Exercises 8.23 and .. - Nielsen and Chuang

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8.23, Amplitude Damping of a Dual-Rail Qubit Let

$$|\psi\rangle = a\,|01\rangle + b\,|10\rangle \tag{1}$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ produces a new state ρ', according to the following operator sum representation:

$$\varepsilon_{AD} \otimes \varepsilon_{AD}(\rho) = \sum_i E_i^{dr} \rho (E_i^{dr})^\dagger = \rho' \tag{2}$$

Where the Kraus operators E_i^{dr} are given by the tensor product of the Kraus operators E_j, which are the operation elements of the Amplitude Damping acting on a single qubit:

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{3}$$

The Kraus operators E_i^{dr} the Amplitude Damping acting on two qubits are then given by:

$$E_{00}^{dr} = E_0 \otimes E_0, \quad , \quad E_{01}^{dr} = E_0 \otimes E_1, \quad E_{10}^{dr} = E_1 \otimes E_0, \quad E_{11}^{dr} = E_1 \otimes E_1 \tag{4}$$

Thus, they are equal to:

$$\begin{aligned} E_{00}^{dr} = |00\rangle\langle 00| + \sqrt{1-\gamma}\,|01\rangle\langle 01| + \sqrt{1-\gamma}\,|10\rangle\langle 10| + (1-\gamma)\,|11\rangle\langle 11| = \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix} \end{aligned} \tag{5}$$

$$E_{01}^{dr} = \sqrt{\gamma}\,|00\rangle\langle 01| + \sqrt{\gamma}\sqrt{1-\gamma}\,|10\rangle\langle 11| = \begin{pmatrix} 0 & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma}\sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$$E_{10}^{dr} = \sqrt{\gamma}\,|00\rangle\langle 01| + \sqrt{\gamma}\sqrt{1-\gamma}\,|01\rangle\langle 11| = \begin{pmatrix} 0 & 0 & \sqrt{\gamma} & 0 \\ 0 & 0 & 0 & \sqrt{\gamma}\sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

$$E_{11}^{dr} = \gamma\,|00\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

If we restrict the operator to the span $|01\rangle$, $|10\rangle$, we get the operators: It can be easily checked that the quantum operation can be described with 3 operators:

$$E_0^{dr} = \sqrt{1-\gamma} I \tag{9}$$

$$E_1^{dr} = \sqrt{\gamma}|00\rangle\langle 01| = \sqrt{\gamma}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10}$$

$$E_2^{dr} = \sqrt{\gamma}|00\rangle\langle 10| = \sqrt{\gamma}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{11}$$

The application of this operator to the state ρ produces the state ρ', which is given by:

$$\rho' = \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (1-\gamma)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{12}$$

So, with probability γ, the state is projected to $|00\rangle$, orthogonal to $|\psi\rangle$, while with probability $1-\gamma$, the state is unchanged. This means that the state $|00\rangle$ can be used to detect amplitude damping errors with measurement operators:

$$M_0 = |00\rangle\langle 00| \quad \text{projector on span}\{|00\rangle\} \tag{13}$$

$$M_1 = |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11| \quad \text{projector on span}\{|01\rangle, |10\rangle, |11\rangle\} \tag{14}$$

We can get the same result algebraically applying amplitude damping on the first qubit and second qubit, respectively using the third and fourth qubit for the environment, which both start from the state 0.

Since the circuit of the amplitude damping is the following:

So the one for the amplitude damping on dual-rail is:

Algebraically, we start from a tensor product state:

And we apply:

1. a controlled rotation on the Y axis (control on first qubit, target on the third qubit)

2. a CNOT, with control on third qubit and target on the first

The rotation on the Y axis of θ is defined as:

It rotates one term of the two present in the starting system, as it is the only one with the first qubit set to 1, producing

Then the CNOT flips only one of the 3 terms, producing

This is ω_1, which is the state of the system after having applied amplitude damping on the first qubit. To get ω_2 we apply again:

1. a controlled rotation on the Y axis (control on 2nd qubit, target on the 4th qubit)

2. a CNOT, with control on 4th qubit and target on the 2nd

Which is:

Now, we have a state the we can call $|\phi\rangle$, and our target is to trace out the environment. To do that, we pass to the density matrix representation $|\phi\rangle\langle\phi|$, which has 16 terms.

Then, we proceed to trace out of the environment.

In order to simplify the calculations, we can do both the operations together, discarding the cross terms that don't have matching qubits for the environment:

Which can be simplified to

If we set $\gamma = sin^2\frac{\theta}{2}$, we get the same result as ?:

So, we proved that the amplitude damping on 2 qubits is described by the operation elements ?

9. ,