Exercises 8.23 and 9.21 - Nielsen and Chuang

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8.23, Amplitude Damping of a Dual-Rail Qubit

A dual-rail qubit is defined as

$$|\psi\rangle = a\,|01\rangle + b\,|10\rangle \tag{1}$$

Applying $\varepsilon_{AD} \otimes \varepsilon_{AD}$ to $\rho = |\psi\rangle\langle\psi|$ produces a new state ρ', according to the following operator sum representation:

$$(\varepsilon_{AD} \otimes \varepsilon_{AD})(\rho) = \sum_i E_i \rho E_i^\dagger = \rho' \tag{2}$$

Where the Kraus operators E_i are given by the tensor product of the Kraus operators E_j^s, which are the operation elements of the Amplitude Damping acting on a single qubit:

$$E_0^s = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, \quad E_1^s = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix} \tag{3}$$

Thus, the Kraus operators E_i the Amplitude Damping acting on two qubits are given by:

$$E_{00} = E_0^s \otimes E_0^s, \quad E_{01} = E_0^s \otimes E_1^s, \quad E_{10} = E_1^s \otimes E_0^s, \quad E_{11} = E_1^s \otimes E_1^s \tag{4}$$

$$\begin{aligned} E_{00} = |00\rangle\langle 00| + \sqrt{1-\gamma}\,|01\rangle\langle 01| + \sqrt{1-\gamma}\,|10\rangle\langle 10| + (1-\gamma)\,|11\rangle\langle 11| = \\ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 & 0 \\ 0 & 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & 0 & 1-\gamma \end{pmatrix} \end{aligned} \tag{5}$$

$$E_{01} = \sqrt{\gamma}\,|00\rangle\langle 01| + \sqrt{\gamma}\sqrt{1-\gamma}\,|10\rangle\langle 11| = \begin{pmatrix} 0 & \sqrt{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\gamma}\sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6}$$

$$E_{10} = \sqrt{\gamma}\,|00\rangle\langle 01| + \sqrt{\gamma}\sqrt{1-\gamma}\,|01\rangle\langle 11| = \begin{pmatrix} 0 & 0 & \sqrt{\gamma} & 0 \\ 0 & 0 & 0 & \sqrt{\gamma}\sqrt{1-\gamma} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{7}$$

$$E_{11} = \gamma\,|00\rangle\langle 11| = \begin{pmatrix} 0 & 0 & 0 & \gamma \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{8}$$

We have that

$$E_{00}\rho E_{00}^{\dagger} = (1-\gamma)\rho \tag{9}$$

$$E_{01}\rho E_{01}^{\dagger} = |a|^2\gamma \left|00\right\rangle \left\langle 00\right| \tag{10}$$

$$E_{10}\rho E_{10}^{\dagger} = |b|^2\gamma \left|00\right\rangle \left\langle 00\right| \tag{11}$$

$$E_{11}\rho E_{11}^{\dagger} = 0 \tag{12}$$

And so, the application of these operators to the state ρ produces the state ρ', which is given by:

$$\rho' = (1-\gamma)\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |a|^2 & ab^* & 0 \\ 0 & a^*b & |b|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{13}$$

Thus, with probability γ, the state is projected to $\left|00\right\rangle$, orthogonal to $\left|\psi\right\rangle$, while with probability $1-\gamma$, the state is unchanged.

Notice that the set of operation elements $\{E_{00}, E_{01}, E_{10}, E_{11}\}$, has the same effect of the operators set

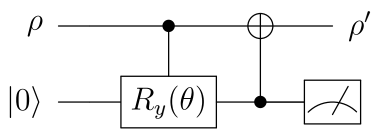
$$E_0 = \sqrt{1-\gamma}(\left|01\right\rangle \left\langle 01\right| + \left|10\right\rangle \left\langle 10\right|) \quad E_1 = \sqrt{\gamma}|00\rangle\langle 01| \quad E_2 = \sqrt{\gamma}|00\rangle\langle 10| \tag{14}$$

which, in the span $\left|01\right\rangle, \left|10\right\rangle$, satisfies the completeness relation $\sum_i E_i^{\dagger} E_i = I$.

Theorem The application of amplitude damping applied on a dual rail qubit gives a process which can be described by the operation elements E_{00}, E_{01}, E_{10}, E_{11}, that is, either nothing happens to the qubit, or the qubit is transformed into the state $\left|00\right\rangle$.

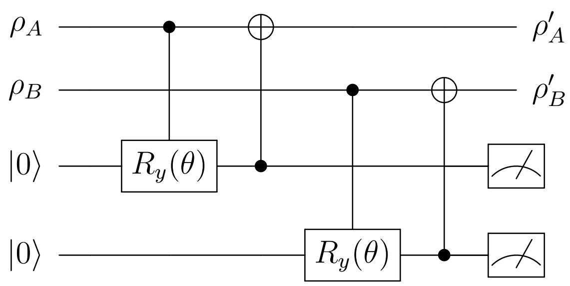
Proof Consider the effect of amplitude damping on the first qubit and second qubit separetely, respectively associated to the third and fourth qubit for the environment, both starting from the state 0.

Since the circuit of the amplitude damping on a single qubit is the following:



(15)

So the one for the amplitude damping on a dual-rail qubit is:



(16)

where ρ_A and ρ_B represent the first and second qubit of the dual-rail qubit, respectively.

Let ω_0 be the starting state of the system, ω_1 the state after the amplitude damping on the first qubit, and ω_2 the state after the amplitude damping on the second qubit.

$$\omega_0 = |\psi\rangle\,|00\rangle = a\,|0100\rangle + b\,|1000\rangle \tag{17}$$

ω_1 is obtained applying, in sequence:

1. a controlled rotation on the Y axis: $CR_y(\theta)_{1,3}$,
2. a controlled not $CX_{3,1}$

onto the state ω_0

$$\omega_1 = CX_{3,1}CR_y(\theta)_{1,3}(\omega_0) \tag{18}$$

Where the rotation on the Y axis of θ is defined as:

$$R_y(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{19}$$

acting on $|0\rangle$ as following:

$$R_y(\theta)\,|0\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\,|1\rangle \tag{20}$$

Only the second term of ω_0 is rotated, as it is the only one with the first qubit set to 1, producing

$$\omega_1 = CX_{3,1}(a\,|0100\rangle + b\cos\frac{\theta}{2}\,|1000\rangle + b\sin\frac{\theta}{2}\,|1010\rangle) \tag{21}$$

Then the CNOT flips only the third term, producing

$$\omega_1 = a\,|0100\rangle + b\cos\frac{\theta}{2}\,|1000\rangle + b\sin\frac{\theta}{2}\,|0010\rangle \tag{22}$$

This is the state of the system after having applied amplitude damping on the first qubit.

ω_2 is obtained applying, in sequence, another pair of gates (controlled rotation and controlled not), on the second and fourth qubit:

$$\omega_2 = CX_{4,2}CR_y(\theta)_{2,4}(\omega_1) \tag{23}$$

$$\omega_2 = CX_{4,2}(a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0101\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle) \tag{24}$$

$$\omega_2 = a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0001\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle \tag{25}$$

Now, let

$$|\phi\rangle = \omega_2 = a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0001\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle \tag{26}$$

Our target is to trace out the environment from this state.

To do that, we pass to the density matrix representation $|\phi\rangle\langle\phi|$, which has 16 terms, deriving from the tensor product of the 4 terms of $|\phi\rangle$ with their conjugates:

$$\begin{aligned}|\phi\rangle\langle\phi| = (&a\cos\frac{\theta}{2}|0100\rangle + a\sin\frac{\theta}{2}|0001\rangle + b\cos\frac{\theta}{2}|1000\rangle + b\sin\frac{\theta}{2}|0010\rangle)\\ &(a^*\cos\frac{\theta}{2}\langle 0100| + a^*\sin\frac{\theta}{2}\langle 0001| + b^*\cos\frac{\theta}{2}\langle 1000| + b^*\sin\frac{\theta}{2}\langle 0010|)\end{aligned} \tag{27}$$

We can simplify calculations by taking the trace of the environment while multiplying the terms: we discard the terms that do not have matching qubits for the environment.

In particular, we have 4 terms coming from the multiplications of the terms by themselves (where obviously the environment qubits are matching), plus 2 more coming from the cross multiplication of the first and the third terms of $|\phi\rangle$:

$$Tr_e(|\phi\rangle\langle\phi|) = |a|^2\cos^2\frac{\theta}{2}|01\rangle\langle 01| + |a|^2\sin^2\frac{\theta}{2}|00\rangle\langle 00| + |b|^2\cos^2\frac{\theta}{2}|10\rangle\langle 10| \tag{28}$$

$$+ |b|^2\sin^2\frac{\theta}{2}|00\rangle\langle 00| + ab^*\cos^2\frac{\theta}{2}|01\rangle\langle 10| + a^*b\cos^2\frac{\theta}{2}|10\rangle\langle 01| \tag{29}$$

Which can be simplified, using $|a|^2 + |b|^2 = 1$, to

$$Tr_e(|\phi\rangle\langle\phi|) = \sin^2\frac{\theta}{2}|00\rangle\langle 00| \tag{30}$$

$$+ |a|^2\cos^2\frac{\theta}{2}|01\rangle\langle 01| + ab^*\cos^2\frac{\theta}{2}|01\rangle\langle 10| \tag{31}$$

$$+ a^*b\cos^2\frac{\theta}{2}|10\rangle\langle 01| + |b|^2\cos^2\frac{\theta}{2}|10\rangle\langle 10| \tag{32}$$

Let $\gamma = sin^2\frac{\theta}{2}$

$$
\begin{aligned}
Tr_e(|\phi\rangle\langle\phi|) &= \gamma\,|00\rangle\langle 00| && (33)\\
&+ (1-\gamma)|a|^2\,|01\rangle\langle 01| + (1-\gamma)ab^*\,|01\rangle\langle 10| && (34)\\
&+ (1-\gamma)a^*b\,|10\rangle\langle 01| + (1-\gamma)|b|^2\,|10\rangle\langle 10| && (35)\\
&= \rho' && (36)
\end{aligned}
$$

thus, we got the same result as eq:13, proving that the amplitude damping on 2 qubits is described by the operation elements E_{00}, E_{01}, E_{10}, E_{11}.

9.21, Relationship between Fidelity and Trace Distance

The relationship between fidelity and trace distance of two mixed states is defined as

$$1 - F(\rho, \sigma) \leq D(\rho, \sigma) \tag{37}$$

Theorem When comparing pure states and mixed states it is possible to make a stronger statement:

$$1 - F(|\psi\rangle, \sigma)^2 \leq D(|\psi\rangle, \sigma) \tag{38}$$

where $|\psi\rangle$ is a pure state and σ is a mixed state.

Proof Consider the definition of fidelity between a pure and a mixed state

$$F(|\psi\rangle, \sigma) = \sqrt{\langle\psi| \sigma |\psi\rangle} \tag{39}$$

Let $\{E_m\}$ set of POVM, such that we have one and only one term where $E_m = |\psi\rangle \langle\psi|$, let it be $m = 0$, so that $\langle\psi| E_m |\psi\rangle = 1$.

We have

$$p_m = Tr(|\psi\rangle \langle\psi| E_m) \qquad q_m = Tr(\sigma E_m) \tag{40}$$

probabilities of obtaining a measurement outcome labeled by m.

We can rewrite p_m as

$$p_m = Tr(|\psi\rangle \langle\psi| E_m) = \sum_i \langle i| |\psi\rangle \langle\psi| E_m |i\rangle = \langle\psi| E_m |\psi\rangle \tag{41}$$

And write the fidelity as

$$F(|\psi\rangle, \rho) = \sum_m \sqrt{p_m q_m} \tag{42}$$

As, according to our definition of the POVM, $\langle\psi| E_m |\psi\rangle = 1$ if and only if $E_m = |\psi\rangle \langle\psi|$, while it is 0 in the other cases; thus:

$$\sum_m \sqrt{p_m q_m} = \sum_m \sqrt{\langle\psi| E_m |\psi\rangle Tr(\sigma E_m)} = \tag{43}$$

$$= \sqrt{\langle\psi| E_0 |\psi\rangle Tr(\sigma E_0)} = \tag{44}$$

$$= \sqrt{\langle\psi| |\psi\rangle \langle\psi| |\psi\rangle Tr(\sigma |\psi\rangle \langle\psi|)} = \tag{45}$$

$$= \sqrt{\langle\psi| \sigma |\psi\rangle} = F(|\psi\rangle, \sigma) \tag{46}$$

Now, consider the trace distance between the probability distributions p_m and q_m:

$$D(p_m, q_m) = \frac{1}{2}\sum_m |p_m - q_m| \tag{47}$$

From Nielsen and Chuang, Theorem 9.1:

$$D(|\psi\rangle, \sigma) = \max_{\{E_m\}} D(p_m, q_m) \tag{48}$$

where the maximization is over all POVMs $\{E_m\}$.

So we have that the trace distance between p_m and q_m is bounded by the trace distance between the correspondent pure state $|\psi\rangle$ and the mixed state ρ:

$$D(p_m, q_m) \leq D(|\psi\rangle, \sigma) \tag{49}$$

We can prove the following:

$$1 - F(|\psi\rangle, \sigma)^2 = D(p_m, q_m) \leq D(|\psi\rangle, \sigma) \tag{50}$$

In particular, we can start from the trace distance between p_m and q_m (eq:47)

$$D(p_m, q_m) = \frac{1}{2}\sum_m |p_m - q_m| = \tag{51}$$

$$= \frac{1}{2}\sum_{m=0}^{r} |\langle\psi| E_m |\psi\rangle - Tr(\sigma E_m)| \tag{52}$$

where r is the number of elements in the POVM set.

We have that:

- For $m = 0$, $p_m = \langle\psi| E_0 |\psi\rangle = 1$, and $q_m = Tr(\sigma E_0) = \langle\psi| \sigma |\psi\rangle$,
- For $m > 0$, $p_m = \langle\psi| E_m |\psi\rangle = 0$, and $q_m = Tr(\sigma E_m) > 0$.

Thus, we can split the sum in two parts, and rewrite the trace distance as

$$D(p_m, q_m) = \frac{1}{2}\left(|\langle\psi| E_0 |\psi\rangle - Tr(\sigma E_0)| + |\sum_{m=1}^{r} -Tr(\sigma E_m)|\right) = \tag{53}$$

$$= \frac{1}{2}\left(|\langle\psi| |\psi\rangle \langle\psi| |\psi\rangle - \langle\psi| \sigma |\psi\rangle | + | -\sum_{m=1}^{r} Tr(\sigma E_m)|\right) = \tag{54}$$

$$= \frac{1}{2}\left(|1 - F(|\psi\rangle, \sigma)^2| + | -\sum_{m=1}^{r} Tr(\sigma E_m)|\right) = \tag{55}$$

$$= \frac{1}{2}\left(1 - F(|\psi\rangle, \sigma)^2 + \sum_{m=1}^{r} Tr(\sigma E_m)\right) \tag{56}$$

We have that

$$\sum_{m=0}^{r} Tr(\sigma E_m) = 1 \tag{57}$$

Thus

$$\sum_{m=1}^{r} Tr(\sigma E_m) = 1 - Tr(\sigma E_0) = 1 - F(|\psi\rangle, \sigma)^2 \tag{58}$$

So

$$D(p_m, q_m) = \frac{1}{2}\left(1 - F(|\psi\rangle, \sigma)^2 + 1 - F(|\psi\rangle, \sigma)^2\right) = \tag{59}$$

$$= 1 - F(|\psi\rangle, \sigma)^2 \tag{60}$$

Which proves the theorem:

$$1 - F(|\psi\rangle, \sigma)^2 = D(p_m, q_m) \leq D(|\psi\rangle, \sigma) \tag{61}$$