

# THE POWER OF MEMORY

In interactions both social and biological is memory size advantageous?

- Both players are better of choosing Cooperation (3)
- there is always a temptation for a player to Defect (5).

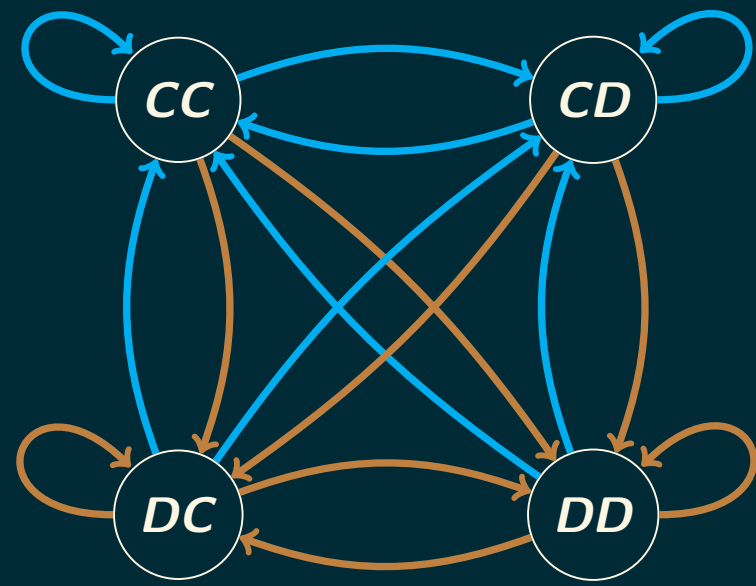
$$\begin{matrix} & C & D \\ \begin{matrix} C \\ D \end{matrix} & \begin{pmatrix} 3, 3 \\ 0, 5 \end{pmatrix} & \begin{pmatrix} 5, 0 \\ 1, 1 \end{pmatrix} \end{matrix} \quad (1)$$

	1	2	3	...	$\underbrace{\dots}_{\text{memory-1}}$ $n-1$	$n$
player $p$	C	C	C	...	D	...
player $q$	C	C	D	...	D	...

memory size =  $n - 1$

## 1. OPTIMAL MEMORY ONE STRATEGY IN A MATCH

Depending on the simultaneous moves of the two players, there are four possibles 'states':



A memory one strategy is denoted by the probabilities of cooperating after each of these states  $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}_{[0,1]}^4$ . A match between two memory one players  $p$  and  $q$  can be modelled as a Markov chain.

$$M = \begin{bmatrix} p_1 q_1 & p_1 (-q_1 + 1) & q_1 (-p_1 + 1) & (-p_1 + 1) (-q_1 + 1) \\ p_2 q_3 & p_2 (-q_3 + 1) & q_3 (-p_2 + 1) & (-p_2 + 1) (-q_3 + 1) \\ p_3 q_2 & p_3 (-q_2 + 1) & q_2 (-p_3 + 1) & (-p_3 + 1) (-q_2 + 1) \\ p_4 q_4 & p_4 (-q_4 + 1) & q_4 (-p_4 + 1) & (-p_4 + 1) (-q_4 + 1) \end{bmatrix}$$

Thus the utility of player  $p$  against an opponent  $q$  is given by:

$$u_q(p) = v \times S_p$$

where  $v$  denotes the stationary vector of  $M$  and  $S_p$  the payoffs of player  $p$  given by equation (1).

Against a single opponent:

$$\begin{aligned} \max_q : u_q(p) &= \frac{\frac{1}{2} p Q p^T + c^T p + a}{\frac{1}{2} p \bar{Q} p^T + \bar{c}^T p + \bar{a}} \\ st : p &\in \mathbb{R}_{[0,1]}^4 \end{aligned}$$

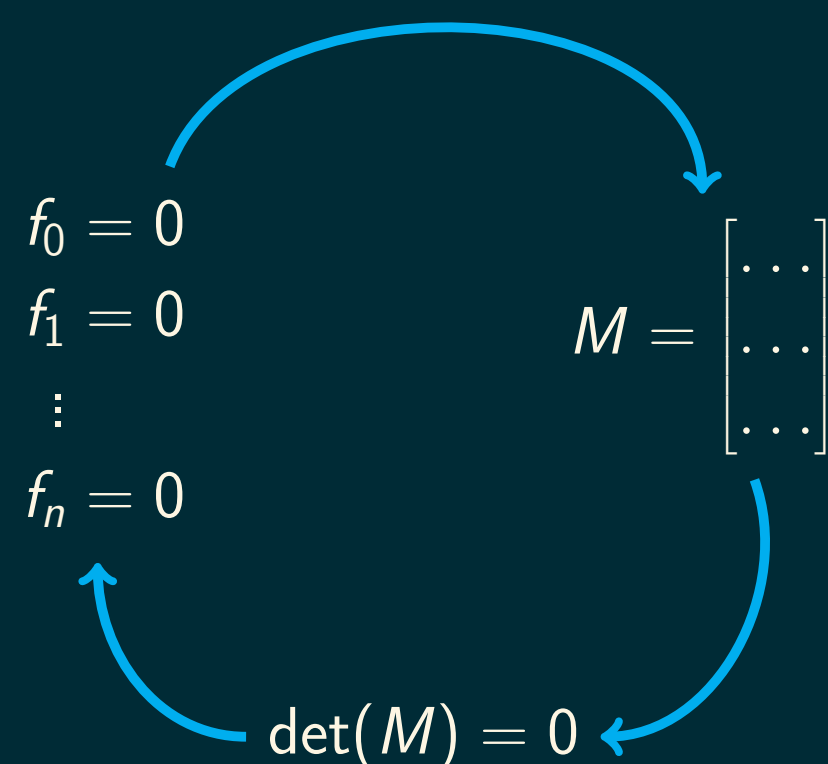
where  $Q, \bar{Q}$  are matrices of  $4 \times 4$ , and  $c, \bar{c}$  are  $4 \times 1$  vectors defined with the transition probabilities of the opponent's transition probabilities  $q_1, q_2, q_3, q_4$ .

## 3. OPTIMAL MEMORY ONE IN A TOURNAMENT

In order to find the optimal memory on player against a set of opponents we need to explore the numeration of the differentiation of:

...

This will be explored using the **resultant**.



- Dixon's resultant;
- Maycalay resultant.

## 2. WHAT IS THE OPTIMAL PURE RANDOM STRATEGY?

A memory one strategy where the transition probabilities of each state are the same is called a **purely random strategy**.

Against a single opponent:

$$\max_q : u_q(p) = \frac{n_2 p^2 + n_1 p + n_0}{d_1 p + d_0}$$

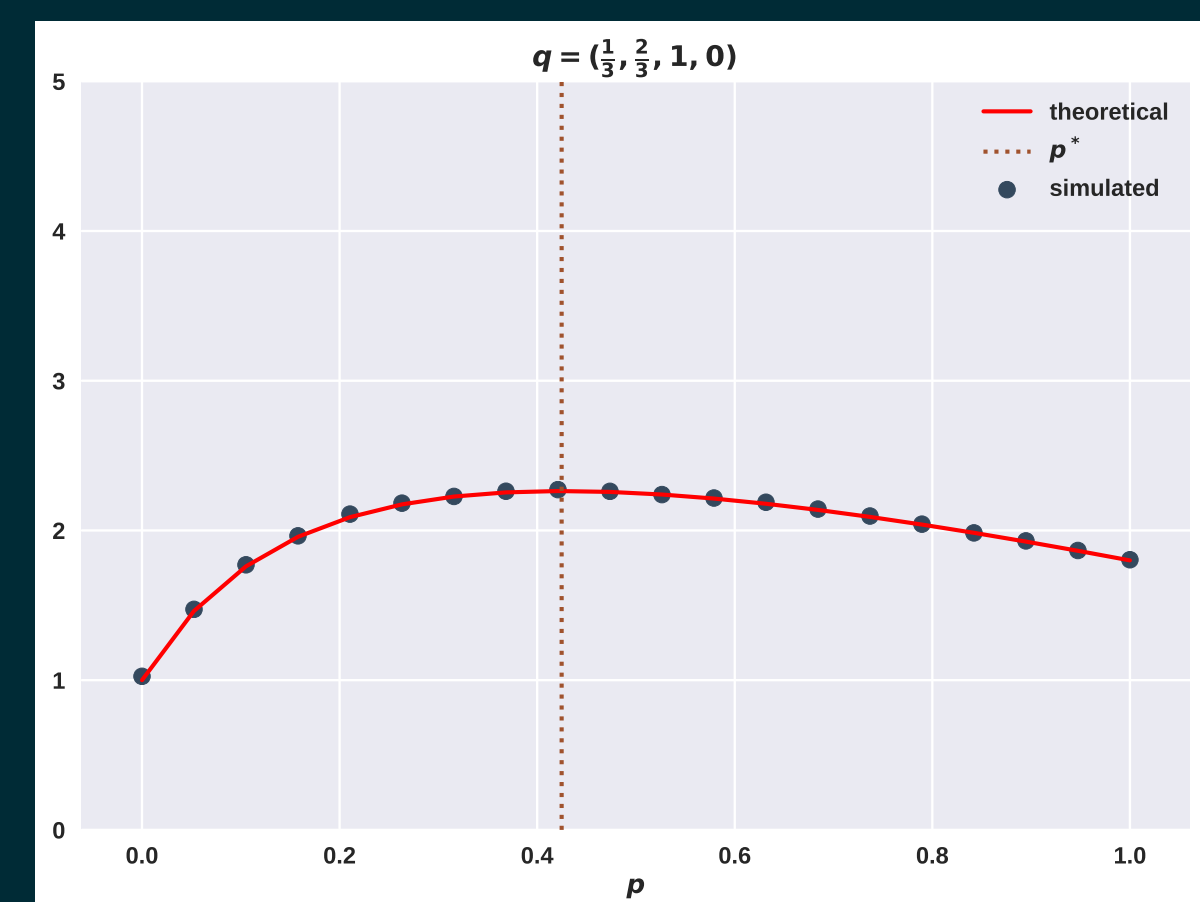
$$\begin{aligned} st : p_1 &= p_2 = p_3 = p_4 = p \\ p &\in \mathbb{R}_{[0,1]} \end{aligned}$$

The optimal behaviour of a **purely random** player  $(p, p, p, p)$  against a memory one opponent  $q$  is given by:

$$p^* = \operatorname{argmax}(u_q(p)), \quad p \in S_q,$$

where the set  $S_q$  is defined as,

$$S_q = \left\{ 0, p_{\pm}, 1 \mid \begin{array}{l} 0 < p_{\pm} < 1, \\ p_{\pm} \neq \frac{-d_0}{d_1} \end{array} \right\}$$



Against multiple opponents:

$$\max_q : \frac{\sum_{i=1}^N u_q^{(i)}(p)}{N}$$

$$\begin{aligned} st : p_1 &= p_2 = p_3 = p_4 = p \\ p &\in \mathbb{R}_{[0,1]} \end{aligned}$$

The optimal behaviour of a **purely random** player  $(p, p, p, p)$  in an  $N$ -memory one player tournament,  $\{q_{(1)}, q_{(2)}, \dots, q_{(N)}\}$  is given by:

$$p^* = \operatorname{argmax}\left(\sum_{i=1}^N u_q^{(i)}(p)\right), \quad p \in S_{q(i)},$$

where the set  $S_{q(i)}$  is defined as:

$$S_{q(i)} = \bigcup_{\substack{i=1 \\ \lambda_i \neq \frac{d_{0i}}{d_{1i}}}}^{2N} \lambda_i \cup \{0, 1\}$$

Note the size of candidate solutions is  $1 \leq |S_{q(i)}| \leq 2N + 2$ .

