Optimisation of short memory strategies in the Iterated Prisoners Dilemma

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Supervised by:

Dr. Vincent Knight   Dr. Jonathan Gillard

$$\begin{bmatrix} (3,3) & (0,5) \\ (5,0) & (1,1) \end{bmatrix}$$

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$$(R, P, S, T) = (3, 1, 0, 5)$$

Tit For Tat

Cycler ***CCD***

	1
Tit For Tat	***C***
Cycler ***CCD***	***C***

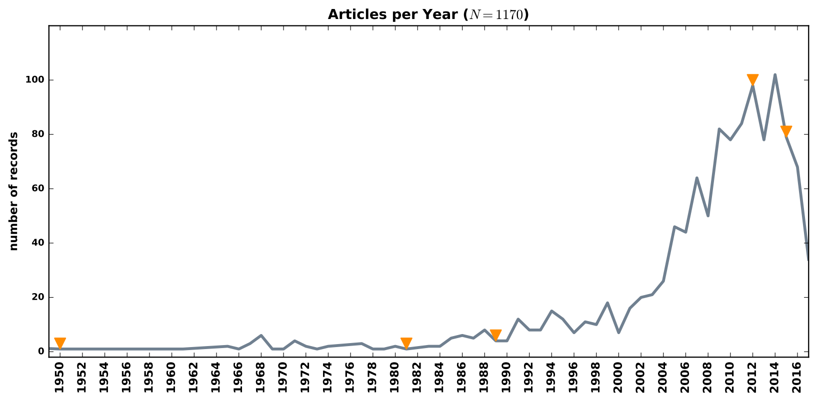
	1	2	3
Tit For Tat	C	C	C
Cycler CCD	C	C	D

	1	2	3
Tit For Tat	***C***	***C***	***C***
Cycler ***CCD***	***C***	***C***	***D***

	1	2	3	4
Tit For Tat	C	C	C	
Cycler CCD	C	C	D	

	1	2	3	4
Tit For Tat	***C***	***C***	***C***	***D***
Cycler ***CCD***	***C***	***C***	***D***	***C***

	1	2	3	4	5	6
Tit For Tat	*C*	*C*	*C*	*D*	*C*	*C*
Cycler ***CCD***	*C*	*C*	*D*	*C*	*C*	*D*



Articles per Year ($N = 1170$)
number of records
0
20
40
60
80
100
1950
1952
1954
1956
1958
1960
1962
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1966
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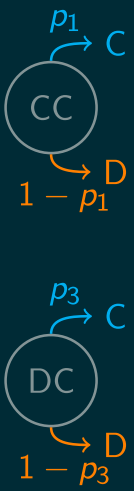
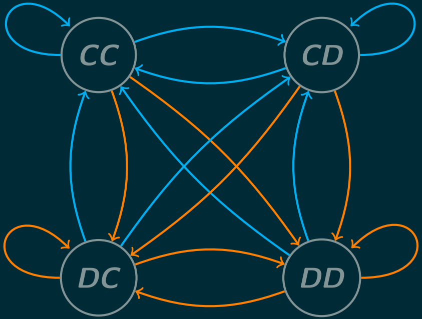


$$p = (p_1, p_2, p_3, p_4) \in \mathbb{R}^4_{[0,1]}$$

- *Christopher Lee, Marc Harper, and Dashiell Fryer.* The art of war: Beyond memory-one strategies in population games. 2015.

How good are memory one strategies?


CC
CD
DC
DD

$$
M = \begin{bmatrix} p_1 q_1 & p_1(-q_1+1) & q_1(-p_1+1) & (-p_1+1)(-q_1+1) \\ p_2 q_3 & p_2(-q_3+1) & q_3(-p_2+1) & (-p_2+1)(-q_3+1) \\ p_3 q_2 & p_3(-q_2+1) & q_2(-p_3+1) & (-p_3+1)(-q_2+1) \\ p_4 q_4 & p_4(-q_4+1) & q_4(-p_4+1) & (-p_4+1)(-q_4+1) \end{bmatrix}
$$

$$\max_p u_q(p) \text{ such that } p \in \mathbb{R}^4_{[0,1]}$$

Lemma

$$u_q(p) = \frac{\frac{1}{2} p Q p^T + c^T p + a}{\frac{1}{2} p \bar{Q} p^T + \bar{c}^T p + \bar{a}}$$

- $Q, \bar{Q} \in \mathbb{R}^{4 \times 4}$
- $c, \bar{c} \in \mathbb{R}^{4 \times 1}$
- $a, \bar{a} \in \mathbb{R}$

$$\max_p u_q(p) \text{ such that } p \in \mathbb{R}^4_{[0,1]}$$

$$\max_p u_q(p) \text{ such that } p \in \mathbb{R}^4_{[0,1]}$$

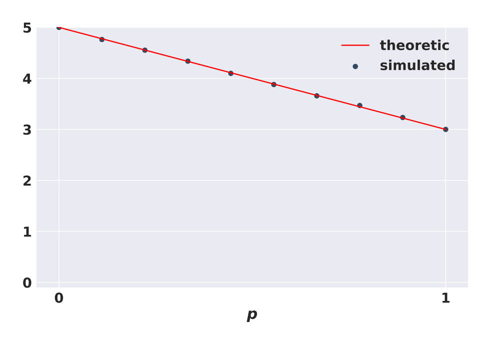
subject to $p_1 = p_2 = p_3 = p_4 = p$

Lemma

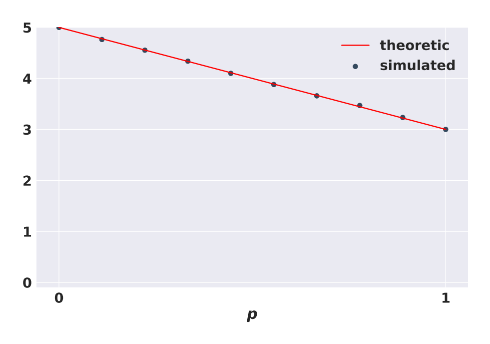
$$u_q(p) = \frac{n_2 p^2 + n_1 p + n_0}{d_1 p + d_0}$$

- $n_2 = -(q_1 - q_2 - 2q_3 + 2q_4)$
- $n_1 = -q_1 + 2q_2 + 5q_3 - 7q_4 - 1$
- $n_0 = q_2 - 5q_4 - 1$
- $d_1 = q_1 - q_2 - q_3 + q_4$
- $d_0 = q_2 - q_4 - 1$

$$q = \left(1, 1, 0, \frac{2}{3}\right)$$

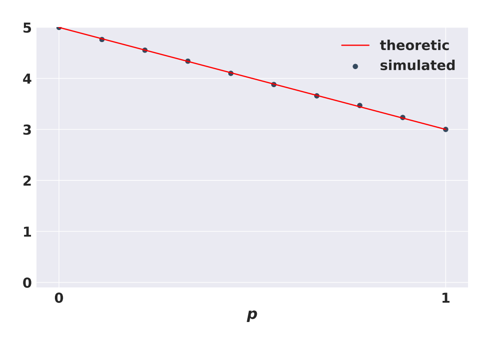


$$q = \left(1, 1, 0, \frac{2}{3}\right)$$



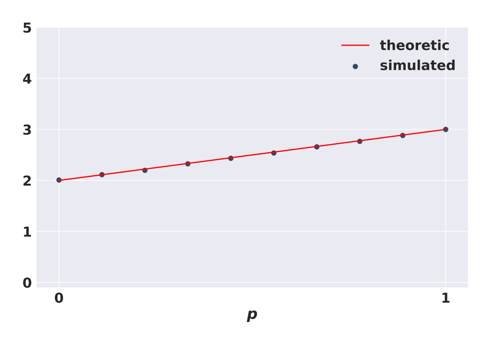
$$u_q(p) = \frac{-\frac{4p^2}{3} + \frac{14p}{3} - \frac{10}{3}}{\frac{2p}{3} - \frac{2}{3}}$$

$$q = \left(1, 1, 0, \frac{2}{3}\right)$$

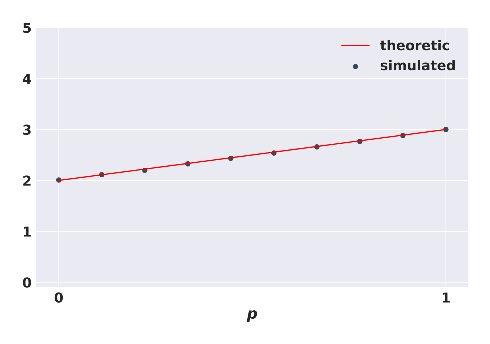


$$u_q(p) = \frac{-\frac{4p^2}{3} + \frac{14p}{3} - \frac{10}{3}}{\frac{2p}{3} - \frac{2}{3}} = -2p + 5$$

$$q = \left(1, 0, 1, \frac{1}{3}\right)$$

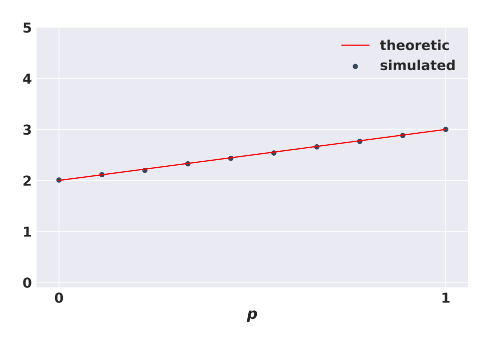


$$q = \left(1, 0, 1, \frac{1}{3}\right)$$



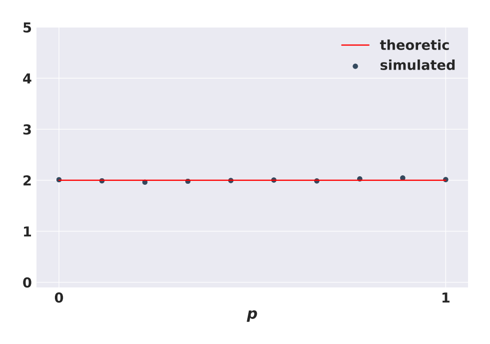
$$u_q(p) = \frac{\frac{p^2}{3} + \frac{8p}{3} - \frac{10}{3}}{\frac{p}{3} - \frac{4}{3}}$$

$$q = \left(1, 0, 1, \frac{1}{3}\right)$$

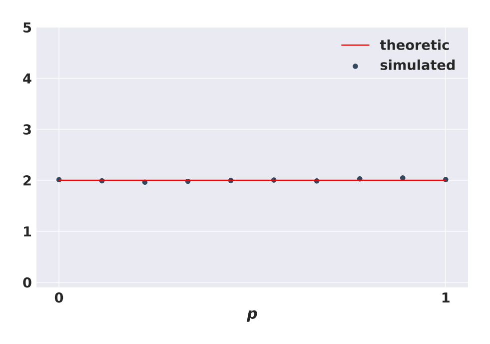


$$u_q(p) = \frac{\frac{p^2}{3} + \frac{8p}{3} - \frac{10}{3}}{\frac{p}{3} - \frac{4}{3}} = p + 2$$

$$q = \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{3}\right)$$

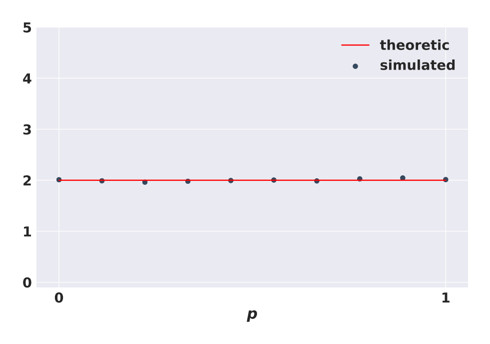


$$q = \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{3}\right)$$



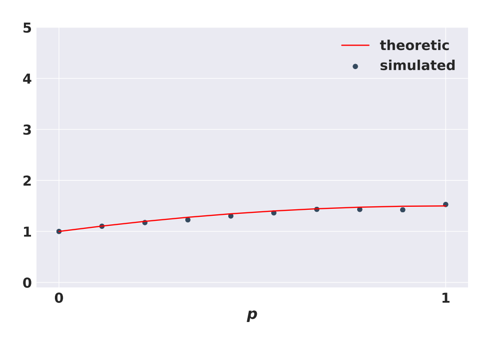
$$u_q(p) = \frac{\frac{2p}{3} - \frac{8}{3}}{\frac{p}{3} - \frac{4}{3}}$$

$$q = \left(\frac{2}{3}, 0, \frac{2}{3}, \frac{1}{3}\right)$$

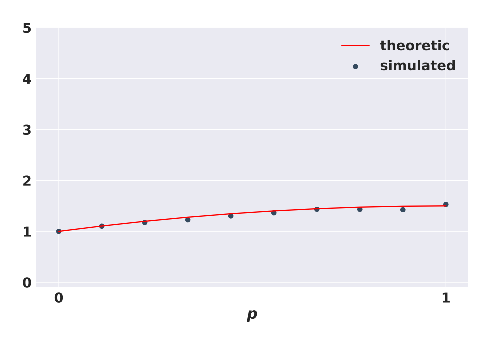


$$u_q(p) = \frac{\frac{2p}{3} - \frac{8}{3}}{\frac{p}{3} - \frac{4}{3}} = 2$$

$$q = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$$

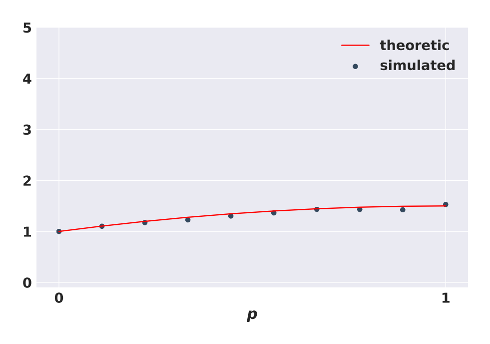


$$q = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$$



$$u_q(p) = \frac{\frac{p^2}{3} - \frac{2p}{3} - \frac{2}{3}}{-\frac{2}{3}}$$

$$q = \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$$



$$u_q(p) = \frac{\frac{p^2}{3} - \frac{2p}{3} - \frac{2}{3}}{-\frac{2}{3}} = -\frac{p^2}{2} + p + 1$$

Lemma (Indifferent)

$-q_1 + q_2 + 2q_3 - 2q_4 = 0$ *and*
$(q_2 - q_4 - 1)(q_1 - 2q_2 - 5q_3 + 7q_4 + 1) - (q_2 - 5q_4 - 1)(q_1 - q_2 - q_3 + q_4) = 0.$

Proof.

$$u_q(p) = \frac{n_2 p^2 + n_1 p + n_0}{d_1 p + d_0} = a_0$$
$$n_2 p^2 + n_1 p + n_0 = a_0 d_1 p + a_0 d_0$$

$$\begin{cases} n_2 = 0 \\ n_1 d_0 = d_1 n_0 \end{cases}$$

□

Lemma (Linear)

$$(q_1 q_4 - q_2 q_3 + q_3 - q_4)(4q_1 - 3q_2 - 4q_3 + 3q_4 - 1) = 0$$

Proof.

$$u_q(p) = \frac{n_2 p^2 + n_1 p + n_0}{d_1 p + d_0} = a_1 p + a_0$$

$$n_2 p^2 + n_1 p + n_0 = a_1 d_1 p^2 + (d_1 a_0 + a_1 d_0) p + a_0 d_0$$

$$\begin{cases} n_2 = d_1 a_1 \\ n_1 d_0 = d_1 n_0 + a_1 d_0 \end{cases}$$

$\square$

Lemma (Quadratic)

$(q_1 - q_2 - q_3 + q_4) = 0,$
$(q_1 q_4 - q_2 q_3 + q_3 - q_4)(4q_1 - 3q_2 - 4q_3 + 3q_4 - 1) \neq 0 \textit{ and } q_2 - q_4 - 1 \neq 0$

Proof.

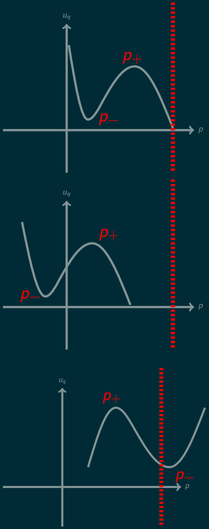
$$u_q(p) = \frac{n_2 p^2 + n_1 p + n_0}{d_1 p + d_0} = a_2 p^2 + a_1 p + a_0$$

$$n_2 p^2 + n_1 p + n_0 = d_1 a_2 p^3 + (a_1 d_1 + d_0 a_2) p^2 + (d_1 a_0 + a_1 d_0) p + a_0 d_0$$

$$\begin{cases} a_1 d_1 = 0 \\ n_2 = d_1 a_1 + d_0 a n_2 \\ n_1 d_0 = d_1 n_0 + a_1 d_0 \end{cases}$$

$\square$

$$\frac{du}{dp} = \frac{m_2 p^2 + m_1 p + m_0}{(d_1 p + d_0)^2}$$

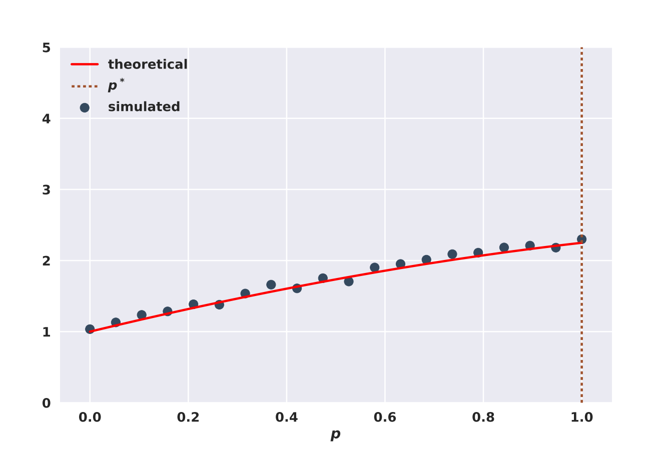


Theorem (Optimization of purely random player)

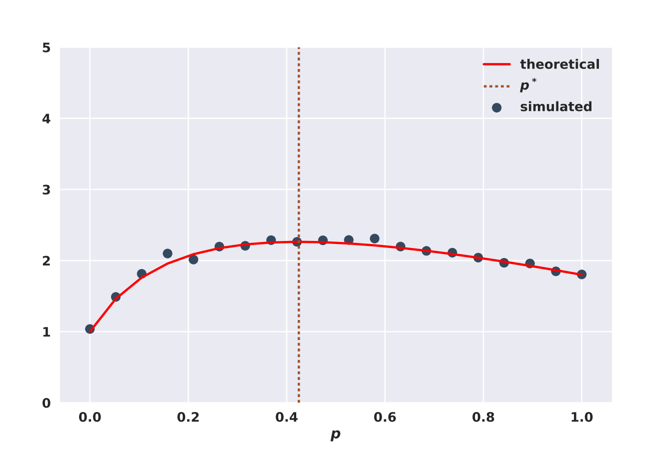
$$S_q = \left\{ 0, p_{\pm}, 1 \;\middle|\; \begin{array}{l} 0 < p_{\pm} < 1, \\ p_{\pm} \neq \frac{-d_0}{d_1} \end{array} \right\}$$

$$p^* = argmax_{\ p \in S_q} u_q(p)$$

$$q = \left(\frac{7}{8}, \frac{7}{16}, \frac{3}{8}, 0\right)$$



$$q = \left(\frac{1}{3}, \frac{2}{3}, 1, 0\right)$$



$$q^{(1)}, q^{(2)}, q^{(3)} \ldots q^{(N)}$$

$$\max_{p} \frac{1}{N} \sum_{i=1}^{N} {u_q}^{(i)}(p)$$

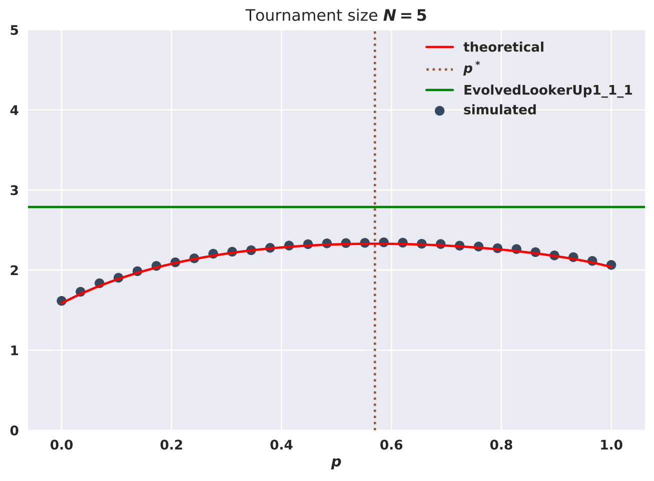
$$\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) = \frac{1}{N}\frac{\sum\limits_{i=1}^{N}\left(n_2^{(i)}p^2 + n_1^{(i)}p + n_0^{(i)}\right)\prod\limits_{\substack{j\,=\,1\\ j\,\neq\,i}}^{N}\left(d_1^{(j)}p + d_0^{(j)}\right)}{\prod\limits_{i=1}^{N}\left(d_1^{(i)}p + d_0^{(i)}\right)}$$

$$\frac{1}{N}\sum_{i=1}^{N} {u_q}^{(i)}(p) = \frac{1}{N}\left(\frac{\sum\limits_{i=0}^{N+1} h_i p^i}{\sum\limits_{i=0}^{N} k_i p^i}\right)$$

$$| S_{q^{(1)},\ldots,q^{(n)}} | \leq 2N + 2$$

$$p^* = \operatorname{argmax}_{S_{q^{(1)},\ldots,q^{(n)}}} u(p)$$

$$S_q = \bigcup_{\substack{i=1 \\ \lambda_i \neq \frac{do_i}{d1_i}}}^{2N} \lambda_i \cup \{0, 1\}$$


Tournament size $N = 5$
theoretical
p^*
EvolvedLookerUp1_1_1
simulated
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
p

Reactive Strategies

$$p = (p_1, p_2, p_1, p_2)$$

Memory One Strategies

$$p = (p_1, p_2, p_3, p_4)$$

Resultant Theory

$$f_0 = 0$$

$$f_1 = 0$$

$$\vdots$$

$$f_n = 0$$

$$M = \begin{bmatrix} \cdots \\ \cdots \end{bmatrix}$$

$$\det(M) = 0$$

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https://github.com/Nikoleta-v3