

Exercise 8.1

Under the transformation $\rho \rightarrow \mathcal{E}(\rho)$, the state transforms as $|\psi\rangle \rightarrow U|\psi\rangle$. Hence, the new density operator is $\rho' = U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger$, and therefore ρ transforms as $\rho \rightarrow U\rho U^\dagger$.

Exercise 8.2

Let $\rho = \sum_i p_i |i\rangle\langle i|$, hence after the measurement for each of the i states will take the form,

$|i'\rangle = \frac{M_m|i\rangle}{\sqrt{\langle i|M_m^\dagger M_m|i\rangle}}$. Therefore, for the final state ρ' we'll have,

$$\rho' = \sum_i p_i \frac{M_m|i\rangle\langle i|M_m^\dagger}{\sqrt{\langle i|M_m^\dagger M_m|i\rangle}\sqrt{\langle i|M_m M_m^\dagger|i\rangle}} = \frac{\mathcal{E}_m(\rho)}{\text{tr}(\mathcal{E}_m(\rho))}$$

For the probability of the m state, using $p(m|i) = \langle i|M_m^\dagger M_m|i\rangle$, we get

$$p(m) = \sum_i p_i p(m|i) = \sum_i p_i \langle i|M_m^\dagger M_m|i\rangle = \sum_i p_i \text{tr}(M_m^\dagger M_m|i\rangle\langle i|) = \text{tr}(\mathcal{E}_m(\rho))$$

Exercise 8.3

Initially we have the state $\rho \otimes |0_{CD}\rangle\langle 0_{CD}|$. Consider the action of \mathcal{E} (i basis for A , j basis for D),

$$\mathcal{E}(\rho) = \text{tr}_A(\text{tr}_D(U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger)) = \sum_i \sum_j \langle i|\langle j|U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger|j\rangle|i\rangle =$$

$$\sum_i \sum_j \langle i|\langle j|U|0_{CD}\rangle\rho\langle 0_{CD}|U^\dagger|j\rangle|i\rangle = \sum_j E_j \rho E_j^\dagger.$$

$$\text{where } E_j = \sum_i \langle i|\langle j|U|0_{CD}\rangle$$

$$\text{Also, (using } \sum_i |i\rangle\langle i| = I)$$

$$\sum_j E_j^\dagger E_j = \sum_i \sum_j \langle 0_{CD}|U^\dagger|j\rangle|i\rangle\langle i|\langle j|U|0_{CD}\rangle = I\langle 0_{CD}|U^\dagger U|0_{CD}\rangle = I\langle 0_{CD}|0_{CD}\rangle = I$$

Exercise 8.4

$E_k = \langle k|U|0\rangle$, hence using the orthogonality of the $|0\rangle$ and $|1\rangle$ states, $E_0 = P_0$, $E_1 = P_1$. Therefore,

$$\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$$

Exercise 8.5

$$E_0 = \frac{X}{\sqrt{2}}, E_1 = \frac{Y}{\sqrt{2}}$$

$$\mathcal{E}(\rho) = \frac{1}{2}(X\rho X^\dagger + Y\rho Y^\dagger) = \frac{1}{2}(X\rho X - Y\rho Y)$$

Exercise 8.6

In general the composition of quantum operations is still a quantum operation, hence we only prove the general case.

Let ρ belong to a Hilbert Space \mathcal{H} and let the quantum operations be given by, $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ and $\mathcal{F}(\rho) = \sum_i F_i \rho F_i^\dagger$.

As by definition, \mathcal{E} and \mathcal{F} are quantum operations, there exist states $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{F}}$ and unitary operators $U_{\mathcal{E}}$ and $U_{\mathcal{F}}$ on Hilbert spaces $\mathcal{K}_{\mathcal{E}}$ and $\mathcal{K}_{\mathcal{F}}$, respectively, such that

$$\mathcal{E}(\rho) = \text{tr}_{\mathcal{K}_{\mathcal{E}}}(U_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}}]U_{\mathcal{E}}^\dagger) \text{ and } \mathcal{F}(\rho) = \text{tr}_{\mathcal{K}_{\mathcal{F}}}(U_{\mathcal{F}}[\rho \otimes \omega_{\mathcal{F}}]U_{\mathcal{F}}^\dagger).$$

Consider the Hilbert space $\mathcal{K} = \mathcal{K}_{\mathcal{E}} \otimes \mathcal{K}_{\mathcal{F}}$ and the state $\omega = \omega_{\mathcal{E}} \otimes \omega_{\mathcal{F}}$. Consider the ampliations $\hat{U}_{\mathcal{E}}$ and $\hat{U}_{\mathcal{F}}$ of $U_{\mathcal{E}}$ and $U_{\mathcal{F}}$ to $\mathcal{H} \otimes \mathcal{K}$, i.e $\hat{U}_{\mathcal{E}} = U_{\mathcal{E}} \otimes \mathcal{I}$ and $\hat{U}_{\mathcal{F}} = \mathcal{I} \otimes U_{\mathcal{F}}$. Lastly, take $U = \hat{U}_{\mathcal{F}}\hat{U}_{\mathcal{E}}$, which is an operator on $\mathcal{H} \otimes \mathcal{K}$. Finally, consider

$$\begin{aligned} \text{tr}_{\mathcal{K}}(U[\rho \otimes \omega]U^\dagger) &= \text{tr}_{\mathcal{K}_{\mathcal{E}} \otimes \mathcal{K}_{\mathcal{F}}}(\hat{U}_{\mathcal{F}}\hat{U}_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}} \otimes \omega_{\mathcal{F}}]\hat{U}_{\mathcal{E}}^\dagger\hat{U}_{\mathcal{F}}^\dagger) \\ &= \text{tr}_{\mathcal{K}_{\mathcal{F}}}(\text{tr}_{\mathcal{K}_{\mathcal{E}}}(\hat{U}_{\mathcal{F}}(U_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}}]U_{\mathcal{E}}^\dagger \otimes \omega_{\mathcal{F}})\hat{U}_{\mathcal{F}}^\dagger)) \\ &= \text{tr}_{\mathcal{K}_{\mathcal{F}}}(U_{\mathcal{F}}(\text{tr}_{\mathcal{K}_{\mathcal{E}}}(U_{\mathcal{E}}[\rho \otimes \omega_{\mathcal{E}}]U_{\mathcal{E}}^\dagger) \otimes \omega_{\mathcal{F}})U_{\mathcal{F}}^\dagger) \\ &= \text{tr}_{\mathcal{K}_{\mathcal{F}}}(U_{\mathcal{F}}(\mathcal{E}(\rho) \otimes \omega_{\mathcal{F}})U_{\mathcal{F}}^\dagger) \\ &= \mathcal{F}(\mathcal{E}(\rho)) \end{aligned}$$

From the trace as previously we can obtain an operator-sum representation, hence the composition even for different input and output spaces is a quantum operation.

Exercise 8.7

Again consider, $\rho^{QE} = \rho \otimes \sigma$. The final state after a general measurement with outcome m is,

$$\frac{M_m U(\rho \otimes \sigma) U^\dagger M_m^\dagger}{\text{tr}(M_m U(\rho \otimes \sigma) U^\dagger M_m^\dagger)}$$

Hence, tracing out E the final state of Q is,

$$\frac{\text{tr}_E(M_m U(\rho \otimes \sigma) U^\dagger M_m^\dagger)}{\text{tr}(M_m U(\rho \otimes \sigma) U^\dagger M_m^\dagger)}$$

Define, $(E)_m(\rho) = \text{tr}_E(M_m U(\rho \otimes \sigma) U^\dagger M_m^\dagger)$. Let $\sigma = \sum_J |j\rangle \langle j|$ and consider an orthonormal

basis $|e_k\rangle$ for the system E . We get,

$$\mathcal{E}_m(\rho) = \sum_{jk} q_j \text{tr}_E(|e_k\rangle \langle e_k| M_m U(\rho \otimes \sigma) U^\dagger M_m^\dagger |e_k\rangle \langle e_k|) = \sum_{jk} E_{jk} \rho E_{jk}^\dagger$$

where $E_{jk} = \sqrt{q_j} \langle e_k| M_m U |j\rangle$

Exercise 8.8

The process will be identical to the trace-preserving method, with the addition of the E_∞ operation element. Additionally, we need to add another orthonormal basis vector $|e_\infty\rangle$ to our basis, i.e amplify the Hilbert Space of the environment.

Exercise 8.9

Consider the action of U on $\rho \otimes |e_0\rangle \langle e_0|$ succeeded by a measurement by P_m . Tracing over this will give the probability of the outcome m .

$$\begin{aligned}
p(m) &= \text{tr}(P_m U(\rho \otimes |e_0\rangle \langle e_0|) U^\dagger P_m) \\
&= \text{tr}\left(\sum_k |m, k\rangle \langle m, k| U |e_0\rangle \rho \langle e_0| U^\dagger |m, k\rangle \langle m, k|\right) \\
&= \text{tr}\left(\sum_{k, m', k'} |m, k\rangle \langle m, k| E_{m'k'} |m', k'\rangle \rho \langle m', k'| E_{m'k'}^\dagger |m, k\rangle \langle m, k|\right) \\
&= \text{tr}\left(\sum_k |m, k\rangle E_{mk} \rho E_{mk}^\dagger \langle m, k|\right) \\
&= \text{tr}_Q(\text{tr}_E\left(\sum_k |m, k\rangle E_{mk} \rho E_{mk}^\dagger \langle m, k|\right)) \\
&= \text{tr}_Q\left(\sum_k E_{mk} \rho E_{mk}^\dagger\right) \\
&= \text{tr}_Q(\mathcal{E}_m(\rho)) = \text{tr}(\mathcal{E}_m(\rho))
\end{aligned}$$

For the state we have, $\frac{\text{tr}_E(P_m U(\rho \otimes |e_0\rangle \langle e_0|) U^\dagger P_m)}{p(m)} = \frac{\mathcal{E}_m(\rho)}{\text{tr}(\mathcal{E}_m(\rho))}$

Exercise 8.10

Exercise 8.15

This is the bit flip channel with $p = 0.5$, hence it deforms into a line on the x-axis.

Exercise 8.16

Exercise 8.17

$$\begin{aligned}
\mathcal{E}(I) &= \frac{I+XX+YY+ZZ}{4} = \frac{4I}{4} = I \\
\mathcal{E}(X) &= \frac{X+XX^4X+YXY+ZXZ}{4} = \frac{X+X-X-X}{4} = 0
\end{aligned}$$

Similarly, $\mathcal{E}(Y) = \mathcal{E}(Z) = 0$

$$\rho = \frac{I + \vec{r} \cdot \vec{\sigma}}{2}$$

$$2\rho = I + r_x X + r_y Y + r_z Z$$

Left and right multiplying by X , Y and Z we get,

$$2X\rho X = I + r_x X - r_y Y - r_z Z$$

$$2Y\rho Y = I - r_x X + r_y Y - r_z Z$$

$$2Z\rho Z = I - r_x X - r_y Y + r_z Z$$

Adding all 4 equations,

$$2(\rho + X\rho X + Y\rho Y + Z\rho Z) = 4I$$

$$\frac{I}{2} = \frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$

Exercise 8.18

We have, $\mathcal{E}(\rho) = \rho' = \frac{pI}{2} + (1-p)\rho$ and $\rho = \frac{I + r \cdot \sigma}{2}$. Hence substituting ρ we get,
 $\rho' = \frac{I}{2} + \frac{(1-p)r \cdot \sigma}{2}$. Now we need to find the eigenvalues of ρ and ρ' . For ρ' we have,

$$\left| \frac{1}{2} + \frac{1-p}{2}r_z - \lambda \quad \frac{1-p}{2}(r_x - ir_y) \right| = 0$$

$$\left| \frac{1-p}{2}(r_x + ir_y) \quad \frac{1}{2} - \frac{1-p}{2}r_z - \lambda \right| = 0$$

Hence, $\lambda = 1 \pm \frac{1-p}{4}|r|$ and similarly for ρ , $\lambda = 1 \pm \frac{1}{4}|r|$.

Therefore we have,

$$tr(\rho) = (1 - \frac{|r|}{4})^k + (1 + \frac{|r|}{4})^k = \sum_n \binom{k}{2n} \left(\frac{|r|}{4}\right)^{2n}$$

$$tr(\rho') = (1 - \frac{(1-p)|r|}{4})^k + (1 + \frac{(1-p)|r|}{4})^k = \sum_n \binom{k}{2n} \left(\frac{(1-p)|r|}{4}\right)^{2n}$$

Therefore, $tr(\rho') \leq tr(\rho)$ with equality for $p = 0$.

Exercise 8.19

We have $\mathcal{E}(\rho) = \frac{pI}{d} + (1-p)\rho$. We know that $tr(\rho) = 1$, hence can write, $\frac{I}{d} = \frac{1}{d}tr(\rho)$. Consider an orthonormal basis $|i\rangle$ for the system. This gives,

$$\frac{I}{d} = \frac{1}{d} \sum_i |i\rangle \langle i| \sum_j \langle j| \rho |j\rangle = \frac{1}{d} \sum_{i,j} |i\rangle \langle j| \rho |j\rangle \langle i|$$

Hence, we can choose as the operation elements $\{\sqrt{\frac{p}{d}} |i\rangle \langle j|\}$

Exercise 8.20

Let the initial state be $|\psi_0\rangle = a|00\rangle + b|10\rangle$. Then applying the controlled- R_y and CNOT gates we get.

After the R_y we have,

$$|\psi_1\rangle = a|00\rangle + b \cos \frac{\theta}{2} |10\rangle + b \sin \frac{\theta}{2} |11\rangle$$

After the CNOT we have,

$$|\psi_2\rangle = a|00\rangle + b \cos \frac{\theta}{2} |10\rangle + b \sin \frac{\theta}{2} |01\rangle$$

Tracing over the environment we get,

$$tr_E(|\psi_2\rangle \langle \psi_2|) = (a|0\rangle + b \cos \frac{\theta}{2} |1\rangle)(a^* \langle 0| + b^* \cos \frac{\theta}{2} \langle 1|) + b b^* \sin^2 \frac{\theta}{2} |0\rangle \langle 0| = \begin{bmatrix} |a|^2 + |b|^2 \sin^2 \frac{\theta}{2} & ab^* \cos \frac{\theta}{2} \\ ba^* \cos \frac{\theta}{2} & |b|^2 \cos^2 \frac{\theta}{2} \end{bmatrix}$$

If we apply amplitude damping to our original state we get,

$$\mathcal{E}_{AD} = E_0 \begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix} E_0^\dagger + E_1 \begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix} E_1^\dagger = \begin{bmatrix} |a|^2 + \gamma |b|^2 & ab^* \sqrt{1-\gamma} \\ ba^* \sqrt{1-\gamma} & |b|^2 (1-\gamma) \end{bmatrix}$$

Comparing with the model above we see that, the circuit does indeed model the quantum operation with $\gamma = \sin^2 \frac{\theta}{2}$.