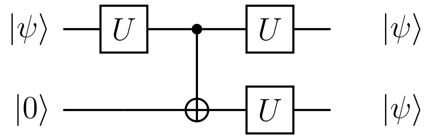
Exercise 12.1

Let $|\psi\rangle = a|0\rangle + b|1\rangle$. Then, $|\varphi\rangle = b|0\rangle - a|1\rangle$.
We require a gate which transforms between the basis of $|\psi\rangle$ and $|\varphi\rangle$, and the computational basis. This can be performed using the unitary, $U = U^dagger = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. Using U we construct the following circuit which performs the cloning.



Exercise 12.2

$\sum_y \sqrt{E_y} \otimes U_y(\sigma \otimes |0\rangle\langle 0|)\sqrt{E_y} \otimes U_y = \sum_y \sqrt{E_y}\sigma\sqrt{E_y} \otimes |0+y\rangle\langle 0+y| = \sum_y \sqrt{E_y}\sigma\sqrt{E_y} \otimes |y\rangle\langle y|$

Exercise 12.3

From the Holevo bound $H(X:Y) \leq S(\rho)$, however $S(\rho)$ is bounded by $\log 2^n = n$, hence no more than n bits can be transferred.

Exercise 12.4

Let $\rho_i = |X_i\rangle\langle X_i|$, and $\rho = \frac{1}{4}\sum_i \rho_i = \frac{I}{2}$, which is the maximally mixed state, hence $S(\rho) = 1$ and the $S(\rho_i) = 0$ as the ρ_i are pure states. Then,
$\chi = S(\rho) - \sum_i p_i S(\rho_i) = 1$
We have tetrahedral symmetry for the states $|X_i\rangle$, hence we consider POVMs with the same symmetry. Consider $E_i = \frac{1}{2}|X_i\rangle\langle X_i|$, and $F_i = \frac{1}{2}|\psi_i\rangle\langle\psi_i|$, where
$|\psi_1\rangle = |1\rangle$
$|\psi_2\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}e^{\pi i/3}|0\rangle + |1\rangle)$
$|\psi_3\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}e^{\pi i}|0\rangle + |1\rangle)$
$|\psi_4\rangle = \sqrt{\frac{1}{3}}(\sqrt{2}e^{5\pi i/3}|0\rangle + |1\rangle)$
Using $H(X:Y) = H(Y) - H(Y|X)$, $P(X=x) = \frac{1}{4}$, $p(Y=y|x=x) = tr(\rho_x E_y)$ and Bayes theorem we get,
$H(Y) = -\sum_y p(Y=y)\log p(Y=y) =$

$$-\sum_y \left(\sum_x p(X=x)p(Y=y|X=x)\right)\log\left(\sum_x p(X=x)p(Y=y|X=x)\right)$$

$H(Y|X) = -\sum_x p(X=x)\sum_y p(Y=y|X=x)\log p(Y=y|X=x)$
The calculation can be found in 12.3.py and gives $H(X:Y)_E \approx 0.208$ and $H(X:Y)_F \approx 0.415$. Therefore, F_i is the desired POVM.
As shown in DOI: 10.1109/TIT.1978.1055941 the maximum information is $\log\frac{4}{3} \approx 0.415$.

Exercise 12.5

Exercise 12.6

Exercise 12.7

Exercise 12.8

Exercise 12.9

Exercise 12.10

Exercise 12.11

Exercise 12.12

Exercise 12.13

Exercise 12.14

Exercise 12.15

Exercise 12.16

Exercise 12.17

Exercise 12.18

Exercise 12.19

Exercise 12.20