UNIVERSITÀ DI PISA

Bachelor of Science in Mathematics
Computational Curriculum

Pricing of European Options Black-Scholes & Heston

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Introduction

The price of a derivative financial product can be obtained using two different methods, one algebraic and the other probabilistic. This is the result of a deep connection that exists between probability and analysis.

The pricing of a derivative product can be carried out by exploiting the theory of Partial Derivative Equations or that of Martingales to achieve the same result. In this thesis we will first address both methods for pricing European options in the continuous case of the Black-Scholes model.

However, this model considers the volatility of the option's underlying to be constant, which is at odds with the empirical analysis of the data.

The solution to this problem presented in this thesis is the stochastic volatility model, whic the Heston model is an example of. It will be calibrated o n the basis of empirical data and an analysis of the computational costs involved in implementing the model will be made.

In the first chapter we will give an introduction to stochastic calculus, in particular we will give the definition of a stochastic Wiener integral for deterministic and random functions. We will then demonstrate the Itō formula and apply it to the Brownian stochastic process of the volatile asset price.

In the second chapter, we will introduce the mathematical model describing a self-financing financial portfolio. We will also give the definition of arbitrage and risk-neutral measure to state the market completeness theorem.

In the third chapter, we will present the Black-scholes theory, referring to the Solution of the Heat Equation to determine the solutions of the Black-scholes equation for calculating the price of a European option.

In chapter four, we will alternatively demonstrate how the price of a European option can be derived using the theory of the stochastic Martingale process.

In the fifth chapter we will analyse the problem of pricing European options with constant volatility in the discrete case. We will show how the Monte Carlo approach is computationally preferable to a deterministic approach.

In the sixth chapter, we will present the issue of considering constant volatility within the Black-Scholes model. An empirical analysis of market data will be made in order to determine what the properties of a model that replicates the price trend of an option as closely as possible should be.

In the seventh chapter we will introduce Mean-reverting stochastic volatility models that have most of the characteristics verified empirically with data. These models are able to replicate the Smile Curve of volatility.

In chapter eight, we will make a detailed analysis of the Heston model. We will describe the parameters required for the calibration and describe some methods for pricing an European option. Finally, we will make a computational comparison of the various methods.

Contents

Brownian Motions and the Itō Formula

We will refer to the book [1]Nicolas Privault *Stochastic Finance.* Chapman & Hall/CRC FINANCIAL MATHEMATICS SERIES to present the famous theory behind financial mathematics.

Let us begin by defining the stochastic process called Brownian Motion. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we are going to construct our Brownian Motion will be $\Omega = \mathcal{C}_0(\mathbb{R}_+)$, i.e. the space of continuous real-valued functions defined in $\mathbb{R}_+$.

1.1 Brownian Motions

Definition 1.1. Standard Brownian Motion is a stochastic process $(B_t)_{t\in\mathbb{R}_+}$ such that:

1. $B_0 = 0$ $\mathbb{P}$-almost certainly
2. It has continuous trajectories $\mathbb{P}$-almost certainly
3. For each finite sequence of times $t_0 < t_1 < \cdots < t_n$, the increments $B_{t_1} - B_{t_0}$, $B_{t_2} - B_{t_1}$, $\ldots, B_{t_n} - B_{t_{n-1}}$ are independent
4. For any fixed time $0 \le s < t$, $B_t - B_s$ has a Gaussian distribution $\mathcal{N}(0, t-s)$ with zero mean and variance $(t-s)$

It can be shown that such a Brownian motion exists and is unique.

We observe in particular that condition 4 implies that

$$\mathbb{E}[B_t - B_s] = 0 \quad \text{and} \quad \operatorname{Var}[B_t - B_s] = t - s, \quad 0 \le s \le t$$

In the following we will use the Filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by Brownian Motion up to time t i.e.

$$\mathcal{F}_t = \sigma(B_s : 0 \le s \le t), \quad t \ge 0$$

A random variable F is called $\mathcal{F}_t$-measurable if knowledge of F depends only on the information up to time t.

Finally, we observe that property 3 shows that $B_t - B_s$ is independent of all Brownian increments up to time s, i.e,

$$(B_t - B_s) \perp \left(B_{t_1} - B_{t_0}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}\right) \quad \forall 0 \le t_0 \le t_1 \le \cdots \le t_n \le s \le t$$

So $B_t - B_s$ is independent of $\mathcal{F}_s, s \ge 0$

We now turn to the construction of the Itō integral for deterministic functions of integrable square with respect to Brownian motion.

1.2 Wiener stochastic integral for deterministic functions

The famous mathematician Bachelier around 1900 tried to model asset prices on the Paris stock exchange by defining a volatile asset with $S_t = \sigma B_t$ where σ represents the implied volatility. The stochastic integral

$$\int_0^T f(t) dS_t = \sigma \int_0^T f(t) dB_t$$

can thus be used to represent the value of the portfolio as the sum of profits and losses $f(t)dS_t$ where dS_t represents the price change and $f(t)$ the quantity invested in the asset S_t in the time interval $[t, t + dt]$.

A naive definition of the stochastic integral with respect to Brownian motion could be

$$\int_0^\infty f(t) dB_t = \int_0^\infty f(t) \frac{dB_t}{dt} dt$$

However, such a definition is not admissible since Brownian Motion is not differentiable. We therefore present the construction of the stocastic Itō integral with the respect to Brownian Motion. The latter will first be approximated by the integral of simple functions of the form

$$f(t) = \sum_{i=1}^{n} a_i \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad t \in \mathbb{R}_+$$

We observe that the set of simple functions f is a dense linear space in $L^2(\mathbb{R}_+)$ for the norm

$$\|f\|_{L^2(\mathbb{R}_+)} := \sqrt{\int_0^\infty |f(t)|^2 dt}.$$

Furthermore, the integral of f is interpreted as the area under the curve f and calculated as

$$\int_0^\infty f(t) dt = \sum_{i=1}^{n} a_i (t_i - t_{i-1}).$$

In our definition we will adapt this construction to the integration with respect to Brownian Motion

Definition 1.2. The stochastic integral with respect to Brownian Motion $(B_t)_{t \in \mathbb{R}_+}$ of a simple function f is defined as:

$$\int_0^\infty f(t) dB_t := \sum_{i=1}^{n} a_i \left(B_{t_i} - B_{t_{i-1}} \right).$$

We now extend the definition to square integrable functions.

Proposizione 1.3. *The definition of the stochastic integral $\int_0^\infty f(t)dB_t$ can be extended to any function $f \in L^2(\mathbb{R}_+)$, i.e. to any function f such that*

$$\int_0^\infty |f(t)|^2 dt < \infty.$$

In that case, $\int_0^\infty f(t)dB_t$ has a Gaussian distribution

$$\int_0^\infty f(t)dB_t \simeq \mathcal{N}\left(0, \int_0^\infty |f(t)|^2 dt\right)$$

with variance $\int_0^\infty |f(t)|^2 dt$ and the Itō isometry is valid.

$$\mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] = \int_0^\infty |f(t)|^2 dt.$$

Proof. Recall that the increments $X_1, \ldots, X_n$ are independent variables with distribution $\mathcal{N}(m_1, \sigma_1^2), \ldots, \mathcal{N}(m_n, \sigma_n^2)$ so $X_1 + \cdots + X_n$ is a Gaussian variable with law $\mathcal{N}(m_1 + \cdots + m_n, \sigma_1^2 + \cdots + \sigma$

If f is a simple function

$$f(t) = \sum_{i=1}^n a_i \mathbf{1}_{(t_{i-1}, t_i]}(t), \quad t \in \mathbb{R}_+,$$

the sum

$$\int_0^\infty f(t)dB_t = \sum_{k=1}^n a_k \left(B_{t_k} - B_{t_{k-1}}\right)$$

is a centred Gaussian variable with variance

$$\sum_{k=1}^n |a_k|^2 (t_k - t_{k-1})$$

since

$$\mathrm{Var}\left[a_k \left(B_{t_k} - B_{t_{k-1}}\right)\right] = a_k^2 \,\mathrm{Var}\left[B_{t_k} - B_{t_{k-1}}\right] = a_k^2 (t_k - t_{k-1}),$$

then the stochastic integral

$$\int_0^\infty f(t)dB_t = \sum_{k=1}^n a_k \left(B_{t_k} - B_{t_{k-1}}\right)$$

of the simple function

$$f(t) = \sum_{k=1}^n a_k \mathbf{1}_{(t_{k-1}, t_k]}(t)$$

has a centred Gaussian distribution with variance

$$\begin{aligned}
\operatorname{Var}\left[\int_0^\infty f(t)dB_t\right] &= \sum_{k=1}^n |a_k|^2 (t_k - t_{k-1}) \\
&= \sum_{k=1}^n |a_k|^2 \int_{t_{k-1}}^{t_k} dt \\
&= \int_0^\infty \sum_{k=1}^n |a_k|^2 1_{(t_{k-1},t_k]}(t)dt \\
&= \int_0^\infty |f(t)|^2 dt.
\end{aligned}$$

Finally, we observe that

$$\begin{aligned}
\operatorname{Var}\left[\int_0^\infty f(t)dB_t\right] &= \mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right] - \left(\mathbb{E}\left[\int_0^\infty f(t)dB_t\right]\right)^2 \\
&= \mathbb{E}\left[\left(\int_0^\infty f(t)dB_t\right)^2\right]
\end{aligned}$$

The extension of the stochastic integral to all integrable square functions can now be achieved by exploiting the vector space density of simple functions, the definition of the Cauchy succession and the Itō isometry.

Let f be such a function and $(f_n)_{n\in\mathbb{N}}$ a sequence of simple functions converging to f for the norm

$$\|f - f_n\|_{L^2(\mathbb{R}_+)} := \left(\int_0^\infty |f(t) - f_n(t)|^2 \, dt\right)^{1/2}$$

or in $L^2(\mathbb{R}_+)$.

The Itō isometry shows that $\left(\int_0^\infty f_n(t)dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy succession in the space $L^2(\Omega)$ of random variable $F : \Omega \to \mathbb{R}$ of integrable square, i.e. such that

$$\|F\|^2_{L^2(\Omega\times\mathbb{R}_+)} := \mathbb{E}\left[F^2\right] < \infty.$$

In fact we have that

$$\begin{aligned}
&\left\|\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right\|_{L^2(\Omega)} \\
&= \left(\mathbb{E}\left[\left(\int_0^\infty f_k(t)dB_t - \int_0^\infty f_n(t)dB_t\right)^2\right]\right)^{1/2} \\
&= \|f_k - f_n\|_{L^2(\mathbb{R}_+)} \\
&\leq \|f - f_k\|_{L^2(\mathbb{R}_+)} + \|f - f_n\|_{L^2(\mathbb{R}_+)},
\end{aligned}$$

which tends to 0 when k and n goes to infinite.

Since $L^2(\Omega)$ is a complete space, $\left(\int_0^\infty f_n(t)dB_t\right)_{n\in\mathbb{N}}$ converges for the norm L^2 and so we will define

$$\int_0^\infty f(t)dB_t := \lim_{n\to\infty}\int_0^\infty f_n(t)dB_t$$

Moreover, thanks to the Itō isometry, this limit is unique. □

Let us now extend the previous definition to processes $\mathcal{F}_t$-adapted and of integrable square. Recall that a process $(X_t)_{t\in\mathbb{R}_+}$ is $\mathcal{F}_t$-adapted if X_t is $\mathcal{F}_t$-measurable for every $t\in\mathbb{R}_+$

1.3 Wiener stochastic integral for random functions

As done above, the stochastic integral for adapted processes will be constructed firstly as the integral of simple processes $(u_t)_{t\in\mathbb{R}_+}$ of the form

$$u_t = \sum_{i=1}^n F_i \mathbf{1}_{(t_{i-1},t_i]}(t), \quad t\in\mathbb{R}_+$$

where F_i is a random variable, $\mathcal{F}_{t_{i-1}}$-measurable for each $i = 1,\dots,n$, which remains constant over the time interval $(t-1,t]$.

By convention, the random function $u:\Omega\times\mathbb{R}_+\to\mathbb{R}$ will be denoted by $u_t(\omega), t\in\mathbb{R}_+, \omega\in\Omega$ and the event ω will often be omitted.

Definition 1.4. The stochastic integral with respect to Brownian Motion $(B_t)_{t\in\mathbb{R}_+}$ of any process $(u_t)_{t\in\mathbb{R}_+}$ of the form

$$u_t = \sum_{i=1}^n F_i \mathbf{1}_{(t_{i-1},t_i]}(t), \quad t\in\mathbb{R}_+$$

is defined as:

$$\int_0^\infty u_t dB_t := \sum_{i=1}^n F_i\left(B_{t_i} - B_{t_{i-1}}\right).$$

The following proposition provides the extension of this integral to any process $(u_t)_{t\in\mathbb{R}_+}$ of integrable square $\mathcal{F}_t$-adapted.

Proposizione 1.5. *The stochastic integral with respect to Brownian Motion $(B_t)_{t\in\mathbb{R}_+}$ extends to any process $(u_t)_{t\in\mathbb{R}_+}$ $\mathcal{F}_t$-adapted such that:*

$$\mathbb{E}\left[\int_0^\infty |u_t|^2\, dt\right] < \infty$$

Furthermore, the Itō isometry is valid

$$\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] = \mathbb{E}\left[\int_0^\infty |u_t|^2\, dt\right]$$

Proof. We begin by showing that the Itō isometry holds for simple processes u, in fact we have:

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_0^\infty u_t dB_t\right)^2\right] &= \mathbb{E}\left[\left(\sum_{i=1}^n F_i\left(B_{t_i}-B_{t_{i-1}}\right)\right)^2\right] \\
&= \mathbb{E}\left[\sum_{i,j=1}^n F_iF_j\left(B_{t_i}-B_{t_{i-1}}\right)\left(B_{t_j}-B_{t_{j-1}}\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2\left(B_{t_i}-B_{t_{i-1}}\right)^2\right] \\
&+ 2\mathbb{E}\left[\sum_{1\le i<j\le n} F_iF_j\left(B_{t_i}-B_{t_{i-1}}\right)\left(B_{t_j}-B_{t_{j-1}}\right)\right] \\
&= \sum_{i=1}^n \mathbb{E}\left[\mathbb{E}\left[|F_i|^2\left(B_{t_i}-B_{t_{i-1}}\right)^2 \mid \mathcal{F}_{t_{i-1}}\right]\right] \\
&+ 2\sum_{1\le i<j\le n} \mathbb{E}\left[\mathbb{E}\left[F_iF_j\left(B_{t_i}-B_{t_{i-1}}\right)\left(B_{t_j}-B_{t_{j-1}}\right) \mid \mathcal{F}_{t_{j-1}}\right]\right] \\
&= \sum_{i=1}^n \mathbb{E}\left[|F_i|^2\,\mathbb{E}\left[\left(B_{t_i}-B_{t_{i-1}}\right)^2 \mid \mathcal{F}_{t_i}\right]\right] \\
&+ 2\sum_{1\le i<j\le n} \mathbb{E}\left[F_iF_j\left(B_{t_i}-B_{t_{i-1}}\right)\mathbb{E}\left[\left(B_{t_j}-B_{t_{j-1}}\right) \mid \mathcal{F}_{t_{j-1}}\right]\right] \\
&= \sum_{i=1}^n \mathbb{E}\left[|F_i|^2\,\mathbb{E}\left[\left(B_{t_i}-B_{t_{i-1}}\right)^2\right]\right] \\
&+ 2\sum_{1\le i<j\le n} \mathbb{E}\left[F_iF_j\left(B_{t_i}-B_{t_{i-1}}\right)\mathbb{E}\left[\left(B_{t_j}-B_{t_{j-1}}\right)\right]\right] \\
&= \sum_{i=1}^n \mathbb{E}\left[|F_i|^2\left(t_i-t_{i-1}\right)\right] \\
&= \mathbb{E}\left[\sum_{i=1}^n |F_i|^2\left(t_i-t_{i-1}\right)\right] = \mathbb{E}\left[\int_0^\infty |u_t|^2\,dt\right],
\end{aligned}
$$

where we used the basic properties of conditional expectation, the fact that $B_{t_i}-B_{t_{i-1}}$ is independent of $\mathcal{F}_{t_{i-1}}$ and the two following properties:

$$
\mathbb{E}\left[B_{t_i}-B_{t_{i-1}}\right]=0, \quad \mathbb{E}\left[\left(B_{t_i}-B_{t_{i-1}}\right)^2\right]=t_i-t_{i-1}, \quad i=1,\ldots,n.
$$

With a similar reasoning as in Proposition 1.3, we can now extend the definition to integrable square processes $(u_t)_{t\in\mathbb{R}_+}$ by exploiting the density of the vector space of simple processes, the definition of Cauchy successions, and concluding as before using the Itō isometry.

Let $L^2(\Omega \times \mathbb{R}_+)$ be the space of integrable square processes $u : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\|u\|^2_{L^2(\Omega\times\mathbb{R}_+)} := \mathbb{E}\left[\int_0^\infty |u_t|^2\, dt\right] < \infty.$$

The set of simple processes form a dense linear space in the subspace $L^2_{ad}(\Omega \times \mathbb{R}_+)$ formed by the adapted integrable square processes in $L^2(\Omega \times \mathbb{R}_+)$. Given an adapted process $(u_t)_{t\in\mathbb{R}}$ of integrable square, there is a sequence $(u^n_t)_{n\in\mathbb{N}}$ of simple processes converging to u_t in $L^2(\Omega \times \mathbb{R}_+)$ and the Itō isometry shows that $\left(\int u^n_t dB_t\right)_{n\in\mathbb{N}}$ is a Cauchy succession inn $L^2(\Omega)$, therefore converges in the complete space $L^2(\Omega)$.

In that case we define

$$\int_0^\infty u_t dB_t := \lim_{n\to\infty} \int_0^\infty u^n_t dB_t$$

and once again, exploiting the Itō isometry, the limit is unique. $\square$

We observe that the Itō integral of an adapted process $(u_t)_{t\in\mathbb{R}_+}$ is still a centred random variable

$$\mathbb{E}\left[\int_0^\infty u_s dB_s\right] = 0$$

Furthermore, the Itō isometry can be rewritten as

$$\mathbb{E}\left[\int_0^\infty u_t dB_t \int_0^\infty v_t dB_t\right] = \mathbb{E}\left[\int_0^\infty u_t v_t dt\right],$$

for all integrable square processes u, v.

In contrast to the previous case, in which the integrand $(u_t)_{t\in\mathbb{R}_+}$ was a deterministic function, the variable $\int_0^\infty u_s dB_s$ now no longer has a Gaussian distribution except in certain cases.

In the following, we will define the price of the volatile asset at time t as

$$dS_t = \mu S_t dt + \sigma S_t dB_t,$$

with $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation can be formally rewritten as

$$S_T = S_0 + \mu \int_0^T S_t dt + \sigma \int_0^T S_t dB_t,$$

from which we understand the need to define a stochastic integral with respect to Brownian motion.

This model will be used in the following to represent the price S_t of a volatile asset at time t. In this case, the gain dS_t/S_t will be given by two components: a constant gain μdt and a random gain σdB_t parametrized by the volatility coefficient σ.

Our objective will be to solve the equation

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t,$$

and to do so we will introduce notions of stochastic calculus and the Itō formula.

1.4 Stochastic calculus and Itō formula

To introduce the Itō formula we start from a generic Itō process

$$X_t = X_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+,$$

or using differential notation

$$dX_t = v_t dt + u_t dB_t,$$

where $(u_t)_{t\in\mathbb{R}_+}$ and $(v_t)_{t\in\mathbb{R}_+}$ are two adapted integrable square processes.

Theorem 1.6. *For every Itō process $(X_t)_{t\in\mathbb{R}_+}$ and for every $f \in \mathcal{C}^{1,2}(\mathbb{R}_+ \times \mathbb{R})$ the Itō formula is valid:*

$$\begin{aligned} f(t, X_t) =& f(0, X_0) + \int_0^t v_s \frac{\partial f}{\partial x}(s, X_s)\, ds + \int_0^t u_s \frac{\partial f}{\partial x}(s, X_s)\, dB_s \\ &+ \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)\, ds. \end{aligned}$$

Proof. cf. [2] □

We observe that using the relationship

$$\int_0^t df(s, X_s) = f(t, X_t) - f(0, X_0),$$

we have

$$\begin{aligned} \int_0^t df(s, X_s) =& \int_0^t v_s \frac{\partial f}{\partial x}(s, X_s)\, ds + \int_0^t u_s \frac{\partial f}{\partial x}(s, X_s)\, dB_s \\ &+ \int_0^t \frac{\partial f}{\partial s}(s, X_s)\, ds + \frac{1}{2}\int_0^t |u_s|^2 \frac{\partial^2 f}{\partial x^2}(s, X_s)\, ds \end{aligned}$$

which allows us to rewrite the Itō formula in differential form as

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + u_t \frac{\partial f}{\partial x}(t, X_t)\, dB_t + v_t \frac{\partial f}{\partial x}(t, X_t)\, dt + \frac{1}{2}|u_t|^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)\, dt$$

or

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t)\, dt + \frac{\partial f}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}|u_t|^2 \frac{\partial^2 f}{\partial x^2}(t, X_t)\, dt.$$

1.5 Itō formula applied to the model

As mentioned earlier, our goal is to solve the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dB_t$$

which will give us, as its solution, the price S_t of the volatile asset at time t, where $\mu \in \mathbb{R}$ and $\sigma > 0$. This equation can be rewritten in integral form as

$$S_t = S_0 + \mu \int_0^t S_s ds + \sigma \int_0^t S_s dB_s, \quad t \in \mathbb{R}_+.$$

and can be solved by applying the Itō formula to $f(S_t) = \log S_t$ with $f(x) = \log x$, in fact

$$\begin{aligned} d \log S_t &= \mu S_t f'(S_t) dt + \sigma S_t f'(S_t) dB_t + \frac{1}{2}\sigma^2 S_t^2 f''(S_t) dt \\ &= \mu dt + \sigma dB_t - \frac{1}{2}\sigma^2 dt \end{aligned}$$

so

$$\begin{aligned} \log S_t - \log S_0 &= \int_0^t d \log S_r \\ &= \int_0^t \left(\mu - \frac{1}{2}\sigma^2\right) dr + \int_0^t \sigma dB_r \\ &= \left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t, \quad t \in \mathbb{R}_+ \end{aligned}$$

which leads to the following solution

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t + \sigma B_t\right), \quad t \in \mathbb{R}_+$$

Portfolio model

Let us now give a formal description of our financial model

2.1 Continuous time market model

We shall denote by $(A_t)_{t \in \mathbb{R}_+}$ the stochastic process representing the risk-free asset defined by the following relationship

$$\frac{dA_t}{A_t} = r dt, \quad t \in \mathbb{R}_+,$$

i.e.,

$$A_t = A_0 e^{rt}, \quad t \in \mathbb{R}_+.$$

We shall further denote by $(S_t)_{t\in\mathbb{R}_+}$ the stochastic process representing the price of the volatile asset defined by the following relation

$$dS_t = \mu S_t dt + \sigma S_t dB_t, \quad t \in \mathbb{R}_+.$$

whose solution is

$$S_t = S_0 \exp\left(\sigma B_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right), \quad t \in \mathbb{R}_+.$$

2.2 "Self-Financing" portfolio

Let ξ_t and η_t be the quantities (possibly fractional) invested at time t, in the assets S_t and A_t, and let us denote by

$$\bar{\xi}_t = (\eta_t, \xi_t), \quad \bar{S}_t = (A_t, S_t), \quad t \in \mathbb{R}_+,$$

respectively the portfolio and the price process associated with it. The value of the portfolio V_t at time t is given by the relation

$$V_t = \bar{\xi}_t \cdot \bar{S}_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+.$$

Definition 2.1. We say that the portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is "Self-financing" if the portfolio value remains constant after updating the portfolio from (η_t, ξ_t) to $(\eta_{t+dt}, \xi_{t+dt})$, i.e.

$$\bar{\xi}_t \cdot \bar{S}_{t+dt} = A_{t+dt}\eta_t + S_{t+dt}\xi_t = A_{t+dt}\eta_{t+dt} + S_{t+dt}\xi_{t+dt} = \bar{\xi}_{t+dt} \cdot \bar{S}_{t+dt}$$

The following lemma states that when a portfolio is 'self-financing', its discounted value equals the difference between the discounted gains and losses.

Lemma 2.2. *Let $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ be a portfolio strategy with value*

$$V_t = \eta_t A_t + \xi_t S_t, \quad t \in \mathbb{R}_+$$

then the following facts are equivalent:

1. *The portfolio strategy $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ is "Self-financing"*
2. $\tilde{V}_t = \tilde{V}_0 + \int_0^t \xi_u dX_u, \quad t \in \mathbb{R}_+$

As a consequence of the lemma, the problem of hedging a derived product reduces to the calculation of $\tilde{C} = e^{-rT}C$ as a stochastic integral:

$$\tilde{C} = \tilde{V}_T = \tilde{V}_0 + \int_0^T \xi_u dX_u.$$

Where we have indicated with

$$\tilde{V}_t = e^{-rt} V_t \quad \text{e} \quad X_t = e^{-rt} S_t$$

respectively the discounted portfolio value and the discounted volatile asset value at time $t \geq 0$.

We have the following relationship

$$\begin{aligned} dX_t &= d\left(e^{-rt} S_t\right) \\ &= -re^{-rt} S_t dt + e^{-rt} dS_t \\ &= -re^{-rt} S_t dt + \mu e^{-rt} S_t dt + \sigma e^{-rt} S_t dB_t \\ &= X_t \left((\mu - r)dt + \sigma dB_t\right) \end{aligned}$$

From which we derive

$$V_t = V_0 + (\mu - r) \int_0^t e^{r(t-u)} \xi_u S_u du + \sigma \int_0^t e^{r(t-u)} \xi_u S_u dB_u, \quad t \in \mathbb{R}_+.$$

2.3 Arbitrage and Risk-neutral Measures

In the following we will only consider portfolio strategies whose total value V_t remains non-negative for each $t \in [0, T]$.

Definition 2.3. A portfolio strategy $(\xi_t, \eta_t)_{t \in [0,T]}$ constitutes an arbitrage opportunity if the following conditions are met:

1. $V_0 \leq 0$
2. $V_T \geq 0$
3. $\mathbb{P}(V_T > 0) > 0$

Definition 2.4. A probability measure $\mathbb{Q}$ on Ω is said to be risk-neutral if, given the filtering $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$ generated by Brownian Motion $(B_t)_{t \in \mathbb{R}_+}$, $\mathbb{Q}$ satisfies

$$\mathbb{E}^{\mathbb{Q}}\left[S_t \mid \mathcal{F}_u\right] = e^{r(t-u)} S_u, \quad 0 \leq u \leq t,$$

Recalling the definition

$$A_t = e^{r(t-u)} A_u, \quad 0 \leq u \leq t,$$

we can think of the above condition as the fact that, under the assumption of a risk-neutral probability measure $\mathbb{Q}$, the expected gain of the volatile asset S_t is equal to that of the riskless asset A_t.

Before giving an equivalent formulation of the previous definitions, let us give the following definition:

Definition 2.5. A continuous-time process $(Z_t)_{t\in\mathbb{R}_+}$ is a Martingale with respect to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}\left[Z_t \mid \mathcal{F}_s\right] = Z_s, \quad 0 \le s \le t.$$

This definition allows us to give a second formulation of the concept of a risk-neutral probability measure.

Proposizione 2.6. . *The probability measure* $\mathbb{Q}$ *is Risk-neutral if and only if* $(X_t)_{t\in\mathbb{R}_+}$ *is a Martingale with the respect to* $\mathbb{Q}$.

2.4 Market Completeness

Let us now give the following definition

Definition 2.7. An option with payoff C is replicable if there exists a carrier strategy $(\eta_t, \xi_t)_{t\in[0,T]}$ such that

$$V_t = \eta_t A_t + \xi_t S_t \quad \forall t \in [0,T] \quad \& \quad V_T = C$$

In this case, the option price at time t is given by the value V_t of the self-financing portfolio whereby we are replicating the option.

With this definition we can give the definition of the Complete Market.

Definition 2.8. A market is said to be complete if every option with payoff C is replicable.

The Black-Scholes Model

We begin this chapter by presenting the partial differential equation of the Black-Scholes model for the price of a volatile asset.

Proposizione 3.1. *Let* $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ *be a portfolio strategy such that*

1. $(\eta_t, \xi_t)_{t\in\mathbb{R}_+}$ *is Self-financing*

2. $V_t := \eta_t A_t + \xi_t S_t, t \in \mathbb{R}_+$ *can be written as*

$$V_t = g\left(t, S_t\right), \quad t \in \mathbb{R}_+,$$

with $g \in \mathcal{C}^{1,2}((0,\infty) \times (0,\infty))$.

Then the function $g(t,x)$ satisfies the Black-Scholes partial derivative equation

$$rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,x), \quad t,x > 0,$$

and ξ_t is given by the relationship

$$\xi_t = \frac{\partial g}{\partial x}(t,S_t), \quad t \in \mathbb{R}_+.$$

Proof. Let us start by observing that the self-financing condition implies that the increase in the value of the portfolio is only given by the increase in the prices of the assets

$$\begin{aligned} dV_t &= \eta_t dA_t + \xi_t dS_t \\ &= r\eta_t A_t dt + \mu\xi_t S_t dt + \sigma\xi_t S_t dB_t \end{aligned}$$

where $t \in \mathbb{R}_+$.

Recall also that the price of the volatile asset is an Itō process and thus can be rewritten as

$$S_t = S_0 + \int_0^t v_s ds + \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+,$$

At this point taking

$$u_t = \sigma S_t, \quad \text{e} \quad v_t = \mu S_t, \quad t \in \mathbb{R}_+.$$

and applying the Itō formula to $g(t,x)$ we obtain that

$$\begin{aligned} &dg(t,S_t) = v_t\frac{\partial g}{\partial x}(t,S_t)dt + u_t\frac{\partial g}{\partial x}(t,S_t)dB_t \\ &+ \frac{\partial g}{\partial t}(t,S_t)dt + \frac{1}{2}|u_t|^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\ &= \frac{\partial g}{\partial t}(t,S_t)dt + \mu S_t\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt + \sigma S_t\frac{\partial g}{\partial x}(t,S_t)dB_t. \end{aligned}$$

We now identify the dB_t and dt terms in the two representations of $V_t = g(t,S_t)$ and obtain the following system

$$\begin{cases} r\eta_t A_t dt + \mu\xi_t S_t dt = \frac{\partial g}{\partial t}(t,S_t)dt + \mu S_t\frac{\partial g}{\partial x}(t,S_t)dt + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,S_t)dt \\ \xi_t S_t\sigma dB_t = S_t\sigma\frac{\partial g}{\partial x}(t,S_t)dB_t \end{cases}$$

whence

$$\begin{cases} rV_t - r\xi_t S_t = \frac{\partial g}{\partial t}(t,S_t) + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,S_t), \\ \xi_t = \frac{\partial g}{\partial x}(t,S_t) \end{cases}$$

or

$$\begin{cases} rg(t,S_t) = \frac{\partial g}{\partial t}(t,S_t) + rS_t\frac{\partial g}{\partial x}(t,S_t) + \frac{1}{2}S_t^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,S_t) \\ \xi_t = \frac{\partial g}{\partial x}(t,S_t) \end{cases}$$

$\square$

The derivative $\frac{\partial g}{\partial x}\left(t, S_t\right)$ that gives the value of ξ_t in the above formula is called the option price delta.

With this value we are also able to determine the amount invested in the risk-less asset thanks to the relationship

$$\eta_t A_t = V_t - \xi_t S_t = g\left(t, S_t\right) - S_t \frac{\partial g}{\partial x}\left(t, S_t\right),$$

whereby

$$\begin{aligned}\eta_t &= \frac{V_t - \xi_t S_t}{A_t} \\ &= \frac{g\left(t, S_t\right) - S_t \frac{\partial g}{\partial x}\left(t, S_t\right)}{A_t} \\ &= \frac{g\left(t, S_t\right) - S_t \frac{\partial g}{\partial x}\left(t, S_t\right)}{A_0 e^{rt}}\end{aligned}$$

We now add a final condition $g(T, x) = f(x)$ to the Black-Scholes equation in order to replicate the option with payoff C of the form $C = f\left(S_T\right)$.

Proposizione 3.2. *The price of a self-financing portfolio of the form $V_t = g\left(t, S_t\right)$ replicating an option with payoff $C = f\left(S_T\right)$ satisfies the following partial derivative equation of Black-Scholes*

$$\begin{cases} rg(t,x) = \frac{\partial g}{\partial t}(t,x) + rx\frac{\partial g}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 g}{\partial x^2}(t,x) \\ g(T,x) = f(x) \end{cases}$$

Recall that in the case of European Call options with strike K, the payoff function is $f(x) = (x-K)^+$ and the Black-Scholes equation become

$$\begin{cases} rg_c(t,x) = \frac{\partial g_c}{\partial t}(t,x) + rx\frac{\partial g_c}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 g_c}{\partial x^2}(t,x) \\ g_c(T,x) = (x-K)^+ \end{cases}$$

which, as we shall show later, admits the solution

$$g_c(t,x) = \mathrm{BS}(K, x, \sigma, r, T-t) = x\Phi\left(d_+\right) - Ke^{-r(T-t)}\Phi\left(d_-\right),$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}dy, \quad x \in \mathbb{R}$$

indicates the distribution function of a Standard Gaussian and

$$d_+ = \frac{\log(x/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_- = \frac{\log(x/K) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

with

$$d_+ = d_- + \sigma\sqrt{T-t}.$$

It can easily be verified that when $t = T$ we have

$$d_+ = d_- = \begin{cases} +\infty, & x > K \\ -\infty, & x < K \end{cases}$$

which allows us to derive the initial condition

$$g_c(T,x) = \left\{ \begin{array}{ll} x\Phi(+\infty) - K\Phi(+\infty) = x - K, & x > K \\ x\Phi(-\infty) - K\Phi(-\infty) = 0, & x < K \end{array} \right\} = (x-K)^+$$

at time $t = T$.

. We now derive the solution of the Black-Scholes partial derivative equation by passing through the Heat Equation.

3.1 The Heat Equation

We study the Heat Equation, which is used to model heat diffusion in solids. We will show that this is equivalent to the Black-Scholes equation after a change of variables and therefore we will be able to find the solution we are looking for.

Proposizione 3.3. *The Heat Equation*

$$\left\{ \begin{array}{l} \frac{\partial g}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y) \\ g(0,y) = \psi(y) \end{array} \right.$$

with initial condition $\psi(y)$ has the solution

$$g(t,y) = \int_{-\infty}^{\infty} \psi(z) e^{-\frac{(y-z)^2}{2t}} \frac{dz}{\sqrt{2\pi t}}.$$

Proof.

$$\begin{aligned}
\frac{\partial g}{\partial t}(t,y) &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\psi(z)e^{-\frac{(y-z)^2}{2t}}\frac{dz}{\sqrt{2\pi t}}\\
&= \int_{-\infty}^{\infty}\psi(z)\frac{\partial}{\partial t}\left(\frac{e^{-\frac{(y-z)^2}{2t}}}{\sqrt{2\pi t}}\right)dz\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\psi(z)\left(\frac{(y-z)^2}{t^2}-\frac{1}{t}\right)e^{-\frac{(y-z)^2}{2t}}\frac{dz}{\sqrt{2\pi t}}\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\psi(z)\frac{\partial^2}{\partial z^2}e^{-\frac{(y-z)^2}{2t}}\frac{dz}{\sqrt{2\pi t}}\\
&= \frac{1}{2}\int_{-\infty}^{\infty}\psi(z)\frac{\partial^2}{\partial y^2}e^{-\frac{(y-z)^2}{2t}}\frac{dz}{\sqrt{2\pi t}}\\
&= \frac{1}{2}\frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty}\psi(z)e^{-\frac{(y-z)^2}{2t}}\frac{dz}{\sqrt{2\pi t}}\\
&= \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y).
\end{aligned}$$

Furthermore, it is verified that at time $t = 0$, the following relationship applies

$$\lim_{t\to 0}\int_{-\infty}^{\infty}\psi(z)e^{-\frac{(y-z)^2}{2t}}\frac{dz}{\sqrt{2\pi t}} = \lim_{t\to 0}\int_{-\infty}^{\infty}\psi(y+z)e^{-\frac{z^2}{2t}}\frac{dz}{\sqrt{2\pi t}} = \psi(y),$$

with $y \in \mathbb{R}$. □

We now turn to the Black-Scholes partial derivative equation.

3.2 Solution of the Black-Scholes partial derivative equation

Proposizione 3.4. *Assume that $f(t,x)$ solves the Black- Scholes partial derivative equation*

$$\begin{cases} rf(t,x) = \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x),\\ f(T,x) = (x-K)^+,\end{cases}$$

with the final condition $h(x) = (x-K)^+$. Then the function $g(t,y)$ defined by the following relation

$$g(t,y) = e^{rt}f\left(T-t, e^{\sigma y + \left(\frac{\sigma^2}{2}-r\right)t}\right)$$

solves the Heat Equation

$$\begin{cases} \frac{\partial g}{\partial t}(t,y) = \frac{1}{2}\frac{\partial^2 g}{\partial y^2}(t,y)\\ g(0,y) = h\left(e^{\sigma y}\right)\end{cases}$$

Proof. Let $s = T - t$ and $x = e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}$ then we have

$$\begin{aligned}\frac{\partial g}{\partial t}(t,y) =& r e^{rt} f(T-t,x) - e^{rt}\frac{\partial f}{\partial s}(T-t,x) + \left(\frac{\sigma^2}{2} - r\right) e^{rt} x \frac{\partial f}{\partial x}(T-t,x) \\ =& \frac{\sigma^2}{2} e^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T-t,x) + \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T-t,x)\end{aligned}$$

Where in the last step we used the fact that $f(t,x)$ solves the Black-Scholes partial derivative equation.

However, it is also true that

$$\frac{\partial g}{\partial y}(t,y) = \sigma e^{rt} e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t} \frac{\partial f}{\partial x}\left(T - t, e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}\right)$$

and thus

$$\begin{aligned}\frac{1}{2}\frac{\partial g^2}{\partial y^2}(t,y) =& \frac{\sigma^2}{2} e^{rt} e^{\sigma y + \frac{1}{2}\left(\frac{\sigma^2}{2} - r\right)t} \frac{\partial f}{\partial x}\left(T - t, e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}\right) \\ &+ \frac{\sigma^2}{2} e^{rt} e^{2\sigma y + 2\left(\frac{\sigma^2}{2} - r\right)t} \frac{\partial^2 f}{\partial x^2}\left(T - t, e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}\right) \\ =& \frac{\sigma^2}{2} e^{rt} x \frac{\partial f}{\partial x}(T-t,x) + \frac{\sigma^2}{2} e^{rt} x^2 \frac{\partial^2 f}{\partial x^2}(T-t,x)\end{aligned}$$

from which we get the thesis thanks to the relation

$$g(0,y) = f\left(T, e^{\sigma y}\right) = h\left(e^{\sigma y}\right).$$

□

We are ready to give a formal justification for the solution of the Black- Scholes equation in the case of a European Call option

Proposizione 3.5. *When $h(x) = (x - K)^+$ the solution of the Black-Scholes partial derivative equation is given by*

$$f(t,x) = x\Phi\left(d_+\right) - K e^{-r(T-t)}\Phi\left(d_-\right)$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbb{R}$$

and

$$d_+ = \frac{\log(x/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d_- = \frac{\log(x/K) + \left(r - \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Proof. From proposition 3.5, assuming $s = T - t$ and $x = e^{\sigma y + \left(\frac{\sigma^2}{2} - r\right)t}$ we have

$$f(s,x) = e^{-r(T-s)} g\left(T - s, \frac{-\left(\frac{\sigma^2}{2} - r\right)(T-s) + \log x}{\sigma}\right)$$

From which using proposition 3.4 and the fact that $g(0,y) = h\left(e^{\sigma y}\right) = \psi(y)$ we have

$$\begin{aligned}
f(t,x) =& e^{-r(T-t)} g\left(T - t, \frac{-\left(\frac{\sigma^2}{2} - r\right)(T-t) + \log x}{\sigma}\right)\\
=& e^{-r(T-t)} \int_{-\infty}^{\infty} \psi\left(\frac{-\left(\frac{\sigma^2}{2} - r\right)(T-t) + \log x}{\sigma} + z\right) e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& e^{-r(T-t)} \int_{-\infty}^{\infty} h\left(x e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)}\right) e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& e^{-r(T-t)} \int_{-\infty}^{\infty} \left(x e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)} - K\right)^{+} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& e^{-r(T-t)} \int_{\frac{(-r+\sigma^2/2)(T-t)+\log(K/x)}{\sigma}}^{\infty} \left(x e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)} - K\right) e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& x e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{\sigma z - \left(\frac{\sigma^2}{2} - r\right)(T-t)} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
&- K e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& x \int_{-d_-\sqrt{T-t}}^{\infty} e^{\sigma z - \frac{\sigma^2}{2}(T-t) - \frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
&- K e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& x \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{1}{2(T-t)}(z - \sigma(T-t))^2} \frac{dz}{\sqrt{2\pi(T-t)}}\\
&- K e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}
\end{aligned}$$

$$\begin{aligned}
=& x \int_{-d_-\sqrt{T-t} - \sigma(T-t)}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
&- K e^{-r(T-t)} \int_{-d_-\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2(T-t)}} \frac{dz}{\sqrt{2\pi(T-t)}}\\
=& x \int_{-d_- - \sigma\sqrt{T-t}}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} - K e^{-r(T-t)} \int_{d_-}^{\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}\\
=& x\left(1 - \Phi\left(-d_+\right)\right) - K e^{-r(T-t)}\left(1 - \Phi\left(-d_-\right)\right)\\
=& x \Phi\left(d_+\right) - K e^{-r(T-t)} \Phi\left(d_-\right),
\end{aligned}$$

where we used

$$1 - \Phi(a) = \Phi(-a), \quad a \in \mathbb{R}$$

□

Martingale for option pricing

We now present the second approach to option pricing which consists of using the stochastic Martingale process theory. The latter allows us to calculate the price of an option through the calculation of a conditional expectation and to determine a portfolio capable of replicating the option.

4.1 Itō Integral properties

Recall that a process $(X_t)_{t\in\mathbb{R}_+}$ is said to be Martingale with respect to filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ if

$$\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] = X_s, \quad 0 \le s \le t$$

With the following proposition, we shall prove that the Itō integral is a Martingale with respect to the filtration given by the Brownian Motion $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$.

Proposizione 4.1. *The stochastic integral* $\left(\int_0^t u_s dB_s\right)_{t\in\mathbb{R}_+}$ *of an integrable square adapted process* $u \in L^2_{ad}\left(\Omega \times \mathbb{R}_+\right)$ *is a Martingale, i.e.,*

$$\mathbb{E}\left[\int_0^t u_\tau dB_\tau \mid \mathcal{F}_s\right] = \int_0^s u_\tau dB_\tau, \quad 0 \le s \le t$$

In fact, for each $u \in L^2_{ad}\left(\Omega \times \mathbb{R}_+\right)$ *we have*

$$\mathbb{E}\left[\int_0^\infty u_s dB_s \mid \mathcal{F}_t\right] = \int_0^t u_s dB_s, \quad t \in \mathbb{R}_+$$

in particular $\int_0^t u_s dB_s$ *is* $\mathcal{F}_t$*-measurable,* $t \in \mathbb{R}_+$.

Proof. To prove the statement, it is sufficient to prove it for simple processes and then extend the result to the general case. □

We observe that every centred stochastic process with independent increments is a Martingale:

$$\begin{aligned}
\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[X_t - X_s + X_s \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[X_t - X_s \mid \mathcal{F}_s\right] + \mathbb{E}\left[X_s \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[X_t - X_s\right] + X_s \\
&= X_s, \quad 0 \le s \le t
\end{aligned}$$

Therefore the Standard Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ is a Martingale.

4.2 Risk-neutral Measure

Recall that a probability measure is said to be risk-neutral if with respect to that measure $(X_t)_{t\in\mathbb{R}_+} = (e^{-rt}S_t)_{t\in\mathbb{R}_+}$ is a Martingale.

We observe that given

$$X_t = X_0 e^{(\mu-r)t+\sigma B_t-\sigma^2 t/2}$$

if $\mu = r$ we have

$$\begin{aligned}
\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[X_0 e^{\sigma B_t-\sigma^2 t/2} \mid \mathcal{F}_s\right] \\
&= X_0 e^{-\sigma^2 t/2}\mathbb{E}\left[e^{\sigma B_t} \mid \mathcal{F}_s\right] \\
&= X_0 e^{-\sigma^2 t/2}\mathbb{E}\left[e^{\sigma(B_t-B_s)+\sigma B_s} \mid \mathcal{F}_s\right] \\
&= X_0 e^{-\sigma^2 t/2+\sigma B_s}\mathbb{E}\left[e^{\sigma(B_t-B_s)} \mid \mathcal{F}_s\right] \\
&= X_0 e^{-\sigma^2 t/2+\sigma B_s}\mathbb{E}\left[e^{\sigma(B_t-B_s)}\right] \\
&= X_0 e^{-\sigma^2 t/2+\sigma B_s} e^{\sigma^2(t-s)/2} \\
&= X_0 e^{\sigma B_s-\sigma^2 s/2} \\
&= X_s, \quad 0 \le s \le t.
\end{aligned}$$

Therefore in this case $\mathbb{Q} = \mathbb{P}$ is a risk-neutral measure.

In this section, we are going to construct a risk-neutral probability measure that also fits the case $\mu \neq r$ and to do so we will exploit Girsanov's Theorem.

We observe that the relationship

$$dX_t = X_t\left((\mu-r)dt + \sigma dB_t\right)$$

can be rewritten as

$$dX_t = \sigma X_t d\tilde{B}_t,$$

where

$$\tilde{B}_t := \frac{\mu-r}{\sigma}t + B_t, \quad t \in \mathbb{R}_+.$$

So, finding a Risk-neutral measure is the same as finding a measure where $\left(\tilde{B}_t\right)_{t\in\mathbb{R}_+}$ is a standard Brownian motion. In fact, if we consider the Brownian motion $\nu t + B_t$ this is no longer centred,

$$\mathbb{E}\left[\nu t + B_t\right] = \nu t + \mathbb{E}\left[B_t\right] = \nu t \neq 0$$

To make it so, we can change the values of $p, q \in [0,1]$, , respectively probability of success and failure of Brownian motion, so that

$$\mathbb{E}\left[\nu t + B_t\right] = 0$$

is verified.

4.3 Girsanov Theorem and Measure Change

Given a probability measure $\mathbb{Q}$ on Ω, with the following notation

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = F$$

we indicate that the probability measure $\mathbb{Q}$ has density F with respect to $\mathbb{P}$. This is equivalent to saying that

$$\int_\Omega \xi(\omega) d\mathbb{Q}(\omega) = \int_\Omega F(\omega)\xi(\omega) d\mathbb{P}(\omega),$$

or using a more compact notation

$$\mathbb{E}^{\mathbb{Q}}[\xi] = \mathbb{E}^{\mathbb{P}}[F\xi].$$

We will also say that $\mathbb{Q}$ is equivalent to $\mathbb{P}$ if $F > 0$ $\mathbb{P}$-almost certainly. We are now ready to state Girsanov's theorem and apply it to our financial model.

Theorem 4.2. *Let $(\psi_t)_{t\in[0,T]}$ be an adapted process satisfying the Novikov integrability condition*

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\psi_t|^2 \, dt\right)\right] < \infty$$

Let $\mathbb{Q}$ also be the probability measure defined by the following relation

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \psi_s dB_s - \frac{1}{2}\int_0^T \psi_s^2 ds\right).$$

then

$$\hat{B}_t := B_t + \int_0^t \psi_s ds, \quad t \in [0, T],$$

is a Standard Brownian Motion with respect to $\mathbb{Q}$.

This theorem applied to

$$\psi_t := \frac{\mu - r}{\sigma},$$

shows that

$$\tilde{B}_t := \frac{\mu - r}{\sigma} t + B_t, \quad t \in \mathbb{R}_+,$$

is a Standard Brownian Motion with respect to the probability measure $\mathbb{Q}$ defined by the relation

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu - r}{\sigma} B_t - \frac{(\mu - r)^2}{2\sigma^2} t\right).$$

Hence the process given by the relationship

$$\frac{dX_t}{X_t} = (\mu - r)dt + \sigma dB_t = \sigma d\tilde{B}_t, \quad t \in \mathbb{R}_+,$$

is a Martingale with respect to the probability measure $\mathbb{Q}$ and is therefore risk-neutral. We observe that in accordance with what was seen above $\mathbb{P} = \mathbb{Q}$ when $\mu = r$.

4.4 Pricing an option using Martingale Theory

The objective of this section will be to recover the solution of the Black-Scholes partial derivative equation derived earlier, but using conditional expectation and Martingale theory.

Recall that a market is without arbitrage opportunities if there is at least one riskneutral probability measure $\mathbb{Q}$ and that this corresponds to proving that the stochastic process

$$X_t := e^{-rt} S_t, \quad t \in \mathbb{R}_+,$$

is a Martingale with respect to $\mathbb{Q}$.

Thanks to the above, in the case where the process $(X_t)_{s\in[t,\infty)}$ satisfies the equation

$$dX_t = (\mu - r)X_t dt + \sigma X_t dB_t = \sigma X_t d\tilde{B}_t, \quad t \in \mathbb{R}_+$$

we have that

$$X_t = S_0 e^{(\mu - r)t + \sigma B_t - \sigma^2 t/2}, \quad t \in \mathbb{R}_+,$$

is a Martingale with respect to $\mathbb{Q}$ and therefore the discounted value $\widetilde{V}_t$ of a self-financing portfolio defined by the relation

$$\begin{aligned} \widetilde{V}_t &= \widetilde{V}_0 + \int_0^t \xi_u dX_u \\ &= \widetilde{V}_0 + \sigma \int_0^t \xi_u X_u d\widetilde{B}_u, \quad t \in \mathbb{R}_+, \end{aligned}$$

turns out to be a Martingale with respect to $\mathbb{Q}$.

Henceforth, we will call the arbitrage price at time t the value V_t of a portfolio $(\xi_t)_{t\in[0,T]}$ capable of replicating the payoff of an option C and we will denote it by $\pi_t(C)$ as it will correspond to the option price at time t.

Proposizione 4.3. *Let $(\xi_t, \eta_t)_{t\in[0,T]}$ be a portfolio strategy whose value is defined by the relation*

$$V_t = \eta_t A_t + \xi_t S_t, \quad t \in [0, T],$$

and let C be the payoff of the option that the latter replicates. Let us assume that the following *statements hold:*

1. $(\xi_t, \eta_t)_{t\in[0,T]}$ *is Self-financing*

2. $(\xi_t, \eta_t)_{t\in[0,T]}$ *replicates the option* C

Then the arbitrage price of option C is given by the relation C *is given by the relation*

$$V_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[C \mid \mathcal{F}_t\right], \quad 0 \leq t \leq T,$$

Proof. Since the portfolio strategy $(\xi_t, \eta_t)_{t\in\mathbb{R}_+}$ is Self-financing, using Lemma 2.2 we have

$$\widetilde{V}_t = \widetilde{V}_0 + \sigma\int_0^t \xi_u X_u d\tilde{B}_u, \quad t \in \mathbb{R}_+,$$

which thanks to previous observations we know is a Martingale with respect to $\mathbb{Q}$ so,

$$\begin{aligned}\widetilde{V}_t &= \mathbb{E}^{\mathbb{Q}}\left[\widetilde{V}_T \mid \mathcal{F}_t\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[V_T \mid \mathcal{F}_t\right] \\ &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[C \mid \mathcal{F}_t\right],\end{aligned}$$

implying

$$V_t = e^{rt}\widetilde{V}_t = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[C \mid \mathcal{F}_t\right]$$

□

Let us now try to recover the solution of the Black-Scholes model through Martingale theory.

Let us start with the following lemma which will be useful in calculating the option price.

Lemma 4.4. *Let* X *be a centred Gaussian variable with variance* σ^2*, in this case we have*

$$\mathbb{E}\left[\left(e^{m+X} - K\right)^+\right] = e^{m+\frac{\sigma^2}{2}}\Phi((\sigma^2 + m - \log K)/\sigma) - K\Phi((m - \log K)/\sigma).$$

Proof.

$$
\begin{aligned}
&\mathbb{E}\left[\left(e^{m+X}-K\right)^{+}\right]=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty}\left(e^{m+x}-K\right)^{+}e^{-\frac{x^2}{2\sigma^2}}dx\\
&=\frac{1}{\sqrt{2\pi\sigma^2}}\int_{-m+\log K}^{\infty}\left(e^{m+x}-K\right)e^{-\frac{x^2}{2\sigma^2}}dx\\
&=\frac{e^m}{\sqrt{2\pi\sigma^2}}\int_{-m+\log K}^{\infty}e^{x-\frac{x^2}{2\sigma^2}}dx-\frac{K}{\sqrt{2\pi\sigma^2}}\int_{-m+\log K}^{\infty}e^{-\frac{x^2}{2\sigma^2}}dx\\
&=\frac{e^{m+\frac{\sigma^2}{2}}}{\sqrt{2\pi\sigma^2}}\int_{-m+\log K}^{\infty}e^{-\frac{\left(\sigma^2-x\right)^2}{2\sigma^2}}dx-\frac{K}{\sqrt{2\pi}}\int_{(-m+\log K)/\sigma}^{\infty}e^{-x^2/2}dx\\
&=\frac{e^{m+\frac{\sigma^2}{2}}}{\sqrt{2\pi\sigma^2}}\int_{-\sigma^2-m+\log K}^{\infty}e^{-\frac{x^2}{2\sigma^2}}dx-K\Phi((m-\log K)/\sigma)\\
&=e^{m+\frac{\sigma^2}{2}}\Phi((\sigma^2+m-\log K)/\sigma)-K\Phi((m-\log K)/\sigma)
\end{aligned}
$$

□

We are now ready to calculate the option price using the theory just presented and find the same result as with the Black-Scholes theory.

Proposizione 4.5. *The price at time t of a European Call option with strike K and maturity T is given by the relation*

$$
C\left(t,S_t\right)=S_t\Phi\left(d_+\right)-Ke^{-r(T-t)}\Phi\left(d_-\right),\quad t\in[0,T]
$$

Proof. Using the relation

$$
S_T=S_te^{r(T-t)+\sigma\left(\tilde{B}_T-\tilde{B}_t\right)-\sigma^2(T-t)/2},\quad t\in[0,T].
$$

thanks to Proposition 4.4 we have

$$
\begin{aligned}
&\pi_t(C)=V_t=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[C\mid\mathcal{F}_t\right]\\
&=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[(S_T-K)^{+}\mid\mathcal{F}_t\right]\\
&=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(S_te^{r(T-t)+\sigma\left(\tilde{B}_T-\tilde{B}_t\right)-\sigma^2(T-t)/2}-K\right)^{+}\mid\mathcal{F}_t\right]\\
&=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(xe^{r(T-t)+\sigma\left(\tilde{B}_T-\tilde{B}_t\right)-\sigma^2(T-t)/2}-K\right)^{+}\right]_{x=S_t}\\
&=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{m(x)+X}-K\right)^{+}\right]_{x=S_t},\quad 0\leq t\leq T,
\end{aligned}
$$

where

$$
m(x)=r(T-t)-\sigma^2(T-t)/2+\log x
$$

and $X=\sigma\left(\tilde{B}_T-\tilde{B}_t\right)$ is a centred Gaussian variable with variance

$$\operatorname{Var}[X]=\operatorname{Var}\left[\sigma\left(\tilde{B}_T-\tilde{B}_t\right)\right]=\sigma^2\operatorname{Var}\left[\tilde{B}_T-\tilde{B}_t\right]=\sigma^2(T-t)$$

with respect to $\mathbb{Q}$. Using Lemma 4.5 we have

$$\begin{aligned}
V_t =& e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\left(e^{m(x)+X}-K\right)^+\right]_{x=S_t} \\
=& e^{-r(T-t)}e^{m(S_t)+\sigma^2(T-t)/2}\Phi\left(\sigma(T-t)+\left(m\left(S_t\right)-\log K\right)/\sigma\sqrt{T-t}\right) \\
& -Ke^{-r(T-t)}\Phi\left(\left(m\left(S_t\right)-\log K\right)/\sigma\sqrt{T-t}\right) \\
=& S_t\Phi\left(\left(\sigma^2(T-t)+m\left(S_t\right)-\log K\right)/\sigma\sqrt{T-t}\right)-Ke^{-r(T-t)}\Phi\left(\left(m\left(S_t\right)-\log K\right)/\sigma\sqrt{T-t}\right) \\
=& S_t\Phi\left(d_+\right)-Ke^{-r(T-t)}\Phi\left(d_-\right),
\end{aligned}$$

$0\leq t\leq T$ □

Before moving on to the volatility considerations of the model and introducing a new, more accurate model, let us analyse the discrete case using the theory presented so far now.

Discrete Case - Data Analysis

Remark. In the following, functions from the package developed for this project will be used, which can be consulted and downloaded at https://github.com/LorenzoLatini13/LabProject

We know that the formula for the price of an option with payoff C in the case of a discrete time model is

$$\pi_t(C)=\frac{1}{(1+r)^{N-t}}\mathbb{E}^{\mathbb{Q}}\left[C\mid\mathcal{F}_t\right],\quad t=0,1,\ldots,N,$$

In our analysis we will study European options and Asian options. We will use two approaches:

1. The first will be a direct approach by deterministic calculation of the option price using a closed formula for conditional expectation.

2. The second will be an approach to approximate the option price using the Monte Carlo method.

5.1 Monte Carlo method

The Monte Carlo method is a broad class of computational methods based on random sampling to obtain numerical results. It can be useful for overcoming computational problems associated with exact tests (e.g. methods based on binomial distribution and combinatorial calculus, which generate an excessive number of permutations for large samples).

This method exploits the "Law of Large Numbers", according to which given a succession of random independent and identically distributed variables $X_1, X_2, \dots, X_M$ with mean (finite) μ, if we consider the sample average $\bar{X}_M = \dfrac{X_1 + X_2 + \cdots + X_M}{M}$, then this relation is valid:

$$\mathrm{P}\left(\lim_{M\to\infty} \bar{X}_M = \mu\right) = 1$$

i.e. the sample mean estimator converges $\mathbb{P}$-almost certainly to the common expected of X_i.

5.2 Deterministic calculation and Monte Carlo approximation

As a first step, it is good to check that the package functions written for both approaches lead to the same results in the case where the difficulty, which in our case corresponds to the time instants $t = 0, ..., N$, is low. Let us recall that M denotes the number of random variables used in the Monte Carlo method.

Take for example the case $N = 4$ and choose $M = 100000$ and we obtain the following results

Option Pricing

Option type	Deterministico	Monte Carlo
European Call Option	0.4268407	0.4266445
European Put Option	0.1206424	0.1204912
Asian Call Option	0.1968680	0.1974798
Asian Put Option	0.0689566	0.0693193

Option Hedging

Option type	Deterministico	Monte Carlo
European Call Option	$(-0.27, 0.494)$	$(-0.27, 0.495)$
European Put Option	$(0.235, -0.19)$	$(0.236, -0.19)$
Asian Call Option	$(-0.367, 0.791)$	$(-0.367, 0.791)$
Asian Put Option	$(0.309, -0.209)$	$(0.309, -0.209)$

Let us now see how the difficulty of the problem to be addressed affects both approaches.

5.3 Difficulty N - European Call Options

Recall that in the case of a European Call option where $C = \left(S_N^{(i)} - K\right)^+$ the following relation is valid:

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \sum_{j=0}^{N-t} \left(\begin{array}{c} N-t \\ j \end{array} \right) (p^*)^j (1-p^*)^{N-t-j} f\left(S_t(1+b)^j(1+a)^{N-t-j}\right)$$

In this case, the problems caused by N in calculating this quantity are twofold:

1. Achieving machine precision, which in the case of R coincides with $double.xmin = 2.225074e-308

2. Reaching the maximum storable digit, which in the case of R coincides with $double.xmax = 1.797693e+308

It is therefore possible to use the closed formula for values of $N \sim 100$, while it would be necessary to use the Monte Carlo method for higher values. The problem with the latter in this case is convergence, which does not occur within a reasonable timeframe. Thus in general for N too large both methods fail, however it is possible to work on improving the convergence speed of the Monte Carlo method.

5.4 Difficulty N - Asian Call Options

In the case of Asian options where $C = f(S_0, ..., S_N) = \left(\frac{1}{N+1}\sum_{t=0}^{N} S_t^{(i)} - K\right)^+$ it is still possible to find a closed formula for calculating $\pi_t(C)$. In particular, if we indicate with p^* the number of occurrences of b in a generic random string $(a, ..., b, b, a) \in \{a, b\}^{(N-t)}$ we have:

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \sum_{(a,a,...,b,a)\in\{a,b\}^{(N-t)}} (p^*)^j (1-p^*)^{N-t-j} f(S_1, ..., S_N)$$

However, we observe that 2^{N-t} terms appear in the summation, so the complexity of the problem grows exponentially as N increases, which leads us to prefer the Monte Carlo method in the case of Asian Options.

This behaviour can also be observed empirically, in fact it is evident how the reproduction time increases exponentially:

```
> tic()
> ECallprice_det = ECallOption_pricingfunction_det(1,20,1.05,1/2,-0.3,0.5,0.1,0.99)
> toc()
6.59 sec elapsed
> tic()
> ECallprice_det = ECallOption_pricingfunction_det(1,21,1.05,1/2,-0.3,0.5,0.1,0.99)
> toc()
11.64 sec elapsed
> tic()
> ECallprice_det = ECallOption_pricingfunction_det(1,22,1.05,1/2,-0.3,0.5,0.1,0.99)
> toc()
26.8 sec elapsed
```

Using the Monte Carlo method, on the other hand, it is possible to generate results in a much shorter time and with excellent precision. For example, with a difficulty $N = 40$ and a number of independent variables $M = 100000$, $M = 1000000$ oppure $M = 1000000$, the following results are obtained:

```
> # N = 40 - Monte Carlo is faster and has also a good precision
>
> tic()
> ECallprice_rand = ECallOption_pricingfunction_rand(1,40,1.05,1/2,-0.3,0.5,0.1,0.99,100000)
> print(ECallprice_rand)
[1] 0.2749529
> toc()
1.46 sec elapsed
> tic()
> ECallprice_rand = ECallOption_pricingfunction_rand(1,40,1.05,1/2,-0.3,0.5,0.1,0.99,1000000)
> print(ECallprice_rand)
[1] 0.2607031
> toc()
19.71 sec elapsed
> tic()
> ECallprice_rand = ECallOption_pricingfunction_rand(1,40,1.05,1/2,-0.3,0.5,0.1,0.99,10000000)
> print(ECallprice_rand)
[1] 0.2616572
> toc()
198.68 sec elapsed
```

5.5 Convergence and Monte Carlo Error

Recall that given a succession of random variables $X_1, X_2, \ldots, X_M$ independent and identically distributed with (finite) mean μ, if we consider the sample mean

$$\bar{X}_M = \frac{X_1 + X_2 + \cdots + X_M}{M}$$

then the following relationship applies:

$$\mathrm{P}\left(\lim_{M\to\infty} \bar{X}_M = \mu\right) = 1$$

i.e. the sample mean estimator converges $\mathbb{P}$-almost certainly to the common expected value of X_i.

In our case, however, there is a further aspect to consider. In fact, for M fixed, the value of $\bar{X}_M = \frac{X_1+X_2+\cdots+X_M}{M}$ fluctuates around an average value that as M varies we expect to be closer and closer to μ. To study this phenomenon, we can therefore study the behaviour of

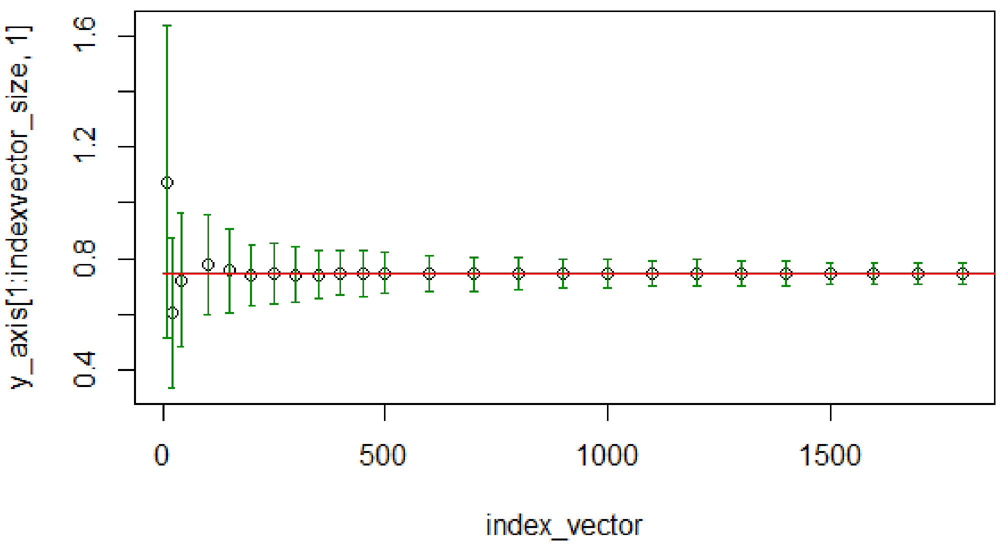
$$\hat{X}_T = \left(\frac{(\frac{X_1+X_2+\cdots+X_M}{M})_1 + \cdots + (\frac{X_1+X_2+\cdots+X_M}{M})_T}{T}\right)$$

and expect there to be convergence at μ of the order of $\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{M}}$. Clearly, as T and M grow, the algorithm's execution time increases and it is reasonable to choose $M = T$ as convergence always occurs in relation to the worst value-

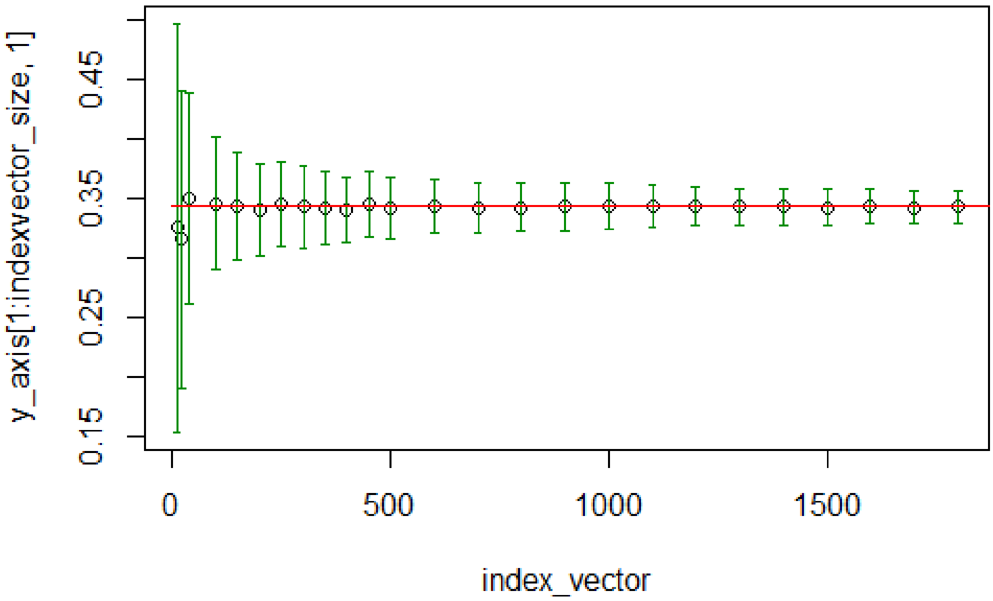
Such an analysis leads us to construct such an algorithm:

```
# European Call Options - N = 10
tic()
N = 11 # Difficoltà
index_vector = c(10,20,40,100,150,200,250,300,350,400,450,500,600,700,800,900,1000,1100,1200,1300,1400,1500,1600,1700,1800)
# vettore degli M per cui eseguiamo il Monte Carlo
indexvector_size = size(index_vector,2) # Numero di indici
y_axis = zeros(indexvector_size,2) # Tabella dei valori attesi e delle deviazioni standard
M_prime = 0 # Contatore
for (M in index_vector) {
  M_prime = M_prime + 1
  vector_of_values = rep(0,M) # Inizializziamo il vettore dei risultati del Monte Carlo per M fissato
  for (i in 1:M) { # T = M
    vector_of_values[i] = VCallOption_pricingfunction_rand(1,N,1.05,1/2,-0.3,0.5,0.1,0.99,M) # Inseriamo i risultati nel vettore
  }
  y_axis[M_prime,1] = mean(vector_of_values) # Compiliamo la tabella dei valori attesi
  y_axis[M_prime,2] = sd(vector_of_values) # Compiliamo la tabella delle deviazioni standard
}

out_lw=y_axis[1:indexvector_size,1]-y_axis[1:indexvector_size,2]
out_up=y_axis[1:indexvector_size,1]+y_axis[1:indexvector_size,2]

ymin=min(out_lw) # estremo inferiore dell'intervallo a 1 deviazione standard
ymax=max(out_up) # estremo superiore dell'intervallo a 1 deviazione standard
plot(index_vector,y_axis[1:indexvector_size,1],ylim=c(ymin,ymax)) # Grafico dei valori attesi del Monte Carlo al variare di M
arrows(index_vector,out_lw,index_vector,out_up,length=0.02,angle=90,code=3,col="green4") # Deviazioni standard del Monte Carlo al variare di M
E = VCallprice_det = VCallOption_pricingfunction_det(1,11,1.05,1/2,-0.3,0.5,0.1,0.99) # Valore Reale dell'opzione
lines(rep(E,2000),col="red") # Linea del valore atteso reale
toc()
```

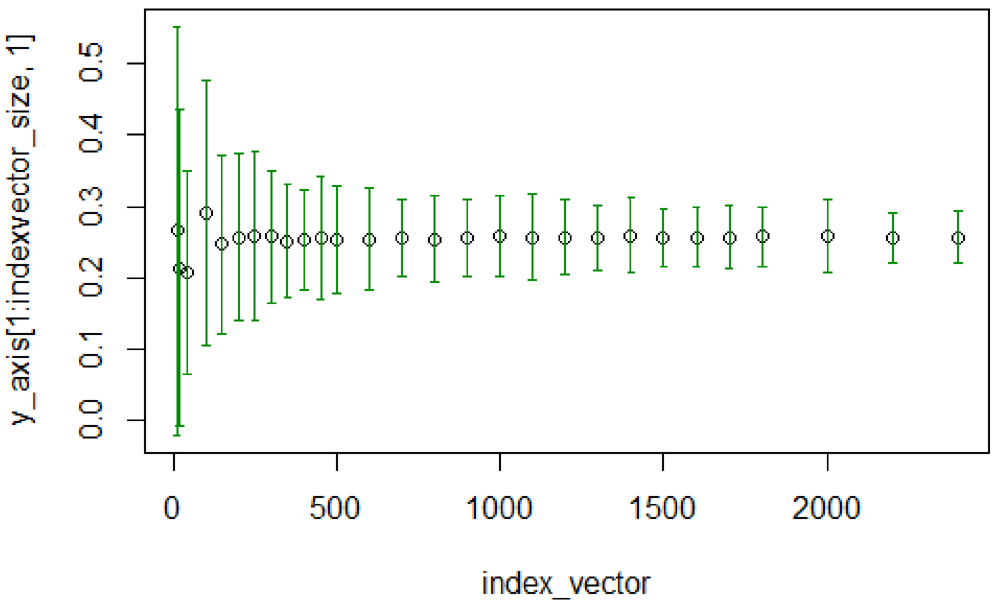
which, once compiled, produces the following graphs:



Pricing of a European Call option where the difficulty is $N = 10$



Pricing of an Asian Call option where the difficulty is $N = 10$



Pricing of an Asian Call option where the difficulty is $N = 40$

1. On the X-axis we have the values of M

2. . On the Y-axis, we have the expected Monte Carlo values corresponding to M

3. The green arrows represent the Monte Carlo standard deviations corresponding to M

4. The red line in the graph represents the price calculated through the deterministic formula when possible

Two important pieces of information also emerge from the graph:

1. The Monte Carlo method converges to the real option price

2. The Monte Carlo dispersion around this value decreases with M growing

5.6 Conclusion - When is Monte Carlo worth it?

In our analysis, the deterministic approach is convenient for the calculation of the option price only in the case of European Call/Put options, as the closed formula can be implemented quickly.

Sometimes, as in the case of Asian options, it is possible to find a closed formula, but the calculation of the latter requires time that increases exponentially as the difficulty N increases. In this case, therefore, using approximation methods such as Monte Carlo is necessary.

We have seen that in general the Monte Carlo method converges to the desired value with speed $\frac{1}{\sqrt{T}} + \frac{1}{\sqrt{M}}$, but as T and M increases, the difficulty in compiling the algoritms for implenting this approach increases too.

To conclude, we can say that in general it will not always be possible to determine a closed formula for calculating the price of an option according to the relation

$$\pi_t(C) = \frac{1}{(1+r)^{N-t}} \mathbb{E}^{\mathbb{Q}} \left[C \mid \mathcal{F}_t \right], \quad t = 0, 1, \dots, N,$$

and it is therefore clear that in this case approximation methods such as Monte Carlo are the only solution and it is therefore worth optimising the speed of convergence.

Volatility Estimation - Smile Curve

In the Black-Scholes model, the volatility parameter σ was considered constant. In reality, this parameter varies with time and depends on factors external and internal to the model.

Referring to the famous article [3] Rama Cont *Empirical properties of asset returns: stylized facts and statistical issues* 28 October 2000, we will present an empirical analysis of market data in order to make considerations on the σ parameter.

Estimating this parameter may be very difficult, but it is essential to determine models that are as close to reality as possible.

6.1 Historical volatility

A first example of an estimate for the volatility parameter is the historical volatility calculated as follows:

$$\hat{\sigma}_N^2 := \frac{1}{N-1} \sum_{k=0}^{N-1} \frac{1}{t_{k+1} - t_k} \left(\frac{S_{t_k+1} - S_{t_k}}{S_{t_k}} - \hat{\mu}_N \left(t_{k+1} - t_k \right) \right)^2 .$$

Clearly, this estimate is based on historical data and requires a large number of examples to be validated.

6.2 Implied volatility

A crucial factor when it comes to volatility in finance is that we cannot measure it directly. In fact, it must be estimated using the data available to us such as the price of the volatile asset, the distance to the strike price and the amount of time to bring the option to expiry.

This data can be entered into the Black-Scholes formula presented above to obtain the following equality between the price for the European call option obtained with the Black-Scholes model $C_{\mathrm{BS}}(t, S_t; K, T; I)$ and the actual market price C_{mkt}:

$$C_{\mathrm{BS}}(t, S_t; K, T; I) = C_{\mathrm{mkt}}.$$

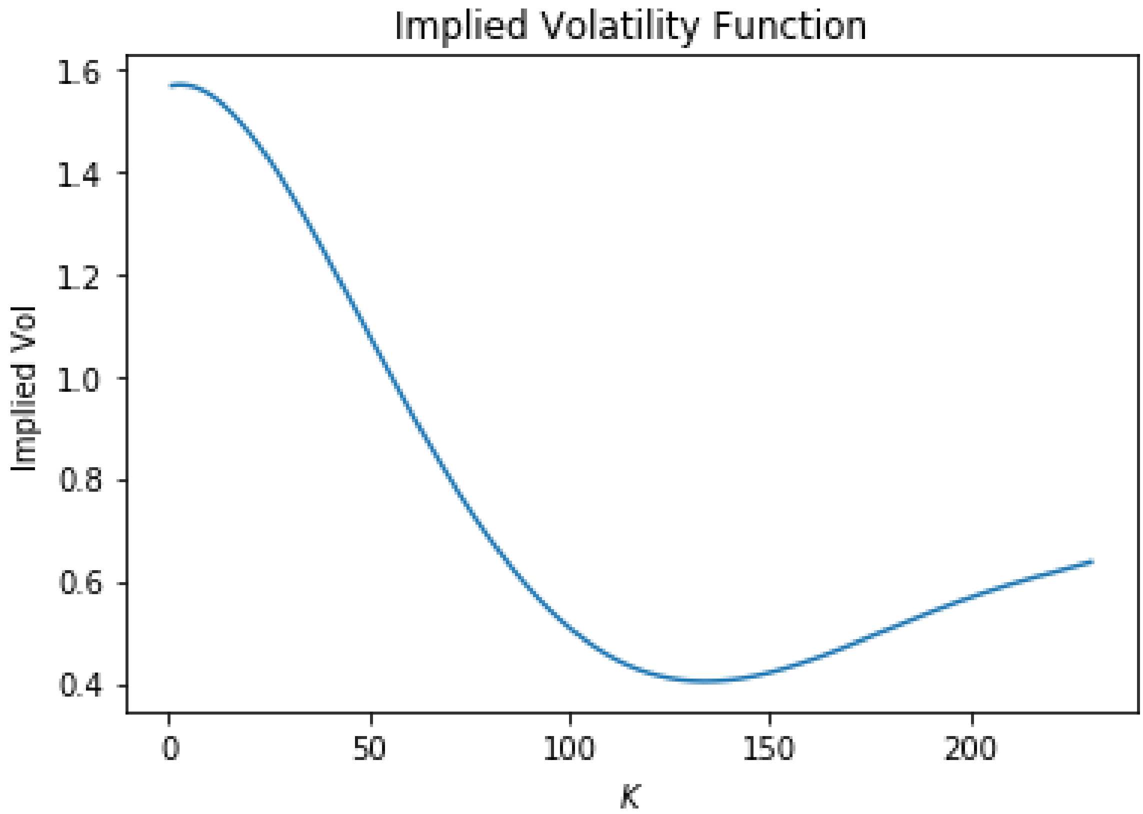
We denote the implied volatility to be calculated using I while we use t and x to denote the present time instant and the asset price at time t, respectively.

With this definition, we are able to plot volatility as a function of distance from the strike price. This brings us to the famous Smile curve, known from option pricing theory, which makes us realise the importance of not keeping the volatility constant in our models.

6.3 Smile Curve

By setting a time t and a price for the volatile asset x, varying the strike price K and calculating the implied volatility as seen above, it is empirically verified that the volatility is the higher the distance from the strike price.

In addition, an asymmetry is evident due to the fact that volatility is higher when the strike price of our European Call option is higher than the current market price and therefore the option is "Out of the money".



Such behaviour, combined with other phenomena such as the decreasing volatility as the option's maturity approaches, leads us to prefer models in which the volatility varies over time.

6.4 Data and Statistical Properties

We now present an empirical analysis of the data, without assuming that it belongs to any kind of statistical-mathematical model, in order to better understand the behaviour of the financial products we want to replicate

For many years, it was thought that using implied volatility was sufficient for an accurate estimate of option pricing. However, empirical studies show that such an approach is insufficient to replicate the pricing behaviour of an option and that there is some information, not present in the historical data, that allows the model to be calibrated more accurately

From the empirical study of the data available to us to date, certain statistical characteristics common to different, even unrelated assets emerge.

1. **Absence of autocorrelation**: autocorrelation in the returns of a volatile asset is absent except for small autocorrelation phenomena when considering "intraday" time intervals ($\simeq 20min$).

2. **Heavy tails**: there are tail events that have a major impact on the price of a volatile

asset and therefore cannot be ignored. In particular, this rules out the possibility of modelling through distributions with infinite moments.

3. **Profit and Loss asymmetry**: there are large drops in the price of securities that are not matched by equal price increases. This also makes it difficult to estimate the effects of the tails of distributions.

4. **Intermittency of returns**: there is a strong intermittency in the returns of volatile assets. This behaviour is characterised by the presence of large price bursts occurring within very short periods of time.

5. **Volatility Clustering**: a strong autocorrelation of volatility emerges that persists over several days. This justifies the fact that high volatility events are clustered.

6. **Leverage**: there appears to be a negative correlation between the returns of a volatile asset and its volatility. This leads investors to demand more guarantees on assets with higher volatility.

Knowing this type of data leads us to search for models with the following characteristics:

1. A central value on which the distribution is concentrated as in the Gaussian case.

2. A scaling parameter measuring the dispersion of the distribution, in our case volatility.

3. A parameter measuring the heaviness of the distribution tails.

4. An asymmetry in the tails such that the left tail behaves differently from the right tail.

6.5 Volatility Clustering

Before presenting a model for describing the behaviour of volatility, let us go deep into the phenomenon of volatility clustering.

The absence of autocorrelation on the returns of a volatile asset supports models in which returns are considered independent random variables.

However, the absence of autocorrelation does not imply independence of the variables in fact the data tell us that this assumption is not verified.

This effect is an expression of the phenomenon of volatility clustering: large price changes are followed by equally large price changes. As a consequence it is clearly incorrect to consider the return of an asset as a random walk as in the BlackScholes model.

The existence of this dependence, as opposed to the absence of autocorrelation of returns, is often interpreted to mean that there is a correlation in the volatility of returns rather than in the returns themselves.

Our task will therefore be to integrate the presence of this phenomenon into our model. This will be pursued by introducing volatility models stochastic.

Stochastic Volatility

A first attempt to correct the Black-Scholes model is to assume that volatility is a positive deterministic function of time t and the price of the volatile asset X_t, i.e. $\sigma = \sigma(t, X_t)$. The stochastic differential equation modelling the price of the volatile asset thus becomes

$$dX_t = \mu X_t dt + \sigma(t, X_t) X_t dB_t^x$$

and can be solved, as done above, by exploiting the absence of arbitrage.

Here, too, there continues to be only one risk-neutral measure and therefore the market is still complete.

The reason for such a choice is that to have an effect similar to that shown by the Smile Curve, σ must depend on both x and t. However, in the case of a deterministic function $\sigma = \sigma(t, X_t)$ what is obtained is a perfect negative correlation with the asset price. As mentioned earlier, however, empirical studies show that there is no complete correlation between volatility and the underlying asset.

This shows the existence of a proper component within volatility, which leads us to consider volatility as a stochastic process σ_t.

7.1 Stochastic volatility models

The need to reformulate the model as

$$dX_t = \mu X_t dt + \sigma_t X_t dB_t^x,$$

where σ_t is a positive stochastic process that is not perfectly correlated with Brownian motion $(B_t)_{t\in\mathbb{R}}$ and thus possesses an independent random component. Such a model is called stochastic volatility model.

7.2 Mean-reverting models

Typically one considers volatility as an Itō process that satisfies a differential equation in which a second Brownian motion appears.

There are several models constructed in this way that share a characteristic called mean-reversion. This characteristic is described by the tendency of a process to return to its mean value defined by its distribution.

Let us assume $\sigma_t = f(v_t)$, where f is a positive function and v_t the process describing the variance behaviour of the volatile asset. A mean-reverting model is characterised by a stochastic differential equation for the process v_t of the type

$$dv_t = \kappa(\theta - v_t)\, dt + \cdots dB_t^v,$$

Where $(B_t^v)_{t\geq 0}$ is a Brownian motion related to $(B_t^x)_{t\geq 0}$, κ is called Mean-reversion rate θ is the mean value of v.

An example of mean-reverting models are the Feller or Cox-Ingersoll-Ross (CIR) processes defined by the following stochastic differential equation for the process v_t

$$dv_t = \kappa(\theta - v_t)\, dt + \epsilon\sqrt{v_t} dB_t^v.$$

In these models, the second Brownian component $(B_t^v)_{t\geq 0}$ is typically related to the Brownian motion $(B_t^x)_{t\geq 0}$ of the price of the volatile asset.

For each $t \geq 0$ we shall denote by $\rho \in [-1, 1]$ the correlation coefficient between B_t^x and B_t^v which will typically be negative.

It is observed that the distribution of these patterns has the characteristics we expected, namely:

1. A concentration of the distribution around the mean value due to Mean- reversion
2. An increased heaviness of the tails due to the impact of 'bursts' caused by flying
3. An asymmetry between the left and right tails due to negative correlation

The success of this theory is due to the fact that stochastic volatility models for European option pricing reproduce the Smile curve and thus come close to the empirical reality of the data.

7.3 Conclusion on Stochastic Volatility Models

To conclude this section, we can summarise what has been said in the following positive aspects concerning stochastic volatility models:

1. Directly modelling the random behaviour of volatility
2. They reproduce more realistic distributions with heavier tails than classical Gaussian distributions
3. They highlight the characteristic asymmetry of volatility and in particular replicate the Smile curve

However, we should not forget that stochastic volatility models do not guarantee market-completeness and therefore not all derivative products can be replicated with these strategies, leading investors to demand more guarantees.

Heston model - Model Calibration

A special case of this type of model is the Heston model where $\rho \neq 0$ and $f(v) = \sqrt{v}$.

In this chapter, we will deal with the study and calibration of the Heston model. Finally, we will study various techniques for discretizing the model and apply them to calculate the price of a European Call option.

Heston's model approximates the Smile curve with very good accuracy, however, it does not guarantee the non-negativity of the variance, which can sometimes be 0, making the process deterministic in such cases. However, as it is a mean-reverting process, the variance only remains equal to 0 for short instants of time, in fact causing a reasonable error in the model.

Let us now give a formal definition of Heston's model:

Definition 8.1. The Heston model is described by the following differential equations stochastic:

$$dS_t = \mu S_t dt + \sqrt{v_t} S_t dB_t^s,$$
$$dv_t = \kappa\left(\theta - v_t\right) dt + \varepsilon\sqrt{v_t} dB_t^v.$$

where

$$dB_t^s = dW_t^s \quad \& \quad dB_t^v = \rho dW_t^s + \sqrt{1-\rho^2} dW_t^v$$

We observe that v_t, defined by the parameters κ, θ, and ε, is a CIR process.

In this model $\kappa, \theta, \varepsilon \geq 0$ while W_t^s and W_t^v are two independent Brownian motions. For each $t \geq 0$ we shall denote by $\rho \in [-1, 1]$ the correlation coefficient between the stochastic component of the volatile asset process and that of the volatility process.

Recall that S_t indicates the price of the underlying asset at time t, and v_t the variance process.

The parameter κ measures the speed with which v_t returns to its mean value θ, ε represents the volatility of the volatility and μ is called the Drift term and represents the intensity with which the value of the volatile asset process changes.

8.1 Feller Condition

Referring back to Chapter 5 of the book [4], let us analyse the conditions for the stochastic process v_t to be greater than zero assuming $v_0 > 0$.

8.1.1 The general case

Suppose we have a process with values in $I = (\ell, r)$ with $-\infty \leq \ell < r \leq \infty$, defined by the following stochastic differential equation

$$dX_t = b\left(X_t\right) dt + \sigma\left(X_t\right) dB_t^x$$

where $b : \mathbb{R} \to \mathbb{R}$ and $\sigma : \mathbb{R} \to \mathbb{R}$ are two Borel-measurable functions. The solution to this stochastic differential equation may not exist globally but may be characterised by breakdown or blow-up phenomena at finite times.

Let us assume that the following conditions of Non-Degeneration (**ND**) and Local Integrability (**LI**) apply

$$(\mathbf{ND}) \qquad \sigma^2(x) > 0; \quad \forall x \in I$$

$$(\mathbf{LI}) \quad \forall x \in I, \exists \varepsilon > 0 \text{ t. c. } \int_{x-\varepsilon}^{x+\varepsilon} \frac{1 + |b(y)|dy}{\sigma^2(y)} < \infty$$

and we define the following scaling function

$$p(x) \triangleq \int_c^x \exp\left\{-2\int_c^\xi \frac{b(\zeta)d\zeta}{\sigma^2(\zeta)}\right\} d\xi; \quad x \in I$$

with $c \in \mathbb{R}$.

We extend p to $[-\infty, \infty]$ so that the new defined function is continuous on the extended topology of the real numbers.

We observe that p depends on c, however the purpose of this function is to determine the behaviour of X_t on the edges of the interval I and in [4] it is shown that the parameter c does not affect the analysis of this behaviour.

Let us now denote by

$$\tau = \inf\left\{t \geq 0 : X_t \notin (\ell, r)\right\}$$

the instant of time when the process X_t exits the interval I and we assume that $X_0 \in I$ so that $P[\tau > 0] = 1$.

Proposition 5.22 in [4] describes the relations between the behaviour of the function p and the stochastic process X_t at the extremes of the interval $I = (\ell, r)$, in particular the following relations apply:

1. If $p(\ell+) = -\infty$ e $p(r-) = \infty$, then

$$P[\tau = \infty] = P\left[\sup_{0 \leq t < \infty} X_t = r\right] = P\left[\inf_{0 \leq t < \infty} X_t = \ell\right] = 1.$$

2. If $p(\ell+) > -\infty$ e $p(r-) = \infty$, then

$$P\left[\lim_{t\uparrow\tau} X_t = \ell\right] = P\left[\sup_{0\le t<\tau} X_t < r\right] = 1.$$

3. If $p(\ell+) = -\infty$ e $p(r-) < \infty$, then

$$P\left[\lim_{t\uparrow\tau} X_t = r\right] = P\left[\inf_{0\le t<\tau} X_t > \ell\right] = 1.$$

4. If $p(\ell+) > -\infty$ e $p(r-) < \infty$, then

$$P\left[\lim_{t\uparrow\tau} X_t = \ell\right] = 1 - P\left[\lim_{t\uparrow\tau} X_t = r\right] = \frac{p(r-) - p(x)}{p(r-) - p(\ell+)}.$$

8.1.2 Feller's condition in Heston's model

In Heston's model, the stochastic process v_t is defined by the stochastic differential equation

$$dv_t = \kappa\left(\theta - v_t\right) dt + \varepsilon\sqrt{v_t} dB_t^v.$$

By analogy with what was presented above, we try to find the condition that guarantees the non-negativity of v_t assuming that $v_0 > 0$.

Let

$$I = (0, +\infty), \quad b(v_t) = \kappa\left(\theta - v_t\right), \quad \sigma(v_t) = \varepsilon\sqrt{v_t}, \quad \tau = \inf\left\{t \ge 0 : v_t \notin (0, +\infty)\right\}$$

then, given $c = 1$, we obtain

$$p(v) \triangleq \int_1^v \exp\left\{-2\int_1^\xi \frac{b(\zeta)d\zeta}{\sigma^2(\zeta)}\right\} d\xi; \quad v \in I.$$

Referring to proposition 5.22 in [4] above, to obtain $v_t > 0$ it is necessary $p(\ell+) = -\infty$.

This boils down to finding conditions on the parameters $\kappa, \theta, \varepsilon \ge 0$ so that

$$\lim_{v\to 0^+} p(v) = -\infty$$

Let us first simplify the term

$$\exp\left\{-2\int_1^\xi \frac{b(\zeta)d\zeta}{\sigma^2(\zeta)}\right\} d\xi$$

by carrying out the following

$$
\begin{aligned}
&\exp\left\{-2\int_1^{\xi}\frac{\kappa\left(\theta-\zeta\right)d\zeta}{\varepsilon^2\zeta}\right\}\\
=&\exp\left\{-2\Big(\frac{\kappa\theta}{\varepsilon^2}\int_1^{\xi}\frac{d\zeta}{\zeta}-\frac{\kappa}{\varepsilon^2}\int_1^{\xi}d\zeta\Big)\right\}\\
=&\exp\left\{-2\Big(\frac{\kappa\theta}{\varepsilon^2}\big[log(\zeta)\big]_1^{\xi}-\frac{\kappa}{\varepsilon^2}\big[\zeta\big]_1^{\xi}\Big)\right\}\\
=&\exp\left\{-2\Big(\frac{\kappa\theta}{\varepsilon^2}\big[log(\xi)\big]-\frac{\kappa}{\varepsilon^2}\big[\xi-1\big]\Big)\right\}\\
=&\exp\left\{\Big(\frac{2\kappa(\xi-1)-2\kappa\theta log(\xi)}{\varepsilon^2}\Big)\right\}\\
=&\exp\left\{\Big(\frac{2\kappa\xi-2\kappa\theta log(\xi)-2\kappa}{\varepsilon^2}\Big)\right\}
\end{aligned}
$$

At this point we have this chain of double implications

$$
\lim_{v\to0^+}p(v)=\lim_{v\to0^+}-\int_v^1\exp\left\{\Big(\frac{2\kappa\xi-2\kappa\theta log(\xi)-2\kappa}{\varepsilon^2}\Big)\right\}d\xi=-\infty
$$

$$
\Updownarrow
$$

$$
\lim_{v\to0^+}p(v)=\lim_{v\to0^+}\int_v^1\exp\left\{\Big(\frac{2\kappa\xi-2\kappa\theta log(\xi)-2\kappa}{\varepsilon^2}\Big)\right\}d\xi=\infty
$$

$$
\Updownarrow
$$

$$
\lim_{v\to0^+}p(v)=\lim_{v\to0^+}\int_v^1\exp\left\{\Big(\frac{2\kappa\xi-2\kappa}{\varepsilon^2}\Big)\right\}\exp\left\{\Big(\frac{-2\kappa\theta log(\xi)}{\varepsilon^2}\Big)\right\}d\xi=\infty
$$

$$
\Updownarrow
$$

$$
\lim_{v\to0^+}p(v)=\lim_{v\to0^+}\int_v^1 c(\xi)\xi^{\frac{-2\kappa\theta}{\varepsilon^2}}d\xi=\infty
$$

The last equality is valid for $\frac{-2\kappa\theta}{\varepsilon^2}\leq-1$ from which we obtain the condition of Feller for our problem, i.e. $2\kappa\theta\geq\varepsilon^2$.

If v reaches zero, the variance is zero, hence the asset price process is deterministic. Model calibration can sometimes return parameters for which such a phenomenon occurs.

In any case, as the process is mean-reverting, there is always an inversion, so the volatility is zero only for brief moments of time.

8.2 Discretization of the Heston model

In order to calculate the price of a European Call option, one possible solution is to use the Monte Carlo method to approximate

$$
C\left(S_T,t,T,K\right)=e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}\left[\max\left(S_T-K,0\right)\right],\quad 0\leq t\leq T,
$$

Where $\mathbb{Q}$ denotes the risk-neutral measure.

In order to apply it, however, it is necessary to discretize the process at continuous times, and to do so we will present two discretization models that we will later use to calibrate Heston's model.

8.2.1 Euler-Maruyama discretization scheme

The first scheme we present is the Euler-Maruyama discretization scheme. To obtain it, it is sufficient to apply the first-order Taylor expansion.

Given a function h that admits the first derivative and fixed a time interval Δ, we obtain the following relation

$$h(t+\Delta) = h(t) + \frac{\partial h}{\partial t}(t)\Delta + \mathcal{O}\left(\Delta^2\right)$$

Applying this pattern to Heston's model

$$\begin{aligned} dS_t &= \mu S_t dt + \sqrt{v_t} S_t dB_t^s, \\ dv_t &= \kappa\left(\theta - v_t\right) dt + \varepsilon\sqrt{v_t} dB_t^v, \end{aligned}$$

which we can think of using integral notation

$$\begin{aligned} S_{t+\Delta} &= S_t + \int_t^{t+\Delta} \mu S_u du + \int_t^{t+\Delta} \sqrt{v_t} S_t dB_u^s, \\ v_{t+\Delta} &= v_t + \int_t^{t+\Delta} \kappa\left(\theta - v_u\right) du + \int_t^{t+\Delta} \varepsilon\sqrt{v_u} dB_u^v \end{aligned}$$

we obtain

$$\begin{aligned} S_{t+\Delta} &= S_t + \mu S_t \Delta + \sqrt{v_t \Delta} S_t \hat{Z}_t^s \\ v_{t+\Delta} &= v_t + \kappa\left(\theta - v_t\right)\Delta + \varepsilon\sqrt{v_t\Delta}\hat{Z}_t^v \end{aligned}$$

where, knowing from the definition of Brownian motion the validity of the relation $B_{t+\Delta} - B_t = \sqrt{\Delta}\hat{Z}_t$ with $\hat{Z}_t$ Standard Gaussian, we can define

$$\hat{Z}_t^s = Z_t^s \quad \& \quad \hat{Z}_t^v = \rho Z_t^s + \sqrt{1-\rho^2} Z_t^v$$

Standard Gaussians with correlation ρ constructed from two independent Standard Gaussians Z_t^s and Z_t^v.

If the Feller's condition is not verified, it should be remembered that v_t can be negative so that calculating the root would cause problems. To avoid such a phenomenon we can replace $v_{t+\Delta}$ with $v_{t+\Delta}^+ = \max\left(v_{t+\Delta}, 0\right)$. Clearly this modification can be applied. to the Euler-Maruyama scheme as well as to all other discretization schemes.

Let us recall that in our case, the order of convergence of an algorithm measures the degree to which the expected value of the modulus of the difference between the real and numerical solution converges to 0. For the Euler-Maruyama discretization scheme, the order of convergence is $\frac{1}{2}$.

8.2.2 Milstein discretization scheme

The second scheme we present is the Milstein scheme. In this scheme, the Itō formula is used to expand the terms $\mu\left(S_t, t\right)$ and $\sigma\left(S_t, t\right)$.

Consider the generic process of the volatile asset

$$dS_t = \mu\left(S_t, t\right) dt + \sigma\left(S_t, t\right) dB_t,$$

which we can formally write as

$$S_{t+\Delta} = S_t \int_t^{t+\Delta} \mu\left(S_s, s\right) ds + \int_t^{t+\Delta} \sigma\left(S_s, s\right) dB_s.$$

Applying the Itō formula to μ and σ we obtain

$$d\mu = \left(\frac{\partial \mu\left(S_u, u\right)}{\partial S_t}\mu_t + \frac{1}{2}\frac{\partial^2 \mu\left(S_u, u\right)}{\partial S_t^2}\sigma^2\left(S_t, t\right)\right) dt + \frac{\partial \mu\left(S_u, u\right)}{\partial S_t}\sigma\left(S_t, t\right) dB_t$$

$$d\sigma = \left(\frac{\partial \sigma\left(S_u, u\right)}{\partial S_t}\mu_t + \frac{1}{2}\frac{\partial^2 \sigma\left(S_u, u\right)}{\partial S_t^2}\sigma^2\left(S_t, t\right)\right) dt + \frac{\partial \sigma\left(S_u, u\right)}{\partial S_t}\sigma\left(S_t, t\right) dB_t.$$

the derivatives with respect to t are null, since we are assuming that there is no dependence on time, i.e. $\mu = \mu(S_t)$ and $\sigma = \sigma(S_t)$. Combining these results we obtain

$$\begin{aligned} S_{t+\Delta} = S_t + &\int_t^{t+\Delta} \Big[\mu\left(S_t, t\right) + \int_t^s \big(\frac{\partial \mu\left(S_u, u\right)}{\partial S_u}\mu\left(S_u, u\right) + \frac{1}{2}\frac{\partial^2 \mu\left(S_u, u\right)}{\partial S_u^2}\sigma^2\left(S_u, u\right)\big) du \\ &+ \int_t^s \frac{\partial \mu\left(S_u, u\right)}{\partial S_t}\sigma\left(S_u, u\right) dB_u\Big] ds + \int_t^{t+\Delta} \Big[\sigma\left(S_t, t\right) + \int_t^s \big(\frac{\partial \sigma\left(S_u, u\right)}{\partial S_t}\mu\left(S_u, u\right) \\ &+ \frac{1}{2}\frac{\partial^2 \sigma\left(S_u, u\right)}{\partial S_u^2}\sigma^2\left(S_u, u\right)\big) du + \int_t^s \frac{\partial \sigma\left(S_u, u\right)}{\partial S_t}\sigma\left(S_u, u\right) dB_u\Big] dB_s \end{aligned}$$

At this point, ignoring terms of order greater than 1, we obtain

$$S_{t+\Delta} = S_t + \mu\left(S_t, t\right)\int_t^{t+\Delta} ds + \sigma\left(S_t, t\right)\int_t^{t+\Delta} dB_s + \int_t^{t+\Delta}\int_t^s \frac{\partial \sigma}{\partial S_u}\sigma\left(S_u, u\right) dB_u dB_s.$$

We observe that the term $dB_u dB_s$ remains, since $dB_u dB_s = \mathcal{O}(\Delta)$ is of order 1.

Let us now exploit the Itō lemma

$$\int_t^{t+\Delta} B_s dB_s = \frac{1}{2}B_{t+\Delta}^2 - \frac{1}{2}B_t^2 - \frac{1}{2}\Delta$$

from which by substituting in the previous formula we obtain

$$\begin{aligned}
\int_t^{t+\Delta}\int_t^s \frac{\partial\sigma}{\partial S_u}\sigma\left(S_u,u\right)dB_u dB_s &\approx \frac{\partial\sigma}{\partial S_t}\sigma\left(S_t,t\right)\int_t^{t+\Delta}\int_t^s dB_u dB_s\\
&= \frac{\partial\sigma}{\partial S_t}\sigma\left(S_t,t\right)\int_t^{t+\Delta}\left(B_s - B_t\right)dB_s\\
&= \frac{\partial\sigma}{\partial S_t}\sigma\left(S_t,t\right)\left(\left(\int_t^{t+\Delta} B_s dB_s\right) - B_t B_{t+\Delta} + B_t^2\right)\\
&= \frac{\partial\sigma}{\partial S_t}\sigma\left(S_t,t\right)\frac{1}{2}\left(\left(B_{t+\Delta}-B_t\right)^2 - \Delta\right)
\end{aligned}$$

We can then rewrite the relationship as

$$S_{t+\Delta} = S_t + S_t\mu_t\Delta + \sigma\left(S_t,t\right)\sqrt{\Delta}Z_t + \frac{1}{2}\frac{\partial\sigma}{\partial S_t}\sigma\left(S_t,t\right)\Delta\left(Z_t^2-1\right).$$

The accuracy of this model is greater than that used in the Euler-Maruyama discretization scheme due to the expansion of the μ e σ.

Applying this technique to Heston's two-dimensional model

$$\begin{aligned}
dS_t &= \mu S_t dt + \sqrt{v_t}S_t dB_t^s,\\
dv_t &= \kappa\left(\theta - v_t\right)dt + \varepsilon\sqrt{v_t}dB_t^v,
\end{aligned}$$

we obtain the following explicit relations for $S_{t+\Delta}$ and $v_{t+\Delta}$:

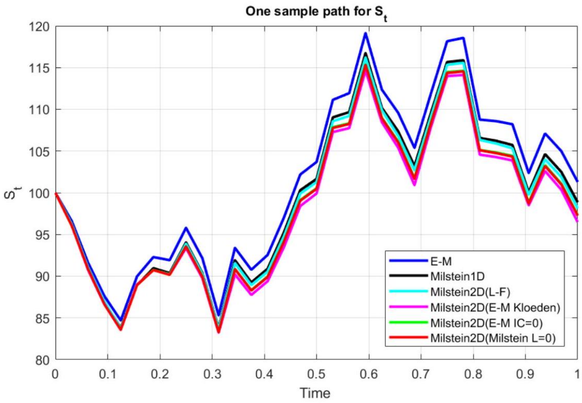
$$\begin{aligned}
S_{t+\Delta} = S_t &+ \mu S_t\Delta + S_t\sqrt{v_t\Delta}\hat{Z}_t^s + \frac{1}{2}\sqrt{1-\rho^2}\varepsilon S_t I_{(2,1);t}\\
&+\left(\frac{1}{2}S_t\Delta v_t + \frac{1}{4}\rho\varepsilon S_t\Delta\right)\left((\hat{Z}_t^s)^2 - 1\right)\\
v_{t+\Delta} = v_t &+ \kappa\left(\theta - v_t\right)\Delta + \varepsilon\sqrt{v_t\Delta}\hat{Z}_t^v\\
&+\frac{1}{4}\varepsilon^2\Delta\left((\hat{Z}_t^v)^2-1\right)
\end{aligned}$$

The term $I_{(2,1);t}$ is derived from the stochastic double integral

$$I_{(2,1)}\left[t,t+\Delta\right] = \int_t^{t+\Delta}\int_t^k dB_l^s dB_k^v$$

which, however, cannot be expressed using correlated Standard Gaussian components $\hat{Z}_t^s$ and $\hat{Z}_t^v$ as in the one-dimensional case, nor is the distribution known.

This problem is addressed in detail in [5] where an analysis of the differences, in terms of accuracy and speed of convergence, between the Euler-Maruyama scheme and the Milstein scheme is also performed. The results can be visualised in the following data extrapolated from [5].



Methods	$log(C)$	γ	$log(C)$	γ
	Asset Price S_T		Variance v_T	
E-M	2.0246	0.5206	-4.1083	0.8475
Milstein1D	1.7387	0.6058	-3.8249	1.0025
Milstein2D(*L-F*)	1.8918	0.5685	-3.8249	1.0025
Milstein2D(*E-M Kloeden*)	1.8667	0.5571	-3.8249	1.0025
Milstein2D(*E-M IC* = 0)	2.3461	1.0385	-3.8249	1.0025
Milstein2D(*Milstein L* = 0)	2.3617	1.0441	-3.8249	1.0025
Methods	Call Options V_T		Put Options V_T	
E-M	1.3511	0.4942	1.3196	0.5537
Milstein1D	1.1647	0.5993	0.9090	0.6143
Milstein2D(*L-F*)	1.1966	0.5402	1.2081	0.6022
Milstein2D(*E-M Kloeden*)	1.2164	0.5419	1.1319	0.5756
Milstein2D(*E-M IC* = 0)	1.8064	1.0463	1.4727	1.0283
Milstein2D(*Milstein L* = 0)	1.8391	1.0564	1.4667	1.0282

We report these data to justify the choice of using one-dimensional Milstein rather than the two-dimensional one. In fact, from the first image it can be seen that, as the number of time instants N increases, the difference between the various Milstein schemes implemented in [5] is less evident in terms of precision than that between the Milstein and Euler-Maruyama schemes.

Furthermore, the second table shows us that the order of convergence γ is always better in Milstein's scheme than in the Euler-Maruyama scheme, however, it is necessary to implement optimised algorithms, such as those studied in [5], in order to achieve order 1 of convergence.

In conclusion, what we are going to do in this study is to ignore the stochastic variation of v_t in S_t and thus use the one-dimensional case of Milstein's scheme.

The equations for $S_{t+\Delta}$ and $v_{t+\Delta}$ in the one-dimensional case are:

$$\begin{aligned} S_{t+\Delta} &= S_t+\mu S_t\Delta + S_t\sqrt{v_t\Delta}\hat{Z}_t^s + \frac{1}{2}S_t\Delta v_t\left((\hat{Z}_t^s)^2 - 1\right) \\ v_{t+\Delta} &= v_t+\kappa\left(\theta - v_t\right)\Delta + \varepsilon\sqrt{v_t\Delta}\hat{Z}_t^v + \frac{1}{4}\varepsilon^2\Delta\left((\hat{Z}_t^v)^2 - 1\right) \end{aligned}$$

We note that the results obtained are similar to those obtained with the Euler-Maruyama scheme except for the presence of an additional term that improves accuracy.

8.3 Calibration of the Heston model

Before using the model to calculate the price of a European option, it is necessary to know the parameters to be entered into it. These can be derived from market data; the model must therefore be calibrated.

Since the market prices are known, the idea will be to compare these prices with those obtained with the Heston model by varying the parameters and minimising the quadratic distance between the estimated and actual option price.

With respect to the risk-neutral measure $\mathbb{Q}$, the following relationship applies, which we will approximate using the Monte Carlo method

$$C\left(S_0, S_T, T, K\right) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\right]$$

Now varying the parameters $\boldsymbol{\theta} = (\kappa, \theta, \varepsilon, \mu, \rho, v_0)$ and analysing a number equal to I of European Call options, we can minimise the error function

$$\min_{\boldsymbol{\theta}\in\Theta} E(\boldsymbol{\theta}) = \sum_{i=1}^{I}\left(C^i\left(S_0, T^i, K^i, \boldsymbol{\theta}\right) - C^i_{mkt}\left(S_0, T^i, K^i\right)\right)^2,$$

where we have denoted by Θ the set of all parameters

Once we have $\hat{\theta} \in \Theta$ realising $\min_{\boldsymbol{\theta}\in\Theta} E(\boldsymbol{\theta})$, this will be an expression of the intrinsic characteristics of the underlying on which our derivative product is based. We can then use the parameters obtained in Heston's model to calculate the price of a derivative product on the same underlying.

We now present an analysis of the algorithms and model calibration results, obtained in this study.

8.3.1 Algorithms and Results

Our first objective is to be able to calculate the following quantity

$$C\left(S_0, S_T, T, K\right) = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\max\left(S_T - K, 0\right)\right]$$

Let us first recall that the Risk-Neutral $\mathbb{Q}$ measurement can be obtained by exploiting Girsanov's theorem applied to Heston's two-dimensional model.

The following relationships apply in the model

$$dB^s_t = dW^s_t \quad \& \quad dB^v_t = \rho dW^s_t + \sqrt{1-\rho^2}dW^v_t$$

and we define what is called the Girsanov core with respect to which we will construct $\mathbb{Q}$

$$d\tilde{W}^s_t = dW^s_t + \frac{\mu - r}{\sqrt{v_t}}dt \quad \& \quad d\tilde{W}^v_t = dW^v_t + \phi(t, S_t, v_t)dt$$

where $\phi(t, S_t, v_t)$ is to be determined and to do so we observe the following chain of equalities

$$
\begin{aligned}
dv_t =& \kappa\left(\theta - v_t\right)dt + \varepsilon\sqrt{v_t}dB_t^v \\
=& \kappa\left(\theta - v_t\right)dt + \varepsilon\sqrt{v_t}\left(\rho(d\tilde{W}_t^s - \frac{\mu - r}{\sqrt{v_t}})\right) + \varepsilon\sqrt{v_t}\left(\sqrt{1-\rho^2}(d\tilde{W}_t^v - \phi(t, S_t, v_t))\right) \\
=& \kappa\left(\theta - v_t\right)dt + \varepsilon\sqrt{v_t}(\rho d\tilde{W}_t^s + \sqrt{1-\rho^2}d\tilde{W}_t^v) + \varepsilon\sqrt{v_t}\left(-\rho\frac{\mu - r}{\sqrt{v_t}} - \sqrt{1-\rho^2}\phi(t, S_t, v_t)\right)dt \\
=& \kappa\left(\theta - v_t\right)dt + \varepsilon\sqrt{v_t}(\rho d\tilde{W}_t^s + \sqrt{1-\rho^2}d\tilde{W}_t^v) + \lambda(t, S_t, v_t)dt
\end{aligned}
$$

Referring back to the theory presented by Heston in [6], let's set $\lambda(t, S_t, v_t) \triangleq \lambda v_t$, with λ the Risk-Premium coefficient of the variance to be determined, so that v_t and s_t have the same distribution with respect to both measures $\mathbb{Q}$ e $\mathbb{P}$. By doing so, we can obtain $\phi(t, S_t, v_t)$, $\mathbb{Q}$ and the following equations of the Heston model with respect to $\mathbb{Q}$

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{v_t}S_t d\tilde{W}_t^s \\
dv_t &= \kappa'\left(\theta' - v_t\right)dt + \varepsilon\sqrt{v_t}d\tilde{W}_t^v
\end{aligned}
$$

where $\kappa' = \kappa + \lambda$, $\theta' = \frac{\kappa\theta}{\kappa+\lambda}$ and the new parameters to be considered will be of the form $\boldsymbol{\theta} = (\kappa, \theta, \varepsilon, \lambda, \rho, v_0)$. Since $\tilde{W}_t^s$ and $\tilde{W}_t^v$ e two Standard Brownian motions with respect to the probability measure $\mathbb{Q}$ obtained from Girsanov's theorem, we can rewrite the equations of the Euler discretization scheme

$$
\begin{aligned}
S_{t+\Delta} &= S_t + rS_t\Delta + S_t\sqrt{v_t\Delta}\hat{Z}_t^s \\
v_{t+\Delta} &= v_t + \kappa'\left(\theta' - v_t\right)\Delta + \varepsilon\sqrt{v_t\Delta}\hat{Z}_t^v
\end{aligned}
$$

and the one-dimensional Milstein discretization scheme

$$
\begin{aligned}
S_{t+\Delta} &= S_t + rS_t\Delta + S_t\sqrt{v_t\Delta}\hat{Z}_t^s + \frac{1}{2}S_t\Delta v_t\left((\hat{Z}_t^s)^2 - 1\right) \\
v_{t+\Delta} &= v_t + \kappa'\left(\theta' - v_t\right)\Delta + \varepsilon\sqrt{v_t\Delta}\hat{Z}_t^v + \frac{1}{4}\varepsilon^2\Delta\left((\hat{Z}_t^v)^2 - 1\right)
\end{aligned}
$$

with $\hat{Z}_t^s$ and $\hat{Z}_t^v$ Standard Gaussians with correlation ρ.

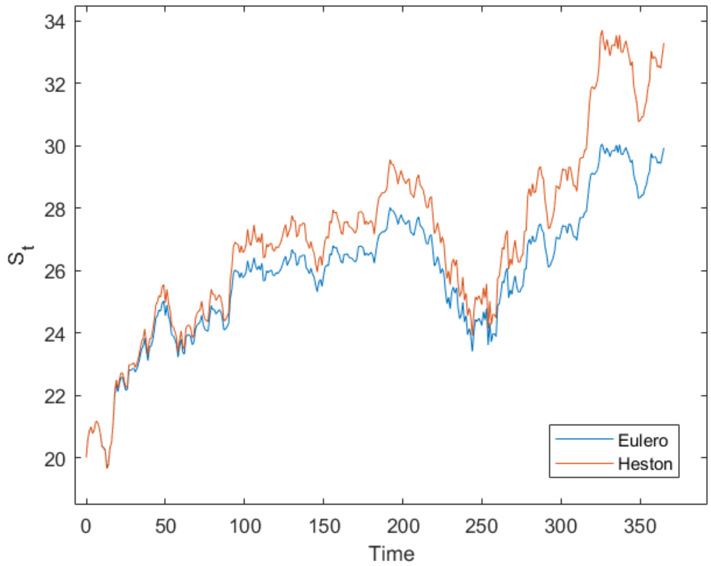
Thanks to these discretisation schemes, we are able to generate S_T starting from S_0 using the following algorithm.

```
function [S_Euler,S_Milstein] = Discretizzazione(k,theta,
    epsilon,lambda,rho,v,r,S,T)
% v il valore iniziale della varianza
% k il tasso di Mean-Reversion
% theta valore medio della varianza
% lambda coefficiente Risk-Premium della varianza
% r il tasso di interesse del Risk-Free asset
```

```
% T il numero di intervalli da considerare
delta = 1/T;
S_Euler = S; % S_T - Schema di Eulero
S_Milstein = S; % S_T - Schema di Milstein % S il prezzo
    iniziale
vE = v;
vM = v;
B_s = normrnd(0,1,T,1);
B_v = normrnd(0,1,T,1);
Z_s = B_s;
Z_v = rho*Z_s + sqrt(1-(rho^2))*B_v;
% Calcolo di S_T con lo Schema di Eulero rispetto a Q
for j = 1:T
S_Euler = S_Euler + r*S_Euler*delta + S_Euler*sqrt(vE*delta)
    *Z_s(j);
vE = max((vE + (k*theta  -(k+lambda)*vE)*delta + epsilon*
    sqrt(vE*delta)*Z_v(j)),0);
end
% Calcolo di S_T con lo Schema di Milstein rispetto a Q
for j = 1:T
S_Milstein = S_Milstein + r*S_Milstein*delta + S_Milstein*
    sqrt(vM*delta)*Z_s(j) + (0.5*S_Milstein*delta*vM)*((Z_s(j)
    ^2)-1);
vM = max((vM + (k*theta  -(k+lambda)*vE)*delta + epsilon*
    sqrt(vM*delta)*Z_v(j) + (0.25*(epsilon^2)*delta)*((Z_v(j)
    ^2)-1)),0);
end
end
```

Let us fix, for example $\bar{\theta} = (3, 0.05, 0.3, 0.03, -0.8, 0.1)$, we choose an expiration at one year $T = 365$, an initial price $S = 20\$$ and an annual interest rate of 3.717% which coincides with a daily rate $r = 0.01\%$.

With these data we are able to calculate a trajectory for S_t like the one in the following graph



By running 100 simulations, using the Monte Carlo method with an accuracy of $M = 10000$, and taking the average value and standard deviation, we obtain the following results:

Euler Mean Value S_T	Milstein Mean Value S_T	Euler Std S_T	Milstein Std S_T
20.0004 $	21.3625 $	0.0460	0.0583

At this point we can calculate

$$C(S_0, S_T, T, K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)]$$

using the Monte Carlo method.

As before, by running 100 simulations with the same parameters, we obtain the following results for $C(S_0, S_T, T, K) = e^{-rT}\mathbb{E}^{\mathbb{Q}}[\max(S_T - K, 0)]$

Euler Mean Value C	Milstein Mean Value C	Euler Std C	Milstein Std C
9.6636 $	10.9869 $	0.0451	0.0548

We now show that we are able to approximate the parameter $\hat{\theta} = (\hat{\kappa}, \hat{\theta}, \hat{\varepsilon}, \hat{\mu}, \hat{\rho}, \hat{v}_0)$ which corresponds to $\min_{\boldsymbol{\theta}\in\Theta}(C(S_0, T, K, \boldsymbol{\theta}) - C_{mkt}(S_0, T, K))$. We are in fact asking ourselves whether, given a market price for the derivative product, we are able to determine those parameters that describe the behaviour of the underlying such that the price of the option on the market is justified.

To do this, we will use MATLAB's *lsqnonlin* d function and discretization schemes to minimise the regularised problem $\min_{\boldsymbol{\theta}\in\Theta}(C(S_0, T, K, \boldsymbol{\theta}) - C_{mkt}(S_0, T, K)) + \alpha||\theta||_2^2$ dove α è un parametro di regolarizzazione che in questo esempio sarà $\alpha = 0.0001$. Di seguito presentiamo gli algoritmi che verranno utilizzati.

Euler discretization scheme

```
function Parameters = function Parameters = Error_function_Euler(k,theta,epsilon,lambda,rho,v,r,S,T,M,strike,mkt_prices,alpha)
p_0 = [k,theta,epsilon,lambda,rho,v];
options.FunctionTolerance = 5e-2;
Parameters = lsqnonlin(@nestedfun,p_0,[0,0,0,-1,-1,0],[Inf,Inf,Inf,1,1,Inf],options);
function Price_error_Euler = nestedfun(p)
rng(4);
[E_Recursive_sum,~] = MonteCarlo(p(1),p(2),p(3),p(4),p(5),p(6),r,S,T,strike,M);
Price_error_Euler = [E_Recursive_sum-mkt_prices,alpha.*p];
end
end
```

Milstein discretization scheme

```
function Parameters = Error_function_Milstein(k,theta,epsilon,lambda,rho,v,r,S,T,M,strike,mkt_prices,alpha)
p = [k,theta,epsilon,lambda,rho,v];
options.FunctionTolerance = 5e-2;
Parameters = lsqnonlin(@nestedfun,p,[0,0,0,-1,-1,0],[Inf,Inf,Inf,1,1,Inf],options);
function Price_error_Milstein = nestedfun(p)
rng(4);
[~,M_Recursive_sum] = MonteCarlo(p(1),p(2),p(3),p(4),p(5),p(6),r,S,T,strike,M);
Price_error_Milstein = [M_Recursive_sum-mkt_prices,alpha.*p];
end
end
```

Let us construct an example showing the correctness of the algorithms. Suppose we have the following parameters $\hat{\theta} = (\hat{\kappa}, \hat{\theta}, \hat{\varepsilon}, \hat{\mu}, \hat{\rho}, \hat{v})$ representing the intrinsic characteristics of the volatile asset on which our derivative product is based.

Suppose that with these characteristics the market price $C_{mkt}(S_0, T, K)$ for the European Call option with $S_0 = 20$, $T = 365$ and $K = 10$ fixed is approximately $C = 15\$$.

We use the parameters from the previous example in the algorithm, knowing that they return a value $C = 10\$$ and are therefore far from $\hat{\theta}$. What we expect is for the minimisation problem to return us parameters $\tilde{\theta}$ that are able to replicate more faithfully the price $C = 15\$$.

With this initial data, the *Error_function_Euler* and *Error_function_Milstein*, algorithms return the following parameters:

$$\tilde{\theta}_{Euler} = (3.12, 5.35, 4.65, -0.43, -0.58, 3.02)$$
$$\tilde{\theta}_{Milstein} = (1.84, 0.93, 0.96, -0.02, -0.92, 0.51)$$

By running 100 simulations, using the Monte Carlo method with an accuracy of $M = 10000$, and taking the mean value and standard deviation, we obtain the following results for S_T

Euler Mean Value S_T	Milstein Mean Value S_T	Euler Std S_T	Milstein Std S_T
19.9655 $	25.1676 $	0.7043	0.1511

We immediately observe that the standard deviation is clearly greater. This less stable behaviour is precisely what we were looking for, as it would justify the higher value of the market price for option C compared to that identified in the initial example. Indeed, let us remember that the products we are going to price are hedging instruments against market volatility.

With these new parameters, we obtain the following results for the price of $C\left(S_0, S_T, T, K\right)$:

Euler Mean Value C	Milstein Mean Value C	Euler Std C	Milstein Std C
15.0595 $	15.1280 $	0.6699	0.1411

The algorithm has therefore found a parameter $\tilde{\theta}$ that can approximate with good accuracy a behaviour of S_t that justifies the $C = 15\$$ price of the option.

At this point with the same strategy we could solve the problem

$$\min_{\boldsymbol{\theta}\in\Theta} E(\boldsymbol{\theta}) = \sum_{i=1}^{I} \left(C^i\left(S_0, T^i, K^i, \boldsymbol{\theta}\right) - C^i_{mkt}\left(S_0, T^i, K^i\right)\right)^2,$$

having available market data of European call options ($C^i_{mkt}\left(S_0, T^i, K^i\right)$) with different strikes and maturities. What we expect is for the algorithm to find a parameter $\tilde{\theta}$ that most closely approximates the behaviour of S_t in all scenarios used for calibration.

This parameter $\tilde{\theta}$ will be an expression, at time $t = 0$, of the intrinsic characteristics of the underlying on which the $C^i\left(S_0, T^i, K^i\right)$ options are based and will thus allow us to determine the price of other options with different strikes and expiry dates.

Clearly, being a primitive model, the claim would be too ambitious. The model,although consistent, needs further detail and optimisation to return accurate values. It certainly remains a good starting point for dealing with the problem of option pricing.

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