Exercise 8.1

Under the transformation $\rho \rightarrow \mathcal{E}(\rho)$, the state transforms as $|\psi\rangle \rightarrow U|\psi\rangle$. Hence, the new density operator is $\rho' = U|\psi\rangle\langle\psi|U^\dagger = U\rho U^\dagger$, and therefore ρ transforms as $\rho \rightarrow U\rho U^\dagger$.

Exercise 8.2

Let $\rho = \sum_i p_i |i\rangle\langle i|$, hence after the measurement for each of the i states will take the form, $|i'\rangle = \frac{M_m|i\rangle}{\sqrt{\langle i|M_m^\dagger M_m|i\rangle}}$. Therefore, for the final state ρ' we'll have,

$$\rho' = \sum_i p_i \frac{M_m|i\rangle\langle i|M_m^\dagger}{\sqrt{\langle i|M_m^\dagger M_m|i\rangle}\sqrt{\langle i|M_m M_m^\dagger|i\rangle}} = \frac{\mathcal{E}_m(\rho)}{tr(\mathcal{E}_m(\rho))}$$

For the probability of the m state, using $p(m|i) = \langle i|M_m^\dagger M_m|i\rangle$, we get

$$p(m) = \sum_i p_i p(m|i) = \sum_i p_i \langle i|M_m^\dagger M_m|i\rangle = \sum_i p_i tr(M_m^\dagger M_m|i\rangle\langle i|) = tr(\mathcal{E}_m(\rho))$$

Exercise 8.3

Initially we have the state $\rho \otimes |0_{CD}\rangle\langle 0_{CD}|$. Consider the action of $\mathcal{E}$(i basis for A, j basis for D),

$$\mathcal{E}(\rho) = tr_A(tr_D(U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger)) = \sum_i\sum_j \langle i|\langle j|U[\rho \otimes |0_{CD}\rangle\langle 0_{CD}|]U^\dagger|j\rangle|i\rangle =$$

$$\sum_i\sum_j \langle i|\langle j|U|0_{CD}\rangle\rho\langle 0_{CD}|U^\dagger|j\rangle|i\rangle = \sum_j E_j\rho E_j^\dagger.$$

where $E_j = \sum_i \langle i|\langle j|U|0_{CD}\rangle$

Also, (using $\sum_i |i\rangle\langle i| = I$)

$$\sum_j E_j^\dagger E_j = \sum_i\sum_j \langle 0_{CD}|U^\dagger|j\rangle|i\rangle\langle i|\langle j|U|0_{CD}\rangle = I\langle 0_{CD}|U^\dagger U|0_{CD}\rangle = I\langle 0_{CD}|0_{CD}\rangle = I$$

Exercise 8.4

$E_k = \langle k|U|0\rangle$, hence using the orthogonality of the $|0\rangle$ and $|1\rangle$ states, $E_0 = P_0$, $E_1 = P_1$. Therefore,

$\mathcal{E}(\rho) = |0\rangle\langle 0|\rho|0\rangle\langle 0| + |1\rangle\langle 1|\rho|1\rangle\langle 1|$

Exercise 8.5

$E_0 = \frac{X}{\sqrt{2}}$, $E_1 = \frac{Y}{\sqrt{2}}$

$\mathcal{E}(\rho) = \frac{1}{2}(X\rho X^\dagger + Y\rho Y^\dagger) = \frac{1}{2}(X\rho X - Y\rho Y)$

Exercise 8.6

In general the composition of quantum operations is still a quantum operation, hence we only prove the general case.
Let ρ belong to a Hilbert Space $\mathcal{H}$ and let the quantum operations be given by, $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger$ and $\mathcal{F}(\rho) = \sum_i F_i \rho F_i^\dagger$.
As by definition, $\mathcal{E}$ and $\mathcal{F}$ are quantum operations, there exist states $\omega_\mathcal{E}$ and $\omega_\mathcal{F}$ and unitary operators $U_\mathcal{E}$ and $U_\mathcal{F}$ on Hilbert spaces $\mathcal{K}_\mathcal{E}$ and $\mathcal{K}_\mathcal{F}$, respectively, such that
$\mathcal{E}(\rho) = tr_{\mathcal{K}_\mathcal{E}}(U_\mathcal{E}[\rho \otimes \omega_\mathcal{E}]U_\mathcal{E}^\dagger)$ and $\mathcal{F}(\rho) = tr_{\mathcal{K}_\mathcal{F}}(U_\mathcal{F}[\rho \otimes \omega_\mathcal{F}]U_\mathcal{F}^\dagger)$.
Consider the Hilbert space $\mathcal{K} = \mathcal{K}_\mathcal{E} \otimes \mathcal{K}_\mathcal{F}$ and the state $\omega = \omega_\mathcal{E} \otimes \omega_\mathcal{F}$. Consider the ampliations $\hat{U}_\mathcal{E}$ and $\hat{U}_\mathcal{F}$ of $U_\mathcal{E}$ and $U_\mathcal{F}$ to $\mathcal{H} \otimes \mathcal{K}$, i.e $\hat{U}_\mathcal{E} = U_\mathcal{E} \otimes \mathcal{I}$ and $\hat{U}_\mathcal{F} = \mathcal{I} \otimes U_\mathcal{F}$. Lastly, take $U = \hat{U}_\mathcal{F}\hat{U}_\mathcal{E}$, which is an operator on $\mathcal{H} \otimes \mathcal{K}$. Finally, consider

$$\begin{aligned}
tr_\mathcal{K}(U[\rho \otimes \omega]U^\dagger) &= tr_{\mathcal{K}_\mathcal{E} \otimes \mathcal{K}_\mathcal{F}}(\hat{U}_\mathcal{F}\hat{U}_\mathcal{E}[\rho \otimes \omega_\mathcal{E} \otimes \omega_\mathcal{F}]\hat{U}_\mathcal{E}\hat{U}_\mathcal{F}) \\
&= tr_{\mathcal{K}_\mathcal{F}}(tr_{\mathcal{K}_\mathcal{E}}(\hat{U}_\mathcal{F}(U_\mathcal{E}[\rho \otimes \omega_\mathcal{E}]U_\mathcal{E}^\dagger \otimes \omega_\mathcal{F})\hat{U}_\mathcal{F}^\dagger)) \\
&= tr_{\mathcal{K}_\mathcal{F}}(U_\mathcal{F}(tr_{\mathcal{K}_\mathcal{E}}(U_\mathcal{E}[\rho \otimes \omega_\mathcal{E}]U_\mathcal{E}^\dagger) \otimes \omega_\mathcal{F})U_\mathcal{F}^\dagger) \\
&= tr_{\mathcal{K}_\mathcal{F}}(U_\mathcal{F}(\mathcal{E}(\rho) \otimes \omega_\mathcal{F})U_\mathcal{F}^\dagger) \\
&= \mathcal{F}(\mathcal{E}(\rho))
\end{aligned}$$

From the trace as previously we can obtain an operator-sum representation, hence the composition even for different input and output spaces is a quantum operation.

Exercise 8.7

Again consider, $\rho^{QE} = \rho \otimes \sigma$. The final state after a general measurement with outcome m is,
$\frac{M_m U(\rho\otimes\sigma)U^\dagger M_m^\dagger}{tr(M_m U(\rho\otimes\sigma)U^\dagger M_m^\dagger)}$
Hence, tracing out E the final state of Q is,
$\frac{tr_E(M_m U(\rho\otimes\sigma)U^\dagger M_m^\dagger)}{tr(M_m U(\rho\otimes\sigma)U^\dagger M_m^\dagger)}$
Define, $(E)_m(\rho) = tr_E(M_m U(\rho\otimes\sigma)U^\dagger M_m^\dagger)$. Let $\sigma = \sum_J |j\rangle\langle j|$ and consider an orthonormal basis $|e_k\rangle$ for the system E. We get,
$\mathcal{E}_m(\rho) = \sum_{jk} q_j tr_E(|e_k\rangle\langle e_k| M_m U(\rho\otimes\sigma)U^\dagger M_m^\dagger |e_k\rangle\langle e_k|) = \sum_{jk} E_{jk}\rho E_{jk}^\dagger$
where $E_{jk} = \sqrt{q_j}\langle e_k| M_m U |j\rangle$

Exercise 8.8

The process will be identical to the trace-preserving method, with the addition of the E_∞ operation element. Additionally, we need to add another orthonormal basis vector $|e_\infty\rangle$ to our basis, i.e ampliate the Hilbert Space of the environment.

Exercise 8.9

Consider the action of U on $\rho \otimes |e_0\rangle \langle e_0|$ succeeded by a measurement by P_m. Tracing over this will give the probability of the outcome m.

$$
\begin{aligned}
p(m) &= tr(P_m U(\rho \otimes |e_0\rangle \langle e_0|)U^\dagger P_m) \\
&= tr(\sum_k |m,k\rangle \langle m,k| U |e_0\rangle \rho \langle e_0| U^\dagger |m,k\rangle \langle m,k|) \\
&= tr(\sum_{k,m',k'} |m,k\rangle \langle m,k| E_{m'k'} |m',k'\rangle \rho \langle m',k'| E^\dagger_{m'k'} |m,k\rangle \langle m,k|) \\
&= tr(\sum_k |m,k\rangle E_{mk}\rho E^\dagger_{mk} \langle m,k|) \\
&= tr_Q(tr_E(\sum_k |m,k\rangle E_{mk}\rho E^\dagger_{mk} \langle m,k|)) \\
&= tr_Q(\sum_k E_{mk}\rho E^\dagger_{mk}) \\
&= tr_Q(\mathcal{E}_m(\rho)) = tr(\mathcal{E}_m(\rho))
\end{aligned}
$$

For the state we have, $\frac{tr_E(P_m U(\rho\otimes|e_0\rangle\langle e_0|)U^\dagger P_m)}{p(m)} = \frac{\mathcal{E}_m(\rho)}{tr(\mathcal{E}_m(\rho))}$

Exercise 8.10

Exercise 8.15

This is the bit flip channel with $p = 0.5$, hence it deforms into a line on the x-axis.

Exercise 8.16

Exercise 8.17

$\mathcal{E}(I) = \frac{I+XX+YY+ZZ}{4} = \frac{4I}{4} = I$
$\mathcal{E}(X) = \frac{X+XXX+YXY+ZXZ}{4} = \frac{X+X-X-X}{4} = 0$
Similarly, $\mathcal{E}(Y) = \mathcal{E}(Z) = 0$
$\rho = \frac{I+\vec{r}\cdot\vec{\sigma}}{2}$
$2\rho = I + r_x X + r_y Y + r_z Z$
Left and right multiplying by X, Y and Z we get,
$2X\rho X = I + r_x X - r_y Y - r_z Z$
$2Y\rho Y = I - r_x X + r_y Y - r_z Z$
$2Z\rho Z = I - r_x X - r_y Y + r_z Z$
Adding all 4 equations,
$2(\rho + X\rho X + Y\rho Y + Z\rho Z) = 4I$
$\frac{I}{2} = \frac{\rho+X\rho X+Y\rho Y+Z\rho Z}{4}$

Exercise 8.18

We have, $\mathcal{E}(\rho) = \rho' = \frac{pI}{2} + (1-p)\rho$ and $\rho = \frac{I + r\cdot\sigma}{2}$. Hence substituting ρ we get,
$\rho' = \frac{I}{2} + \frac{(1-p)r\cdot\sigma}{2}$. Now we need to find the eigenvalues of ρ and ρ'. For ρ' we have,
$\begin{vmatrix} \frac{1}{2} + \frac{1-p}{2}r_z - \lambda & \frac{1-p}{2}(r_x - ir_y) \\ \frac{1-p}{2}(r_x + ir_y) & \frac{1}{2} - \frac{1-p}{2}r_z - \lambda \end{vmatrix} = 0$
Hence, $\lambda = 1 \pm \frac{1-p}{4}|r|$ and similarly for ρ, $\lambda = 1 \pm \frac{1}{4}|r|$.
Therefore we have,
$tr(\rho) = (1 - \frac{|r|}{4})^k + (1 + \frac{|r|}{4})^k = \sum_n \binom{k}{2n}\left(\frac{|r|}{4}\right)^{2n}$
$tr(\rho') = (1 - \frac{(1-p)|r|}{4})^k + (1 + \frac{(1-p)|r|}{4})^k = \sum_n \binom{k}{2n}\left(\frac{(1-p)|r|}{4}\right)^{2n}$
Therefore, $tr(\rho') \le tr(\rho)$ with equality for $p = 0$.

Exercise 8.19

We have $\mathcal{E}(\rho) = \frac{pI}{d} + (1-p)\rho$. We know that $tr(\rho) = 1$, hence can write, $\frac{I}{d} = \frac{I}{d}tr(\rho)$. Consider an orthonormal basis $|i\rangle$ for the system. This gives,
$\frac{I}{d} = \frac{1}{d}\sum_i |i\rangle\langle i| \sum_j \langle j|\rho|j\rangle = \frac{1}{d}\sum_{i,j} |i\rangle\langle j|\rho|j\rangle\langle i|$
Hence, we can choose as the operation elements $\{\sqrt{\frac{p}{d}}\,|i\rangle\langle j|\}$

Exercise 8.20

Let the initial state be $|\psi_0\rangle = a|00\rangle + b|10\rangle$. Then applying the controlled-R_y and CNOT gates we get.
After the R_y we have,
$|\psi_1\rangle = a|00\rangle + b\cos\frac{\theta}{2}|10\rangle + b\sin\frac{\theta}{2}|11\rangle$
After the CNOT we have,
$|\psi_2\rangle = a|00\rangle + b\cos\frac{\theta}{2}|10\rangle + b\sin\frac{\theta}{2}|01\rangle$
Tracing over the environment we get,
$tr_E(|\psi_2\rangle\langle\psi_2|) = (a|0\rangle + b\cos\frac{\theta}{2}|1\rangle)(a^*\langle 0| + b^*\cos\frac{\theta}{2}\langle 1|) + bb^*\sin^2\frac{\theta}{2}|0\rangle\langle 0| = \begin{bmatrix} |a|^2 + |b|^2\sin^2\frac{\theta}{2} & ab^*\cos\frac{\theta}{2} \\ ba^*\cos\frac{\theta}{2} & |b|^2\cos^2\frac{\theta}{2} \end{bmatrix}$
If we apply amplitude damping to our original state we get,
$\mathcal{E}_{AD} = E_0\begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix}E_0^\dagger + E_1\begin{bmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{bmatrix}E_1^\dagger = \begin{bmatrix} |a|^2 + \gamma|b|^2 & ab*\sqrt{1-\gamma} \\ ba^*\sqrt{1-\gamma} & |b|^2(1-\gamma) \end{bmatrix}$
Comparing with the model above we see that, the circuit does indeed model the quantum operation with $\gamma = \sin^2\frac{\theta}{2}$.

Exercise 8.21

Exercise 8.22

$\mathcal{E}_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix}\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ b^* & c \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \sqrt{\gamma} & 0 \end{bmatrix} =$
$\begin{bmatrix} a + \gamma c & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} = \begin{bmatrix} a + \gamma(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix} = \begin{bmatrix} 1 - (1-\gamma)(1-a) & b\sqrt{1-\gamma} \\ b^*\sqrt{1-\gamma} & c(1-\gamma) \end{bmatrix}$

Exercise 8.23

Exercise 8.24

Exercise 8.25

Exercise 8.26

Exercise 8.27

Exercise 8.28