Exercise 6.1

The phase shift performs the following,
$|0\rangle \rightarrow |0\rangle$
$|x\rangle \rightarrow -|x\rangle$ for $x > 0$
Hence, checking,
$(2|0\rangle\langle 0| - I)|0\rangle = 2|0\rangle - |0\rangle = |0\rangle$
$(2|0\rangle\langle 0| - I)|x\rangle = 0 - |x\rangle = -|x\rangle$
Also, $(2|0\rangle\langle 0| - I)^{\dagger}(2|0\rangle\langle 0| - I) = I$
Therefore, $2|0\rangle\langle 0| - I$ is the phase shift unitary operator.

Exercise 6.2

$(2|\psi\rangle\langle\psi| - I)\sum_k \alpha_k |k\rangle = \sum_k 2\alpha_k |\psi\rangle\langle\psi|k\rangle - \alpha_k|k\rangle = \sum_{x_1}\sum_{x_2}\sum_k 2\frac{\alpha_k}{N}|x_1\rangle\langle x_2|k\rangle - \alpha_k|k\rangle =$
$\sum_{x_1} 2\langle\alpha\rangle|x_1\rangle - \sum_k \alpha_k|k\rangle = 2\langle\alpha\rangle\sum_{x_1}|x_1\rangle - \sum_k \alpha_k|k\rangle = 2\langle\alpha\rangle\sum_k|k\rangle - \sum_k\alpha_k|k\rangle =$
$\sum_k[-\alpha_k + 2\langle\alpha\rangle]|k\rangle$
Where we have relabeled, x_1 to k as both are the basis vectors of the vector space.

Exercise 6.3

Let $|\psi\rangle = \sqrt{\frac{N-M}{N}}|\alpha\rangle + \sqrt{\frac{M}{N}}|\beta\rangle = \cos\frac{\theta}{2}|\alpha\rangle + \sin\frac{\theta}{2}|\beta\rangle = \begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix}$
$G|\psi\rangle = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\frac{\theta}{2}\\ \sin\frac{\theta}{2}\end{bmatrix} = \begin{bmatrix}\cos\frac{\theta}{2}\cos\theta - sin\frac{\theta}{2}\sin\theta\\ \cos\frac{\theta}{2}\sin\theta + sin\frac{\theta}{2}\cos\theta\end{bmatrix} = \begin{bmatrix}\cos\frac{3\theta}{2}\\ \sin\frac{3\theta}{2}\end{bmatrix} =$
$\cos\frac{3\theta}{2}|\alpha\rangle + \sin\frac{3\theta}{2}|\beta\rangle$
Hence, G does indeed implement the Grover iteration.

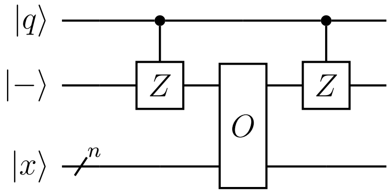
Exercise 6.4

Input: (1) a black box oracle O which performs the transformation $O|x\rangle|q\rangle = |x\rangle|q \oplus f(x)\rangle$, where $f(x) = 0\ \forall 0 \leq x < 2^n$ except x_i for $1 \leq i \leq m$ for which $f(x_i) = 1$; (2) $n+1$ qubits in the state $|0\rangle$.
Output: One of the x_i.
Runtime: $O(\sqrt{2^n})$ operations. Succeeds with probability $O(1)$.
Procedure:

1. $|0\rangle^{\otimes n}|0\rangle$	initial state
2. $\rightarrow \frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$	apply $H^{\otimes}n$ to the first n qubits, and HX to the last qubit
3. $[2(|\psi\rangle\langle\psi| - I)O]^R\frac{1}{\sqrt{2^n}}\sum_{x=0}^{2^n-1}|x\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$	apply the Grover iteration $R \approx \lceil\frac{\pi}{4}\sqrt{\frac{2^n}{M}}\rceil$ times
$\approx \frac{1}{\sqrt{2^m}}\sum_{i=1}^{m}|x_i\rangle\left[\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right]$	
4. $\rightarrow x_k$	measure the first n qubits

Exercise 6.5

The following circuit can be used,



If $|q\rangle$ is set then on the second qubit we have $|-\rangle \rightarrow |+\rangle$, hence the oracle performs $|x\rangle |+\rangle \rightarrow |x\rangle |+ \oplus f(x)\rangle = |x\rangle |+\rangle$, i.e. no items are marked. If the $|q\rangle$ is not set then the oracle functions as intended, i.e marking the correct solutions.

Exercise 6.6

The gates in the dotted box perform,

$U = X_1X_2H_2\text{CNOT}_{12}H_2X_1X_2 = \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} =$

$\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} 0 & HX \\ XHX & 0 \end{bmatrix} = \begin{bmatrix} XHXHX & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -Z & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$

$I - 2\,|00\rangle\,\langle 00| = -(2\,|00\rangle\,\langle 00| - I)$

Exercise 6.7

The oracle is the operator

$O = \sum_{y \neq x} |y\rangle\,\langle y|\, I_2 + |x\rangle\,\langle x|\, X_2 = (I_1 - |x\rangle\,\langle x|)I_2 + |x\rangle\,\langle x|\, X_2 = I + |x\rangle\,\langle x|\,(X_2 - I_2)$

Let P_2 denote the phase gate, then the circuit implements,

$OPO = (I + |x\rangle\,\langle x|\,(X_2 - I_2))P_2(I + |x\rangle\,\langle x|\,(X_2 - I_2)) = P + |x\rangle\,\langle x|\,(P_2X_2 - P_2 + X_2 - P_2 + P_2X_2 - P_2X_2 - P_2X_2 + P_2 + XP_2X) = P_2 + |x\rangle\,\langle x|\,(-P_2 + e^{i\Delta t}P_2^\dagger)$

Hence, applying the circuit to a state $|y\rangle\,|0\rangle$, the circuit performs,

$e^{i\Delta t}\left(I - |x\rangle\,\langle x| + e^{-i\Delta t}\,|x\rangle\,\langle x|\right)|y\rangle\,|0\rangle = \left(I - |x\rangle\,\langle x| + \sum_{n=0} \frac{(-i\Delta t)^n}{n!}\,|x\rangle\,\langle x|\right)|y\rangle\,|0\rangle =$

$\left(I + \sum_{n=1} \frac{(-i\Delta t)^n}{n!}\,|x\rangle\,\langle x|\right)|y\rangle\,|0\rangle = \left(\sum_{n=0} \frac{(-i\Delta t\,|x\rangle\,\langle x|)^n}{n!}\right)|y\rangle\,|0\rangle = e^{-i|x\rangle\langle x|\Delta t}\,|y\rangle\,|0\rangle$

The first two gates of the second circuit performs the action,

$H^{\otimes n}\left(\sum_{y \neq |0\rangle^{\otimes n}} |y\rangle\,\langle y|\, I_2 + |0\rangle\,\langle 0|\, X_2\right) = H^{\otimes n}((I_1 - |0\rangle\,\langle 0|)I_2 + |0\rangle\,\langle 0|\, X_2) = H^{\otimes n}(I + |0\rangle\,\langle 0|\,(X_2 - I_2))$

Hence, the total action of the circuit, similar to above is,

$H^{\otimes n}(I + |0\rangle\,\langle 0|\,(X_2 - I_2))P_2(I + |0\rangle\,\langle 0|\,(X_2 - I_2))H^{\otimes n} = P_2 + H^{\otimes n}\,|0\rangle\,\langle 0|\, H^{\otimes n}(-P_2 + e^{i\Delta t}P_2^\dagger) = P_2 + |\psi\rangle\,\langle\psi|\,(-P_2 + e^{i\Delta t}P_2^\dagger)$

Therefore, as above applying the circuit on a state $|y\rangle\,|0\rangle$, results in

$e^{-i|\psi\rangle\langle\psi|\Delta t}\,|y\rangle\,|0\rangle$

Exercise 6.8

For an accuracy of $O(\Delta t^r)$ the cumulative error is $O(\Delta t^r \times \sqrt{N}/\Delta t) = O(\Delta t^{r-1}\sqrt{N})$, hence we choose $\Delta t = \Theta(N^{-1/(2(r-1))})$, therefore the number of steps is $\frac{t}{\Delta t} = O(\sqrt{N}N^{1/(2(r-1))}) = O(N^{r/(2(r-1))})$.

Exercise 6.9

Using Exercise 4.15 (b),
$c_{12} = \cos^2\left(\frac{\Delta}{2}\right) - \sin^2\left(\frac{\Delta}{2}\right)\vec{\psi}.\hat{z}$
$s_{12}\hat{n}_{12} = \sin\left(\frac{\Delta}{2}\right)\cos\left(\frac{\Delta}{2}\right)(\vec{\psi} + \hat{z}) + \sin^2\left(\frac{\Delta}{2}\right)\vec{\psi} \times \hat{z}$
Hence,
$U(\Delta t) = \left(\cos^2\left(\frac{\Delta}{2}\right) - \sin^2\left(\frac{\Delta}{2}\right)\vec{\psi}.\hat{z}\right) I - 2i\sin\left(\frac{\Delta t}{2}\right)\left(\cos\left(\frac{\Delta}{2}\right)\frac{\vec{\psi}+\hat{z}}{2} + \sin\left(\frac{\Delta}{2}\right)\frac{\vec{\psi}\times\hat{z}}{2}\right).\hat{\sigma}$

Exercise 6.10

Exercise 6.11

Exercise 6.12

Exercise 6.13

Exercise 6.14

Exercise 6.15

Exercise 6.16

Exercise 6.17

Exercise 6.18

Exercise 6.19

Exercise 6.20