POLITECNICO MILANO 1863

055738 – STRUCTURAL DYNAMICS AND AEROELASTICITY

07 Structural Dynamics: Modal Analysis

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Material

Preumont: Chapter 3 (up to section 2.2) and section 5.5

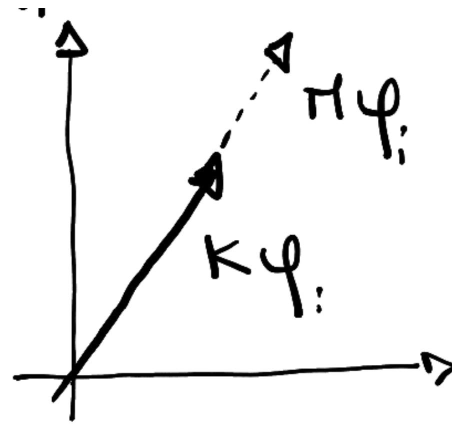
Masarati: Sections 5.3 and D.1

Eigenvalues and eigenvectors

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \quad \mathbf{q} = \boldsymbol{\varphi}_i e^{j\omega_i t}$$

$$\left(-\mathbf{M}\omega_i^2 + \mathbf{K}\right)\boldsymbol{\varphi}_i = \mathbf{0}$$

$$\begin{aligned} \mathbf{K}\boldsymbol{\varphi}_i &= \omega_i^2 \mathbf{M}\boldsymbol{\varphi}_i \\ \mathbf{K}\boldsymbol{\Phi} &= \boldsymbol{\lambda}\mathbf{M}\boldsymbol{\Phi} \end{aligned}$$



$$\boldsymbol{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_N]$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \ddots & & \\ & \omega_i^2 & \\ & & \ddots \end{bmatrix}$$

The eigenmode is a vector that multiplied by K is equal to itself multiplied by matrix M and a constant $\lambda = \omega_i^2$

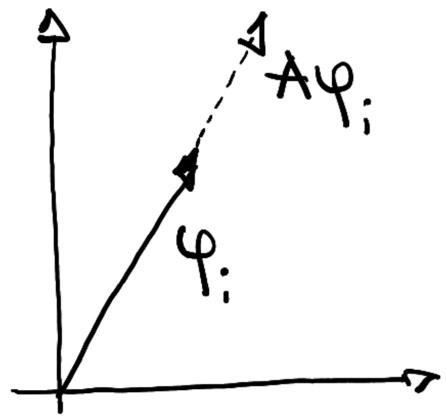
The eigenvalue is the value that allows to scale $M\phi$ to $K\phi$

Eigenvalues and eigenvectors (state-space approach)

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}, \quad \mathbf{q} = \boldsymbol{\varphi}\mathrm{e}^{\lambda t}$$

$$\left(\mathbf{A} - \lambda\mathbf{I}\right)\boldsymbol{\varphi} = \mathbf{0}$$

$$\mathbf{A}\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$$



$$\boldsymbol{\Phi} = \left[\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1, \ldots, \boldsymbol{\varphi}_N\right]$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \ddots & & \\ & \lambda_i & \\ & & \ddots \end{bmatrix}$$

$$\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda}, \qquad \boldsymbol{\Lambda} = \boldsymbol{\Phi}^{-1}\mathbf{A}\boldsymbol{\Phi}$$

The eigenmode is a vector that multiplied by A is equal to itself multiplied by a constant λ

Normalization of eigenvectors or modal shapes

Modal shapes are defined unless for a constant. Several normalization are often used

1) Maximum displacement $\max(\phi_i) = 1.0$
2) Modal mass $\phi_i^T M \phi_i = \mu_i = 1$

Eigenmodes to decouple

$$\begin{aligned}\omega_i^2 \boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i &= \boldsymbol{\varphi}_j^T \mathbf{K} \boldsymbol{\varphi}_i \\ \omega_j^2 \boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_j &= \boldsymbol{\varphi}_i^T \mathbf{K} \boldsymbol{\varphi}_j\end{aligned}$$

$$\left(\omega_i^2 - \omega_j^2\right) \boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i = 0$$

Since $\boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i$ is a scalar so is equal to its traspose. Same for $\boldsymbol{\varphi}_j^T \mathbf{K} \boldsymbol{\varphi}_i$

$$\omega_j^2 \boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i = \boldsymbol{\varphi}_j^T \mathbf{K} \boldsymbol{\varphi}_i$$

1. Orthogonality with respect to mass matrix

$$\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_j = 0 \quad \text{if } i \neq j$$

2. Orthogonality with respect to stiffness matrix

$$\boldsymbol{\varphi}_i^T \mathbf{K} \boldsymbol{\varphi}_j = 0 \quad \text{if } i \neq j$$

$$2\boldsymbol{\varphi}_j^T \mathbf{K} \boldsymbol{\varphi}_i = \left(\omega_i^2 + \omega_j^2\right) \boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i$$

WARNING!! In general

$$\boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_j \neq 0 \, \forall i, j$$

Eigenmodes to decouple

$$\mathbf{q} = \mathbf{\Phi z} \quad \text{with} \mathbf{\Phi} = [\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_N]$$

$$\mathbf{M\Phi\ddot{z}} + \mathbf{K\Phi z} = \mathbf{0} \rightarrow \mathbf{\Phi}^T \left(\mathbf{M\Phi\ddot{z}} + \mathbf{K\Phi z}\right) = \mathbf{0}$$

$$\mathbf{\Phi}^T\mathbf{M\Phi} = \begin{bmatrix} \ddots & & \\ & \mu_i & \\ & & \ddots \end{bmatrix}, \; \mathbf{\Phi}^T\mathbf{K\Phi} = \begin{bmatrix} \ddots & & \\ & \mu_i\omega_i^2 & \\ & & \ddots \end{bmatrix}$$

$$\mu_i \left(\ddot{z}_i + \omega_i^2 z_i\right) = 0 \; \forall i$$

The N-dimensional second order differential system is decomposed in N independent scalar linear second order differential equations.

Transformation to first order state-space form

Define a new state variable as the first derivative of states

$$\dot{\mathbf{q}} = \mathbf{q}_d$$

The equation becomes

$$\mathbf{M}\dot{\mathbf{q}}_d + \mathbf{K}\mathbf{q} = \mathbf{0}$$

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{q}}_d \end{Bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{K} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{q} \\ \mathbf{q}_d \end{Bmatrix}$$

$$\check{\mathbf{M}}\dot{\check{\mathbf{q}}} = \check{\mathbf{K}}\check{\mathbf{q}}$$

if $\check{\mathbf{M}} > 0$ i.e., is positive definite, then

$$\dot{\check{\mathbf{q}}} = \check{\mathbf{M}}^{-1}\check{\mathbf{K}}\check{\mathbf{q}}, \rightarrow \ \dot{\check{\mathbf{q}}} = \check{\mathbf{A}}\check{\mathbf{q}}$$

Eigenmodes to decouple (state space approach)

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q}$$

$$\begin{aligned} \boldsymbol{\varphi}_j^T \mathbf{A} \boldsymbol{\varphi}_i &= \lambda_i \boldsymbol{\varphi}_j^T \boldsymbol{\varphi}_i \\ \boldsymbol{\varphi}_i^T \mathbf{A} \boldsymbol{\varphi}_j &= \lambda_j \boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_j \end{aligned} \qquad (\lambda_i - \lambda_j)\, \boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_j = 0$$

Consequently,

$$\boldsymbol{\varphi}_i^T \boldsymbol{\varphi}_j = 0 \quad \text{if } i \neq j \;\rightarrow\; \boldsymbol{\Phi}^T \boldsymbol{\Phi} = \mathbf{I} \;\Rightarrow\; \boldsymbol{\Phi}^T = \boldsymbol{\Phi}^{-1}$$

The matrix of eigenvectors is an orthogonal matrix

$$\mathbf{A}\boldsymbol{\Phi} = \boldsymbol{\Phi}\boldsymbol{\Lambda} \;\Rightarrow\; \boldsymbol{\Lambda} = \boldsymbol{\Phi}^T \mathbf{A} \boldsymbol{\Phi}$$

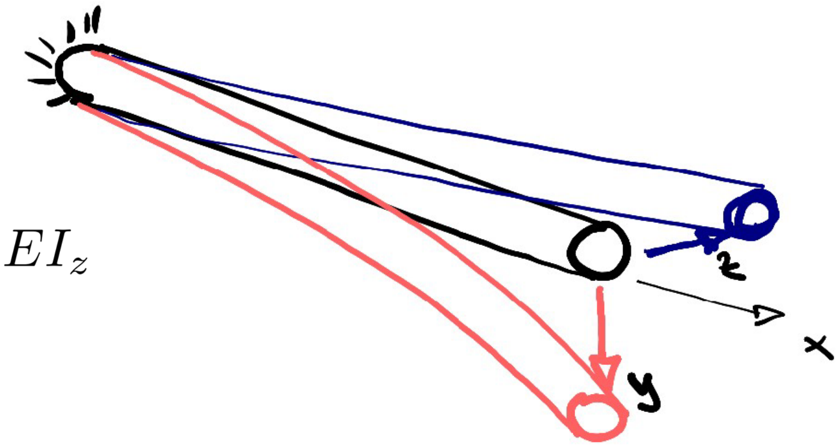
$$\text{If } \mathbf{q} = \boldsymbol{\Phi}\mathbf{z} \;\Rightarrow\; \dot{\mathbf{z}} = \boldsymbol{\Phi}^T \mathbf{A} \boldsymbol{\Phi} \mathbf{z} \rightarrow \dot{z}_i = \lambda z_i \,\forall\, i$$

The N-dimensional differential system is decomposed in N idependent scalar linear differential equations.

Coincident eigenvalues

They typically appear when there are symmetries in the structures under analysis

$$EI_y = EI_z$$



$$E_v = \{\mathbf{v}_i : (\mathbf{A} - \lambda_i \mathbf{I})\,\mathbf{v}_i = 0\}$$

Algebraic multiplicity: the number of times an eigenvalue is repeted

Geometric multiplicity: is the dimension of the eigenspace E_v associated with an eigenvalue λ_i

Coincident eigenvalues

If the algebraic multiplicity is = geometric multiplicity the matrix is still diagonalizable because all eigenvectors associated to the same eigenvalue are linearly independent

$$\mathbf{\Lambda} = \mathbf{\Phi}^T \mathbf{A} \mathbf{\Phi}$$

or

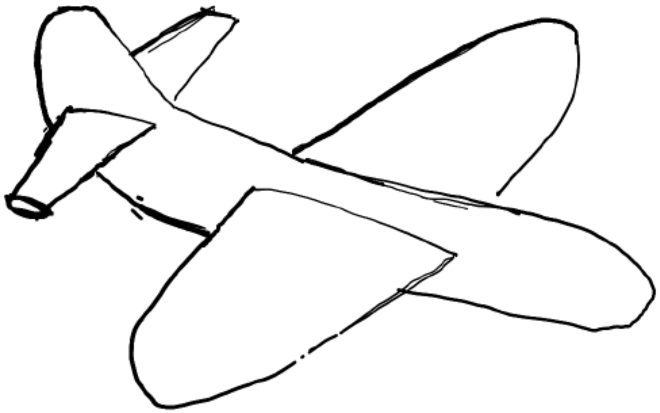
$$\mathbf{\Lambda} \mathbf{\Phi}^T \mathbf{K} \mathbf{\Phi} = \mathbf{\Phi}^T \mathbf{M} \mathbf{\Phi}$$

Rigid body modes

Rigid modes

$$\omega_i = 0$$

$$\omega_i^2 \mathbf{M}\boldsymbol{\varphi}_i = \mathbf{K}\boldsymbol{\varphi}_i \;\Rightarrow\; \mathbf{K}\boldsymbol{\varphi}_i = \mathbf{0}$$



It follows that the strain energy is $\boldsymbol{\varphi}_i^T \mathbf{K}\boldsymbol{\varphi}_i = 0$ that means that NO DEFORMATION is generated.

For a free flying aircraft 3 rigid translation and 3 rigid rotations are possible, so 6 rigid body modes with null frequency

$\mathbf{K}$ matrix is six times singular for a free flying aircraft

The coincidence between algebraic and geometric multiplicity is ensured

$$\left(\omega_i^2 - \omega_j^2\right) \boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_j = 0$$

$$E_\varphi = \{\boldsymbol{\varphi}_i : \mathbf{K}\boldsymbol{\varphi}_i = \mathbf{0}\} \equiv \mathbb{R}^6$$

$\omega_i = 0$ for $i = 1, \dots, 6$ so

$$\Rightarrow \boldsymbol{\varphi}_i^T \mathbf{M}\boldsymbol{\varphi}_j \neq 0 \quad \text{when } i, j = 1, \dots, 6$$

Rigid body modes

If the initially computed rigid body modes are not linearly independent, it is possible to compute those who diagonalize the matrices through the GRAM-SCHMIDT orthotgonalization

If $\boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i \neq 0$ define a modified j mode shape

$$\tilde{\boldsymbol{\varphi}}_j = \boldsymbol{\varphi}_j - \alpha \boldsymbol{\varphi}_i$$

so that

$$\begin{aligned} \tilde{\boldsymbol{\varphi}}_j^T \mathbf{M} \boldsymbol{\varphi}_i &= 0 \\ \boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i - \alpha \boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i &= 0 \end{aligned}$$

This allows to compute α as

$$\alpha = \frac{\boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i} \Rightarrow \tilde{\boldsymbol{\varphi}}_j = \boldsymbol{\varphi}_j - \frac{\boldsymbol{\varphi}_j^T \mathbf{M} \boldsymbol{\varphi}_i}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i} \boldsymbol{\varphi}_i$$

Computation of the solution to initial conditions

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \quad \mathbf{q} \in \mathbb{R}^n, \quad \mathbf{q}(0) = \underline{q},\ \dot{\mathbf{q}}(0) = \underline{\dot{q}}$$

The general solution is obtained as superimposition of $\boldsymbol{\varphi}_i \mathrm{e}^{\mathrm{j}\omega_i t}$ and $\boldsymbol{\varphi}_i \mathrm{e}^{-\mathrm{j}\omega_i t}$ for all i

$$\mathbf{q} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i \left(A_i \cos\omega_i t + B_i \sin\omega_i t \right)$$

It is easy to verify that

$$\begin{aligned} \boldsymbol{\varphi}_k^T \mathbf{M}\mathbf{q} &= \mu_k \left(A_k \cos\omega_k t + B_k \sin\omega_k t \right) \\ \boldsymbol{\varphi}_k^T \mathbf{M}\dot{\mathbf{q}} &= \omega_k \mu_k \left(-A_k \sin\omega_k t + B_k \cos\omega_k t \right) \end{aligned}$$

Using $t = 0$

$$\begin{aligned} A_k &= \frac{\boldsymbol{\varphi}_k^T \mathbf{M}\underline{\mathbf{q}}}{\mu_k}, \\ B_k &= \frac{\boldsymbol{\varphi}_k^T \mathbf{M}\underline{\dot{\mathbf{q}}}}{\omega_k \mu_k} \end{aligned}$$

$$\mathbf{q} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i \left(\frac{\boldsymbol{\varphi}_k^T \mathbf{M}\underline{\mathbf{q}}}{\mu_k} \cos\omega_i t + \frac{\boldsymbol{\varphi}_k^T \mathbf{M}\underline{\dot{\mathbf{q}}}}{\omega_k \mu_k} \sin\omega_i t \right)$$

Computation of the solution to initial conditions

$$\mathbf{q} = \sum_{i=1}^{n} \boldsymbol{\varphi}_i \left(\frac{\boldsymbol{\varphi}_k^T \mathbf{M} \underline{\underline{\mathbf{q}}}}{\mu_k} \cos \omega_i t + \frac{\boldsymbol{\varphi}_k^T \mathbf{M} \underline{\underline{\dot{\mathbf{q}}}}}{\omega_k \mu_k} \sin \omega_i t \right)$$

If there are rigid modes for which $\omega_i = 0$ it results

$$\cos \omega_i t \to 1$$
$$\frac{\sin \omega_i t}{\omega_i} \to t$$

$$\mathbf{q} = \sum_{k=1}^{r} \boldsymbol{\varphi}_k \left(A_k + B_k t \right) + \sum_{i=r+1}^{n} \boldsymbol{\varphi}_i \left(\frac{\boldsymbol{\varphi}_k^T \mathbf{M} \underline{\underline{\mathbf{q}}}}{\mu_k} \cos \omega_i t + \frac{\boldsymbol{\varphi}_k^T \mathbf{M} \underline{\underline{\dot{\mathbf{q}}}}}{\omega_k \mu_k} \sin \omega_i t \right)$$

Polynomial
response for rigid
body modes

Rayleigh quotient

Consider a generic mode shape $\mathbf{u}_i$ compatible with kinematic BC

$$\omega_i^2 = \frac{\int EJ_2 (w_i'')^2 \mathrm{d}x_1}{\int m w_i^2 \mathrm{d}x_1} \rightarrow \omega_i^2 = \frac{\boldsymbol{\varphi}_i^T \mathbf{K} \boldsymbol{\varphi}_i}{\boldsymbol{\varphi}_i^T \mathbf{M} \boldsymbol{\varphi}_i}$$

$$R(\mathbf{u}_i) = \frac{\mathbf{u}_i^T \mathbf{K} \mathbf{u}_i}{\mathbf{u}_i^T \mathbf{M} \mathbf{u}_i} \rightarrow \mathbf{u}_i = \sum_{k=1}^{\infty} \alpha_k \boldsymbol{\varphi}_k = \boldsymbol{\Phi} \boldsymbol{\alpha}$$

$$R(\mathbf{u}_i) = \frac{\boldsymbol{\alpha}^T \boldsymbol{\Phi}^T \mathbf{K} \boldsymbol{\Phi} \boldsymbol{\alpha}}{\boldsymbol{\alpha}^T \boldsymbol{\Phi}^T \mathbf{M} \boldsymbol{\Phi} \boldsymbol{\alpha}} = \frac{\sum_{k=1}^{n} \alpha_k^2 \omega_k^2 \mu_k}{\sum_{k=1}^{n} \alpha_k^2 \mu_k}$$

if $\omega_1 > \omega_2 > \cdots > \omega_n$ then

$$R(\mathbf{u}_i) = \omega_1 \frac{1 + \sum_{k=2}^{n} \frac{\alpha_k^2}{\alpha_1^2} \frac{\omega_k^2}{\omega_1^2} \frac{\mu_k}{\mu_1}}{1 + \sum_{k=2}^{n} \frac{\alpha_k^2}{\alpha_1^2} \frac{\mu_k}{\mu_1}}$$

Through the use of the Rayleigh quotient, it is possible to obtain an approximation of an eigenvalue

The close is u to an eigenvector the closer will be the approximation of the eigenvalue

The higher is the separation between the eigenvalues the faster is the convergence

Rayleigh quotient: error on the eigenvalue

$$R(\mathbf{u}) = \frac{\mathbf{u}^T\mathbf{K}\mathbf{u}}{\mathbf{u}^T\mathbf{M}\mathbf{u}}, \qquad \mathbf{u} = \boldsymbol{\varphi}_i + \delta\mathbf{u}$$

$$R(\mathbf{u}) = \frac{\boldsymbol{\varphi}_i^T\mathbf{K}\boldsymbol{\varphi}_i + 2\boldsymbol{\varphi}_i^T\mathbf{K}\delta\mathbf{u} + \delta\mathbf{u}^T\mathbf{K}\delta\mathbf{u}}{\boldsymbol{\varphi}_i^T\mathbf{M}\boldsymbol{\varphi}_i + 2\boldsymbol{\varphi}_i^T\mathbf{M}\delta\mathbf{u} + \delta\mathbf{u}^T\mathbf{M}\delta\mathbf{u}}$$

Since

$$\delta\mathbf{u} = \sum_j \alpha_j\boldsymbol{\varphi}_j = \boldsymbol{\Phi}\boldsymbol{\alpha} \text{ with } \alpha_i = 0$$

Consequently,

$$\begin{aligned}
\boldsymbol{\varphi}_i^T\mathbf{K}\delta\mathbf{u} &= \boldsymbol{\varphi}_i^T\mathbf{K}\boldsymbol{\Phi}\boldsymbol{\alpha} = 0 \\
\boldsymbol{\varphi}_i^T\mathbf{M}\delta\mathbf{u} &= \boldsymbol{\varphi}_i^T\mathbf{M}\boldsymbol{\Phi}\boldsymbol{\alpha} = 0
\end{aligned}$$

$$R(\mathbf{u}) = \frac{\boldsymbol{\varphi}_i^T\mathbf{K}\boldsymbol{\varphi}_i}{\boldsymbol{\varphi}_i^T\mathbf{M}\boldsymbol{\varphi}_i} + O\left(\|\delta\mathbf{u}\|^2\right)$$

If the eigenvector has an error of order $O(\delta u)$ then the eigenvalue approximated with the Rayleigh quotient $R(u)$ has an error of order $O(\delta u^2)$

Computation of a group of eigenvalues/eigenvectors

$$\mathbf{u} = \mathbf{u}_0 + \delta\mathbf{u} \quad \mathbf{u} = \tilde{\mathbf{\Phi}}\boldsymbol{\alpha}$$
$$R(\mathbf{u}) = R(\mathbf{u}_0) + O(\|\delta\mathbf{u}\|^2)$$

where $\tilde{\mathbf{\Phi}}$ is an approximation of the proper orthogonal modes matrix. If we consider a Taylor expansion it is easy to infer that

$$\frac{\mathrm{d}R(\mathbf{u})}{\mathrm{d}\mathbf{u}} = 0 \Rightarrow \frac{\partial R(\mathbf{u})}{\partial \boldsymbol{\alpha}} = \frac{\mathrm{d}R(\mathbf{u})}{\mathrm{d}\mathbf{u}}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\boldsymbol{\alpha}} = \mathbf{0}$$

$$R(\mathbf{u}) = \frac{\mathbf{u}^T\mathbf{K}\mathbf{u}}{\mathbf{u}^T\mathbf{M}\mathbf{u}} = \frac{\boldsymbol{\alpha}^T\tilde{\mathbf{\Phi}}^T\mathbf{K}\tilde{\mathbf{\Phi}}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\tilde{\mathbf{\Phi}}^T\mathbf{M}\tilde{\mathbf{\Phi}}\boldsymbol{\alpha}} = \frac{\boldsymbol{\alpha}^T\tilde{\mathbf{K}}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\tilde{\mathbf{M}}\boldsymbol{\alpha}}$$

$$\frac{\partial R(\mathbf{u})}{\partial \boldsymbol{\alpha}} = \frac{2\tilde{\mathbf{K}}\boldsymbol{\alpha}\boldsymbol{\alpha}^T\tilde{\mathbf{M}}\boldsymbol{\alpha} - 2\boldsymbol{\alpha}^T\tilde{\mathbf{K}}\boldsymbol{\alpha}\tilde{\mathbf{M}}\boldsymbol{\alpha}}{\left(\boldsymbol{\alpha}^T\tilde{\mathbf{M}}\boldsymbol{\alpha}\right)^2} = \mathbf{0}$$

$$\frac{1}{\boldsymbol{\alpha}^T\tilde{\mathbf{M}}\boldsymbol{\alpha}}\left(\tilde{\mathbf{K}}\boldsymbol{\alpha} - \frac{\boldsymbol{\alpha}^T\tilde{\mathbf{K}}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\tilde{\mathbf{M}}\boldsymbol{\alpha}}\tilde{\mathbf{M}}\boldsymbol{\alpha}\right) = \mathbf{0}$$

Since $\boldsymbol{\alpha}^T\tilde{\mathbf{M}}\boldsymbol{\alpha} \neq \mathbf{0}$

$$\tilde{\mathbf{K}}\boldsymbol{\alpha} = \tilde{\lambda}_i\tilde{\mathbf{K}}\boldsymbol{\alpha}$$

Computing the eigenvalues of the reduced and approximated mass and stiffness matrix it is possible to obtain and approx. of eigenvalues

Bloch-Stodola block iteration

1. start with $\tilde{\Phi}_0$
2. $k = 0, \dots n$

   $K \tilde{\Phi}_{k+1} = M \tilde{\Phi}_k$ (power method)
3. $\bar{K} = \tilde{\Phi}_{k+1}^T K \Phi_{k+1}$ $\quad \bar{M} = \Phi_{k+1}^T M \Phi_{k+1}$
4. solve $\bar{K} \alpha = \lambda \bar{M} \alpha$ $\to \lambda_i, \alpha_i$
5. $\tilde{\Phi}_N = \tilde{\Phi}_{k+1} \alpha$ restart.