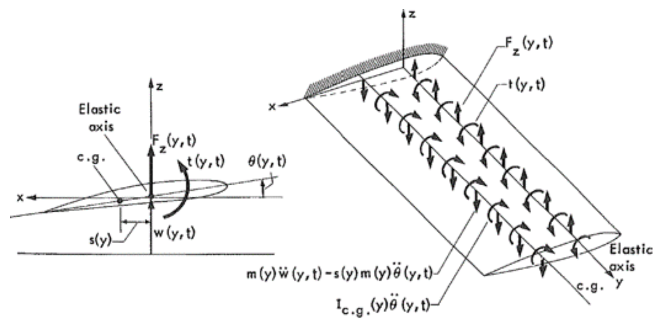

Figure 2.2: Airplane straight wing for coupled bending torsion problem.

2.3 Setting up of the coupled bending and torsion problem for a wing

Write the coupled equations for bending and torsion of the wing shown in figure ??. Use a Ritz-Galerkin approach to build the approximation and compute the mass and stiffness matrix of the coupled system. As modal shapes use the uncoupled bending and torsional proper orthogonal modes. The two PDEs that have to be considered are the following

$$(EJ_2 w'')'' = p_3 \tag{2.20}$$
$$(GJ\theta')' = -m_t \tag{2.21}$$

Consider the case where the wing section at x moves in the transversal direction with an acceleration $\ddot{w}(x)$. In this case the distributed loads on the wings sections will be

$$p_3 = -m\ddot{w} \tag{2.22}$$
$$m_t = sm\ddot{w} \tag{2.23}$$

Considering now the case where the wing section at x moves in twist with an acceleration $\ddot{\theta}(x)$. In this case the distributed loads on the wings sections will be

$$p_3 = sm\ddot{\theta} \tag{2.24}$$
$$m_t = -I_\theta\ddot{\theta} \tag{2.25}$$

with $I_\theta = I_0 + ms^2$. Considering the superimposition of the two effects the equation of motion to be discretized will read

$$(EJ_2 w'')'' + m\ddot{w} - sm\ddot{\theta} = 0 \tag{2.26}$$
$$(GJ\theta')' - I_\theta\ddot{\theta} + sm\ddot{w} = 0 \tag{2.27}$$

Use the following Ritz-Galekin approximations for the two unknown displacement fields

$$\hat{w}(x,t) = \sum_{i=1}^{n_w} N_{wi}(x) q_w(t) = \mathbf{N}_w \mathbf{q}_w \tag{2.28}$$

$$\hat{\theta}(x,t) = \sum_{i=1}^{n_t} N_{ti}(x) q_t(t) = \mathbf{N}_t \mathbf{q}_t \tag{2.29}$$

The PVW writing all bending and torsion elements for internal and external work is

$$\int_0^L \delta w''^T E J_2 w'' \,\mathrm{d}x_1 + \int_0^L \delta\theta'^T G J \theta' \,\mathrm{d}x_1 + \int_0^L \delta w^T \left(m\ddot{w} - sm\ddot{\theta}\right) \mathrm{d}x_1 + \int_0^L \delta\theta^T \left(ms\ddot{w} - I_\theta \ddot{\theta}\right) \mathrm{d}x_1 = 0 \tag{2.30}$$

Using the Ritz-Galerkin approximations chosen it results

$$\begin{aligned} &\delta\mathbf{q}_w \int_0^L \mathbf{N}_w''^T E J_2 \mathbf{N}_w'' \,\mathrm{d}x_1 \, \mathbf{q}_w + \delta\mathbf{q}_t \int_0^L \mathbf{N}_t'^T G J \mathbf{N}_t' \,\mathrm{d}x_1 \, \mathbf{q}_t + \\ &\delta\mathbf{q}_w \int_0^L \mathbf{N}_w{}^T m \mathbf{N}_w \,\mathrm{d}x_1 \, \ddot{\mathbf{q}}_w - \delta\mathbf{q}_w \int_0^L \mathbf{N}_w{}^T sm \mathbf{N}_t \,\mathrm{d}x_1 \, \ddot{\mathbf{q}}_t + \\ &\delta\mathbf{q}_t \int_0^L \mathbf{N}_t{}^T ms \mathbf{N}_w \,\mathrm{d}x_1 \, \ddot{\mathbf{q}}_w - \delta\mathbf{q}_t \int_0^L \mathbf{N}_t{}^T I_\theta \mathbf{N}_t \,\mathrm{d}x_1 \, \ddot{\mathbf{q}}_t = 0 \end{aligned} \tag{2.31}$$

Considering the arbitrariness of $\delta\mathbf{q}_w, \delta\mathbf{q}_t$ the following matrix system will result

$$\begin{bmatrix} \mathbf{M}_{ww} & \mathbf{M}_{wt} \\ \mathbf{M}_{tw} & \mathbf{M}_{tt} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}}_w \\ \ddot{\mathbf{q}}_t \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ww} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_{tt} \end{bmatrix} \begin{Bmatrix} \mathbf{q}_w \\ \mathbf{q}_t \end{Bmatrix} = \mathbf{0} \tag{2.32}$$

where the meaning of the different matrices could be obtained by comparing this last equation with (2.31). In particular it is possible to verify that $\mathbf{M}_{tw} = \mathbf{M}_{wt}^T$ and so that the mass matrix is symmetric. The stiffness matrix is block-diagonal, so the coupling between bending and torsion, as expected, is due only to inertial forces. In addition, considering the fact that proper orthogonal modes are chosen as base function it is possible to verify that

$$\mathbf{M}_{ww} = \int_0^L \mathbf{N}_w{}^T m \mathbf{N}_w \,\mathrm{d}x_1 = \begin{bmatrix} \ddots & & \\ & \mu_{wi} & \\ & & \ddots \end{bmatrix}, \tag{2.33}$$

$$\mathbf{M}_{tt} = \int_0^L \mathbf{N}_t{}^T I_\theta \mathbf{N}_t \,\mathrm{d}x_1 = \begin{bmatrix} \ddots & & \\ & \mu_{ti} & \\ & & \ddots \end{bmatrix}, \tag{2.34}$$

$$\mathbf{K}_{ww} = \int_0^L \mathbf{N}_w''^T EJ_2 \mathbf{N}_w'' \,\mathrm{d}x_1 = \begin{bmatrix} \ddots & & \\ & \mu_{wi}\omega_{wi}^2 & \\ & & \ddots \end{bmatrix}, \tag{2.35}$$

$$\mathbf{K}_{tt} = \int_0^L \mathbf{N}_t'^T GJ \mathbf{N}_t' \,\mathrm{d}x_1 = \begin{bmatrix} \ddots & & \\ & \mu_{ti}\omega_{ti}^2 & \\ & & \ddots \end{bmatrix}, \tag{2.36}$$

Nothing instead could be said about the extra-diagonal mass matrices elements beccause in general bending and torsional models are not expected to be orthogonal.