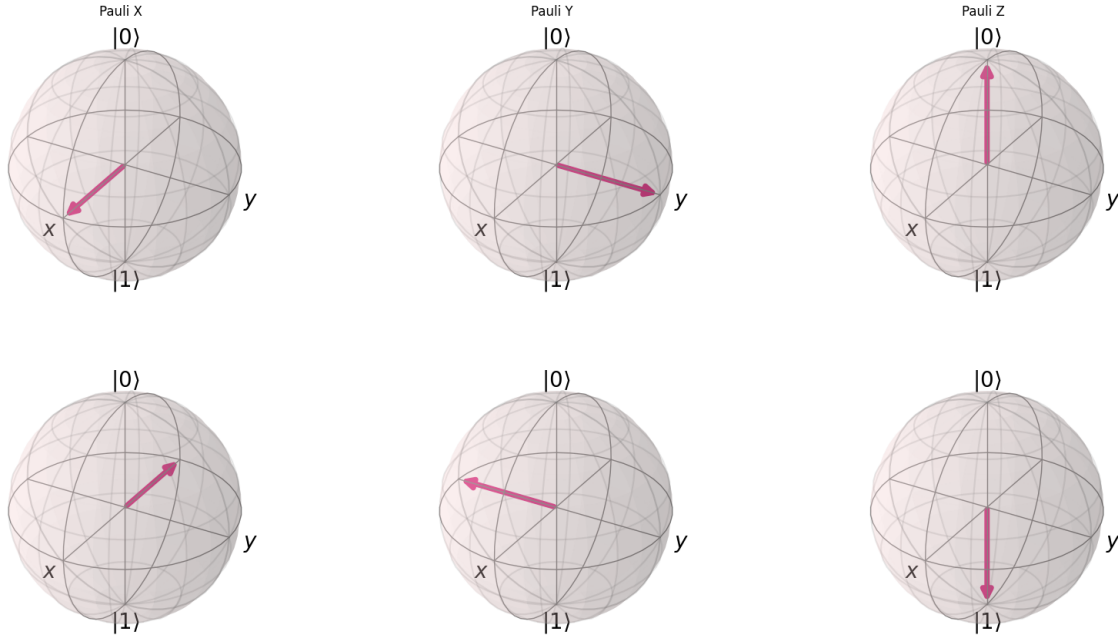## Exercise 4.1

The eigenvectors are as follows:
Pauli Z: $|0\rangle$, $|1\rangle$
Pauli X: $|0\rangle + |1\rangle$, $|0\rangle - |1\rangle$
Pauli Y: $|0\rangle + i\,|1\rangle$, $|0\rangle - i\,|1\rangle$
Bloch sphere representations:

Exercise 4.2

$$\exp(iAx) = \sum_n (iAx)^n = \sum_n (-1)^n x^{2n} I + \sum_n (-1)^n i x^n A = \cos(x)I + i\sin xA$$

Exercise 4.3

Up to a global phase:
$$T = \begin{bmatrix} e^{-i\pi/8} & 0 \\ 0 & e^{i\pi/8} \end{bmatrix} = \begin{bmatrix} e^{-i\frac{\pi}{4}/2} & 0 \\ 0 & e^{i\frac{\pi}{4}/2} \end{bmatrix} = R_z(\pi/4)$$

Exercise 4.4

First consider $R_z R_x R_z$:
$$R_z R_x R_z = \begin{bmatrix} \cos\frac{\theta}{2} e^{-i\theta} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} e^{i\theta} \end{bmatrix}$$
For $\theta = \frac{\pi}{2}$:
$$R_z R_x R_z = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{-i\frac{\pi}{2}} & e^{-i\frac{\pi}{2}} \\ e^{-i\frac{\pi}{2}} & e^{i\frac{\pi}{2}} \end{bmatrix}$$
Hence, by multiplying by $e^{i\frac{\pi}{2}}$ we get,

$$e^{i\frac{\pi}{2}} R_z R_x R_z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H.$$

Exercise 4.5

We have $n_x^2 + n_y^2 + n_z^2 = 1$

$$\hat{n} \cdot \vec{\sigma} = \begin{bmatrix} n_z & n_x - in_y \\ n_x + in_y & n_z \end{bmatrix}$$

Therefore,

$$(\hat{n} \cdot \vec{\sigma})^2 = \begin{bmatrix} n_x^2 + n_y^2 + n_z^2 & 0 \\ 0 & n_x^2 + n_y^2 + n_z^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Consider, $R_n(\theta) R_n(-\theta)$

$I = R_n(\theta) R_n(-\theta) = (\cos(\frac{\theta}{2})I - \sin(\frac{\theta}{2})\hat{n} \cdot \vec{\sigma})(\cos(\frac{\theta}{2})I + \sin(\frac{\theta}{2})\hat{n} \cdot \vec{\sigma}) = \cos^2(\frac{\theta}{2})I + \sin^2(\frac{\theta}{2})(\hat{n} \cdot \vec{\sigma})^2 = (\cos^2(\frac{\theta}{2}) + \sin^2(\frac{\theta}{2}))I = I$

Exercise 4.6

First, let's show that $R_Z(x)$ rotates around the Z-axis by an angle x. Consider the general state $|\psi\rangle = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix}$. Then,

$$R_Z(x)|\psi\rangle = (\cos \frac{x}{2} I - i \sin \frac{x}{2} Z) \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} = \cos \frac{x}{2} \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} - i \sin \frac{x}{2} \begin{pmatrix} \cos \frac{\theta}{2} \\ -e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} e^{-ix/2} \cos \frac{\theta}{2} \\ e^{ix/2} e^{i\phi} \sin \frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} \cos \frac{\theta}{2} \\ e^{i(\phi+x)} \sin \frac{\theta}{2} \end{pmatrix}$$

Hence, the state has been rotated by x around the Z-axis. Similarly, we get that $R_X(x)$ and $R_Y(x)$ rotate around the X and Y axis respectively.

We also have that,

$R_n(x) = \cos \frac{x}{2} I - i \sin \frac{x}{2}(n_x X + n_y Y + n_Z Z) = \cos \frac{x}{2} I - i \sin \frac{x}{2}(\sin \theta_n \cos \phi_n X + \sin \theta_n \sin \theta_n Y + \cos \theta_n Z) = R_Z(\phi_n) R_X(\theta_n)(\cos \frac{x}{2} I - i \sin \frac{x}{2} Z) R_X(\theta_n)^\dagger R_Z(\phi_n)^\dagger = R_Z(\phi_n) R_X(\theta_n) R_Z(x) R_X(\theta_n)^\dagger R_Z(\phi_n)^\dagger$

Therefore, $R_n(x)$ rotates the axis of rotation to the Z axis performs the rotations by angle x and then returns the axis back to n, which is the same as rotating around n by an angle x.

Exercise 4.7

$\{X, Y\} = 1$ therefore, $XYX = -XXY = -Y$.
$X R_Y(\theta) X = X(\cos \frac{\theta}{2} I - i \sin \frac{\theta}{2} Y) X = \cos \frac{\theta}{2} I + i \sin \frac{\theta}{2} Y = R_Y(-\theta)$

Exercise 4.8

Any 2x2 unitary matrix for $a^2 + b^2 + c^2 + d^2 = 1$ can be written as,

1)$U = e^{i\alpha} \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}$

Consider, the given form for U,

$$U = e^{i\alpha} R_n(\theta) = e^{i\alpha} \begin{bmatrix} \cos \frac{\theta}{2} - i \sin \frac{\theta}{2} n_z & -\sin \frac{\theta}{2}(n_y + in_x) \\ \sin \frac{\theta}{2}(n_y - in_x) & \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} n_z \end{bmatrix}$$

As, $n_x^2 + n_y^2 + n_z^2 = 1$ this has the same form as the general U, hence any arbitrary 2x2 unitary matrix can be written as $U = e^{i\alpha} R_n(\theta)$.

2) $n_z = \frac{1}{\sqrt{2}}$, $n_y = 0$, $n_x = \frac{1}{\sqrt{2}}$, $\alpha = 0$ and $\theta = \pi$.

3) $n_x, n_y = 0$, $n_z = 1$, $\alpha = \theta = \frac{\pi}{4}$.

Exercise 4.9

We can write,

$$U = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix} = e^{i\alpha}\begin{bmatrix} e^{i(-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -e^{i(-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ e^{i(\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix} = $$
$$e^{i\alpha}\begin{bmatrix} a & b \\ -b^* & a^* \end{bmatrix}$$

which is the general form of a 2x2 unitary matrix as $det(U) = \cos^2\frac{\gamma}{2} + \sin^2\frac{\gamma}{2} = 1$.

Exercise 4.10

$$U = e^{i\alpha}R_Z(\beta)R_X(\gamma)R_Z(\delta) = \begin{bmatrix} e^{i(\alpha-\beta/2-\delta/2)}\cos\frac{\gamma}{2} & -ie^{i(\alpha-\beta/2+\delta/2)}\sin\frac{\gamma}{2} \\ -ie^{i(\alpha+\beta/2-\delta/2)}\sin\frac{\gamma}{2} & e^{i(\alpha+\beta/2+\delta/2)}\cos\frac{\gamma}{2} \end{bmatrix}$$

which once again is a unitary matrix.

Exercise 4.11

Exercise 4.12

From the proof of Corollary 4.2 we have, $AXBXC = R_Z(\beta)R_Y(\gamma)R_Z(\delta)$. We can see that $\alpha = \gamma = \delta = \frac{\pi}{2}$ and $\beta = 0$ gives H. Hence, we can take $A = R_Y(\frac{\pi}{4})$, $B = R_Y(-\frac{\pi}{4})R_Z(-\frac{\pi}{4})$ and $C = R_Z(\frac{\pi}{4})$.

Exercise 4.13

$$HXH = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix} = Z$$

$$HYH = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & 2i \\ -2i & 0 \end{bmatrix} = -Y \quad HZH = HHXHH = X$$

Exercise 4.14

$T = R_Z(\frac{\pi}{4})$
$HTH = HR_Z(\frac{\pi}{4})H = H(\cos\frac{\pi}{8} - i\sin\frac{\pi}{8}Z)H = \cos\frac{\pi}{8} - i\sin\frac{\pi}{8}X = R_X(\frac{\pi}{4})$

Exercise 4.15

(Check Errata for the sign in the second equation) 1)$R_{\hat{n}_2}(\beta_2)R_{\hat{n}_1}(\beta_1) = (c_2 I - is_2\hat{n}_2.\sigma)(c_1 I - is_1\hat{n}_1.\sigma) = c_1c_2 I - s_1s_2(\hat{n}_2.\sigma)(\hat{n}_1.\sigma) - ic_2s_1\hat{n}_1.\sigma - ic_1s_2\hat{n}_2.\sigma = c_1c_2 I - s_1s_2(\hat{n}_1.\hat{n}_2 I + i(\hat{n}_2 \times \hat{n}_1).\sigma) - ic_2s_1\hat{n}_1.\sigma - ic_1s_2\hat{n}_2.\sigma = (c_1c_2 - s_1s_2\hat{n}_1.\hat{n}_2)I - i(c_2s_1\hat{n}_1 + c_1s_2\hat{n}_2 + s_1s_2\hat{n}_2 \times \hat{n}_1).\sigma$
Therefore,
$c_{12} = c_1c_2 - s_1s_2\hat{n}_1.\hat{n}_2$
$s_{12}\hat{n}_{12} = c_2s_1\hat{n}_1 + c_1s_2\hat{n}_2 + s_1s_2\hat{n}_2 \times \hat{n}_1$
2)$\beta_1 = \beta_2$ and $\hat{n}_1 = \hat{z}$, hence $c_1 = c_2 = c$ and $s_1 = s_2 = s$. Therefore,
$c_{12} = c^2 - s^2\hat{z}.\hat{n}_2$
$s_{12}\hat{n}_{12} = cs(\hat{z} + \hat{n}_2) + s^2\hat{n}_2 \times \hat{z}$

Exercise 4.16

$H_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} I & I \\ I & -I \end{bmatrix}$

$H_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix}$

Exercise 4.17

$$\begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix} \begin{bmatrix} H & 0 \\ 0 & H \end{bmatrix} = \begin{bmatrix} HH & 0 \\ 0 & HZH \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = CNOT$$

Exercise 4.18

When the second qubit is the control we have the following representation,

$$CZ_2 = |00\rangle\langle 00| + |01\rangle\langle 01| - |11\rangle\langle 11| + |10\rangle\langle 10| = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Exercise 4.19

$$\text{CNOT}\rho\text{CNOT} = \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} = \begin{bmatrix} A & XB \\ XC & XDX \end{bmatrix}$$

X only rearranges elements, hence the CNOT only rearranges the elements of ρ.

Exercise 4.20

$$(H_1 \otimes H_2)\text{CNOT}(H_1 \otimes H_2) = \frac{1}{2} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & X \end{bmatrix} \begin{bmatrix} H & H \\ H & -H \end{bmatrix} = \frac{1}{2} \begin{bmatrix} HH + HXH & HH - HXH \\ HH - HXH & HH + HXH \end{bmatrix} =$$

$$\frac{1}{2} \begin{bmatrix} I + Z & I - Z \\ I - Z & I + Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \text{CNOT}(2^{\text{nd}} \text{ qubit control})$$

The left circuit transforms between the $|0\rangle,|1\rangle$ and $|+\rangle,|-\rangle$ basis applies a CNOT and transforms back. Hence, the effect of the CNOT on the $|\pm\rangle|\pm\rangle$ is the same as applying the CNOT with the 2nd qubit as target to the state in the $|0\rangle,|1\rangle$ basis and then replacing 0 with $+$ and 1 with $-$. This process does indeed give the equations 4.24-4.27.
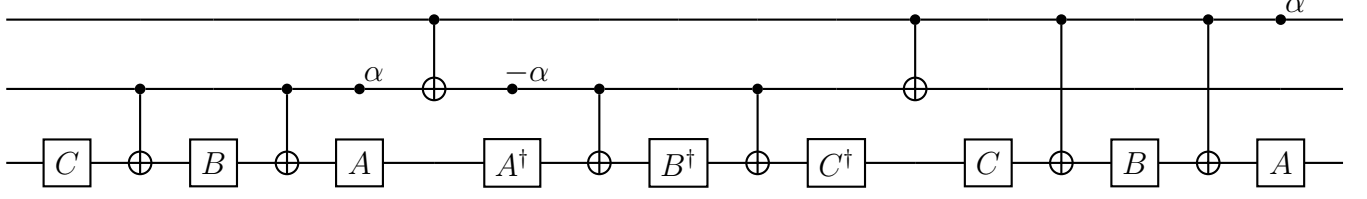
Exercise 4.21

Consider all the possible inputs,
$|00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle \to |00\psi\rangle$
$|01\psi\rangle \to |01(V\psi)\rangle \to |01(V\psi)\rangle \to |01(V^\dagger V\psi)\rangle \to |01\psi\rangle \to |01\psi\rangle$
$|10\psi\rangle \to |10\psi\rangle \to |11\psi\rangle \to |11(V^\dagger\psi)\rangle \to |10(VV^\dagger\psi)\rangle \to |10\psi\rangle$
$|11\psi\rangle \to |11(V\psi)\rangle \to |10(V\psi)\rangle \to |11(V\psi)\rangle \to |11(VV\psi)\rangle \to |11U\psi\rangle$
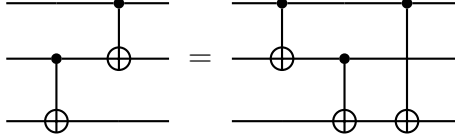Hence, the circuit does perform the $C^2(U)$ operation.
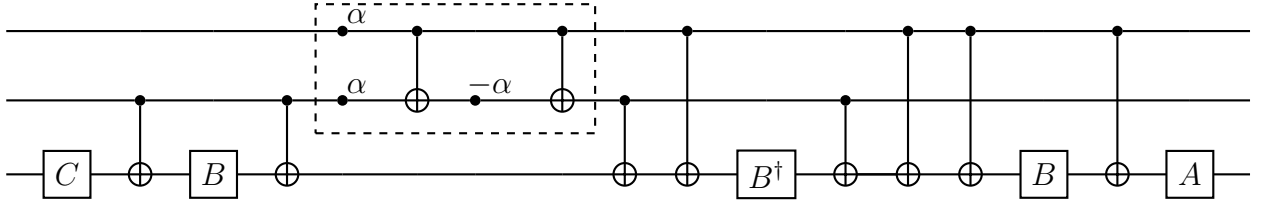
Exercise 4.22

Firstly, we apply the circuit in figure 4.6 to the circuit in figure 4.8 for $V = e^{i\alpha}AXBXC$, which gives,


We move the 6$^{\text{th}}$ CNOT left from the 4$^{\text{th}}$ one, which involves adding CNOTs after the 4$^{\text{th}}$ and 5$^{\text{th}}$ CNOTs from first to third qubit, which is due to,
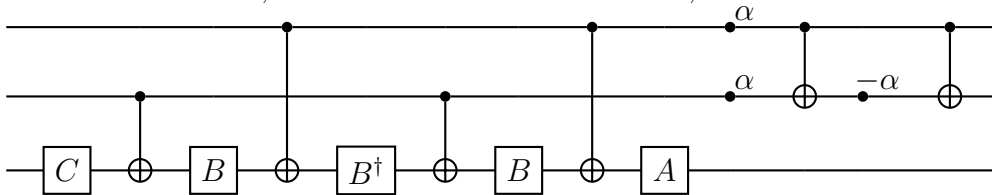

and can be checked by considering all the possible inputs. We also see that $AA^\dagger = CC^\dagger = I$. Hence we get the following circuit.

The dashed section is diagonal hence commutes with the rest of the components therefore can be moved to the end for convenience.
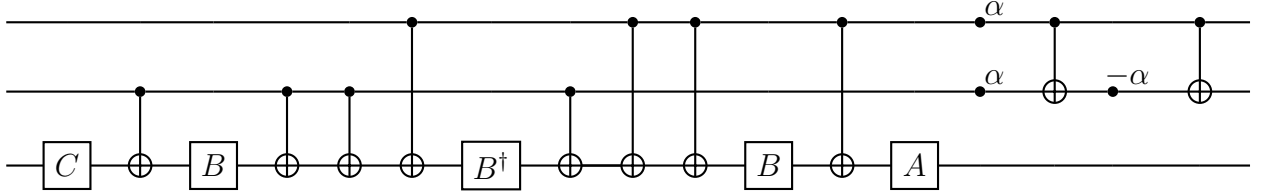
CNOTs 2 and 3, and 6 and 7 cancel each other, hence we're left with the circuit.

This has 8 single qubit gates and 6 CNOTs as desired.

The circuit performs the operation, $AXBXB^\dagger XBXC = (VC^\dagger)B^\dagger(A^\dagger V) = V(ABC)^\dagger = V^2 = U$. Therefore, the circuit performs the $C^2(U)$ operation.

Exercise 4.23