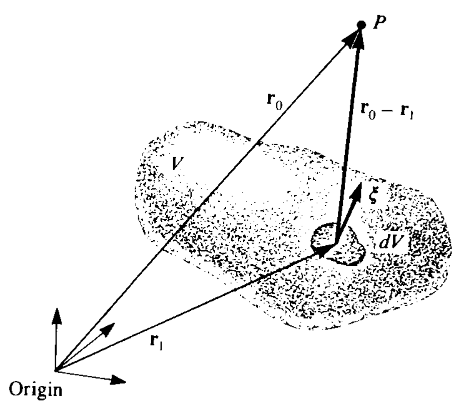

FIGURE 2.12
Velocity at point P due to a vortex distribution.

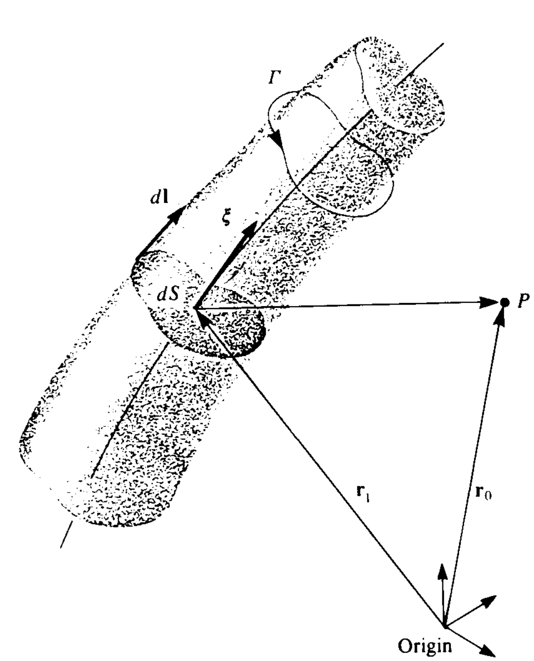

FIGURE 2.13
The velocity at point P induced by a vortex segment.

$$\mathbf{q} = \frac{1}{4\pi} \int_V \nabla \times \frac{\zeta}{|\mathbf{r}_0 - \mathbf{r}_1|} dV \tag{2.67}$$

Before proceeding with this integration, let us consider an infinitesimal piece of the vorticity filament ζ, as shown in Fig. 2.13. The cross section area dS is selected such that it is normal to ζ and the direction $d\mathbf{l}$ on the filament is

$$d\mathbf{l} = \frac{\zeta}{\zeta} dl$$

Also the circulation Γ is

$$\Gamma = \zeta \, dS$$

and

$$dV = dS \, dl$$

so that

$$\nabla \times \frac{\zeta}{|\mathbf{r}_0 - \mathbf{r}_1|} dV = \nabla \times \Gamma \frac{d\mathbf{l}}{|\mathbf{r}_0 - \mathbf{r}_1|}$$

and carrying out the curl operation while keeping $\mathbf{r}_1$ and $d\mathbf{l}$ fixed we get

$$\nabla \times \Gamma \frac{d\mathbf{l}}{|\mathbf{r}_0 - \mathbf{r}_1|} = \Gamma \frac{d\mathbf{l} \times (\mathbf{r}_0 - \mathbf{r}_1)}{|\mathbf{r}_0 - \mathbf{r}_1|^3}$$

Substitution of this result back into Eq. (2.67) results in the Biot–Savart law, which states

$$\mathbf{q} = \frac{\Gamma}{4\pi} \int \frac{d\mathbf{l} \times (\mathbf{r}_0 - \mathbf{r}_1)}{|\mathbf{r}_0 - \mathbf{r}_1|^3} \tag{2.68}$$

or in differential form

$$\Delta\mathbf{q} = \frac{\Gamma}{4\pi} \frac{d\mathbf{l} \times (\mathbf{r}_0 - \mathbf{r}_1)}{|\mathbf{r}_0 - \mathbf{r}_1|^3} \tag{2.68a}$$

A similar manipulation of Eq. (2.67) leads to the following result for the velocity due to a volume distribution of vorticity:

$$\mathbf{q} = \frac{1}{4\pi} \int_V \frac{\boldsymbol{\zeta} \times (\mathbf{r}_0 - \mathbf{r}_1)}{|\mathbf{r}_0 - \mathbf{r}_1|^3} dV \tag{2.67a}$$

2.12 THE VELOCITY INDUCED BY A STRAIGHT VORTEX SEGMENT

In this section, the velocity induced by a straight vortex line segment is derived, based on the Biot–Savart law. It is clear that a vortex line cannot start or end in a fluid, and the following discussion is aimed at developing the contribution of a segment that is a section of a continuous vortex line. The vortex segment is placed at an arbitrary orientation in the (x, y, z) frame with constant circulation Γ, as shown in Fig. 2.14. The velocity induced by this

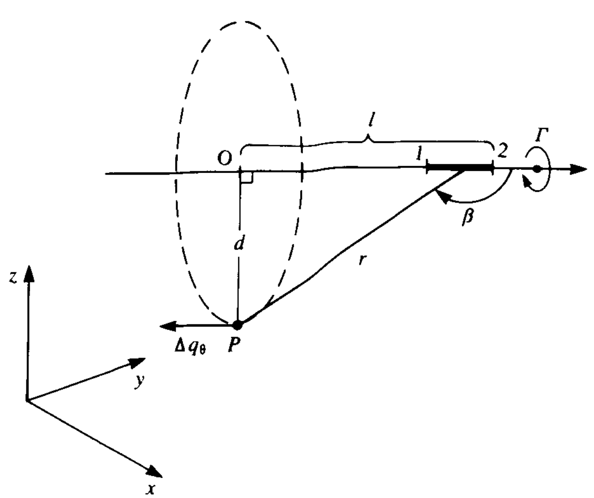

FIGURE 2.14
Velocity induced by a straight vortex segment.

vortex segment will have tangential components only, as indicated in the figure. Also, the difference $\mathbf{r}_0 - \mathbf{r}_1$ between the vortex segment and the point P is $\mathbf{r}$. According to the Biot–Savart law (Eq. 2.68*a*) the velocity induced by a segment $d\mathbf{l}$ on this line, at a point P, is

$$\Delta\mathbf{q} = \frac{\Gamma}{4\pi} \frac{d\mathbf{l} \times \mathbf{r}}{r^3} \tag{2.68b}$$

This may be rewritten in scalar form

$$\Delta q_\theta = \frac{\Gamma}{4\pi} \frac{\sin\beta}{r^2} dl \tag{2.68c}$$

From the figure it is clear that

$$d = r\cos\beta$$

$$l = d\tan\beta \qquad \text{and} \qquad dl = \frac{d}{\cos^2\beta} d\beta$$

Substituting these into Δq_θ

$$\Delta q_\theta = \frac{\Gamma}{4\pi} \frac{\cos^2\beta}{d^2} \sin\beta \frac{d}{\cos^2\beta} d\beta = \frac{\Gamma}{4\pi d} \sin\beta\, d\beta$$

This equation can be integrated over a section $(1 \rightarrow 2)$ of the straight vortex segment of Fig. 2.15

$$(q_\theta)_{1,2} = \frac{\Gamma}{4\pi d} \int_{\beta_1}^{\beta_2} \sin\beta\, d\beta = \frac{\Gamma}{4\pi d} (\cos\beta_1 - \cos\beta_2) \tag{2.69}$$

The results of this equation are shown schematically in Fig. 2.15. Thus, the velocity induced by a straight vortex segment is a function of its strength Γ, the distance d, and the two view angles β_1, β_2.

For the two-dimensional case (infinite vortex length) $\beta_1 = 0$, $\beta_2 = \pi$ and

$$q_\theta = \frac{\Gamma}{4\pi d} \int_0^\pi \sin\beta\, d\beta = \frac{\Gamma}{2\pi d} \tag{2.70}$$

For the semi-infinite vortex line that starts at point O in Fig. 2.14, $\beta_1 = \pi/2$ and $\beta_2 = \pi$ and the induced velocity is

$$q_\theta = \frac{\Gamma}{4\pi d} \tag{2.71}$$

which is exactly half of the previous value.

Equation (2.68*b*) can be modified to a form that is more convenient for numerical computations by using the definitions of Fig. 2.16. For the general three-dimensional case the two edges of the vortex segment will be located by $\mathbf{r}_1$ and $\mathbf{r}_2$ and the vector connecting the edges is

$$\mathbf{r}_0 = \mathbf{r}_2 - \mathbf{r}_1$$

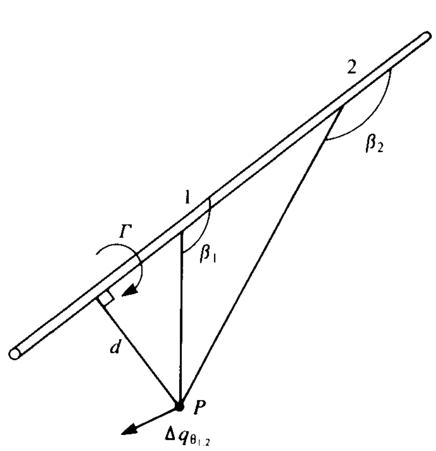

FIGURE 2.15
Definition of the view angles used for the vortex-induced velocity calculations.

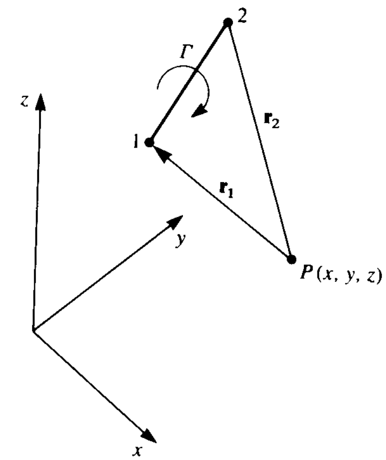

FIGURE 2.16
Nomenclature used for the velocity induced by a three-dimensional, straight vortex segment.

as shown in Fig. 2.16. The distance d, and the cosines of the angles β are then

$$d = \frac{|\mathbf{r}_1 \times \mathbf{r}_2|}{|\mathbf{r}_0|}$$

$$\cos\beta_1 = \frac{\mathbf{r}_0 \cdot \mathbf{r}_1}{|\mathbf{r}_0|\,|\mathbf{r}_1|}$$

$$\cos\beta_2 = \frac{\mathbf{r}_0 \cdot \mathbf{r}_2}{|\mathbf{r}_0|\,|\mathbf{r}_2|}$$

The direction of the velocity $\mathbf{q}_{1,2}$ is normal to the plane created by the point P and the vortex edges 1, 2 and is given by

$$\frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}$$

and by substituting these quantities, and by multiplying with this directional vector the induced velocity is

$$\mathbf{q}_{1,2} = \frac{\Gamma}{4\pi} \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|^2} \mathbf{r}_0 \cdot \left(\frac{\mathbf{r}_1}{r_1} - \frac{\mathbf{r}_2}{r_2}\right) \tag{2.72}$$

A more detailed procedure for using this formula when the (x, y, z) values of the points 1, 2, and P are known is provided in Section 10.4.5.

2.13 THE STREAM FUNCTION

Consider two arbitrary streamlines in a two-dimensional steady flow, as shown in Fig. 2.17. The velocity $\mathbf{q}$ along these lines $\mathbf{l}$ is tangent to them

$$\mathbf{q} \times d\mathbf{l} = u\,dz - w\,dx = 0 \tag{1.5}$$

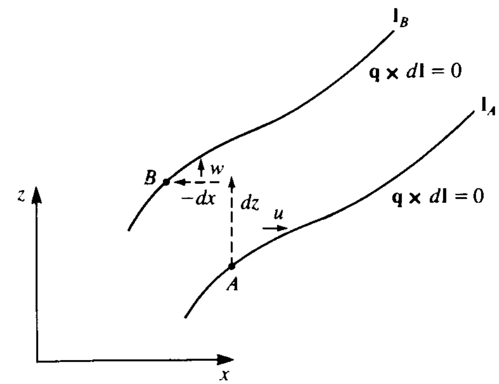

FIGURE 2.17
Flow between two two-dimensional streamlines.

and, therefore, the flux (volumetric flow rate) between two such lines is constant. This flow rate between these two curves is

$$\text{Flux} = \int_A^B \mathbf{q} \cdot \mathbf{n}\,dl = \int_A^B u\,dz + w(-dx) \tag{2.73}$$

where A and B are two arbitrary points on these lines. If a scalar function $\Psi(x, z)$ for this flux is to be introduced, such that its variation along a streamline will be zero (according to Eq. (1.5)), then based on these two equations (Eqs. (1.5) and (2.73)), its relation to the velocity is

$$u = \frac{\partial \Psi}{\partial z} \qquad w = -\frac{\partial \Psi}{\partial x} \tag{2.74}$$

Substituting this into Eq. (1.5) for the streamline results in

$$d\Psi = \frac{\partial \Psi}{\partial x} dx + \frac{\partial \Psi}{\partial z} dz = -w\,dx + u\,dz = 0 \tag{2.75}$$

Therefore, $d\Psi$ along a streamline is zero, and between two different streamlines $d\Psi$ represents the volume flux (Eq. (2.73)). Integration of this equation results in

$$\Psi = \text{const.} \qquad \text{on streamlines} \tag{2.76}$$

Substituting Eqs. (2.74) into the continuity equation yields

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \frac{\partial^2 \Psi}{\partial x\,\partial z} - \frac{\partial^2 \Psi}{\partial x\,\partial z} = 0 \tag{2.77}$$

and therefore the continuity equation is automatically satisfied. Note that the stream function is valid for viscous flow, too, and if the irrotational flow requirement is added then $\zeta_y = 0$. Recall that the y component of the vorticity is

$$\zeta_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \nabla^2 \Psi$$