Exercise 11.1

For a fair coin $H(X) = -2 \times \frac{1}{2} \log \frac{1}{2} = 1$
For a fair die $H(X) = -6 \times \frac{1}{6} \log \frac{1}{6} = 1 + \log 3$
For an unfair coin we can write $H(X) = -p \log p - (1-p) \log (1-p)$ and for the unfair die $H(X) = -p_1 \log p_1 - p_2 \log p_2 - p_3 \log p_3 - p_4 \log p_4 - p_5 \log p_5 - (1 - p_1 - p_2 - p_3 - p_4 - p_5) \log (1 - p_1 - p_2 - p_3 - p_4 - p_5)$.
Differentiating both of these we see that for both the global maxima is when all the probabilities are equal, therefore for the unfair coin and die the entropy will decrease.

Exercise 11.2

$I(p) = k \log p$ is a function of probability alone.
$\log p$ is smooth for $0 < p \leq 1$
$I(pq) = k \log (pq) = k(\log p + \log q) = I(p) + I(q)$

Exercise 11.3

$H_{bin}(p) = -p \log p - (1-p) \log (1-p)$
$\frac{dH_{bin}}{dp} = -\frac{1}{\ln 2} - \log p + \frac{1}{\ln 2} + \log (1-p) = 0$
$\frac{1-p}{p} = 1$
Therefore, $p = \frac{1}{2}$.

Exercise 11.4

For a function $f(x)$ to be concave we require $f''(x) < 0$.
$\frac{d^2 H_{bin}}{dp^2} = \frac{d}{dp}(\log (1-p) - \log p) = \frac{1}{\ln 2 (1-p)p} < 0$
Hence, H_{bin} is concave.

Exercise 11.5

$H(p(x,y)||p(x)p(y)) = \sum_{xy} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = \sum_{xy} p(x,y) \log p(x,y) - \sum_{xy} p(x,y) \log p(x) - \sum_{xy} p(x,y) \log p(y) = \sum_{xy} p(x,y) \log p(x,y) - \sum_{x} p(x) \log p(x) - \sum_{y} p(y) \log p(y) = H(p(x)) + H(p(y)) - H(p(x,y))$
$H(p(x,y)||p(x)p(y)) \geq 0$
Therefore,
$H(p(x)) + H(p(y)) - H(p(x,y)) = H(X) + H(Y) - H(X,Y) \geq 0$
$H(X,Y) \leq H(X) + H(Y)$
If X and Y are independent then $p(x,y) = p(x)p(y)$. Therefore,
$H(X,Y) = -\sum_{xy} p(x,y) \log p(x,y) = -\sum_{xy} p(x)p(y) \log p(x)p(y) = -\sum_{x} p(x) \log p(x) - \sum_{y} p(y) \log p(y) = H(X) + H(Y)$
Therefore, equality hold if and only if X and Y are independent.

Exercise 11.6

Exercise 11.7

Exercise 11.8

Exercise 11.9

Exercise 11.10

Exercise 11.12

Exercise 11.13

Exercise 11.14

Exercise 11.15

Exercise 11.16

Exercise 11.17

Exercise 11.18

Exercise 11.19

Exercise 11.20

Exercise 11.21

Exercise 11.22

Exercise 11.23

Exercise 11.24

Exercise 11.25

Exercise 11.26