1.2 Dynamic Aeroelasticity

1.2.1 Equation of motion of the typical section using the Lagrange approach

It is possible to start by writing the kinetic energy

$$T = \frac{1}{2} m \dot{y}_{CG}^2 + \frac{1}{2} I_0 \dot{\theta}_{CG}^2 \tag{1.71}$$

In this case the chosen free coordinates are the plunge h and pitch θ i.e., the downward translation and clockwise rotation of the elastic centre. So, it is necessary to write the kinematic relationship between the degrees of freedom of the centre of gravity and the the free coordinates. In this case it is easy to write the relationship between speeds (time derivatives)

$$\begin{cases} \dot{y}_{CG} & = \dot{h} + d\dot{\theta} \\ \dot{\theta}_{CG} & = \dot{\theta} \end{cases} \tag{1.72}$$

So, the kinetic energy expressed as function of the free coordinates is

$$T = \frac{1}{2} m (\dot{h} + d\dot{\theta})^2 + \frac{1}{2} I_0 \dot{\theta}^2 \tag{1.73}$$

The elastic component could be considered by writing the associated potential

$$U = \frac{1}{2} k_h h^2 + \frac{1}{2} k_\theta \theta^2. \tag{1.74}$$

The effect of gravity will be neglected. Calling the Lagrangian function $\mathcal{L} = T - U$, it is possible to write the Lagrange equations that allow to study the dynamics of this system

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k} \right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k \tag{1.75}$$

where Q_k are components that keep into account all not-conservative forces $\mathbf{F}_i$ and moments $\mathbf{M}_i$. If the displacements of the points of application of forces are $\mathbf{r}_i$ and the rotations of the points of moments of forces are $\boldsymbol{\varphi}_i$, the Q_k components are defined as

$$Q_k = \sum_{i=0}^{n} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_k} + \sum_{j=0}^{m} \mathbf{M}_j \cdot \frac{\partial \boldsymbol{\varphi}_j}{\partial q_k} \tag{1.76}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{h}} = \frac{\partial T}{\partial \dot{h}} = m(\dot{h} + d\dot{\theta}) \tag{1.77}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = md(\dot{h} + d\dot{\theta}) + I_0 \dot{\theta} \tag{1.78}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{h}}\right) = m(\ddot{h} + d\ddot{\theta}) \tag{1.79}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = md(\ddot{h} + d\ddot{\theta}) + I_0\ddot{\theta} \tag{1.80}$$

$$\frac{\partial \mathcal{L}}{\partial h} = -\frac{\partial U}{\partial h} = -k_h h \tag{1.81}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -\frac{\partial U}{\partial \theta} = -k_\theta \theta \tag{1.82}$$

To compute the components Q_k we have to consider the aerodynamic forces (lift L and moment with respect to the aerodynamic centre M_{AC}) and the associated displacements at the points of application. The movements of the aerodynamic centre are

$$\begin{cases} y_{AC} & = h - e\theta \\ \theta_{AC} & = \theta \end{cases} \tag{1.83}$$

$$Q_h = -L\frac{\partial y_{AC}}{\partial h} + M_{AC}\frac{\partial \theta_{AC}}{\partial h} = -L \tag{1.84}$$

$$Q_\theta = -L\frac{\partial y_{AC}}{\partial \theta} + M_{AC}\frac{\partial \theta_{AC}}{\partial \theta} = Le + M_{AC} \tag{1.85}$$

Putting all together the final system of equation is

$$\begin{bmatrix} m & md \\ md & I_o + md^2 \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} -L \\ eL + M_{CA} \end{Bmatrix} \tag{1.86}$$

The quantity md is called the static moment S that is null when the centre of gravity is coincident with the elastic axis, while $I_\theta = I_0 + md^2$ is the moment of inertia with respect to the elastic axis. So the system of equation could be written as

$$\begin{bmatrix} m & S \\ S & I_\theta \end{bmatrix} \begin{Bmatrix} \ddot{h} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_h & 0 \\ 0 & k_\theta \end{bmatrix} \begin{Bmatrix} h \\ \theta \end{Bmatrix} = \begin{Bmatrix} -L \\ eL + M_{CA} \end{Bmatrix} \tag{1.87}$$