Exercise 11.1

For a fair coin $H(X) = -2 \times \frac{1}{2} \log \frac{1}{2} = 1$
For a fair die $H(X) = -6 \times \frac{1}{6} \log \frac{1}{6} = 1 + \log 3$
For an unfair coin we can write $H(X) = -p \log p - (1-p) \log (1-p)$ and for the unfair die $H(X) = -p_1 \log p_1 - p_2 \log p_2 - p_3 \log p_3 - p_4 \log p_4 - p_5 \log p_5 - (1 - p_1 - p_2 - p_3 - p_4 - p_5) \log (1 - p_1 - p_2 - p_3 - p_4 - p_5)$.
Differentiating both of these we see that for both the global maxima is when all the probabilities are equal, therefore for the unfair coin and die the entropy will decrease.

Exercise 11.2

$I(p) = k \log p$ is a function of probability alone.
$\log p$ is smooth for $0 < p \leq 1$
$I(pq) = k \log (pq) = k(\log p + \log q) = I(p) + I(q)$

Exercise 11.3

$H_{bin}(p) = -p \log p - (1-p) \log (1-p)$
$\frac{dH_{bin}}{dp} = -\frac{1}{\ln 2} - \log p + \frac{1}{\ln 2} + \log (1-p) = 0$
$\frac{1-p}{p} = 1$
Therefore, $p = \frac{1}{2}$.

Exercise 11.4

For a function $f(x)$ to be concave we require $f''(x) < 0$.
$\frac{d^2 H_{bin}}{dp^2} = \frac{d}{dp}(\log (1-p) - \log p) = \frac{1}{\ln 2(1-p)p} < 0$
Hence, H_{bin} is concave.

Exercise 11.5

$H(p(x,y)||p(x)p(y)) = \sum_{xy} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = \sum_{xy} p(x,y) \log p(x,y) - \sum_{xy} p(x,y) \log p(x) - \sum_{xy} p(x,y) \log p(y) = \sum_{xy} p(x,y) \log p(x,y) - \sum_{x} p(x) \log p(x) - \sum_{y} p(y) \log p(y) = H(p(x)) + H(p(y)) - H(p(x,y))$
$H(p(x,y)||p(x)p(y)) \geq 0$
Therefore,
$H(p(x)) + H(p(y)) - H(p(x,y)) = H(X) + H(Y) - H(X,Y) \geq 0$
$H(X,Y) \leq H(X) + H(Y)$
If X and Y are independent then $p(x,y) = p(x)p(y)$. Therefore,
$H(X,Y) = -\sum_{xy} p(x,y) \log p(x,y) = -\sum_{xy} p(x)p(y) \log p(x)p(y) = -\sum_{x} p(x) \log p(x) - \sum_{y} p(y) \log p(y) = H(X) + H(Y)$
Therefore, equality hold if and only if X and Y are independent.

Exercise 11.6

$H(X,Y,Z) = -\sum_{xyz} p(x,y,z) \log p(x,y,z) = -\sum_{xyz} p(x,y,z) \log p(x|y,z)p(y,z) = H(Y,Z) - \sum_{xyz} p(x,y,z) \log p(x|y,z)$

$H(X,Y) - H(Y) = H(X|Y) = -\sum_{xy} p(x,y) \log p(x|y)$

Then, using $\log x \leq (x-1)/\ln 2$ we have,

$H(X,Y,Z) - H(Y,Z) - H(X,Y) + H(Y) = -\sum_{xyz} p(x,y,z) \log p(x|y,z) + \sum_{xyz} p(x,y,z) \log p(x|y) = \sum_{xyz} p(x,y,z) \log \frac{p(x|y)}{p(x|y,z)} \leq \frac{1}{\ln 2} \sum_{xyz} p(x,y,z) \left(\frac{p(x|y)}{p(x|y,z)} - 1 \right) = \frac{1}{\ln 2} \sum_{xyz} (p(x|y)p(y,z) - p(x,y,z)) = \frac{1}{\ln 2} \left(\sum_{xy} p(x|y)p(y) - 1 \right) = \frac{1}{\ln 2} \left(\sum_{xy} p(x,y) - 1 \right) = \frac{1}{\ln 2}(1-1) = 0$

Hence,

$H(X,Y,Z) - H(Y,Z) \leq H(X,Y) - H(Y)$

with equality when $p(x|y,z) = p(x,y)$ which is the definition for a $Z \to Y \to X$ Markov chain.

Exercise 11.7

$H(Y|X) = H(Y) - H(Y:X) = -\sum_{y} p(y) \log p(y) - \sum_{xy} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = -\sum_{xy} p(x,y) \log p(y) - \sum_{xy} p(x,y) \log \frac{p(x,y)}{p(x)p(y)} = \sum_{xy} p(x,y) \log \frac{p(x,y)}{p(x)} = -H(p(x,y)||p(x)) \geq 0$

Equality when $p(x,y) = p(x)$, therefore when Y is a function of X, $Y = f(X)$.

Exercise 11.8

For $p(x,y,z)$ we have $p(0,0,0) = p(0,1,1) = p(1,0,1) = p(1,1,0) = \frac{1}{4}$ and 0 otherwise. Hence,

$H(X,Y,Z) = -4\frac{1}{4} \log \frac{1}{4} = 2$

$H(X,Y) = H(X,Z) = H(Y,Z) = -4\frac{1}{4} \log \frac{1}{4} = 2$

$H(X) = H(Y) = H(Z) = -2\frac{1}{2} \log \frac{1}{2} = 1$

Therefore,

$H(X,Y:Z) = H(X,Y) + H(Z) - H(X,Y,Z) = 1$

$H(X:Z) = H(Y:Z) = H(Y) + H(Z) - H(Y,Z) = 0$

Exercise 11.9

For $p(x_1,x_2,y_1,y_2)$ we have $p(0,0,0,0) = p(1,1,1,1) = 1/2$ and 0 otherwise. Hence,

$H(X_1,X_2,Y_1,Y_2) = H(X_1,X_2) = H(X_1,Y_1) = H(X_2,Y_2) = H(Y_1,Y_2) = H(X_1) = H(X_2) = H(Y_1) = H(Y_2) = -2\frac{1}{2} \log \frac{1}{2} = 1$

Therefore,

$H(X_1:Y_1) + H(X_2:Y_2) = 2H(X_1:Y_1) = 2(H(X_1) + H(Y_1) - H(X_1,Y_1)) = 2$

$H(X_1,X_2:Y_1,Y_2) = H(X_1,X_2) + H(Y_1,Y_2) - H(X_1,X_2,Y_1,Y_2) = 1$

Exercise 11.10

If $X \to Y \to Z$ is a Markov chain then,
$p(Z|Y,X) = p(Z|Y)$
Using $p(X|Y) = \frac{p(X,Y)}{p(Y)}$ on both sides,

$$\frac{p(Z,Y,X)}{p(Y,X)} = \frac{p(Z,Y)}{p(Y)}$$

$$\frac{p(Z,Y,X)}{p(Z,Y)} = \frac{p(Y,X)}{p(Y)}$$

$p(X|Y,Z) = p(X|Y)$
Therefore, $Z \to Y \to X$ is also a Markov chain.

Exercise 11.11

$$\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$S(\rho) = -1 \log 1 = 0$

$$\rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$\lambda = 1$ or 0
$S(\rho) = -1 \log 1 = 0$

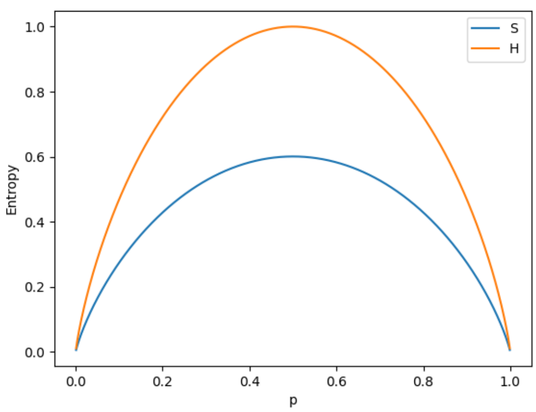
$$\rho = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{6}$
$S(\rho) = -(\frac{1}{2} + \frac{\sqrt{5}}{6}) \log(\frac{1}{2} + \frac{\sqrt{5}}{6}) - (\frac{1}{2} - \frac{\sqrt{5}}{6}) \log(\frac{1}{2} - \frac{\sqrt{5}}{6}) \approx 0.55$

Exercise 11.12

$$\rho = \frac{1}{2}\begin{bmatrix} 1+p & 1-p \\ 1-p & 1-p \end{bmatrix}$$

$\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{1-2p(1-p)}$
$S(\rho) = -\frac{1}{2}((1+\sqrt{1-2p(1-p)}) \log \frac{1}{2}(1+\sqrt{1-2p(1-p)})$
$+ (1-\sqrt{1-2p(1-p)}) \log \frac{1}{2}(1-\sqrt{1-2p(1-p)}))$
$H(p, 1-p) = -p \log p - (1-p) \log(1-p)$



Therefore, $S(\rho) \leq H(p, 1-p)$.

Exercise 11.13

First note that for $\rho = \sum_i p_i |i\rangle \langle i|$, $H(p_i) = -\sum_i p_i \log p_i = S(\rho)$. Then using the joint entropy theorem for $\rho_i = \sigma \; \forall i$ we have,
$S(\rho \otimes \sigma) = S(\sum_i p_i \rho |i\rangle \langle i| \otimes \sigma) = H(p_i) + \sum_i p_i S(\sigma) = S(\rho) + S(\sigma)$
Otherwise from the definition of the entropy for $\rho = \sum_i p_i |i\rangle \langle i|$ and $\sigma = \sum_j q_i |j\rangle \langle j|$ we have,
$S(\rho \otimes \sigma) = S\left(\sum_{ij} p_i q_j |i\rangle |j\rangle \langle i| \langle j|\right) = -\sum_{ij} p_i q_j \log p_i q_j = -\sum_i p_i \log p_i - \sum_j q_j \log q_j = S(\rho) + S(\sigma)$

Exercise 11.14

If $|AB\rangle$ is a pure state of the composite system then $|A\rangle$ is a pure state if and only if there's no entanglement. Hence, $S(A) \neq 0$ if and only if $|AB\rangle$ is entangled. As $|AB\rangle$ is a pure state $S(A, B) = 0$, and therefore $S(B|A) = -S(A)$. As $S(A) \geq 0$, $S(B|A) < 0$ if and only if $|AB\rangle$ is entangled.

Exercise 11.15

Let $\rho = I + r.\sigma 2$. Then
$\rho' = M_1 \rho M_1^\dagger + M_2 \rho M_2^\dagger = \frac{1 + r_z}{2} |0\rangle \langle 0| + \frac{1 - r_z}{2} |0\rangle \langle 0| = |0\rangle \langle 0|$
Hence,
$S(\rho') = -\log 1 = 0$
Therefore, $S(\rho) \geq S(\rho')$.

Exercise 11.16

ρ^{AB} is a mixed state, hence
$\rho^A = \sum_i \lambda_i \rho_i^A$ and $\rho^B = \sum_i \lambda_j \rho_i^B$
Introduce purification R of AB
$|ABR\rangle = \sum_i \sqrt{\lambda_i} |i\rangle |i^R\rangle$
$\rho^{ABR} = \sum_{ij} \sqrt{\lambda_i \lambda_j} |i\rangle \langle j| \otimes |i^R\rangle \langle j^R|$
$\rho^R = \sum_i \lambda_i |i\rangle \langle i|$
Trace over B,
$\rho^{AR} = \sum_{ij} \sqrt{\lambda_i \lambda_j} tr_B(|i\rangle \langle j|) \otimes |i^R\rangle \langle j^R|$
Equality condition is, $\rho^{AR} = \rho^A \otimes \rho^R$, hence
$\sum_{ij} \sqrt{\lambda_i \lambda_j} tr_B(|i\rangle \langle j|) \otimes |i^R\rangle \langle j^R| = \sum_{ij} \lambda_i \lambda_j \rho_i^A \otimes |j^R\rangle \langle j^R|$
Multiplying on both sides by $\langle k|$ and $|k\rangle$ we get,
$\sum_{ij} \sqrt{\lambda_i \lambda_j} tr_B(|i\rangle \langle j|) \delta_{ij} = \sum_{ij} \lambda_i \lambda_j \rho_i^A \delta_{jk}$

$\sum_j \lambda_j \rho_i^A = \sum_i \lambda_i \lambda_k \rho_i^A$
$\sum_{ij} \lambda_i \lambda_j \rho_j^A = \sum_{ij} \lambda_i \lambda_j \lambda_k \rho_i^A$
$\sum_{ij} \lambda_i \lambda_j (\rho_i^A - \lambda_k \rho_i^A) = 0$
Hence, ρ_i^A have a common eigenbasis.

Exercise 11.17

Consider $\rho^{AB} = \frac{1}{2}(|10\rangle\langle 10| + |11\rangle\langle 11|)$. Then,
$\rho^A = |1\rangle\langle 1|$
$\rho^B = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$
Therefore, $S(A,B) = S(B) = 1$ and $S(A) = 0$, i.e $S(A,B) = S(B) - S(A)$.

Exercise 11.18

The equality condition is the same as for subadditivity inequality, i.e. $\rho^{AB} = \rho^A \otimes \rho^B$. Hence we have,
$\sum_i p_i \rho_i \otimes |i\rangle\langle i| = \sum_{ij} p_i p_j \rho_i \otimes |j\rangle\langle j|$
Multiplying on both sides by $\langle k|$ and $|k\rangle$ we get,
$\sum_i p_i \rho_i \delta_{ik} = \sum_{ij} p_i p_j \rho_i \delta jk$
$p_k \rho_k = \sum_i p_i p_k \rho_i$
$\rho_k = \sum_i p_i \rho_i$
$\sum_i p_i(\rho_k - \rho_i) = 0$
Hence, we have equality if and only if $\rho_k = \rho_i \ \forall i$. However, as this is true $\forall k$ we conclude that we have equality if and only if all the ρ_i are equal.

Exercise 11.19

Consider the case of A being a 2x2 matrix. Let $p_i = \frac{1}{4}$, $U_i = I, X, Y, Z$ and $A = c_1 I + c_2 X + c_2 Y + c_4 Z$. Hence, $tr(A) = 2c_1$ as X, Y, Z are traceless. Then,
$\frac{1}{4}\sum_i U_i A U_i^\dagger = \frac{1}{4}(4c_1 I) = 2tr(A)\frac{I}{4} = tr(A)\frac{I}{2}$
We can expand this to any matrix dxd size matrix by choosing U_i to be the Sylvester's generalized Pauli matrices with $p_i = \frac{1}{d^2}$.
$tr(\rho) = 1$, hence
$S(\frac{I}{d}) = S(tr(\rho)\frac{I}{d}) = S(\sum_i p_i U_i \rho U_i^\dagger) \geq \sum_i p_i S(U_i \rho U_i^\dagger) = \sum_i p_i S(\rho) = S(\rho)$
As this is true for any ρ, the completely mixed state is the unique state of maximal entropy.

Exercise 11.20

We consider a unitary matrix $U = I - 2P$. Then $P = \frac{1}{2}(I - U)$ and $Q = \frac{1}{2}(I + U)$. We get,
$P\rho P + Q\rho Q = \frac{1}{4}(I - U)\rho(I - U) + \frac{1}{4}(I + U)\rho(I + U) = \frac{1}{2}\rho + \frac{1}{2}U\rho U$

Hence, $p = \frac{1}{2}$, $U_1 = I$ and $U_2 = U$.
We can generalize this to n projectors P_i, by taking $U = 1 - 2P_i$ this leads to the equality,
$\sum_i U_i \rho U_i^\dagger = \sum_i (\rho - 2P_i\rho - 2\rho P_i + 4P_i\rho P_i) = (n-4)\rho + 4\sum_i P_i \rho P_i$
Hence,
$\rho' = \sum_i P_i \rho P_i = \frac{1}{4}\sum_i U_i\rho U_i^\dagger + \frac{n-4}{4}\rho$
Therefore using concavity,
$S(\rho') = S\left(\frac{1}{4}\sum_i U_i\rho U_i^\dagger + \frac{n-4}{4}\rho\right) \geq \frac{1}{4}\sum_i S(U_i\rho U_i^\dagger) + \frac{n-4}{4}S(\rho) = \frac{1}{4}\sum_i S(\rho) + \frac{n-4}{4}S(\rho) = S(\rho)$

Exercise 11.21

Consider density matrices $\rho = \sum_i p_i |i\rangle\langle i|$ and $\sigma = \sum_i q_i |i\rangle\langle i|$. As $|i\rangle\langle i|$ are pure states we have,
$H(\lambda p_i + (1-\lambda)q_i) = S(\lambda\rho + (1-\lambda)\sigma) \geq \lambda S(\rho) + (1-\lambda)S(\sigma) = \lambda H(p_i) + (1-\lambda)H(q_i)$

Exercise 11.22

For concavity $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)\ \forall \lambda \in [0,1], p, q$.
Let $\lambda = \frac{1}{2}$, $x = p - h$ and $y = p + h$. Then
$f(p) \geq \frac{1}{2}(f(p-h) + f(p+h))$
$\frac{1}{h}(f(p-h) - 2f(p) + f(p+h)) \leq 0$
Taking the limit as h goes to zero gives,
$f''(p) \leq 0$
We can write $\rho = \sum_i p_i \rho_i$ and $\sigma = \sum_i q_i \sigma_i$ $f''(p) = -\sum_i \frac{(p_i - q_i)^2}{p_i p + (1-p)q_i} \leq 0$

Exercise 11.23

Fix B. Then,
$f(\lambda A_1 + (1-\lambda)A_2, \lambda B_1 + (1-\lambda)B_2) = q(\lambda A_1 + (1-\lambda)A_2) \geq \lambda f(A_1, B_1) + (1-\lambda)f(A_2, B_2) = \lambda q(A_1) + (1-\lambda)q(A_2)$
Consider $f(x,y) = y\log x$, both y and $\log x$ are concave. If $f(x,y)$ is concave then the function $f(x,x)$ should also be concave.
$f(x,x) = x\log x$, however $f'' = \frac{1}{x} \geq 0$, hence it's not concave. Therefore we have a contradiction and $f(x,y)$ is not jointly concave.

Exercise 11.24

Let R be the purification for the system ABC. Then from strong subadditivity,
$S(R,B,C) + S(B) \leq S(R,B) + S(B,C)$,
however $S(R,B,C) = S(A)$ and $S(R,B) = S(A,C)$, therefore
$S(A) + S(B) \leq S(A,C) + S(B,C)$

Exercise 11.25

Consider the state $\lambda\rho \otimes |0\rangle\langle 0| + (1-\lambda)\sigma \otimes |1\rangle\langle 1|$, where ρ and σ are density matrices of the system AB and the rest of system C. Then by the strong subadditivity inequality, equation (11.57), exercise 11.13 and using $S(A|B) = S(A,B) - S(B)$ we get,
$S(A|B)_{\lambda\rho+(1-\lambda)\sigma} \geq S(A,B,C) - S(B,C) = S(A,B,C) - S(A) = H(\lambda) + \lambda S(\rho \otimes |0\rangle\langle 0|) + (1-\lambda)S(\sigma \otimes |1\rangle 1) - H(\lambda) - \lambda S(tr_B(\rho)) - (1-\lambda)S(tr_B(\sigma)) = \lambda(S(\rho) - S(tr_B(\rho))) + (1-\lambda)(S(\sigma) - S(tr_B(\sigma))) = \lambda S(A|B)_\rho + (1-\lambda)S(A|B)_\sigma$
As this is true for all λ, ρ and σ, $S(A|B)$ is concave.

Exercise 11.26

Using, $S(B) + S(C) \leq S(A,B) + S(A,C)$,
$S(A:B) + S(A:C) = S(A) + S(B) - S(A,B) + S(A) + S(C) - S(A,C) \leq 2S(A) + S(B) + S(C) - S(B) - S(C) = 2S(A)$
Consider $|AB\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$,
$S(A,B) = 0$, $S(A) = S(B) = 1$. Hence, $S(A:B) = 2 > S(A)$.