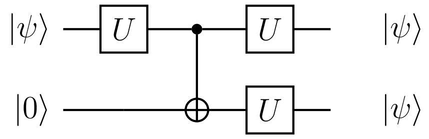
Exercise 12.1

Let $|\psi\rangle = a|0\rangle + b|1\rangle$. Then, $|\varphi\rangle = b|0\rangle - a|1\rangle$.
We require a gate which transforms between the basis of $|\psi\rangle$ and $|\varphi\rangle$, and the computational basis. This can be performed using the unitary, $U = U^d agger = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Using U we construct the following circuit which performs the cloning.



Exercise 12.2

$\sum_y \sqrt{E_y} \otimes U_y(\sigma \otimes |0\rangle\langle 0|)\sqrt{E_y} \otimes U_y = \sum_y \sqrt{E_y}\sigma\sqrt{E_y} \otimes |0+y\rangle\langle 0+y| = \sum_y \sqrt{E_y}\sigma\sqrt{E_y} \otimes |y\rangle\langle y|$

Exercise 12.3

From the Holevo bound $H(X:Y) \leq S(\rho)$, however $S(\rho)$ is bounded by $\log 2^n = n$, hence no more than n bits can be transferred.

Exercise 12.4

Let $\rho_i = |X_i\rangle\langle X_i|$, and $\rho = \frac{1}{4}\sum_i \rho_i = \frac{I}{2}$, which is the maximally mixed state, hence $S(\rho) = 1$ and the $S(\rho_i) = 0$ as the ρ_i are pure states. Then,
$\chi = S(\rho) - \sum_i p_i S(\rho_i) = 1$
We have tetrahedral symmetry for the states $|X_i\rangle$, hence we consider POVMs with the same symmetry. Consider $E_i = \frac{1}{2}|X_i\rangle\langle X_i|$, and $F_i = \frac{1}{2}|\psi_i\rangle\langle\psi_i|$, where
$|\psi_1\rangle = |1\rangle$
$|\psi_2\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}e^{\pi i/3}|0\rangle + |1\rangle)$
$|\psi_3\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}e^{\pi i}|0\rangle + |1\rangle)$
$|\psi_4\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}e^{5\pi i/3}|0\rangle + |1\rangle)$
Using $H(X:Y) = H(Y) - H(Y|X)$, $P(X=x) = \frac{1}{4}$, $p(Y=y|x=x) = tr(\rho_x E_y)$ and Bayes theorem we get,
$H(Y) = -\sum_y p(Y=y)\log p(Y=y) =$

$-\sum_y \left(\sum_x p(X=x)p(Y=y|X=x)\right)\log\left(\sum_x p(X=x)p(Y=y|X=x)\right)$
$H(Y|X) = -\sum_x p(X=x)\sum_y p(Y=y|X=x)\log p(Y=y|X=x)$
The calculation can be found in 12.3.py and gives $H(X:Y)_E \approx 0.208$ and $H(X:Y)_F \approx 0.415$. Therefore, F_i is the desired POVM.
As shown in DOI: 10.1109/TIT.1978.1055941 the maximum information is $\log\frac{4}{3} \approx 0.415$.

Exercise 12.5

We know that $p(x \in T(n, \epsilon)) \geq 1 - \delta$, hence $1 - p(x \in T(n, \epsilon)) \leq \delta$. The expected number of bits required for a single symbol when the xs are ϵ-typical will be,
$E(n, \epsilon) = H(X)p(x \in T(n, \epsilon)) + \frac{\log d^n}{n} p(x \notin T(n, \epsilon)) \leq$
$Rp(x \in T(n, \epsilon)) + \log d(1 - p(x \in T(n, \epsilon))) \leq R + \delta \log \delta$
Hence,
$E(n, \epsilon) - R \leq \delta \log d$

Exercise 12.6

$C_X = p^{n - wt(X)}(1-p)^{wt(X)}$
Where, $wt(X)$ is the Hamming weight.

Exercise 12.7

This is the circuit outlined in Box 12.4 but with the basis change gate V set to I.

Exercise 12.8

Taking $p_J = p_{j_1} p_{j_2} \ldots p_{j_n}$ and using Exercise 9.20 for the concavity of fidelity, from Equation 9.141 we have,
$\bar{F} \geq F\left(\sum_J p_J \rho_J, D^n \circ C^n\right) = F\left(\sum_J p_{j_1} |\psi_{j_1}\rangle \langle\psi_{j_1}| \otimes p_{j_2} |\psi_{j_2}\rangle \langle\psi_{j_2}| \ldots p_{j_n} |\psi_{j_n}\rangle \langle\psi_{j_n}|, D^n \circ C^n\right) =$
$F(\rho^{\otimes n}, D^n \circ C^n) = F(\rho, D^n \circ C^n)^n \geq (1 - 2\delta)^n$
Where the last inequality is only true for $R > S(\rho)$.

Exercise 12.9

Exercise 12.10

Exercise 12.11

Exercise 12.12

Exercise 12.13

Exercise 12.14

Exercise 12.15

Exercise 12.16

Exercise 12.17

Exercise 12.18

Exercise 12.19

Exercise 12.20