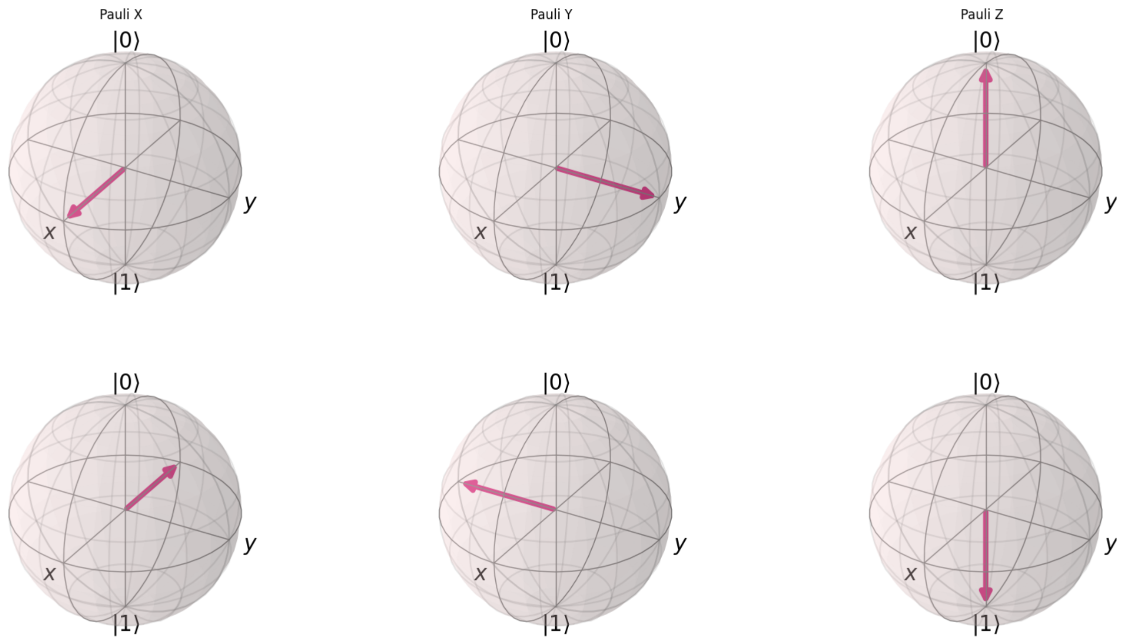
Exercise 4.1

The eigenvectors are as follows:
Pauli Z: $|0\rangle$, $|1\rangle$
Pauli X: $|0\rangle + |1\rangle$, $|0\rangle - |1\rangle$
Pauli Y: $|0\rangle + i\,|1\rangle$, $|0\rangle - i\,|1\rangle$
Bloch sphere representations:



Exercise 4.2

$$\exp(iAx) = \sum_n (iAx)^n = \sum_n (-1)^n x^{2n} I + \sum_n (-1)^n i x^n A = \cos(x) I + i \sin x A$$

Exercise 4.3

Up to a global phase:

$$T = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\pi}{4}/2} & 0 \\ 0 & e^{i\frac{\pi}{4}/2} \end{bmatrix} = R_z(\pi/4)$$

Exercise 4.4

First consider $R_z R_x R_z$:

$$R_z R_x R_z = \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\theta} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} e^{i\theta} \end{bmatrix}$$

For $\theta = \frac{\pi}{2}$:

$$R_z R_x R_z = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\frac{\pi}{2}} & e^{-i\frac{\pi}{2}} \\ e^{-i\frac{\pi}{2}} & e^{i\frac{\pi}{2}} \end{bmatrix}$$

Hence, by multiplying by $e^{i\frac{\pi}{2}}$ we get,

$e^{i\frac{\pi}{2}}R_zR_xR_z = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix} = H$

Exercise 4.5

We have $n_x^2 + n_y^2 + n_z^2 = 1$
$\hat{n}\cdot\vec{\sigma} = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & n_z \end{bmatrix}$
Therefore,
$(\hat{n}\cdot\vec{\sigma})^2 = \begin{bmatrix} n_x^2+n_y^2+n_z^2 & 0 \\ 0 & n_x^2+n_y^2+n_z^2 \end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$
Consider, $R_n(\theta)R_n(-\theta)$
$I = R_n(\theta)R_n(-\theta) = (\cos(\frac{\theta}{2})I - \sin(\frac{\theta}{2})\hat{n}\cdot\vec{\sigma})(\cos(\frac{\theta}{2})I + \sin(\frac{\theta}{2})\hat{n}\cdot\vec{\sigma}) = \cos^2(\frac{\theta}{2})I + \sin^2(\frac{\theta}{2})(\hat{n}\cdot\vec{\sigma})^2 = (\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}))I = I$

Exercise 4.6

First, let's show that $R_Z(x)$ rotates around the Z-axis by an angle x. Consider the general state $|\psi\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$. Then,
$R_Z(x)|\psi\rangle = (\cos\frac{x}{2}I - i\sin\frac{x}{2}Z)\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \cos\frac{x}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} - i\sin\frac{x}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ -e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}e^{-ix/2}\cos\frac{\theta}{2}\\ e^{ix/2}e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i(\phi+x)}\sin\frac{\theta}{2}\end{pmatrix}$
Hence, the state has been rotated by x around the Z-axis. Similarly, we get that $R_X(x)$ and $R_Y(x)$ rotate around the X and Y axis respectively.
We also have that,
$R_n(x) = \cos\frac{x}{2}I - i\sin\frac{x}{2}(n_xX + n_yY + n_ZZ) = \cos\frac{x}{2}I - i\sin\frac{x}{2}(\sin\theta_n\cos\phi_nX + \sin\theta_n\sin\theta_nY + \cos\theta_nZ) = R_Z(\phi_n)R_X(\theta_n)(\cos\frac{x}{2}I - i\sin\frac{x}{2}Z)R_X(\theta_n)^\dagger R_Z(\phi_n)^\dagger = R_Z(\phi_n)R_X(\theta_n)R_Z(x)R_X(\theta_n)^\dagger R_Z(\phi_n)^\dagger$
Therefore, $R_n(x)$ rotates the axis of rotation to the Z axis performs the rotations by angle x and then returns the axis back to n, which is the same as rotating around n by an angle x.

Exercise 4.7

$\{X, Y\} = 1$ therefore, $XYX = -XXY = -Y$.
$XR_Y(\theta)X = X(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y)X = \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}Y = R_Y(-\theta)$

Exercise 4.8

Any 2x2 unitary matrix for $a^2 + b^2 + c^2 + d^2 = 1$ can be written as,
1)$U = e^{i\alpha}\begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$
Consider, the given form for U,
$U = e^{i\alpha}R_n(\theta) = e^{i\alpha}\begin{bmatrix}\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z & -\sin\frac{\theta}{2}(n_y + in_x) \\ \sin\frac{\theta}{2}(n_y - in_x) & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}n_z\end{bmatrix}$
As, $n_x^2 + n_y^2 + n_z^2 = 1$ this has the same form as the general U, hence any arbitrary 2x2 unitary matrix can be written as $U = e^{i\alpha}R_n(\theta)$.
2) $n_z = \frac{1}{\sqrt{2}}$, $n_y = 0$, $n_x = \frac{1}{\sqrt{2}}$, $\alpha = 0$ and $\theta = \pi$.
3) $n_x, n_y = 0$, $n_z = 1$, $\alpha = \theta = \frac{\pi}{4}$.

Exercise 4.9

We can write,
$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix} = e^{i\alpha}\begin{bmatrix} e^{i(-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix} =$
$e^{i\alpha}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$
which is the general form of a 2x2 unitary matrix as $det(U) = \cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2} = 1$.

Exercise 4.10

$U = e^{i\alpha}R_Z(\beta)R_X(\gamma)R_Z(\delta) = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -ie^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ -ie^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix}$
which once again is a unitary matrix.

Exercise 4.11

Exercise 4.12

From the proof of Corollary 4.2 we have, $AXBXC = R_Z(\beta)R_Y(\gamma)R_Z(\delta)$. We can see that $\alpha = \gamma = \delta = \frac{\pi}{2}$ and $\beta = 0$ gives H. Hence, we can take $A = R_Y(\frac{\pi}{4})$,$B = R_Y(-\frac{\pi}{4})R_Z(-\frac{\pi}{4})$ and $C = R_Z(\frac{\pi}{4})$.

Exercise 4.13

$HXH = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = Z$
$HYH = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} = -Y$ $HZH = HHXHH = X$

Exercise 4.14

$T = R_Z(\frac{\pi}{4})$
$HTH = HR_Z(\frac{\pi}{4})H = H(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}Z)H = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}X = R_X(\frac{\pi}{4})$

Exercise 4.15

(Check Errata for the sign in the second equation)
1)$R_{\hat{n}_2}(\beta_2)R_{\hat{n}_1}(\beta_1) = (c_2I - is_2\hat{n}_2.\sigma)(c_1I - is_1\hat{n}_1.\sigma) = c_1c_2I - s_1s_2(\hat{n}_2.\sigma)(\hat{n}_1.\sigma) - ic_2s_1\hat{n}_1.\sigma - ic_1s_2\hat{n}_2.\sigma = c_1c_2I - s_1s_2(\hat{n}_1.\hat{n}_2I + i(\hat{n}_2\times\hat{n}_1).\sigma) - ic_2s_1\hat{n}_1.\sigma - ic_1s_2\hat{n}_2.\sigma = (c_1c_2 - s_1s_2\hat{n}_1.\hat{n}_2)I - i(c_2s_1\hat{n}_1 + c_1s_2\hat{n}_2 + s_1s_2\hat{n}_2\times\hat{n}_1).\sigma$
Therefore,
$c_{12} = c_1c_2 - s_1s_2\hat{n}_1.\hat{n}_2$
$s_{12}\hat{n}_{12} = c_2s_1\hat{n}_1 + c_1s_2\hat{n}_2 + s_1s_2\hat{n}_2\times\hat{n}_1$
2)$\beta_1 = \beta_2$ and $\hat{n}_1 = \hat{z}$, hence $c_1 = c_2 = c$ and $s_1 = s_2 = s$. Therefore,
$c_{12} = c^2 - s^2\hat{z}.\hat{n}_2$
$s_{12}\hat{n}_{12} = cs(\hat{z} + \hat{n}_2) + s^2\hat{n}_2\times\hat{z}$

Exercise 4.16

$H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$

$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$

Exercise 4.17

$$\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} = \begin{bmatrix} HH & 0 \\ 0 & HZH \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = CNOT$$

Exercise 4.18

When the second qubit is the control we have the following representation,

$$CZ_2 = |00\rangle\langle 00| + |01\rangle\langle 01| - |11\rangle\langle 11| + |10\rangle\langle 10| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Exercise 4.19

$$\text{CNOT}\rho\text{CNOT} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} A & XB \\ XC & XDX \end{bmatrix}$$

X only rearranges elements, hence the CNOT only rearranges the elements of ρ.

Exercise 4.20

$$(H_1 \otimes H_2)\text{CNOT}(H_1 \otimes H_2) = \frac{1}{2} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2} \begin{bmatrix} HH + HXH & HH - HXH \\ HH - HXH & HH + HXH \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} I + Z & I - Z \\ I - Z & I + Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \text{CNOT}(2^{\text{nd}} \text{ qubit control})$$

The left circuit transforms between the $|0\rangle$,$|1\rangle$ and $|+\rangle$,$|-\rangle$ basis applies a CNOT and transforms back. Hence, the effect of the CNOT on the $|\pm\rangle|\pm\rangle$ is the same as applying the CNOT with the 2^{nd} qubit as target to the state in the $|0\rangle$,$|1\rangle$ basis and then replacing 0 with $+$ and 1 with $-$. This process does indeed give the equations 4.24-4.27.

Exercise 4.21

Consider all the possible inputs,

$|00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle$

$|01\psi\rangle \to |01(V\psi)\rangle \to |01(V\psi)\rangle \to |01(V^\dagger V\psi)\rangle \to |01\psi\rangle \to |01\psi\rangle$

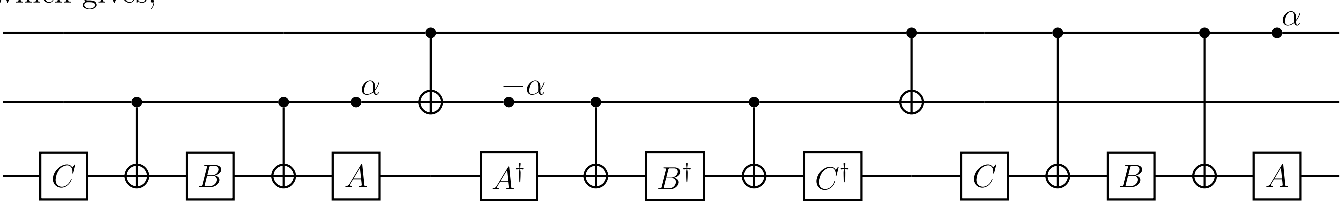
$|10\psi\rangle \to |10\psi\rangle \to |11\psi\rangle \to |11(V^\dagger\psi)\rangle \to |10(VV^\dagger\psi)\rangle \to |10\psi\rangle$

$|11\psi\rangle \to |11(V\psi)\rangle \to |10(V\psi)\rangle \to |11(V\psi)\rangle \to |11(VV\psi)\rangle \to |11U\psi\rangle$

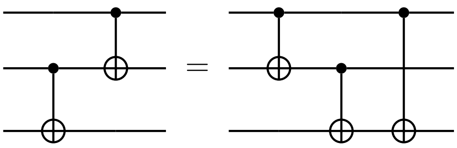
Hence, the circuit does perform the $C^2(U)$ operation.

Exercise 4.22

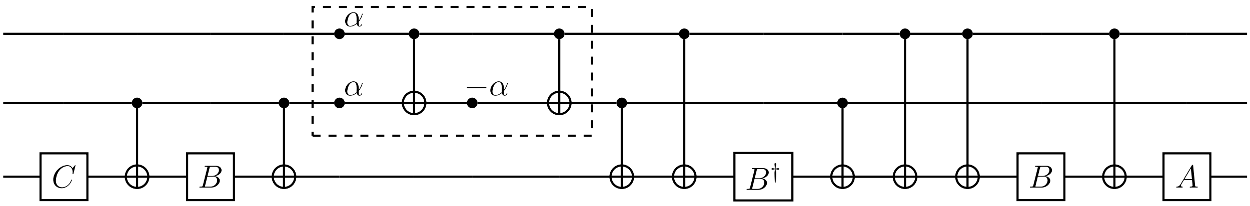
Firstly, we apply the circuit in figure 4.6 to the circuit in figure 4.8 for $V = e^{i\alpha}AXBXC$, which gives,



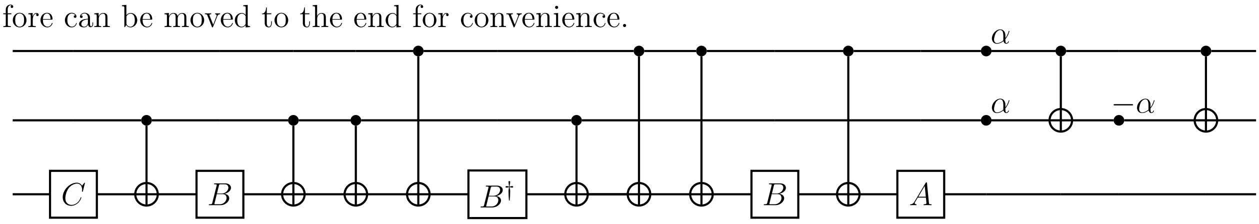
We move the 6^{th} CNOT left from the 4^{th} one, which involves adding CNOTs after the 4^{th} and 5^{th} CNOTs from first to third qubit, which is due to,



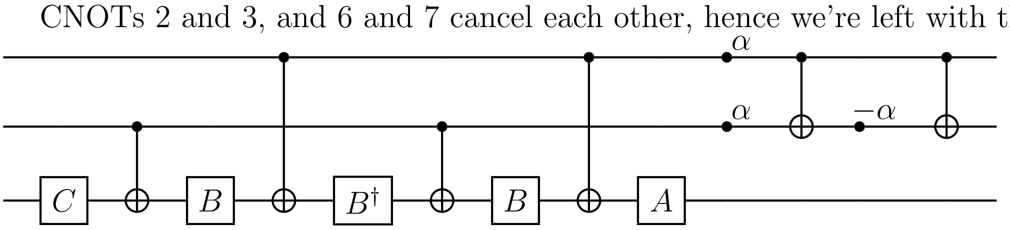
and can be checked by considering all the possible inputs. We also see that $AA^\dagger = CC^\dagger = I$. Hence we get the following circuit.



The dashed section is diagonal hence commutes with the rest of the components therefore can be moved to the end for convenience.



CNOTs 2 and 3, and 6 and 7 cancel each other, hence we're left with the circuit.



This has 8 single qubit gates and 6 CNOTs as desired.
The circuit performs the operation, $AXBXB^\dagger XBXC = (VC^\dagger)B^\dagger(A^\dagger V) = V(ABC)^\dagger = V^2 = U$. Therefore, the circuit performs the $C^2(U)$ operation.

Exercise 4.23

For $U = R_X(\theta)$ from corollary 4.2 we can see that $A = H$, $B = R_Z(-\frac{\theta}{2})$, $C = R_Z(\frac{\theta}{2})H$, which gives $ABC = I$ and $AXBXC = HXR_Z(-\frac{\theta}{2})XR_Z(\frac{\theta}{2})H = HXXR_Z(\frac{\theta}{2})R_Z(\frac{\theta}{2})H = R_X(\theta)$.
For $U = R_Y(\theta)$ we can take $A = I$, $B = R_Y(-\frac{\theta}{2})$ and $C = R_Y(\frac{\theta}{2})$, which gives $ABC = I$ and $AXBXC = XR_Y(-\frac{\theta}{2})XR_Y(\frac{\theta}{2}) = XXR_Y(\frac{\theta}{2})R_Y(\frac{\theta}{2}) = R_Y(\theta)$.

Exercise 4.24

To verify the circuit we consider the state after each operation. Let the initial state be $|xyz\rangle$, where $x, y, z \in \{0, 1\}$.

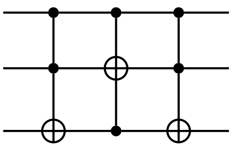
$$
\begin{gathered}
|xyz\rangle \\
\Big\downarrow H_3 \\
\frac{1}{\sqrt{2}}(|xy0\rangle + (-1)^z |xy1\rangle) \\
\Big\downarrow \mathrm{CNOT}_{23} \\
\frac{1}{\sqrt{2}}(|xyy\rangle + (-1)^z |xy\bar{y}\rangle) \\
\Big\downarrow T_3^\dagger \\
\frac{1}{\sqrt{2}}(e^{-iy\pi/4} |xyy\rangle + e^{-i\bar{y}\pi/4}(-1)^z |xy\bar{y}\rangle) \\
\Big\downarrow \mathrm{CNOT}_{13} \\
\frac{1}{\sqrt{2}}(e^{-iy\pi/4} |xy(y\oplus x)\rangle + e^{-i\bar{y}\pi/4}(-1)^z |xy(\bar{y}\oplus x)\rangle) \\
\Big\downarrow T_3 \\
\frac{1}{\sqrt{2}}(e^{i(y\oplus x-y)\pi/4} |xy(y\oplus x)\rangle + e^{i(\bar{y}\oplus x-\bar{y})\pi/4}(-1)^z |xy(\bar{y}\oplus x)\rangle) \\
\Big\downarrow \mathrm{CNOT}_{23} \\
\frac{1}{\sqrt{2}}(e^{i(y\oplus x-y)\pi/4} |xyx\rangle + e^{i(\bar{y}\oplus x-\bar{y})\pi/4}(-1)^z |xy\bar{x}\rangle) \\
\Big\downarrow T_3^\dagger \\
\frac{1}{\sqrt{2}}(e^{i(y\oplus x-y-x)\pi/4} |xyx\rangle + e^{i(\bar{y}\oplus x-\bar{y}-\bar{x})\pi/4}(-1)^z |xy\bar{x}\rangle) \\
\Big\downarrow \mathrm{CNOT}_{13} \\
\frac{1}{\sqrt{2}}(e^{i(y\oplus x-y-x)\pi/4} |xy0\rangle + e^{i(\bar{y}\oplus x-\bar{y}-\bar{x})\pi/4}(-1)^z |xy1\rangle) \\
\Big\downarrow T_2^\dagger T_3 \\
\frac{1}{\sqrt{2}}(e^{i(y\oplus x-2y-x)\pi/4} |xy0\rangle + e^{i(\bar{y}\oplus x-\bar{x})\pi/4}(-1)^z |xy1\rangle) \\
\Big\downarrow \mathrm{CNOT}_{12} H_3 \\
\frac{1}{2}(e^{i(y\oplus x-2y-x)\pi/4} + e^{i(\bar{y}\oplus x-\bar{x})\pi/4}) |x(y\oplus x)0\rangle + \frac{1}{2}(e^{i(y\oplus x-2y-x)\pi/4} - e^{i(\bar{y}\oplus x-\bar{x})\pi/4}(-1)^z) |x(y\oplus x)1\rangle \\
\Big\downarrow T_2^\dagger \\
\frac{1}{2}(e^{i(-2y-x)\pi/4} + e^{i((1-2y)(1-2x)-\bar{x})\pi/4}) |x(y\oplus x)0\rangle + \frac{1}{2}(e^{i(-2y-x)\pi/4} - e^{i((1-2y)(1-2x)-\bar{x})\pi/4}(-1)^z) |x(y\oplus x)1\rangle \\
\Big\downarrow \mathrm{CNOT}_{23} \\
\frac{1}{2}(e^{i(-2y-x)\pi/4} + e^{i((1-2y)(1-2x)-\bar{x})\pi/4}) |xy0\rangle + \frac{1}{2}(e^{i(-2y-x)\pi/4} - e^{i((1-2y)(1-2x)-\bar{x})\pi/4}(-1)^z) |xy1\rangle \\
\Big\downarrow T_1 S_2
\end{gathered}
$$

$$\frac{1}{2}(1+e^{i((1-2y)(1-2x)-1+2x+2y)\pi/4})\,|xy0\rangle + \frac{1}{2}(1-e^{i((1-2y)(1-2x)-1+2x+2y)\pi/4}(-1)^z)\,|xy1\rangle$$
$$=$$
$$\frac{1}{2}(1+(-1)^{xy+z})\,|xy0\rangle + \frac{1}{2}(1-(-1)^{(xy+z)})\,|xy1\rangle$$

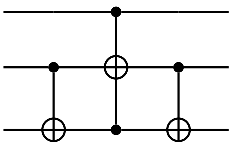
If $x, y = 1$ then we get $|xy\bar{z}\rangle$ and $|xyz\rangle$ otherwise, hence the circuit does indeed implement the Toffoli gate.

Exercise 4.25

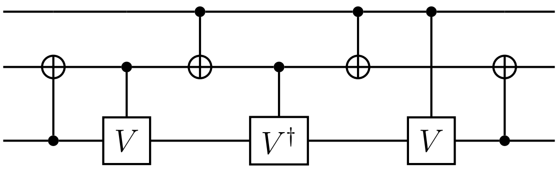
1)



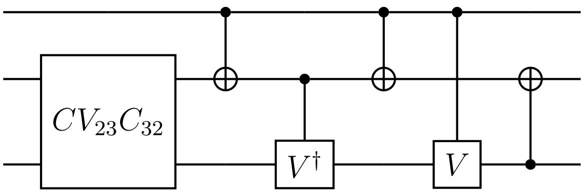
2)



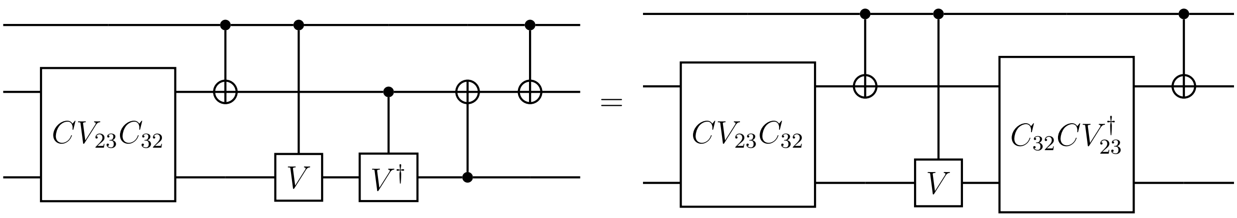
If the first qubit is 0, then the Toffoli just performs the identity, hence the CNOTs cancel leading to an overall identity. If the first qubit is 1, then the Toffoli performs a CNOT on the last 2 qubits, which overall performs the SWAP operation.
3) For the Toffoli we take $V = \frac{(1-i)(1+iX)}{2}$, which gives $V^2 = X$. Then, after changing the order of the last 2 qubits the circuit is,



Which simplifies to,



4)As V is unitary the CV_{13} gate commutes with the $CV_{23}^\dagger$ and C_{12} gates, hence we can move it to the left of $CV_{23}^\dagger$. Afterwards, the last to CNOTs commute as well, hence we get



which contains 5 two-qubit gates.

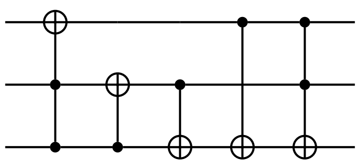
Exercise 4.26

Consider all the possible controls,
$|00t\rangle \rightarrow |00(R_Y(\pi/4)R_Y(\pi/4)R_Y(-\pi/4)R_Y(-\pi/4))t\rangle = |00t\rangle$
$|01t\rangle \rightarrow |01(R_Y(\pi/4)XR_Y(\pi/4)R_Y(-\pi/4)XR_Y(-\pi/4))t\rangle =$
$|01(R_Y(\pi/4)R_Y(-\pi/4)XXR_Y(\pi/4)R_Y(-\pi/4))t\rangle = |01t\rangle$
$|10t\rangle \rightarrow |10(R_Y(\pi/4)R_Y(\pi/4)XR_Y(-\pi/4)R_Y(-\pi/4))t\rangle =$
$|10(R_Y(\pi/4)R_Y(\pi/4)R_Y(\pi/4)R_Y(\pi/4)X)t\rangle = |10(R_Y(\pi)X)t\rangle = -|10(Zt)\rangle$
$|11t\rangle \rightarrow |11(R_Y(\pi/4)XR_Y(\pi/4)XR_Y(-\pi/4)XR_Y(-\pi/4))t\rangle =$
$|11(R_Y(\pi/4)R_Y(-\pi/4)R_Y(-\pi/4)R_Y(\pi/4)X)t\rangle = |11(Xt)\rangle$
Hence, the circuit does indeed implement the Toffoli gate, with the angle for the phase factor being,

$$\theta(c_1, c_2, t) = \begin{cases} \pi, & \text{for } (c_1, c_2, t) = (1,0,0) \\ 0, & \text{otherwise} \end{cases}$$

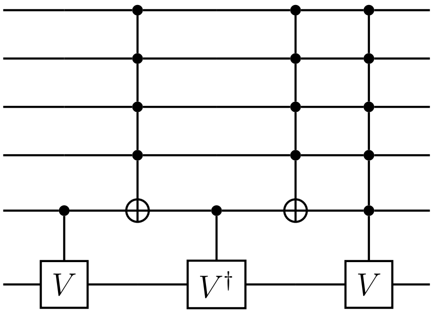
Exercise 4.27

We can write the matrix as,
$U = |000\rangle\langle 000| + |010\rangle\langle 001| + |011\rangle\langle 010| + |100\rangle\langle 011| + |101\rangle\langle 100| + |110\rangle\langle 101| + |111\rangle\langle 110| + |001\rangle\langle 111|$
Which implies the following inputs and outputs
$|000\rangle \rightarrow |000\rangle$
$|001\rangle \rightarrow |010\rangle$
$|101\rangle \rightarrow |110\rangle$
$|010\rangle \rightarrow |011\rangle$
$|011\rangle \rightarrow |100\rangle$
$|100\rangle \rightarrow |101\rangle$
$|110\rangle \rightarrow |111\rangle$
$|111\rangle \rightarrow |001\rangle$
Firstly, we can notice that the first qubit only changes when both the other ones are set, hence we start the circuit with a Toffoli(2,3,1) with the first qubit as target. Looking at $|001\rangle \rightarrow |010\rangle$, $|101\rangle \rightarrow |110\rangle$, $|010\rangle \rightarrow |011\rangle$, $|110\rangle \rightarrow |111\rangle$ and $|111\rangle \rightarrow |001\rangle$ we can see that $C_{23}C_{32}$ after the Toffoli gives the desired results. For $|011\rangle \rightarrow |100\rangle$ and $|100\rangle \rightarrow |101\rangle$ we require a C_{13} after the rest of the gates. However, this changes $|110\rangle \rightarrow |111\rangle$ to $|110\rangle \rightarrow |110\rangle$, hence we apply a Toffoli(1,2,3), as no other final state has both first and second qubit set. Hence the circuit will be,



Exercise 4.28

Analogous to the circuit for $C^2(U)$ we have,



Exercise 4.29