Exercises 8.23 and 9.21 - Nielsen and Chuang

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8.23, Amplitude Damping of a Dual-Rail Qubit

A dual-rail qubit is defined as

$$|\psi\rangle = a\,|01\rangle + b\,|10\rangle \tag{1}$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ produces a new state ρ', according to the following operator sum representation:

$$\varepsilon_{AD} \otimes \varepsilon_{AD}(\rho) = \sum_i E_i \rho E_i^\dagger = \rho' \tag{2}$$

Where the Kraus operators E_i are given by the tensor product of the Kraus operators E_j^s, which are the operation elements of the Amplitude Damping acting on a single qubit:

$$E_0^s = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1^s = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{3}$$

Thus, the Kraus operators E_i the Amplitude Damping acting on two qubits are given by:

$$E_{00} = E_0^s \otimes E_0^s, \quad , \quad E_{01} = E_0^s \otimes E_1^s, \quad E_{10} = E_1^s \otimes E_0^s, \quad E_{11} = E_1^s \otimes E_1^s \tag{4}$$

$$\begin{aligned} E_{00} = |00\rangle\langle 00| + \sqrt{1-\gamma}\,|01\rangle\langle 01| + \sqrt{1-\gamma}\,|10\rangle\langle 10| + (1-\gamma)\,|11\rangle\langle 11| = \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix} \end{aligned} \tag{5}$$

$$E_{01} = \sqrt{\gamma}\,|00\rangle\langle 01| + \sqrt{\gamma}\sqrt{1-\gamma}\,|10\rangle\langle 11| = \begin{pmatrix} 0 & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma}\sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$$E_{10} = \sqrt{\gamma}\,|00\rangle\langle 01| + \sqrt{\gamma}\sqrt{1-\gamma}\,|01\rangle\langle 11| = \begin{pmatrix} 0 & 0 & \sqrt{\gamma} & 0 \\ 0 & 0 & 0 & \sqrt{\gamma}\sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

$$E_{11} = \gamma\,|00\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

We have that

$$E_{00}\rho E_{00}^{\dagger} = (1-\gamma)\rho \tag{9}$$

$$E_{01}\rho E_{01}^{\dagger} = |a|^2\gamma\,|00\rangle\langle 00| \tag{10}$$

$$E_{10}\rho E_{10}^{\dagger} = |b|^2\gamma\,|00\rangle\langle 00| \tag{11}$$

$$E_{11}\rho E_{11}^{\dagger} = 0 \tag{12}$$

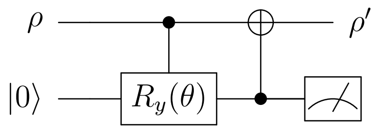
And so, the application of these operators to the state ρ produces the state ρ', which is given by:

$$\rho' = \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + (1-\gamma) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

So, with probability γ, the state is projected to $|00\rangle$, orthogonal to $|\psi\rangle$, while with probability $1-\gamma$, the state is unchanged.

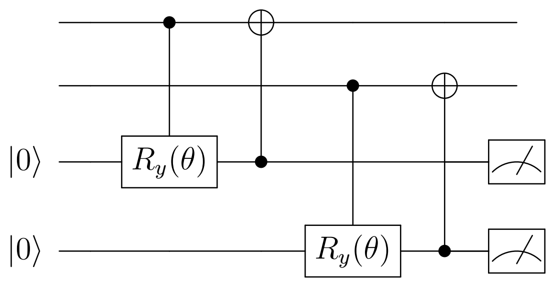
We can get the same result algebraically applying amplitude damping on the first qubit and second qubit, respectively using the third and fourth qubit for the environment, which both start from the state 0.

Since the circuit of the amplitude damping is the following:



(14)

So the one for the amplitude damping on dual-rail is:



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Algebraically, we start from a tensor product state:

$$\omega_0 = |\psi\rangle|00\rangle = a\,|01\rangle|00\rangle + b\,|10\rangle|00\rangle = a\,|0100\rangle + b\,|1000\rangle \tag{16}$$

And we apply:

1. a controlled rotation on the Y axis (control on first qubit, target on the third qubit)
2. a CNOT, with control on third qubit and target on the first

So that

$$\omega_1 = CX_{3,1} CR_y(\theta)_{1,3} \omega_0 \tag{17}$$

The rotation on the Y axis of θ is defined as:

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{18}$$

And acts on $|0\rangle$ as following:

$$R_y(\theta)|0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle \tag{19}$$

It rotates one term of the two present in the starting system, as it is the only one with the first qubit set to 1, producing

$$\omega_1 = CX_{3,1}(a|0100\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|1010\rangle) \tag{20}$$

Then the CNOT flips only the third term, producing

$$\omega_1 = a|0100\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle \tag{21}$$

This is the state of the system after having applied amplitude damping on the first qubit.

To get ω_2 we apply again:

1. a controlled rotation on the Y axis (control on 2nd qubit, target on the 4th qubit)
2. a CNOT, with control on 4th qubit and target on the 2nd

Which is:

$$\omega_2 = CX_{4,2} CR_y(\theta)_{2,4} \omega_1 \tag{22}$$

$$\omega_2 = CX_{4,2}(a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0101\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle) \tag{23}$$

$$\omega_2 = a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0001\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle \tag{24}$$

Now, we have a state the we can call $|\phi\rangle$, and our target is to trace out the environment.

$$|\phi\rangle = \omega_2 = a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0001\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle \tag{25}$$

To do that, we pass to the density matrix representation $|\phi\rangle\langle\phi|$, which has 16 terms, deriving from the tensor product of the 4 terms of $|\phi\rangle$ with their conjugates:

$$\begin{aligned}|\phi\rangle\langle\phi| = (&a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0001\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle)\\ &(a^*\cos\frac{\theta}{2}\langle 0100| + a^*\sin\frac{\theta}{2}\langle 0001| + b^*\cos\frac{\theta}{2}\langle 1000| + b^*\sin\frac{\theta}{2}\langle 0010|)\end{aligned} \tag{26}$$

We can simplify calculations by taking the trace of the environment while multiplying the terms: doing that, we can discard the terms that don't have matching qubits for the environment, as they will be zero when taking the trace.

In particular, we have 4 terms coming from the multiplications of the terms by themselves (where obviously the environment bits are matching), plus 2 more coming from the multiplication of the first and the third terms of ϕ, which have the same bits from the environment.

So, as a result:

$$Tr_e(|\phi\rangle\langle\phi|) = |a|^2\cos^2\frac{\theta}{2}|01\rangle\langle 01| + |a|^2\sin^2\frac{\theta}{2}|00\rangle\langle 00| + |b|^2\cos^2\frac{\theta}{2}|10\rangle\langle 10| \tag{27}$$

$$+ |b|^2\sin^2\frac{\theta}{2}|00\rangle\langle 00| + ab^*\cos^2\frac{\theta}{2}|01\rangle\langle 10| + a^*b\cos^2\frac{\theta}{2}|10\rangle\langle 01| \tag{28}$$

Which can be simplified to

$$Tr_e(|\phi\rangle\langle\phi|) = \sin^2\frac{\theta}{2}|00\rangle\langle 00| \tag{29}$$

$$+ |a|^2\cos^2\frac{\theta}{2}|01\rangle\langle 01| + ab^*\cos^2\frac{\theta}{2}|01\rangle\langle 10| \tag{30}$$

$$+ a^*b\cos^2\frac{\theta}{2}|10\rangle\langle 01| + |b|^2\cos^2\frac{\theta}{2}|10\rangle\langle 10| \tag{31}$$

If we set $\gamma = sin^2\frac{\theta}{2}$, we get the same result as eq.13.

So, we proved that the amplitude damping on 2 qubits is described by the operation elements E_{00}, E_{01}, E_{10}, E_{11}.

9.21, Relationship between Fidelity and Trace Distance

When comparing pure states and mixed states it is possible to make a stronger statement with respect to the bound between fidelity and trace distance of two mixed states:

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \tag{32}$$

As a matter of fact, prove that:

$$1 - F(|\psi\rangle, \sigma)^2 \leq D(|\psi\rangle, \sigma) \tag{33}$$

Where $|\psi\rangle$ is a pure state and σ is a mixed state.