Exercise 8.1

Under the transformation $\rho \to \mathcal{E}(\rho)$, the state transforms as $|\psi\rangle \to U|\psi\rangle$. Hence, the new density operator is $\rho' = U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger$, and therefore ρ transforms as $\rho \to U\rho U^\dagger$.

Exercise 8.2

Let $\rho = \sum_i p_i |i\rangle\langle i|$, hence after the measurement for each of the i states will take the form, $|i'\rangle = \frac{M_m|i\rangle}{\sqrt{\langle i|M_m^\dagger M_m|i\rangle}}$. Therefore, for the final state ρ' we'll have,

$$\rho' = \sum_i p_i \frac{M_m|i\rangle\langle i|M_m^\dagger}{\sqrt{\langle i|M_m^\dagger M_m|i\rangle}\sqrt{\langle i|M_m M_m^\dagger|i\rangle}} = \frac{\mathcal{E}_m(\rho)}{tr(\mathcal{E}_m(\rho))}$$

For the probability of the m state, using $p(m|i) = \langle i|M_m^\dagger M_m|i\rangle$, we get

$$p(m) = \sum_i p_i p(m|i) = \sum_i p_i \langle i|M_m^\dagger M_m|i\rangle = \sum_i p_i tr(M_m^\dagger M_m|i\rangle\langle i|) = tr(\mathcal{E}_m(\rho))$$

Exercise 8.3

Initially we have the state $\rho \otimes |0_{CD}\rangle\langle 0_{CD}|$. Consider the action of $\mathcal{E}$(i basis for A, j basis for D),

$$\mathcal{E}(\rho) = tr_A(tr_D(U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger)) = \sum_i\sum_j \langle i|\langle j|U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger|j\rangle|i\rangle =$$

$$\sum_i\sum_j \langle i|\langle j|U|0_{CD}\rangle\rho\langle 0_{CD}|U^\dagger|j\rangle|i\rangle = \sum_j E_j\rho E_j^\dagger.$$

where $E_j = \sum_i \langle i|\langle j|U|0_{CD}\rangle$

Also, (using $\sum_i |i\rangle\langle i| = I$)

$$\sum_j E_j^\dagger E_j = \sum_i\sum_j \langle 0_{CD}|U^\dagger|j\rangle|i\rangle\langle i|\langle j|U|0_{CD}\rangle = I\langle 0_{CD}|U^\dagger U|0_{CD}\rangle = I\langle 0_{CD}|0_{CD}\rangle = I$$

Exercise 8.4

$E_k = \langle k|U|0\rangle$, hence using the orthogonality of the $|0\rangle$ and $|1\rangle$ states, $E_0 = P_0$, $E_1 = P_1$. Therefore,

$\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$

Exercise 8.5

$E_0 = \frac{X}{\sqrt{2}}$, $E_1 = \frac{Y}{\sqrt{2}}$

$\mathcal{E}(\rho) = \frac{1}{2}(X\rho X^\dagger + Y\rho Y^\dagger) = \frac{1}{2}(X\rho X - Y\rho Y)$

Exercise 8.6

In general the composition of quantum operations is still a quantum operation, hence we only prove the general case.

Let ρ belong to a Hilbert Space $\mathcal{H}$ and let the quantum operations be given by, $\mathcal{E}(\rho) = \sum_i E_i\rho E_i^\dagger$ and $\mathcal{F}(\rho) = \sum_i F_i\rho F_i^\dagger$.

As by definition, $\mathcal{E}$ and $\mathcal{F}$ are quantum operations, there exist states $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{F}}$ and unitary operators $U_{\mathcal{E}}$ and $U_{\mathcal{F}}$ on Hilbert spaces $\mathcal{K}_{\mathcal{E}}$ and $\mathcal{K}_{\mathcal{F}}$, respectively, such that $\mathcal{E}(\rho) = tr_{\mathcal{K}_{\mathcal{E}}}(U_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}}]U_{\mathcal{E}}^{\dagger})$ and $\mathcal{F}(\rho) = tr_{\mathcal{K}_{\mathcal{F}}}(U_{\mathcal{F}}[\rho \otimes \omega_{\mathcal{F}}]U_{\mathcal{F}}^{\dagger})$.
Consider the Hilbert space $\mathcal{K} = \mathcal{K}_{\mathcal{E}} \otimes \mathcal{K}_{\mathcal{F}}$ and the state $\omega = \omega_{\mathcal{E}} \otimes \omega_{\mathcal{F}}$. Consider the ampliations $\hat{U}_{\mathcal{E}}$ and $\hat{U}_{\mathcal{F}}$ of $U_{\mathcal{E}}$ and $U_{\mathcal{F}}$ to $\mathcal{H} \otimes \mathcal{K}$, i.e $\hat{U}_{\mathcal{E}} = U_{\mathcal{E}} \otimes \mathcal{I}$ and $\hat{U}_{\mathcal{F}} = \mathcal{I} \otimes U_{\mathcal{F}}$. Lastly, take $U = \hat{U}_{\mathcal{F}}\hat{U}_{\mathcal{E}}$, which is an operator on $\mathcal{H} \otimes \mathcal{K}$. Finally, consider

$$\begin{aligned}
tr_{\mathcal{K}}(U[\rho \otimes \omega]U^{\dagger}) &= tr_{\mathcal{K}_{\mathcal{E}} \otimes \mathcal{K}_{\mathcal{F}}}(\hat{U}_{\mathcal{F}}\hat{U}_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}} \otimes \omega_{\mathcal{F}}]\hat{U}_{\mathcal{E}}\hat{U}_{\mathcal{F}}) \\
&= tr_{\mathcal{K}_{\mathcal{F}}}(tr_{\mathcal{K}_{\mathcal{E}}}(\hat{U}_{\mathcal{F}}(U_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}}]U_{\mathcal{E}}^{\dagger} \otimes \omega_{\mathcal{F}})\hat{U}_{\mathcal{F}}^{\dagger})) \\
&= tr_{\mathcal{K}_{\mathcal{F}}}(U_{\mathcal{F}}(tr_{\mathcal{K}_{\mathcal{E}}}(U_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}}]U_{\mathcal{E}}^{\dagger}) \otimes \omega_{\mathcal{F}})U_{\mathcal{F}}^{\dagger}) \\
&= tr_{\mathcal{K}_{\mathcal{F}}}(U_{\mathcal{F}}(\mathcal{E}(\rho) \otimes \omega_{\mathcal{F}})U_{\mathcal{F}}^{\dagger}) \\
&= \mathcal{F}(\mathcal{E}(\rho))
\end{aligned}$$

From the trace as previously we can obtain an operator-sum representation, hence the composition even for different input and output spaces is a quantum operation.