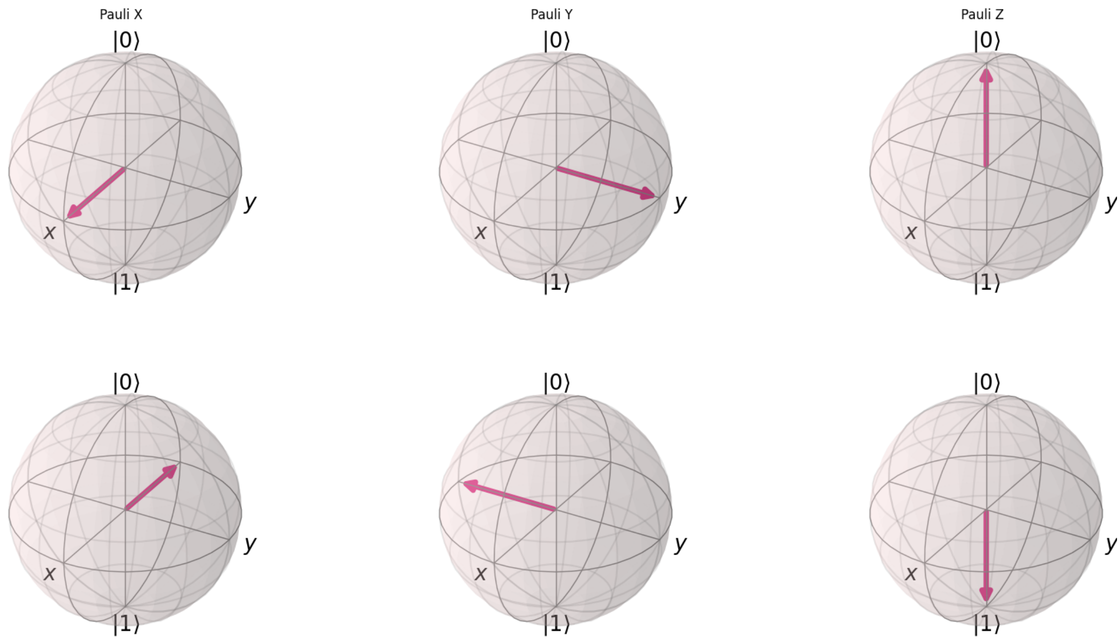
Exercise 4.1

The eigenvectors are as follows:
Pauli Z: $|0\rangle$, $|1\rangle$
Pauli X: $|0\rangle + |1\rangle$, $|0\rangle - |1\rangle$
Pauli Y: $|0\rangle + i\,|1\rangle$, $|0\rangle - i\,|1\rangle$
Bloch sphere representations:



Exercise 4.2

$$\exp(iAx) = \sum_n (iAx)^n = \sum_n (-1)^n x^{2n} I + \sum_n (-1)^n i x^n A = \cos(x) I + i \sin x A$$

Exercise 4.3

Up to a global phase:
$$T = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\pi}{4}/2} & 0 \\ 0 & e^{i\frac{\pi}{4}/2} \end{bmatrix} = R_z(\pi/4)$$

Exercise 4.4

First consider $R_z R_x R_z$:
$$R_z R_x R_z = \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\theta} & -i \sin\frac{\theta}{2} \\ -i \sin\frac{\theta}{2} & \cos\frac{\theta}{2} e^{i\theta} \end{bmatrix}$$
For $\theta = \frac{\pi}{2}$:
$$R_z R_x R_z = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\frac{\pi}{2}} & e^{-i\frac{\pi}{2}} \\ e^{-i\frac{\pi}{2}} & e^{i\frac{\pi}{2}} \end{bmatrix}$$
Hence, by multiplying by $e^{i\frac{\pi}{2}}$ we get,

$e^{i\frac{\pi}{2}}R_zR_xR_z = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} = H$

Exercise 4.5

We have $n_x^2 + n_y^2 + n_z^2 = 1$
$\hat{n}\cdot\vec{\sigma} = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & n_z \end{bmatrix}$
Therefore,
$(\hat{n}\cdot\vec{\sigma})^2 = \begin{bmatrix} n_x^2 + n_y^2 + n_z^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2 \end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$
Consider, $R_n(\theta)R_n(-\theta)$
$I = R_n(\theta)R_n(-\theta) = (\cos(\frac{\theta}{2})I - \sin(\frac{\theta}{2})\hat{n}\cdot\vec{\sigma})(\cos(\frac{\theta}{2})I + \sin(\frac{\theta}{2})\hat{n}\cdot\vec{\sigma}) = \cos^2(\frac{\theta}{2})I + \sin^2(\frac{\theta}{2})(\hat{n}\cdot\vec{\sigma})^2 = (\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}))I = I$

Exercise 4.6

First, let's show that $R_Z(x)$ rotates around the Z-axis by an angle x. Consider the general state $|\psi\rangle = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix}$. Then,
$R_Z(x)|\psi\rangle = (\cos\frac{x}{2}I - i\sin\frac{x}{2}Z)\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \cos\frac{x}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} - i\sin\frac{x}{2}\begin{pmatrix}\cos\frac{\theta}{2}\\ -e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}e^{-ix/2}\cos\frac{\theta}{2}\\ e^{ix/2}e^{i\phi}\sin\frac{\theta}{2}\end{pmatrix} = \begin{pmatrix}\cos\frac{\theta}{2}\\ e^{i(\phi+x)}\sin\frac{\theta}{2}\end{pmatrix}$
Hence, the state has been rotated by x around the Z-axis. Similarly, we get that $R_X(x)$ and $R_Y(x)$ rotate around the X and Y axis respectively.
We also have that,
$R_n(x) = \cos\frac{x}{2}I - i\sin\frac{x}{2}(n_xX + n_yY + n_ZZ) = \cos\frac{x}{2}I - i\sin\frac{x}{2}(\sin\theta_n\cos\phi_nX + \sin\theta_n\sin\theta_nY + \cos\theta_nZ) = R_Z(\phi_n)R_X(\theta_n)(\cos\frac{x}{2}I - i\sin\frac{x}{2}Z)R_X(\theta_n)^\dagger R_Z(\phi_n)^\dagger = R_Z(\phi_n)R_X(\theta_n)R_Z(x)R_X(\theta_n)^\dagger R_Z(\phi_n)^\dagger$
Therefore, $R_n(x)$ rotates the axis of rotation to the Z axis performs the rotations by angle x and then returns the axis back to n, which is the same as rotating around n by an angle x.

Exercise 4.7

$\{X, Y\} = 1$ therefore, $XYX = -XXY = -Y$.
$XR_Y(\theta)X = X(\cos\frac{\theta}{2}I - i\sin\frac{\theta}{2}Y)X = \cos\frac{\theta}{2}I + i\sin\frac{\theta}{2}Y = R_Y(-\theta)$

Exercise 4.8

Any 2x2 unitary matrix for $a^2 + b^2 + c^2 + d^2 = 1$ can be written as,
1)$U = e^{i\alpha}\begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}$
Consider, the given form for U,
$U = e^{i\alpha}R_n(\theta) = e^{i\alpha}\begin{bmatrix}\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}n_z & -\sin\frac{\theta}{2}(n_y + in_x) \\ \sin\frac{\theta}{2}(n_y - in_x) & \cos\frac{\theta}{2} + i\sin\frac{\theta}{2}n_z\end{bmatrix}$
As, $n_x^2 + n_y^2 + n_z^2 = 1$ this has the same form as the general U, hence any arbitrary 2x2 unitary matrix can be written as $U = e^{i\alpha}R_n(\theta)$.
2) $n_z = \frac{1}{\sqrt{2}}$, $n_y = 0$, $n_x = \frac{1}{\sqrt{2}}$, $\alpha = 0$ and $\theta = \pi$.
3) $n_x, n_y = 0$, $n_z = 1$, $\alpha = \theta = \frac{\pi}{4}$.

Exercise 4.9