Exercise 6.1

The phase shift performs the following,
$|0\rangle \to |0\rangle$
$|x\rangle \to -|x\rangle$ for $x > 0$
Hence, checking,
$(2|0\rangle\langle 0| - I)|0\rangle = 2|0\rangle - |0\rangle = |0\rangle$
$(2|0\rangle\langle 0| - I)|x\rangle = 0 - |x\rangle = -|x\rangle$
Also, $(2|0\rangle\langle 0| - I)^\dagger(2|0\rangle\langle 0| - I) = I$
Therefore, $2|0\rangle\langle 0| - I$ is the phase shift unitary operator.

Exercise 6.2

$(2|\psi\rangle\langle\psi| - I)\sum_k \alpha_k|k\rangle = \sum_k 2\alpha_k|\psi\rangle\langle\psi|k\rangle - \alpha_k|k\rangle = \sum_{x_1}\sum_{x_2}\sum_k 2\frac{\alpha_k}{N}|x_1\rangle\langle x_2|k\rangle - \alpha_k|k\rangle =$
$\sum_{x_1} 2\langle\alpha\rangle|x_1\rangle - \sum_k \alpha_k|k\rangle = 2\langle\alpha\rangle\sum_{x_1}|x_1\rangle - \sum_k \alpha_k|k\rangle = 2\langle\alpha\rangle\sum_k|k\rangle - \sum_k \alpha_k|k\rangle =$
$\sum_k[-\alpha_k + 2\langle\alpha\rangle]|k\rangle$
Where we have relabeled, x_1 to k as both are the basis vectors of the vector space.

Exercise 6.3

Let $|\psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}$
$G|\psi\rangle = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} = \begin{bmatrix}\cos\frac{\theta}{2}\cos\theta - sin\frac{\theta}{2}\sin\theta\\ \cos\frac{\theta}{2}\sin\theta + sin\frac{\theta}{2}\cos\theta\end{bmatrix} = \begin{bmatrix}\cos\frac{3\theta}{2}\\ \sin\frac{3\theta}{2}\end{bmatrix} =$
$\cos\frac{3\theta}{2}|\alpha\rangle + \sin\frac{3\theta}{2}|\beta\rangle$
Hence, G does indeed implement the Grover iteration.

Exercise 6.4

Input: (1) a black box oracle O which performs the transformation $O|x\rangle|q\rangle = |x\rangle|q \oplus f(x)\rangle$, where $f(x) = 0\ \forall 0 \le x < 2^n$ except x_i for $1 \le i \le m$ for which $f(x_i) = 1$; (2) $n+1$ qubits in the state $|0\rangle$.
Output: One of the x_i.
Runtime: $O(\sqrt{2^n})$ operations. Succeeds with probability $O(1)$.
Procedure:

1. $|0\rangle^{\otimes n}|0\rangle$ — initial state

2. $\to \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$ — apply $H^{\otimes}n$ to the first n qubits, and HX to the last qubit

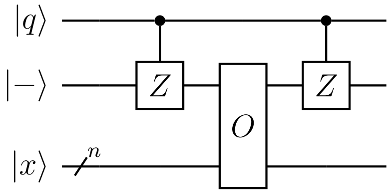
3. $[2(|\psi\rangle\langle\psi| - I)O]^R\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$ — apply the Grover iteration $R \approx \lceil\frac{\pi}{4}\sqrt{\frac{2^n}{M}}\rceil$ times

$\approx \frac{1}{\sqrt{2^m}}\sum_{i=1}^{m}|x_i\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$

4. $\to x_k$ — measure the first n qubits

Exercise 6.5

The following circuit can be used,



If $|q\rangle$ is set then on the second qubit we have $|-\rangle \to |+\rangle$, hence the oracle performs $|x\rangle |+\rangle \to |x\rangle |+ \oplus f(x)\rangle = |x\rangle |+\rangle$, i.e. no items are marked. If the $|q\rangle$ is not set then the oracle functions as intended, i.e marking the correct solutions.

Exercise 6.6

The gates in the dotted box perform,

$U = X_1 X_2 H_2 \mathrm{CNOT}_{12} H_2 X_1 X_2 = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} =$

$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} 0 & HX \\ XHX & 0 \end{bmatrix} = \begin{bmatrix} XHXHX & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -Z & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$

$I - 2\,|00\rangle \langle 00| = -(2\,|00\rangle \langle 00| - I)$

Exercise 6.7

The oracle is the operator

$O = \sum_{y \neq x} |y\rangle \langle y| \, I_2 + |x\rangle \langle x| \, X_2 = (I_1 - |x\rangle \langle x|) I_2 + |x\rangle \langle x| \, X_2 = I + |x\rangle \langle x| \, (X_2 - I_2)$

Let P_2 denote the phase gate, then the circuit implements,

$OPO = (I + |x\rangle \langle x| \, (X_2 - I_2)) P_2 (I + |x\rangle \langle x| \, (X_2 - I_2)) = P + |x\rangle \langle x| \, (P_2 X_2 - P_2 + X_2 - P_2 + P_2 X_2 - P_2 X_2 - P_2 X_2 + P_2 + X P_2 X) = P_2 + |x\rangle \langle x| \, (-P_2 + e^{i\Delta t} P_2^\dagger)$

Hence, applying the circuit to a state $|y\rangle |0\rangle$, the circuit performs,

$e^{i\Delta t} \left(I - |x\rangle \langle x| + e^{-i\Delta t} \, |x\rangle \langle x|\right) |y\rangle |0\rangle = \left(I - |x\rangle \langle x| + \sum_{n=0} \frac{(-i\Delta t)^n}{n!} \, |x\rangle \langle x| \right) |y\rangle |0\rangle =$

$\left(I + \sum_{n=1} \frac{(-i\Delta t)^n}{n!} \, |x\rangle \langle x| \right) |y\rangle |0\rangle = \left(\sum_{n=0} \frac{(-i\Delta t \, |x\rangle \langle x|)^n}{n!} \right) |y\rangle |0\rangle = e^{-i|x\rangle\langle x|\Delta t} \, |y\rangle |0\rangle$

The first two gates of the second circuit performs the action,

$H^{\otimes n} \left(\sum_{y \neq |0\rangle^{\otimes n}} |y\rangle \langle y| \, I_2 + |0\rangle \langle 0| \, X_2 \right) = H^{\otimes n} ((I_1 - |0\rangle \langle 0|) I_2 + |0\rangle \langle 0| \, X_2) = H^{\otimes n} (I + |0\rangle \langle 0| \, (X_2 - I_2))$

Hence, the total action of the circuit, similar to above is,

$H^{\otimes n} (I + |0\rangle \langle 0| \, (X_2 - I_2)) P_2 (I + |0\rangle \langle 0| \, (X_2 - I_2)) H^{\otimes n} = P_2 + H^{\otimes n} \, |0\rangle \langle 0| \, H^{\otimes n} (-P_2 + e^{i\Delta t} P_2^\dagger) = P_2 + |\psi\rangle \langle \psi| \, (-P_2 + e^{i\Delta t} P_2^\dagger)$

Therefore, as above applying the circuit on a state $|y\rangle |0\rangle$, results in

$e^{-i|\psi\rangle\langle\psi|\Delta t} \, |y\rangle |0\rangle$

Exercise 6.8

For an accuracy of $O(\Delta t^r)$ the cumulative error is $O(\Delta t^r \times \sqrt{N}/\Delta t) = O(\Delta t^{r-1}\sqrt{N})$, hence we choose $\Delta t = \Theta(N^{-1/(2(r-1))})$, therefore the number of steps is $\frac{t}{\Delta t} = O(\sqrt{N}N^{1/(2(r-1))}) = O(N^{r/(2(r-1))})$.

Exercise 6.9

Using Exercise 4.15 (b),
$c_{12} = \cos^2\left(\frac{\Delta}{2}\right) - \sin^2\left(\frac{\Delta}{2}\right)\vec{\psi}.\hat{z}$
$s_{12}\hat{n}_{12} = \sin\left(\frac{\Delta}{2}\right)\cos\left(\frac{\Delta}{2}\right)(\vec{\psi}+\hat{z}) + \sin^2\left(\frac{\Delta}{2}\right)\vec{\psi}\times\hat{z}$
Hence,
$U(\Delta t) = \left(\cos^2\left(\frac{\Delta}{2}\right) - \sin^2\left(\frac{\Delta}{2}\right)\vec{\psi}.\hat{z}\right) I - 2i\sin\left(\frac{\Delta t}{2}\right)\left(\cos\left(\frac{\Delta}{2}\right)\frac{\vec{\psi}+\hat{z}}{2} + \sin\left(\frac{\Delta}{2}\right)\frac{\vec{\psi}\times\hat{z}}{2}\right).\hat{\sigma}$

Exercise 6.10

$\cos\left(\frac{\theta}{2}\right) = 1 - \frac{2}{N}\sin^2\left(\frac{\Delta t}{2}\right)$
$\theta = 2\arccos\left(1 - \frac{2}{N}\sin^2\left(\frac{\Delta t}{2}\right)\right)$
Choose the largest Δt such that $\theta = \pi/k$ for $k \in \mathbb{Z}$.

Exercise 6.11

$$H = \sum_{i=1}^{m} |x_i\rangle\langle x_i| + |\psi\rangle\langle\psi|$$

Exercise 6.12

1) Let $|\psi\rangle = \alpha|x\rangle + \beta|y\rangle$ $H = |x\rangle\langle\psi| + |\psi\rangle\langle x| = \begin{bmatrix} 2\alpha & \beta \\ \beta & 0 \end{bmatrix} = \alpha I + \alpha Z + \beta X$
Hence,
$e^{-iHt}|\psi\rangle = e^{-i\alpha t}(\cos t|\psi\rangle - i\sin t(\alpha Z + \beta X)|\psi\rangle) = e^{-i\alpha t}(\cos t|\psi\rangle - i\sin t(\alpha^2|x\rangle - \alpha\beta|y\rangle + \alpha\beta|y\rangle + \beta^2|x\rangle)) = e^{-i\alpha t}(\cos t|\psi\rangle - i\sin t|x\rangle)$
Therefore, we can let $t = \pi/2$ to get $|x\rangle$, which is $O(1)$ in time.
2) We simulated the Hamiltonian H by alternating application of the Hamiltonians $H_1 = |x\rangle\langle\psi|$ and $H_2 = H_1^\dagger = |\psi\rangle|x\rangle$. Similar to the Hamiltonian simulated by figures 6.4 and 6.5, each iteration will require 2 oracle calls, hence we will require $O(N^{r/(2(r-1))})$ oracle calls for a Δt of accuracy $O(\Delta t^r)$.

Exercise 6.13

$\frac{k}{N}S$ $B(k,p)$ with $p = \frac{M}{N}$
$\mathrm{Var}[\frac{k}{N}S] = \frac{k^2}{N^2}\mathrm{Var}[S] = kp(1-p) = k\frac{M(N-M)}{N^2}$
$\mathrm{Var}[S] = \frac{M(N-M)}{k}$

$\Delta S = \sqrt{\frac{M(N-M)}{k}}$
From Chebyshev's inequality, $2\Delta S \leq \sqrt{M}$, hence $k \geq 4(N-M)$. Therefore, $k = \Omega(N)$.

Exercise 6.14

Assume we sample A values to get an estimate $M + c\sqrt{M}$. The expectation is $A\frac{M}{N}$ and the standard deviation $\sqrt{A\frac{M}{N}}$. Therefore, the estimate of M with 75% probability is, $M_{est} = \frac{N}{A}(A\frac{M}{N} \pm 2\sqrt{A\frac{M}{N}}) = M + 2\sqrt{\frac{N}{A}}\sqrt{M}$
Hence,
$M + c\sqrt{M} = M_{est} = M + 2\sqrt{\frac{N}{A}}\sqrt{M}$
$c^2 = \frac{4N}{A}$
$A = \frac{4N}{c^2} = \Omega(N)$

Exercise 6.15

From Cauchy-Schwarz,
$$\left(\sum_x |\langle x|y\rangle| \times 1\right)^2 \leq \sum_x |\langle x|y\rangle|^2 \sum_x 1 = N$$
Hence,
$$\sum_x ||\psi - x||^2 = ||\psi||^2 + ||x||^2 - 2\mathcal{R}(\langle x|y\rangle) = 2N - 2\mathcal{R}(\langle x|y\rangle) \geq 2N - 2\sqrt{N}$$

Exercise 6.16

Instead of $|\langle x|\psi_k^x\rangle|^2 \geq 1/2$, we have $\dfrac{1}{N}\displaystyle\sum_x |\langle x|\psi_k^x\rangle|^2 \geq 1/2$.
Hence,
$$\frac{1}{N}\sum_x ||\psi_k^x - x||^2 = 2 - 2\frac{1}{N}\sum_x |\langle x|\psi_k^x\rangle| \leq 2 - \sqrt{2}$$
Therefore,
$$E_k = \sum_x ||\psi_k^x - x||^2 \leq (2-\sqrt{2})N$$
And the rest follows as in the text.

Exercise 6.17

Exercise 6.18

Exercise 6.19

Exercise 6.20