Exercise 10.1

Let $|\psi\rangle = a|0\rangle + b|1\rangle$ and the initial state be $|\psi_0\rangle = a|000\rangle + b|100\rangle$.
Applying a CNOT to the first two qubits we get,
$|\psi_1\rangle = a|000\rangle + b|110\rangle$
Applying a CNOT to the first and last qubits we get,
$|\psi_2\rangle = a|000\rangle + b|111\rangle$

Exercise 10.2

$P_\pm = \frac{1}{2}(|0\rangle \pm |1\rangle)(\langle 0| \pm \langle 1|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1| \pm |1\rangle\langle 0| \pm |0\rangle\langle 1|) = \frac{1}{2}(I \pm X)$
Therefore,
$\mathcal{E}(\rho) = (1-2p)\rho + 2pP_+\rho P_+ + 2pP_-\rho P_- = (1-2p)\rho + \frac{1}{2}p(I+X)\rho(I+X) + \frac{1}{2}p(I-X)\rho(I-X) = (1-2p)\rho + p\rho + pX\rho X = (1-p)\rho + pX\rho X$

Exercise 10.3

$Z_2Z_3Z_1Z_2 = [I \otimes (|00\rangle\langle 00| + |11\rangle\langle 11|) - I \otimes (|01\rangle\langle 01| + |10\rangle\langle 10|)][(|00\rangle\langle 00| + |11\rangle\langle 11|) \otimes I - (|01\rangle\langle 01| + |10\rangle\langle 10|) \otimes I] = \underbrace{|000\rangle\langle 000| + |111\rangle\langle 111|}_{P_0} - (\underbrace{|100\rangle\langle 100| + |011\rangle\langle 011|}_{P_1})$

$+ \underbrace{|010\rangle\langle 010| + |101\rangle\langle 101|}_{P_2} - (\underbrace{|001\rangle\langle 001| + |110\rangle\langle 110|}_{P_3})$

Exercise 10.4

1)$|000\rangle\langle 000|$, $|111\rangle\langle 111|$: no bit flip
$|100\rangle\langle 100|$, $|011\rangle\langle 011|$: first bit flipped
$|010\rangle\langle 010|$, $|101\rangle\langle 101|$: second bit flipped
$|001\rangle\langle 001|$, $|110\rangle\langle 110|$: third bit flipped
2) If our state is $|\psi\rangle = a|000\rangle + b|111\rangle$, then the measurement will collapse the state into $|000\rangle$ or $|111\rangle$ with probabilities $|a|^2$ or $|b|^2$, respectively. Hence, only the computational basis states $|000\rangle$ and $|111\rangle$ can be corrected.
3)Assuming the initial state is $|000\rangle$ the probability that one or fewer bit flips occur is $(1-p)^3 + p(1-p)^2$, hence $F \geq \sqrt{(1-p)^3 + p(1-p)^2}$.

Exercise 10.5

Assuming no more than one error has occurred, $X_1X_2X_3X_4X_5X_6$ will be 1 if no phase flip occurred and -1 and if one occurred on the first or second block. Identically for $X_4X_5X_6X_7X_8X_9$. Hence, if both give -1 the error is on the second block, otherwise it's on the first block if $X_1X_2X_3X_4X_5X_6$ gives -1 and on the third block if $X_4X_5X_6X_7X_8X_9$ gives -1. If both give 1 then no error has occurred.

Exercise 10.6

The eigenvalues of Z are ± 1, hence
$Z_1Z_2Z_3(|000\rangle - |111\rangle) = |000\rangle - (-1)^3|111\rangle = |000\rangle + |111\rangle$

Exercise 10.7

Need to prove that $PE_i^\dagger E_jP = \alpha_{ij}P$. I and X are Hermitian, hence suffices to show for IX_1,II,X_1X_1 and X_1X_2.
$P\sqrt{(1-p)^3}I\sqrt{p(1-p)^2}X_1P = (1-p)^2\sqrt{p(1-p)}(|000\rangle\langle000| + |111\rangle\langle111|)X_1(|000\rangle\langle000| + |111\rangle\langle111|) = (1-p)^2\sqrt{p(1-p)}(|000\rangle\langle000| + |111\rangle\langle111|)(|100\rangle\langle000| + |011\rangle\langle111|) = 0$
$P\sqrt{(1-p)^3}I\sqrt{(1-p)^3}IP = (1-p)^3PP = (1-p)^3P$
$P\sqrt{p(1-p)^2}X_1\sqrt{p(1-p)^2}X_1P = p(1-p)^2PIP = p(1-p)^2P$
$P\sqrt{p(1-p)^2}X_1\sqrt{p(1-p)^2}X_2 = p(1-p)^2(|000\rangle\langle000|+|111\rangle\langle111|)(|110\rangle\langle000|+|001\rangle\langle111|) = 0$
Hence, the quantum error-correction conditions are satisfied.

Exercise 10.8

$P = |+++\rangle\langle+++| + |---\rangle\langle---|$, hence like in the previous exercise.
$PE_i^\dagger E_jP = 0$, $i \neq j$
$PE_i^\dagger E_jP = P$, $i = j$
Hence, the quantum error-correction conditions are satisfied.

Exercise 10.9

$PIIP = P$
$PIP_1P = (|+++\rangle\langle+++|+|---\rangle\langle---|)(|0\rangle\langle0|\otimes I\otimes I)(|+++\rangle\langle+++|+|---\rangle\langle---|) = (|+++\rangle\langle+++|+|---\rangle\langle---|)\frac{1}{\sqrt{2}}(|0++\rangle\langle+++|+|0--\rangle\langle---|) = \frac{1}{2}(|+++\rangle\langle+++|+|---\rangle\langle---|) = \frac{1}{2}P$
Identically,
$PIQ_1P = \frac{1}{2}P$
$PP_1Q_1 = 0$
$PP_1P_1P = PP_1P = \frac{1}{2}P$
$PQ_1Q_1P = PQ_1P = \frac{1}{2}P$
$PP_1P_2P = (|+++\rangle\langle+++| + |---\rangle\langle---|)(|0\rangle\langle0| \otimes |0\rangle\langle0| \otimes I)(|+++\rangle\langle+++| + |---\rangle\langle---|) = (|+++\rangle\langle+++|+|---\rangle\langle---|)\frac{1}{2}(|00+\rangle\langle+++|+|00-\rangle\langle---|) = \frac{1}{4}(|+++\rangle\langle+++| + |---\rangle\langle---|) = \frac{1}{4}P$
$PP_1Q_2P = (|+++\rangle\langle+++| + |---\rangle\langle---|)(|0\rangle\langle0| \otimes |1\rangle\langle1| \otimes I)(|+++\rangle\langle+++| + |---\rangle\langle---|) = (|+++\rangle\langle+++|+|---\rangle\langle---|)\frac{1}{2}(|01+\rangle\langle+++|-|01-\rangle\langle---|) = \frac{1}{4}(|+++\rangle\langle+++| + |---\rangle\langle---|) = \frac{1}{4}P$
Hence, the quantum error-correction conditions are satisfied.

Exercise 10.10

$P = |0_L\rangle\langle0_L| + |1_L\rangle\langle1_L|$
Due to phase and bit flips,
$PIX_iP = PIY_iP = PIZ_iP = 0$
$PIIP = PX_iX_iP = PY_iY_iP = PZ_iZ_iP = P$
The X_i and Y_i change the individual qubits, hence if $i \neq j$ $PX_iY_jP = 0$, e.g. for PX_1Y_2P looking at the first triplet, we have
$(\langle000| + \langle111|)i(|110\rangle - |001\rangle) = 0$
$X_iY_i = iZ_i$, hence $PX_iY_iP = 0$
For Z_iZ_j if i and j belong to different triplets then we have a phase flip on 2 separate triplets,

hence $PZ_iZ_jP = 0$.
However, if i and j are in the same triplet, then we apply 2 phase shifts to the triplet which is equivalent to no change, hence $PZ_iZ_jP = P$.
For X_iZ_j and Y_iZ_j we perform a bit and phase flip, hence for all i and j $PX_iZ_jP = PY_iZ_jP = 0$.

Exercise 10.11

$\mathcal{E}(\rho) = \frac{I}{2}$
Consider the operation elements found for the general depolarizing channel in Exercise 8.19 $\{\sqrt{\frac{p}{d}}\,|i\rangle\langle j|\}$. Taking $p = 1$ and $d = 2$, we get $\{\frac{1}{2}|0\rangle\langle 0|, \frac{1}{2}|1\rangle\langle 1|, \frac{1}{2}|0\rangle\langle 1|, \frac{1}{2}|1\rangle\langle 0|\}$.

Exercise 10.12

$F(|0\rangle, \mathcal{E}(|0\rangle\langle 0|)) = \sqrt{\langle 0|\mathcal{E}(|0\rangle\langle 0|)|0\rangle}$
$= \sqrt{\langle 0|((1-p)|0\rangle\langle 0| + \frac{p}{3}(X|0\rangle\langle 0|X + Y|0\rangle\langle 0| + Z|0\rangle\langle 0|Z))|0\rangle} = \sqrt{1-p+\frac{p}{3}} = \sqrt{1-\frac{2p}{3}}$
As the depolarizing channel is symmetric, for any pure state $|\psi\rangle$,
$F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)) = \sqrt{1-\frac{2p}{3}}$.
As fidelity is jointly concave, for any ρ and some $|\psi\rangle$ we have,
$F(\rho, \mathcal{E}(\rho)) \geq F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)) = \sqrt{1-\frac{2p}{3}}$

Exercise 10.13

Let $|\psi\rangle = a|0\rangle + b|1\rangle$
$F(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)) = \sqrt{\langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle}$
$\sqrt{|\langle\psi|E_0|\psi\rangle|^2 + |\langle\psi|E_1|\psi\rangle|^2} = \sqrt{||a|^2 + |b|^2\sqrt{1-\gamma}|^2 + |a|b|^2\sqrt{\gamma}|^2}$
Minimum will occur when $a = 0$ and $b = 1$, hence
$F_{min}(|\psi\rangle, \mathcal{E}(|\psi\rangle\langle\psi|)) = F(|1\rangle, \mathcal{E}(|1\rangle\langle 1|)) = \sqrt{1-\gamma}$

Exercise 10.14

$$G = rk\left\{\underbrace{\begin{bmatrix}1&0&\dots&0\\ {}_r\vdots&\vdots&\vdots&\vdots\\1&0&\dots&0\\0&1&\dots&0\\ \vdots&\vdots&\vdots&\vdots\\0&1&\dots&0\\ \vdots&\vdots&\vdots&\vdots\\0&0&\dots&1\\ \vdots&\vdots&\vdots&\vdots\\0&0&\dots&1\end{bmatrix}}_{k}\right.$$

Exercise 10.15

Let c_1 and c_2 be columns of G. Then
$G = [c_1|c_2|G']$
$G'' = [c_1|c_1 + c_2|G']$
Let $x = (x_1, x_2, \ldots, x_n)$.
$Gx = c_1x_1 + c_2x_2 + \ldots$
$G''x = c_1x_1 + (c_1 + c_2)x_2 + \ldots$
$G''x - Gx = c_1x_2 \in C$
Therefore, as C is linear with G as generator, G'' is a generator for C as well, as the difference of the two codes is still in C.

Exercise 10.16

Let r_1 and r_2 be rows of H. Then

$$H = \left[\begin{array}{c} r_1 \\ \hline r_2 \\ \hline H' \end{array}\right]$$

$$H'' = \left[\begin{array}{c} r_1 \\ \hline r_1 + r_2 \\ \hline H' \end{array}\right]$$

Let $x = (x_1, x_2, \ldots, x_n)$.

$$Hx = \begin{bmatrix} r_1x \\ r_2x \\ \vdots \end{bmatrix} = 0$$

Therefore, $r_1x = r_2x = 0$. Hence,

$$H''x = \begin{bmatrix} r_1x \\ r_1x + r_2x \\ \vdots \end{bmatrix} = 0$$

Hence, H'' is a parity check matrix for the same code.

Exercise 10.17

$y_1 = (1, 1, 1, 0, 0, 0)$, $y_2 = (0, 0, 0, 1, 1, 1)$, hence we can take y_3 to y_6 as,
$y_3 = (1, 1, 0, 0, 0, 0)$
$y_4 = (1, 0, 1, 0, 0, 0)$
$y_5 = (0, 0, 0, 0, 1, 1)$
$y_6 = (0, 0, 0, 1, 0, 1)$
Therefore,

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Exercise 10.18

Let x be an arbitrary message to be encoded. Then,
$y = Gx \in C$
Hence, $HGx = Hy = 0$ for $\forall x$
Hence, $HG = 0$

Exercise 10.19

Using that $HG = 0$ we have,

$$HG = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1k} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \ddots & \\ a_{(n-k)1} & a_{(n-k)2} & \dots & a_{(n-k)k} & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1k} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nk} \end{bmatrix} = 0$$

Hence,

$$\sum_{i \leq k} a_{1i} b_{i1} + b_{(k+1)1} = 0 \dots \sum_{i \leq k} a_{(n-k)i} b_{i1} + b_{n1} = 0$$

$\vdots$

$$\sum_{i \leq k} a_{1i} b_{ik} + b_{(k+1)k} = 0 \dots \sum_{i \leq k} a_{(n-k)i} b_{ik} + b_{nk} = 0$$

We see that for example, taking for $2 \leq i \leq k$ $b_{i1} = 0$, $b_{11} = 1$ and $b_{(k+1)1} = -a_{11}$ gives a solution.
Therefore for $i, j \leq k$ $b_{ij} = \delta_{ij}$ and for $i, j > k$ $b_{ij} = -a_{(i-k)j}$, i.e.

$$G = \left[\frac{I_k}{-A} \right]$$

Exercise 10.20

Let x be a codeword such that $\text{wt}(x) \leq d - 1$. Let $H = c_1 | c_2 \dots c_n$ for code C. Consider Hx,
$Hx = \sum_i c_i x_i$ for $d - 1$ columns. Therefore, as any $d - 1$ columns are linearly independent, this sum cannot equal 0. Hence, $d(C) \geq d$. However, as any d columns are linearly dependant there exists a codeword y with $\text{wt}(y) = d$ such that $Hy = 0$. Therefore, $d(C) = d$.

Exercise 10.21

The parity check matrix is a $n - k$ by n matrix, hence the maximum number of linearly independent columns is $n - k$. Therefore, from Exercise 10.20 $n - k \geq d - 1$.

Exercise 10.22

The Hamming parity check matrix is constructed from columns which are all the possible $n - k$ bit strings, of which there are $2^r - 1$ of excluding the 0 string. Hence, any two columns will be linearly independent as all are different, however there always will be 3 linearly dependant columns, e.g. $(1, 0, 0, \dots)$, $(0, 1, 0, \dots)$ and $(1, 1, 0, \dots)$. Therefore, as per exercise 10.20 the code will have distance 3.

Exercise 10.23

Exercise 10.24

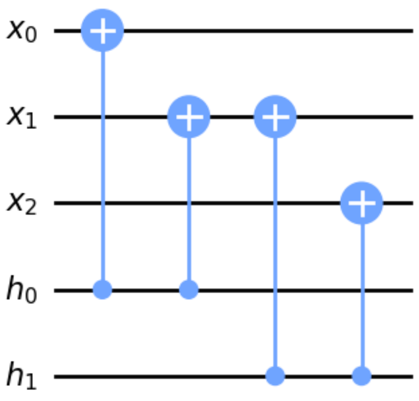
If $C^\perp \subseteq C$, $\forall x\ y = Gx \in C^\perp$ and $G^T = H^\perp$. Hence, $\forall x\ G^TGx = H^\perp y = 0$, i.e. $G^TG = 0$.
If $G^TG = 0$, $\forall x\ G^TGx = H^\perp y = 0$, therefore $y \in C^\perp$, hence $C^\perp \subseteq C$.

Exercise 10.25

$x = H^T z_0$
If $x \in C^\perp$,
$$\sum_{y\in C}(-1)^{x.y} = \sum_z (-1)^{(H^Tz_0)^TGz} = \sum_z (-1)^{z_0^T HGz} = \sum_z (-1)^0 = |C|$$
If $x \notin C^\perp$,
$$\sum_{y\in C}(-1)^{x.y} = \sum_z (-1)^{x^TGz}$$
Let, $x^TG = z_1^T$, then
$$\sum_{y\in C}(-1)^{x.y} = \sum_z (-1)^{z_1.z}$$
As we're summing over all z, $z_1.z = 0$ or 1 both with probability $\frac{1}{2}$. Hence,
$$\sum_{y\in C}(-1)^{x.y} = 0$$

Exercise 10.26

To perform the transformation $|x\rangle|0\rangle \rightarrow |x\rangle|Hx\rangle$ we perform the following. Let $|x\rangle = |x_1, x_2, \ldots, x_n\rangle$ and $|0\rangle = |0_1, 0_2, \ldots, 0_m\rangle$. For each 0_i, consider the i^{th} row of H and for each column j which is 1 apply a CNOT between x_j and the 0_i with x_j the control. After, applying this for all the qubits of $|0\rangle$ we obtain the desired transformation. As an example here's the circuit for $H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$,



Exercise 10.27

Consider a bit error e_1 and flip error e_2. We get,

$$\frac{1}{\sqrt{|C_2|}}\sum_{y\in C_2}(-1)^{u.y}(-1)^{(x+y+v).e_2}\left|x+y+v+e_1\right\rangle$$

Applying the parity matrix H_1 to $\left|x+C_2\right\rangle\left|0\right\rangle$ we get,

$$\frac{1}{\sqrt{|C_2|}}\sum_{y\in C_2}(-1)^{u.y}(-1)^{(x+y+v).e_2}\left|x+y+v\right\rangle\left|H_1(v+e_1)\right\rangle$$

As v is known so is H_1v, hence we can calculate the syndrome H_1e_1. Therefore, removing the bit error we get,

$$\frac{1}{\sqrt{|C_2|}}\sum_{y\in C_2}(-1)^{u.y}(-1)^{(x+y+v).e_2}\left|x+y+v\right\rangle$$

Applying Hadamard gates to each qubit we get,

$$\frac{1}{\sqrt{|C_2|2^n}}\sum_{z}\sum_{y\in C_2}(-1)^{u.y}(-1)^{(x+y+v).(z+e_2)}\left|z\right\rangle=\frac{1}{\sqrt{|C_2|2^n}}\sum_{z}\sum_{y\in C_2}(-1)^{(u+z+e_2).y}(-1)^{(x+v).(z+e_2)}\left|z\right\rangle$$

Let $e_2+z=z'+u$, then we have,

$$\frac{1}{\sqrt{|C_2|2^n}}\sum_{z'}\sum_{y\in C_2}(-1)^{z'.y}(-1)^{(x+v).(z'+u)}\left|z'+e_2+u\right\rangle$$

Using Exercise 10.25 we get,

$$\frac{1}{\sqrt{2^n/|C_2|}}\sum_{z'\in C_2^{\perp}}(-1)^{(x+v).(z'+u)}\left|z'+e_2+u\right\rangle$$

Once again by knowing H_2u we calculate the syndrome H_2e_2, where H_2 is the parity check matrix for $C_2^{\perp}$, and hence correct the error e_2 to get,

$$\frac{1}{\sqrt{2^n/|C_2|}}\sum_{z'\in C_2^{\perp}}(-1)^{(x+v).(z'+u)}\left|z'+u\right\rangle$$

Applying the Hadamards again we get,

$$\frac{1}{\sqrt{|C_2|}}\sum_{y\in C_2}(-1)^{u.y}\left|x+y+v\right\rangle$$

Hence, this has the same error-correcting properties as the $CSS(C_1,C_2)$.

Exercise 10.28

For the $[7,4,3]$ Hamming code we have,

$$H=\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}$$

$$HH[C_2]^T=\begin{bmatrix}0&0&0&1&1&1&1\\0&1&1&0&0&1&1\\1&0&1&0&1&0&1\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\\0&1&1&1\\1&0&1&1\\1&1&0&1\end{bmatrix}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

Hence, $H[C_2]^T=G[C_1]$.

Exercise 10.29

Let $|x\rangle, |y\rangle \in V_S$, i.e. $\forall g \in S\ g|x\rangle = |x\rangle$ and $g|y\rangle = |y\rangle$. Consider $a|x\rangle + b|y\rangle$ for some a and b. As g are linear operators we have,
$g(a|x\rangle + b|y\rangle) = ag|x\rangle + bg|y\rangle = a|x\rangle + b|y\rangle$
Hence, $a|x\rangle + b|y\rangle \in V_S$.
Let $|x\rangle \in V_S \implies \forall g \in S\ g|x\rangle = |x\rangle \implies \forall g \in S\ |x\rangle \in V_g \implies |x\rangle \in \bigcap_{g \in S} V_G$

Exercise 10.30

Let $\pm iI \in S$ then as S is a group $(\pm iI)(\pm iI) \in S$, hence $-I \in S$, which is a contradiction therefore $\pm iI \notin S$.

Exercise 10.31