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Lecture 9: Markov and Regime Switching Models

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20192– Financial Econometrics

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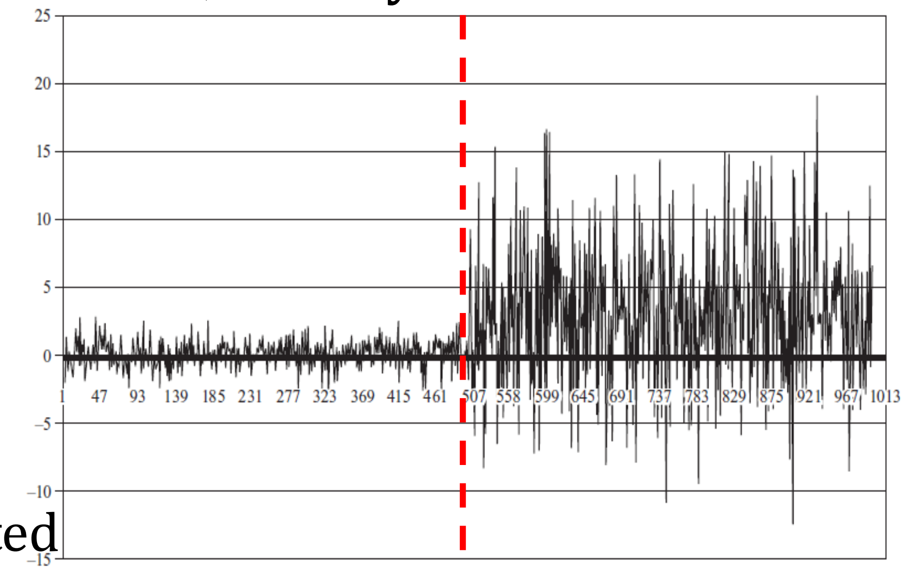
Overview

- Motivation
- Deterministic vs. Endogeneous, Stochastic Switching
- Dummy Regressiom Switching Models
- Markov Switching Models
- Threshold and Self-Exciting Threshold Models
- Specification Tests in Regime Switching Models
- Non-Normalities under Regime Switching Models

Overview and Motivation

- Financial time series are typically subject to structural instability, in the form of either breaks or regimes

- Many financial and economic time series undergo episodes in which the behavior of the series changes dramatically
  - The behavior of a series could change over time in terms of its mean value, its volatility, or its persistence
- The behavior may change once and for all, usually known as a structural break



- Or it may change for a period of time before reverting back to its original behavior or switching to yet another style of behavior; this is a regime shift or regime switch
  - Substantial changes in the properties of a series are attributed to large-scale events, such as wars, financial panics, changes in government policy (e.g., the introduction of an inflation target), etc.

Overview and Motivation

- Piece-wise linear regressions fit to alternative sub-samples appear to be inadequate to capture breaks and regimes
  - These are all examples of breaks but such changes may occur as a result of more subtle factors
  - E.g., consider the intraday patterns observed in equity market bid--ask spreads that appear to start with high values at the open, gradually narrowing throughout the day, before widening at the close
- In the face of such structural instabilities, **a linear model estimated over the whole sample covering the change is not appropriate**
  - One possible approach would be to split the data around the time of the change and to estimate separate models on each portion
  - E.g., if it was thought an AR(1) process was appropriate to capture the relevant features of a particular series whose behavior changed at observation 500, say, two models could be estimated:

$$y_t = \mu_1 + \phi_1 y_{t-1} + u_{1t} \quad \text{before observation 500}$$

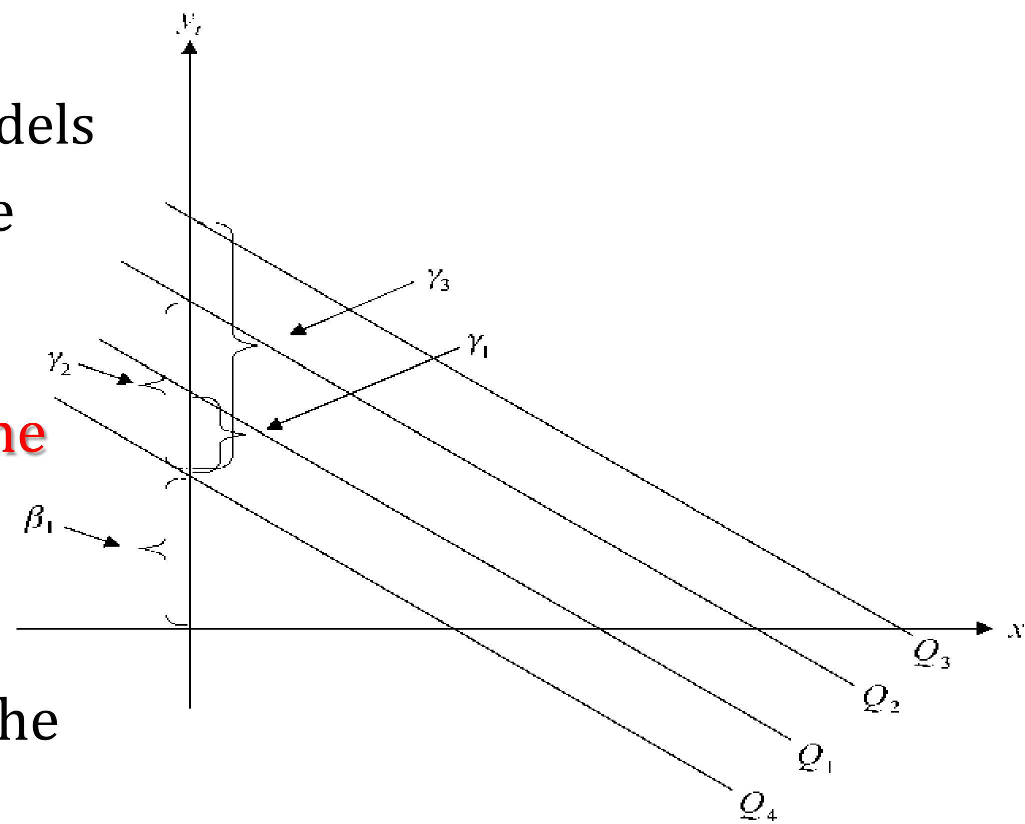
$$y_t = \mu_2 + \phi_2 y_{t-1} + u_{2t} \quad \text{after observation 500}$$

Overview and Motivation

- **Piece-wise linear models require the dates of structural instabilities to be observable and are in general inefficient**
  - This involves focusing on the mean shift only
  - In a piecewise linear model although the model is globally (i.e. taken as a whole) non-linear, each of the components is a linear model
- Problem: how do we know ex-ante when the structural break has occurred if not analyzing the data, which does require a model?
- This method may be valid, but it is inefficient
  - It may be the case that only one property of the series has changed
  - If the (unconditional) mean of the series has changed, leaving its other properties unaffected, it would be sensible to try to keep all of the observations together when it comes to estimate variance
- Required a set of models that allow all of the observations to be used in estimation, but also that the model is sufficiently flexible to allow different types of behavior at different points in time

Dummy Switching Models

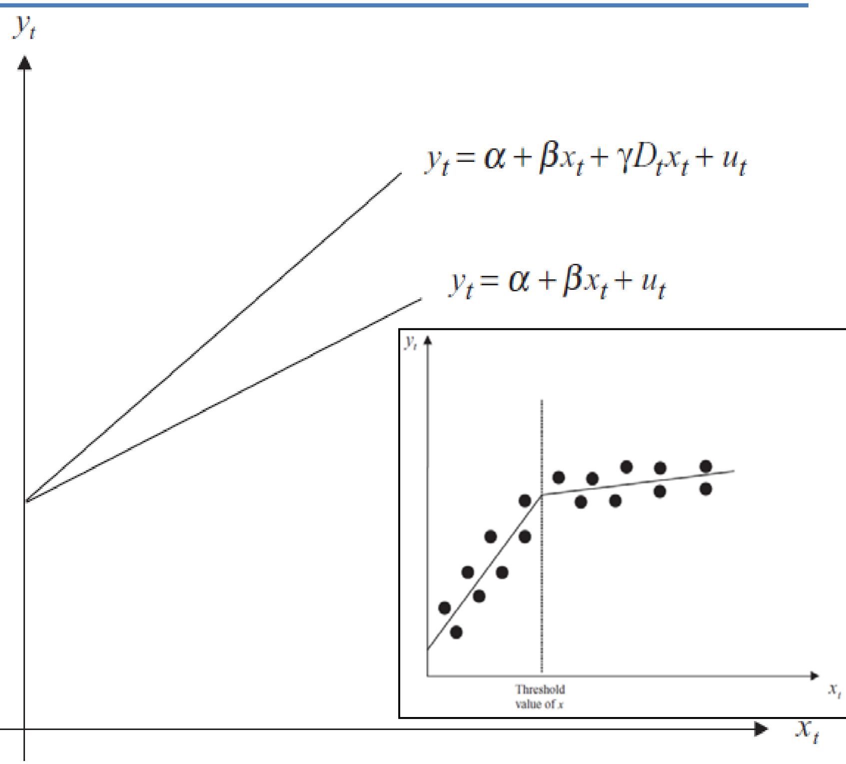
- Three classes of models
  1. Deterministic dummy regression/ARMA models
  2. Threshold AR models
  3. Markov switching AR models
- In the first case, switches are deterministic and pre-determined
- In the other two cases, regime switches are stochastic and endogenously determined from the data
- These operate by changing the slope of the regression line, leaving the intercept unchanged:



$$y_t = \alpha + \beta x_t + \gamma_1 D1_t x_t + \gamma_2 D2_t x_t + \gamma_3 D3_t x_t + u_t$$

Markov Switching Models

- In a MS model, the process followed by y_t switches over time according to one of k values taken by a discrete variable S_t



- A slope dummy changes the slope of the regression line, leaving the intercept unchanged
- For periods where the value of the dummy is zero, the slope will be β, while for periods where the dummy is one, the slope will be β + γ
- Of course, intercept and slope dummies may also be combined
- Markov switching models (MSMs) are the most popular class of non-linear models that can be found in finance and (macro) economics
- Under a MSM there are k regimes: y_t switches regime according to some (possibly unobserved) variable, z_t , that takes integer values

Markov Switching Models

- In MSMs, the state variable follows a *qth order Markov process* and is often assumed to be unobservable
  - If $z_t = 1$, the process is in regime 1 at time t, and if $z_t = 2$, in regime 2
- Movements of state btw. regimes are governed by a Markov process such that: $P[a < y_t \leq b \mid y_1, y_2, \ldots, y_{t-1}] = P[a < y_t \leq b \mid y_{t-1}]$
  - The probability distribution of the state at t depends only on the state at t −1 and not on the states that were passed through at t −2, t −3, . . .
  - Markov processes are not path-dependent
  - The model's strength lies in its flexibility, being capable of capturing changes in the variance btw. states, as well as changes in the mean
- In the most typical implementation, the unobserved state variable, z_t, follows a first-order Markov process with transition probs.:
- p_{ij} = probability of being in regime *j*, give that the system was in regime *i* during the previous period

$$\boxed{\begin{aligned}
\text{prob}[z_t = 1 | z_{t-1} = 1] &= p_{11} \\
\text{prob}[z_t = 2 | z_{t-1} = 1] &= 1 - p_{11} \\
\text{prob}[z_t = 2 | z_{t-1} = 2] &= p_{22} \\
\text{prob}[z_t = 1 | z_{t-1} = 2] &= 1 - p_{22}
\end{aligned}}$$

Markov Switching Models

- E.g., $1-p_{11}$ defines the probability that z_t will change from state 1 in period t −1 to state 2 in period t
- z_t evolves as AR(1): $z_t = (1 - p_{11}) + \rho z_{t-1} + \eta_t$ where $\rho = p_{11} + p_{22} - 1$
- Under the MS approach, there can be multiple shifts

- In this framework, the observed returns series evolves as

$$\boxed{y_t = \mu_1 + \mu_2 z_t + (\sigma_1^2 + \phi z_t)^{1/2} u_t} \qquad u_t \sim \mathrm{N}(0, 1)$$

- The expected values and variances of the series are μ_1 and $\sigma^2{}_1$, respectively in state 1, and $(\mu_1 + \mu_2)$ and $\sigma^2{}_1 + \phi$ in state 2
  - If a variable follows a MSM, all that is required to forecast the probability that it will be in a regime in the next period is the current period's probability and a set of transition probabilities collected in

$$P = \begin{bmatrix} P_{11} & P_{12} & \dots & P_{1m} \\ P_{21} & P_{22} & \dots & P_{2m} \\ \dots & \dots & \dots & \dots \\ P_{m1} & P_{m2} & \dots & P_{mm} \end{bmatrix} \qquad \sum_{j=1}^{m} P_{ij} = 1 \forall i \qquad \text{(Here } m = k\text{)}$$

Markov Switching Models

- Given a first-order Markov chain process with transition matrix P and a vector of state probabilities π_t, the H-step ahead vector of probabilities π_{t+H} is given by $\pi_{t+\mathrm{H}} = \pi_t \mathrm{P}^{\mathrm{H}}$

  - This is called the transition matrix of the MSM
  - A vector of current state probabilities is then defined as

  $$\pi_t = [\pi_1 \quad \pi_2 \quad \ldots \quad \pi_m]$$

  where π_i is the probability that the variable y is currently in state i . Given π_t and P, the probability that the variable y will be in a given regime next period can be forecast using: $\pi_{t+1} = \pi_t P$

- The probabilities for S steps into the future will be given by

$$\pi_{t+\mathrm{H}} = \pi_t \mathrm{P}^{\mathrm{H}}$$

where $\mathrm{P}^{\mathrm{H}} \equiv \prod_{\mathrm{h}=1}^{\mathrm{H}} \mathrm{P}$, i.e., the product of P with itself H times

  - Why a simple first-order Markov chain? Because it can be shown that by expanding the number of regimes $k = m$, a qth order Markov process may be represented as a 2^q first-order Markov chain

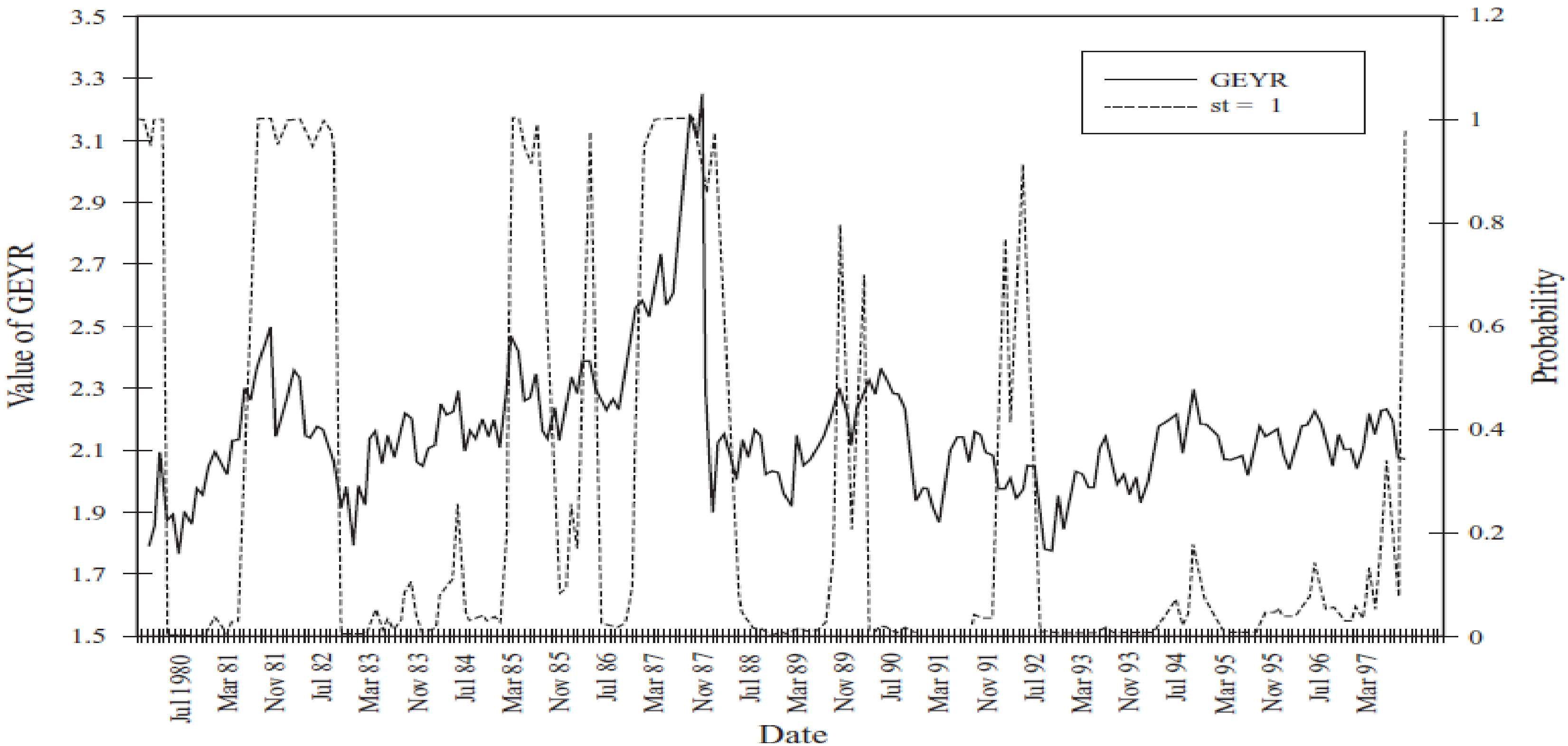
Markov Switching Models: Typical Outputs

Statistic	μ_1 (1)	μ_2 (2)	σ_1^2 (3)	σ_2^2 (4)	p_{11} (5)	p_{22} (6)	N_1 (7)	N_2 (8)
UK	2.4293	2.0749	0.0624	0.0142	0.9547	0.9719	102	170
	(0.0301)	(0.0367)	(0.0092)	(0.0018)	(0.0726)	(0.0134)		
US	2.4554	2.1218	0.0294	0.0395	0.9717	0.9823	100	172
	(0.0181)	(0.0623)	(0.0604)	(0.0044)	(0.0171)	(0.0106)		
Germany	3.0250	2.1563	0.5510	0.0125	0.9816	0.9328	200	72
	(0.0544)	(0.0154)	(0.0569)	(0.0020)	(0.0107)	(0.0323)		

Notes: Standard errors in parentheses; N_1 and N_2 denote the number of observations deemed to be in regimes 1 and 2, respectively.

- What are the typical outputs you may expect from a **univariate** MS AR(q) model?
  - First, **regime-specific estimates of** means (μ_1 and μ_2), variance (σ^2_1 and σ^2_2), and transition probabilities (p_{11} and p_{22}); these are all parameters
    - There is one set of such parameters per each series (e.g., countries)
  - Second, **plots of the state probabilities** inferred from the data

Threshold AR Models



- Threshold autoregressive (TAR) models are one class of non-linear autoregressive that allow for a locally linear approximation over alternative states, for instance:

$$y_t = \begin{cases} \mu_1 + \phi_1 y_{t-1} + u_{1t} & \textit{if } s_{t-k} < r \\ \mu_2 + \phi_2 y_{t-1} + u_{2t} & \textit{if } s_{t-k} \geq r \end{cases}$$

Threshold AR Models

- In TAR models, the switches are governed by observable variables while in MSMs by a latent Markov state

- The key difference between TAR and Markov switching models is that, under the former, the state variable is assumed known and observable, while it is latent in MSMs
  - For instance in the case of
    $$y_t = \begin{cases} \mu_1 + \phi_1 y_{t-1} + u_{1t} & \textit{if } s_{t-k} < r \\ \mu_2 + \phi_2 y_{t-1} + u_{2t} & \textit{if } s_{t-k} \geq r \end{cases}$$
    the model contains a first order AR in each of two regimes, and there is only one threshold value, *r*
  - The number of thresholds will always be the number of regimes minus one
    - The dependent variable y_t follows an AR process with intercept μ_1 and autoregressive coefficient ϕ_1 if the value of the state-determining variable lagged *k* periods, denoted s_{t-k} is lower than some threshold value r
    - If the value of the state-determining variable lagged k periods, is equal to or greater than that threshold value *r* , y_t is an AR(1) with intercept μ_2 and autoregressive coefficient ϕ_2

Threshold AR and STAR Models

- s_{t-k} , the state-determining variable, can be any variable that is thought to make y_t shift from one set of behavior to another
  - If $k = 0$, it is the current value of the state-determining variable that influences the regime that y is in at time t
  - In many applications k is set to 1, so that the immediately preceding value of s is the one that determines the current value of y
- The simplest case for the state determining variable is where it is the variable under study, i.e. $s_{t-k} = y_{t-k}$
- This situation is known as a self-exciting TAR, or a SETAR

$$y_t = \begin{cases} \mu_1 + \phi_1 y_{t-1} + u_{1t} \ \text{if} \ y_{t-k} < r \\ \mu_2 + \phi_2 y_{t-1} + u_{2t} \ \text{if} \ y_{t-k} \geq r \end{cases}$$

  - The number of lags of the dependent variable used in each regime may be higher than one, and the number of lags need not be the same for both regimes
  - The number of states can also be increased to more than two

Threshold AR and STAR Models

- A general threshold autoregressive model is:

$$x_t = \sum_{j=1}^{J} I_t^{(j)} \left(\phi_0^{(j)} + \sum_{i=1}^{p_j} \phi_i^{(j)} x_{t-i} + u_t^{(j)} \right), \; r_{j-1} \leq z_{t-d} \leq r_j$$

  - $I^{(j)}_t$ is an indicator function for the *j*th regime taking the value 1 if the underlying variable is in state j and zero otherwise
  - z_{t-d} is an observed variable determining the switching
  - $u^{(j)}_t$ is a zero mean IID error process
- Estimation of the model parameters is considerably more difficult than for a standard linear autoregressive process, because in general they cannot be determined simultaneously in a simple way
  - It may be preferable to endogenously estimate the values of the threshold(s) as part of the non-linear least squares (NLS) optimisation procedure, but this is not feasible
  - The underlying functional relationship between the variables is discontinuous in the thresholds, such that the thresholds cannot be estimated at the same time as the other components

Specification Tests in Regime Switching Models

- Testing for the number of regimes in MSMs and (S)TAR models is subject to a nuisance parameter problem
  - One solution sometimes used in empirical work is to use a grid search procedure that seeks the minimal residual sum of squares over a range of values of the threshold(s) for an assumed model
- In the context of both Markov switching and (S)TAR models, it is of interest to determine whether the threshold models represent a superior fit to the data relative to a comparable linear model
  - A tempting, but incorrect, way to examine this issue would be to do something like the following: estimate the desired threshold model and the linear counterpart, and compare the residual sums of squares using an F- or LR test
- However, such an approach is not valid in this instance owing to unidentified nuisance parameters under the null hypothesis
  - In other words, the null hypothesis would be that the additional parameters were zero so that the model collapsed to the linear specification, but under the linear model, there is no threshold

Specification Tests in Regime Switching Models

Model	For regime	Number of observations
$\hat{E}_t = \underset{(0.0458)}{0.0222} + \underset{(0.0079)}{0.9962} E_{t-1}$	$E_{t-1} < 5.8306$	344
$\hat{E}_t = \underset{(0.2391)}{0.3486} + \underset{(0.0889)}{0.4394} E_{t-1} + \underset{(0.1098)}{0.3057} E_{t-2} + \underset{(0.0866)}{0.1951} E_{t-3}$	$E_{t-1} \geq 5.8306$	103

- The conditions required to show that the test statistics follow a standard asymptotic distribution do not apply
- Analytical critical values are not available, and critical values must be obtained via simulation for each individual case

- This a typical example of STAR outputs concerning exchange rates
- It is possible to generalize (S)TAR and MSMs to include variance process that contain GARCH of various forms

$$R_{t+1} = \mu_{St+1} + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \omega_{St+1} + \alpha_{St+1}\epsilon^2_t + \beta_{St+1}\sigma^2_t), \; S_{t+1} = 1, 2$$

$$\Sigma_{t+1} = D_{S_{t+1}} \Gamma_{S_{t+1}} D_{S_{t+1}}$$

$$Q_{t+1} = E\left[z_t z_t'\right](1 - \alpha_{S_{t+1}} - \beta_{S_{t+1}}) + \alpha_{S_{t+1}}\left(z_t z_t'\right) + \beta_{S_{t+1}} Q_{S_{t+1}} \; S_{t+1} = 1, 2$$

Implied Kurtosis and Asymmetries from MSMs

- Models with regimes capture fait tails, asymmetries and multi-modalities in the unconditional density of returns

- First claim: MS models can be useful in active risk management and they do capture deviations from normality
  - For instance, consider the simple case in which $\pi_{1t} = \pi_1 \equiv \Pr(S_t = 1)$ and $\pi_{2t} = \pi_2 \equiv \Pr(S_t = 2)$
  - This is not Markov chain probabilities of each of the two regimes are just independent of the past
  - We talk about IID Mixture Distributions
  - Yet, even in this case combining two normal densities delivers arbitrary skewness and excess kurtosis

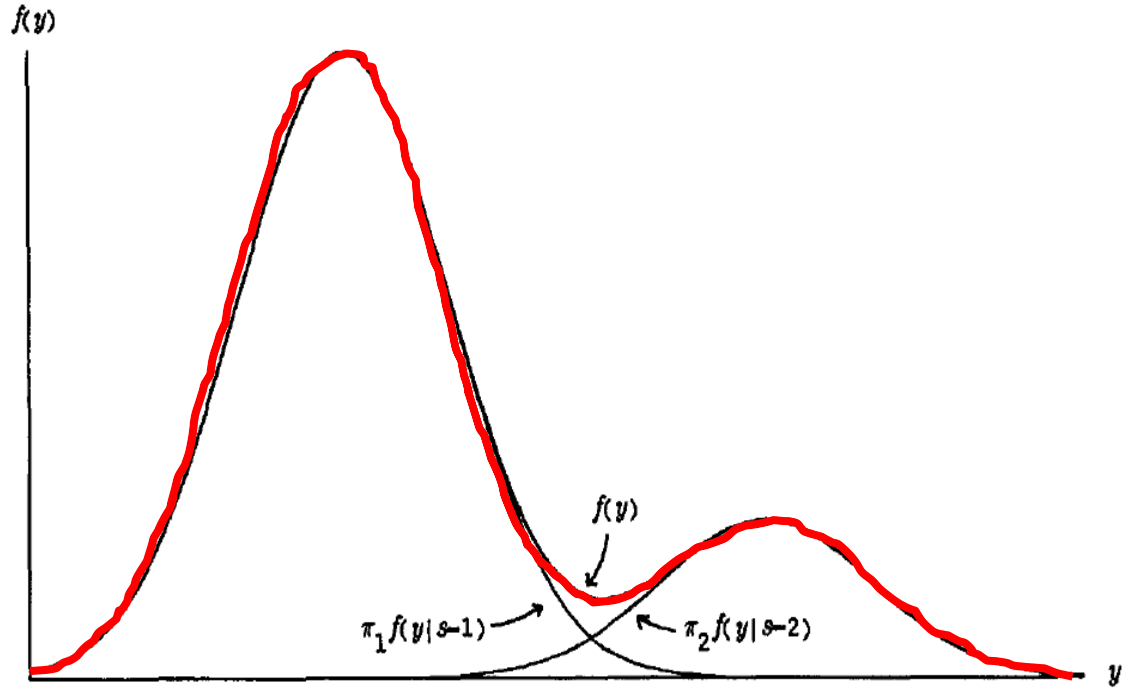


FIGURE 22.2 Density of mixture of two Gaussian distributions with $y_t|s_t = 1 \sim N(0, 1)$, $y_t|s_t = 2 \sim N(4, 1)$, and $P\{s_t = 1\} = 0.8$.

Implied Kurtosis and Asymmetries from MSMs

- However, a mixture of two Gaussian variables need not have the bimodal appearance: Gaussian mixtures can also produce a unimo-dal density, allowing skew or kurtosis diffe-rent from that of a single Gaussian variabl
- Therefore Markov mo dels can clearly capture non-normalities in the data

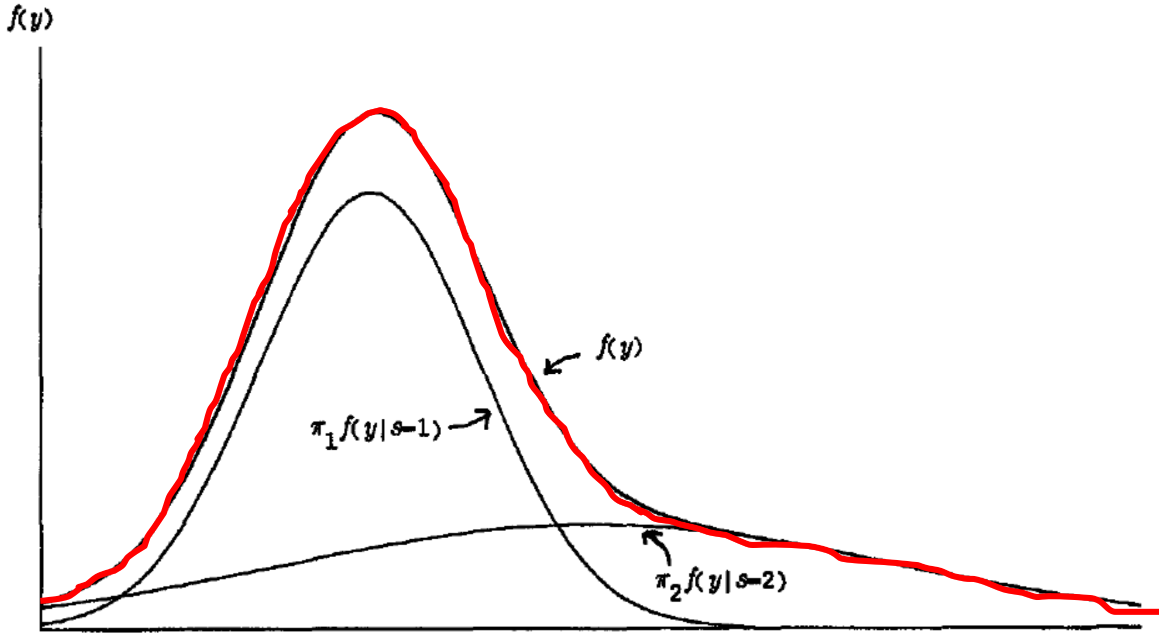


FIGURE 22.3 Density of mixture of two Gaussian distributions with $y_t|s_t = 1 \sim N(0, 1)$, $y_t|s_t = 2 \sim N(2, 8)$, and $P\{s_t = 1\} = 0.6$.

- <u>Second claim</u>: Although at some frequencies, MS directly competes with GARCH, at high (daily, weekly) frequencies MS, ARCH, DCC, and t-student variants are compatible

Reading List/How to prepare the exam

- Carefully read these Lecture Slides + class notes
- Possibly read BROOKS, chapter 10
- Lecture Notes are available on Prof. Guidolin's personal web page
- *Guidolin, M. (2012) "Markov Switching Models in Empirical Finance", in Advances in Econometrics (D. Drukker et al., editors), Emerald Publishers Ltd.
- *Hamilton, J. (2005) "Regime Switching Models", in New Palgrave Dictionary of Economics.