Exercise 9.1

$D((1,0),(\frac{1}{2},\frac{1}{2})) = \frac{1}{2} * 2 * \frac{1}{2} = \frac{1}{2}$

$D((\frac{1}{2},\frac{1}{3},\frac{1}{6}),(\frac{3}{4},\frac{1}{8},\frac{1}{8})) = \frac{1}{2}(\frac{1}{4} + \frac{5}{24} + \frac{1}{24}) = \frac{1}{4}$

Exercise 9.2

$D((p,1-p),(q,1-q)) = \frac{1}{2}(|p-q| + |1-p-1+q|) = \frac{1}{2}(|p-q| + |p-q|) = |p-q|$

Exercise 9.3

$F((1,0),(\frac{1}{2},\frac{1}{2})) = \frac{1}{\sqrt{2}}$

$F((\frac{1}{2},\frac{1}{3},\frac{1}{6}),(\frac{3}{4},\frac{1}{8},\frac{1}{8})) = \sqrt{\frac{3}{8}} + \sqrt{\frac{1}{24}} + \sqrt{\frac{1}{48}} = 0.96$

Exercise 9.4

$$D(p_x,q_x) = \tfrac{1}{2}\sum_x |p_x - q_x| = \frac{1}{2}\left(\sum_{p_x>q_x}(p_x-q_x) - \sum_{p_x<q_x}(p_x-q_x)\right)$$

$$\sum_{p_x<q_x}(p_x-q_x) = \sum_{p_x<q_x} p_x - \sum_{p_x<q_x} q_x = 1 - \sum_{p_x>q_x} p_x - 1 + \sum_{p_x>q_x} q_x = -\sum_{p_x>q_x}(p_x-q_x)$$

Therefore,

$$D(p_x,q_x) = \sum_{p_x>q_x}(p_x-q_x)$$

Looking at the last term, if we add an other $(p_{x'}, q_{x'})$ pair to the sum, the overall sum will decrease as $(p_{x'} - q_{x'})$ is negative. Hence,

$$D(p_x,q_x) = \sum_{p_x>q_x}(p_x-q_x) = \max_S \left|\sum_{x\in S}(p_x-q_x)\right|$$

Exercise 9.5

Exercise 9.6

$D\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|\right) = \frac{1}{2}tr\left|\frac{1}{12}|0\rangle\langle 0| - \frac{1}{12}|1\rangle\langle 1|\right| = \frac{1}{12}$

$D\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \frac{2}{3}|+\rangle\langle +| + \frac{1}{3}|-\rangle\langle -|\right) =$

$= D\left(\frac{3}{4}|0\rangle\langle 0| + \frac{1}{4}|1\rangle\langle 1|, \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right) =$

$= \frac{1}{2}tr\left|\frac{1}{4}|0\rangle\langle 0| - \frac{1}{4}|1\rangle\langle 1| + \frac{1}{6}(|0\rangle\langle 1| + |1\rangle\langle 0|)\right| = \frac{\sqrt{13}}{12}$

Exercise 9.7

Let $\rho - \sigma = UDU^\dagger = U(\Lambda_+ + \Lambda_-)U^\dagger$, where Λ_+ and Λ_- are the diagonal matrices of the positive and negative eigenvalues of $\rho - \sigma$.
Hence, we can write
$\rho - \sigma = U\Lambda_+U^\dagger + U\Lambda_-U^\dagger = Q - S$, where $Q = U\Lambda_+U^\dagger$ and $S = -U\Lambda_-U^\dagger$ are positive operators, with their support being the partial eigenbasis of $\rho - \sigma$, which is orthogonal.

Exercise 9.8

Using $\sum_i p_i = 1$ we have,

$D\left(\sum_i p_i\rho_i, \sigma\right) = D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right)$

From eq 9.50$\left(D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right) \leq \sum_i p_i D(\rho_i, \sigma_i)\right)$, it follows that,

$D\left(\sum_i p_i\rho_i, \sigma\right) = D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right) \leq \sum_i p_i D(\rho_i, \sigma)$

Exercise 9.9

The set of the density matrices(positive, trace one, Hermitian) is convex and compact. Hence, as the CPTP maps are continuous, they have a fixed point.

Exercise 9.10

Let ρ and σ,$\rho \neq \sigma$ both be fixed points of $\mathcal{E}$. Therefore, $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) = D(\rho, \sigma)$ from the definition of a fixed point. However, $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$, hence we have a contradiction, therefore, $\rho = \sigma$, i.e there's a unique fixed point.

Exercise 9.11

$$\begin{aligned}
D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= D(p\rho_0 + (1-p)\mathcal{E}'(\rho), p\rho_0 + (1-p)\mathcal{E}'(\sigma)) \\
&\leq pD(\rho_0, \rho_0) + (1-p)D(\mathcal{E}'(\rho), \mathcal{E}'(\sigma)) \\
&\leq (1-p)D(\rho, \sigma)
\end{aligned}$$

Therefore, as $0 \leq (1-p) < 1$, we have $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$, i.e. $\mathcal{E}$ is strictly contractive.

Exercise 9.12

$$\begin{aligned}
D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) &= \frac{1}{2}tr\left|\frac{pI}{2} - (1-p)\rho - \frac{pI}{2} + (1-p)\sigma\right| \\
&= \frac{1}{2}(1-p)tr|\rho - \sigma| \\
&= (1-p)D(\rho, \sigma)
\end{aligned}$$

Therefore, as $0 \leq (1-p) < 1$, we have $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) < D(\rho, \sigma)$.

Exercise 9.13

$\mathcal{E}(\rho) = p\rho + (1-p)X\rho X$

Using that $D(X\rho X, X\sigma X) = D(\rho, \sigma)(X$ unitary) and Theorem 9.3, i.e.

$$D\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \leq \sum_i p_i D(\rho_i, \sigma_i)$$

we have,

$$\begin{aligned}D(\mathcal{E}(\rho),\mathcal{E}(\sigma)) &= D(p\rho+(1-p)X\rho X, p\sigma+(1-p)X\sigma X)\\ &\leq pD(\rho,\sigma)+(1-p)D(X\rho X, X\sigma X)\\ &= pD(\rho,\sigma)+(1-p)D(\rho,\sigma) = D(\rho,\sigma)\end{aligned}$$

Hence, $\mathcal{E}$ is contractive but not strictly contractive.

Exercise 9.14

Using the fact that density matrices are positive operators and the given identity, we have,

$$\begin{aligned}F(U\rho U^\dagger, U\sigma U^\dagger) &= tr\sqrt{(U\rho U^\dagger)^{1/2}U\sigma U^\dagger(U\rho U^\dagger)^{1/2}}\\ &= tr\sqrt{U\rho^{1/2}U^\dagger U\sigma U^\dagger U\rho^{1/2}U^\dagger}\\ &= tr\sqrt{U\rho^{1/2}\sigma\rho^{1/2}U^\dagger}\\ &= tr(U\sqrt{\rho^{1/2}\sigma\rho^{1/2}}U^\dagger) = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}} = F(\rho,\sigma)\end{aligned}$$

Exercise 9.15

Let $|\psi\rangle = (U_R\otimes\sqrt{\rho}U_Q)|m\rangle$ be a fixed purification for ρ with fixed U_R and U_M. Repeating the steps in Uhlmann's theorems proof, we get,
$|\langle\psi|\phi\rangle| = |tr(V_R^\dagger U_R U_Q^\dagger\sqrt{\rho}\sqrt{\sigma}V_Q)|$
Where, V_R and V_Q define an arbitrary purification of σ. Hence, letting $U = V_Q V_R^\dagger U_R U_Q^\dagger$, from Lemma 9.5 we have
$|\langle\psi|\phi\rangle| = |tr(\sqrt{\rho}\sqrt{\sigma}U) \leq tr|\sqrt{\rho}\sqrt{\sigma}| = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$
Let, $\sqrt{\rho}\sqrt{\sigma} = |\sqrt{\rho}\sqrt{\sigma}|V$ be the polar decomposition of $\sqrt{\rho}\sqrt{\sigma}$. Choosing, $V_R = U_R$ and $V_Q = V^\dagger U_Q$, we get $U = V^\dagger$. Therefore, for such V_R and V_Q by Lemme 9.5,
$|\langle\psi|\phi\rangle| = tr\sqrt{\rho^{1/2}\sigma\rho^{1/2}}$
i.e
$F(\rho,\sigma) = \max_{|\phi\rangle}|\langle\psi|\phi\rangle|$

Exercise 9.16

$$\langle m|(A\otimes B)|m\rangle = \sum_{i,j} = \langle i_R|A|j_R\rangle\langle i_Q|B|j_Q\rangle = \sum_{i,j}A^\dagger_{ji}B_{ij} = tr(A^\dagger B)$$

Exercise 9.17

$0\leq F(\rho,\sigma)\leq 1$, hence for $A(\rho,\sigma) = \arccos F(\rho,\sigma)$ we have $0\leq A(\rho,\sigma)\leq\frac{\pi}{2}$
$A(\rho,\sigma) = 0$ if and only if $F(\rho,\sigma) = 1$, which is only true if and only if $\rho = \sigma$.

Exercise 9.18

In the range $0\leq x\leq 1$ $\arccos x$ is a decreasing function, hence from $F(\mathcal{E}(\rho),\mathcal{E}(\sigma))\geq F(\rho,\sigma)$ we have
$\arccos F(\mathcal{E}(\rho),\mathcal{E}(\sigma))\leq\arccos F(\rho,\sigma)$
Therefore,
$A(\mathcal{E}(\rho),\mathcal{E}(\sigma)) = \arccos F(\mathcal{E}(\rho),\mathcal{E}(\sigma))\leq F(\rho,\sigma) = A(\rho,\sigma)$

Exercise 9.19

Using Theorem 9.7 and letting $q_i = p_i$ we have
$$F\left(\sum_i p_i\rho_i, \sum_i p_i\sigma_i\right) \geq \sum_i \sqrt{p_i p_i} F(\rho_i, \sigma_i) = \sum_i p_i F(\rho_i, \sigma_i)$$

Exercise 9.20

Using Theorem 9.7 and letting $\sigma_i = \sigma$ we have,
$$F\left(\sum_i p_i\rho_i, \sigma\right) = F\left(\sum_i p_i\rho_i, \sum_i p_i\sigma\right) \geq \sum_i p_i F(\rho_i, \sigma)$$

Exercise 9.21