# THE POWER OF MEMORY

## In interactions both social and biological is memory size advantageous?

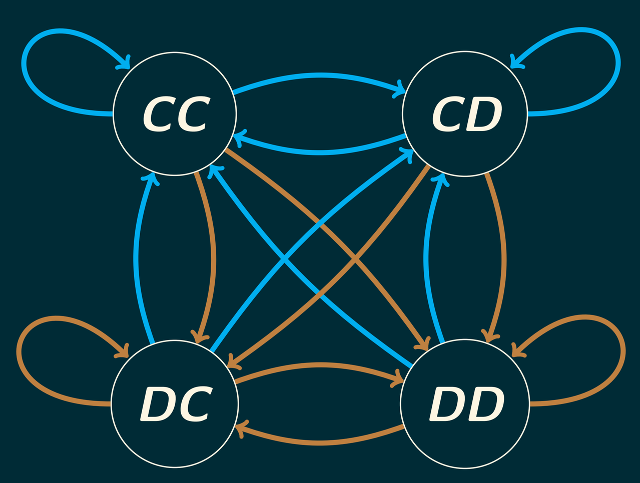
- Both players are better of choosing Cooperation (3)
- there is always a temptation for a player to Defect (5).

$$\begin{array}{ccc} & C & D \\ \begin{array}{c} C \\ D \end{array} & \multicolumn{2}{c}{\begin{pmatrix} (3,3) & (5,0) \\ (0,5) & (1,1) \end{pmatrix}} \end{array} \quad (1)$$

$$\begin{array}{lcccccc} & 1 & 2 & 3 & \cdots & \overbrace{n-1}^{\text{memory-1}} & n \\ \text{player } p & C & C & C & \cdots & D & \cdots \\ \text{player } q & \underbrace{C \quad C \quad D \quad \cdots \quad D}_{\text{memory size-}n-1} & & & & & \cdots \end{array}$$

### 1. OPTIMAL MEMORY ONE STRATEGY IN A MATCH

Depending on the simultaneous moves of the two players, there are four possibles 'states':




A memory one strategy is denoted by the probabilities of cooperating after each of these states $p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_{[0,1]}$ [1]. A match between two memory one players $p$ and $q$ can be modelled as a Markov chain.

$$M = \begin{bmatrix} p_1q_1 & p_1(-q_1+1) & q_1(-p_1+1) & (-p_1+1)(-q_1+1) \\ p_2q_3 & p_2(-q_3+1) & q_3(-p_2+1) & (-p_2+1)(-q_3+1) \\ p_3q_2 & p_3(-q_2+1) & q_2(-p_3+1) & (-p_3+1)(-q_2+1) \\ p_4q_4 & p_4(-q_4+1) & q_4(-p_4+1) & (-p_4+1)(-q_4+1) \end{bmatrix}$$

Thus the utility of player $p$ against an opponent $q$ is given by:

$$u_q(p) = v \times S_p$$

where $v$ denotes the stationary vector of $M$ and $S_p$ the payoffs of player $p$ given by equation (1).

Against a single opponent:

$$\max_q : u_q(p) = \frac{\frac{1}{2}\, pQp^T + c^Tp + a}{\frac{1}{2}\, p\bar{Q}p^T + \bar{c}^Tp + \bar{a}}$$
$$st: \ p \in \mathbb{R}^4_{[0,1]}$$

where $Q, \bar{Q}$ are matrices of $4 \times 4$, and $c, \bar{c}$ are $4 \times 1$ vectors defined with the transition probabilities of the opponent's transition probabilities $q_1, q_2, q_3, q_4$.

### 3. OPTIMAL MEMORY ONE IN A TOURNAMENT

In order to find the optimal memory on player against a set of opponents we need to explore the numeration of the differentiation of:
This will be explored using Resultant Theory. The resultant will equal zero if and only if the polynomials of a multivariate system have at least one common root.

- Dixon's resultant;
- Maycalay resultant.

### 2. WHAT IS THE OPTIMAL PURE RANDOM STRATEGY?

A set of memory one strategies where the transition probabilities of each state are the same, are called **purely random strategies**.

Against a single opponent:

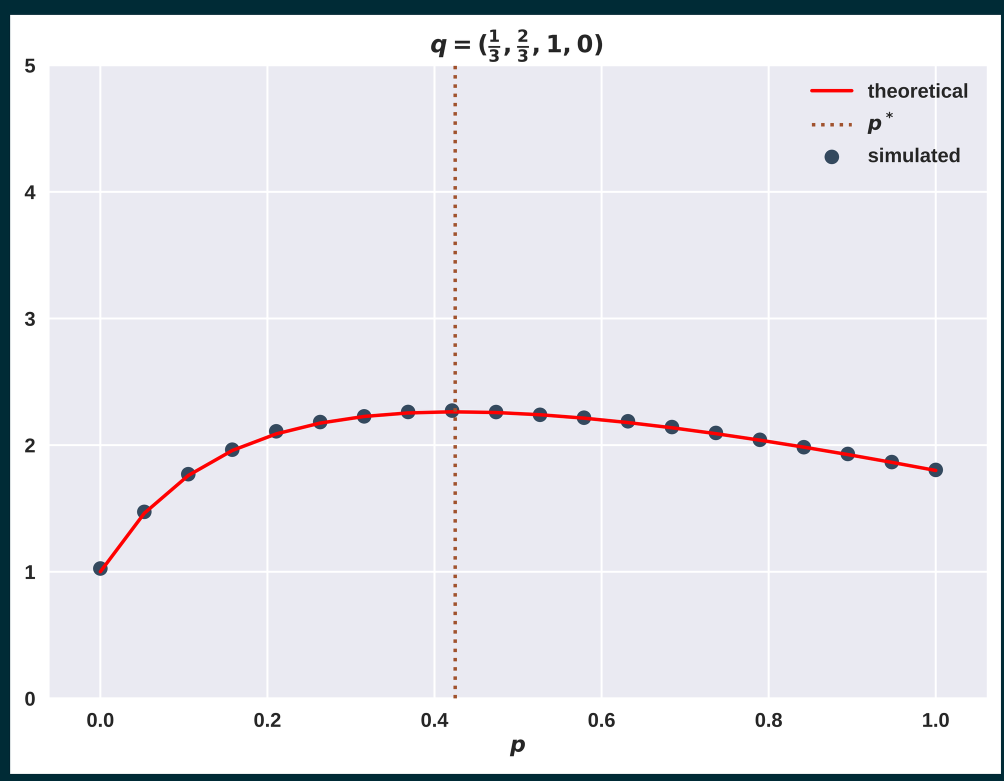
$$\max_q : u_q(p) = \frac{n_2p^2 + n_1p + n_0}{d_1p + d_0}$$
$$st: \ p_1 = p_2 = p_3 = p_4 = p$$
$$p \in \mathbb{R}_{[0,1]}$$

The optimal behaviour of a **purely random** player $(p, p, p, p)$ against a memory one opponent $q$ is given by:

$$p^* = \text{argmax}(u_q(p)), \ \ p \in S_q,$$

where the set $S_q$ is defined as

$$S_q = \left\{0, p_\pm, 1 \,\middle|\, \begin{array}{l} 0 < p_\pm < 1, \\ p_\pm \neq \frac{-d_0}{d_1} \end{array} \right\}$$




Against multiple opponents:

$$\max_q : \frac{\sum_{i=1}^{N} {u_q}^{(i)}(p)}{N}$$
$$st: \ p_1 = p_2 = p_3 = p_4 = p$$
$$p \in \mathbb{R}_{[0,1]}$$

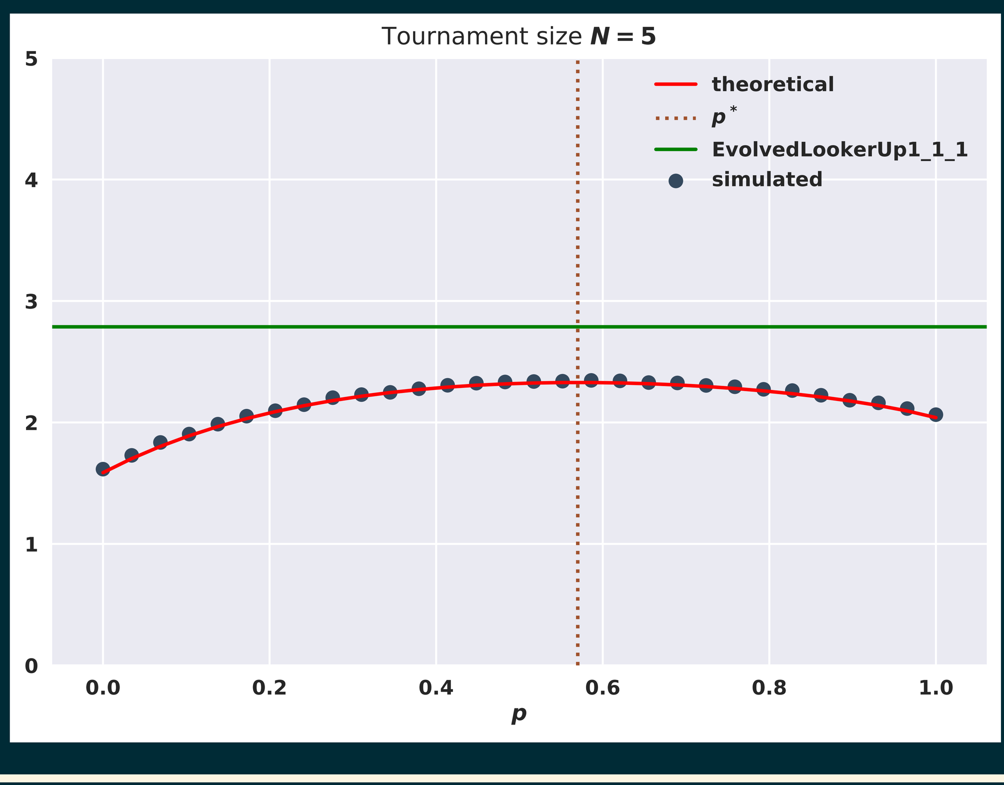
The optimal behaviour of a **purely random** player $(p, p, p, p)$ in an $N-$memory one player tournament, $\{q_{(1)}, q_{(2)} \dots, q_{(N)}\}$ is given by:

$$p^* = \text{argmax}(\sum_{i=1}^{N} {u_q}^{(i)}(p)), \ \ p \in S_{q(i)},$$

where the set $S_{q(i)}$ is defined as:

$$S_{q(i)} = \bigcup_{\substack{i=1 \\ \lambda_i \neq \frac{do_i}{d1_i}}}^{2N} \lambda_i \cup \{0, 1\}$$

Note the size of candidate solutions is $1 \leq |S_{q(i)}| \leq 2N + 2$.





CARDIFF UNIVERSITY
PRIFYSGOL CAERDYDD

NikoletaGlyn
Nikoleta-v3



Martin Nowak and Karl Sigmund.
The evolution of stochastic strategies in the Prisoner's Dilemma.
1990.