QUESTION 2

a) To find the geometric buckling B_g we need to solve the Helmholtz equation for B_g:

$$\nabla^2 \phi(r) + B_g^2 \phi(r) = 0$$

In spherical coordinates: $\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\,\cdot) + \underbrace{\cdots}_{\text{angular terms}}$

Therefore the eq. becomes:

$$\frac{1}{r}\frac{d^2}{dr^2}(r\phi) + B_g^2\phi = 0 \longrightarrow \frac{d^2}{dr^2}[u] + B_g^2 \cdot u = 0 \quad \text{with} \quad u(r) := \phi(r)\cdot r$$

The solution is: $u(r) = A\cdot\cos(B_g\cdot r) + B\cdot\sin(B_g\cdot r) \rightsquigarrow \phi(r) = \frac{u(r)}{r} = \frac{1}{r}\left[A\cos(B_g\cdot r) + B\sin(B_g\cdot r)\right]$

We now have to consider B.C.: (since the notation in the assignment is confusing I call R the radius of the small absorber sphere)

① $\phi(R) = 0$

② $\phi(3R) = 0$

① $0 = A\cdot\frac{\cos(B_g\cdot R)}{R} + B\cdot\frac{\sin(B_g\cdot R)}{R} \rightsquigarrow A = -B\cdot\tan(B_g\cdot R)$

so $\phi(r) = \frac{B}{r}\left[-\tan(B_g\cdot R)\cos(B_g\cdot r) + \sin(B_g\cdot r)\right]$

② $0 = \frac{B}{3R}\cdot\left[-\tan(B_g R)\cdot\cos(3B_g\cdot R) + \sin(3B_g R)\right] \rightsquigarrow$ $B=0$ would yield the solution $\phi = 0$ (NOT ACCEPTABLE) $\rightsquigarrow \sin(3B_g R) = \tan(B_g R)\cdot\cos(3B_g\cdot R)$

The solution to this equation is periodic:

$B_g = \frac{\pi n}{R}$ with $n \in \mathbb{Z}$ $\rightsquigarrow$ the smallest positive value is found for $n=1$ $\rightsquigarrow$ $\boxed{B_g = \frac{\pi}{R}}$

<u>NOTE</u>: this is the same geometric buckling of a full sphere with the radius of the cavity which is predictably bigger than the B_g for a full sphere of equivalent radius without cavity $\left(\frac{\pi}{3R}\right)$

b) For a general 2-group diffusion problem the equations to solve are:

$$\begin{cases} -D_1\nabla^2\phi_1(\vec{r}) + \Sigma_{r1}\phi_1(\vec{r}) = \frac{\chi_1}{k}\left[\nu\Sigma_{f1}\phi_1(\vec{r}) + \nu\Sigma_{f2}\phi_2(\vec{r})\right] \\ -D_2\nabla^2\phi_2(\vec{r}) + \Sigma_{r2}\phi_2(\vec{r}) = \Sigma_{s,1\to2}\phi_1(\vec{r}) + \frac{\chi_2}{k}\left[\nu\Sigma_{f1}\phi_1(\vec{r}) + \nu\Sigma_{f2}\phi_2(\vec{r})\right] \end{cases}$$

where
- $\Sigma_{r1} = \Sigma_{t1} - \Sigma_{s,1\to1} = \Sigma_{a,1} + \Sigma_{s,1\to2}$
- $\Sigma_{r2} = \Sigma_{t2} - \Sigma_{s,2\to2} = \Sigma_{a,2} + \Sigma_{s,2\to1}$

In our case:
1) $\chi_2 = 0$ $(\chi_1 = 1)$
2) $\Sigma_{s,2\to1} = 0$ (no upscattering)

Substituting $\nabla^2\phi = -B_g^2\phi$ the eq. become:

$$\begin{cases} D_1 B_g^2 \phi_1(\vec{r}) + \Sigma_{r1} \phi_1(\vec{r}) = \frac{1}{k}\left[\nu\Sigma_{f1}\phi_1(\vec{r}) + \nu\Sigma_{f2}\phi_2(\vec{r})\right] \\ D_2 B_g^2 \phi_2(\vec{r}) + \Sigma_{a2}\phi_2(\vec{r}) = \Sigma_{s,1\to2}\phi_1(\vec{r}) \end{cases}$$

The matrix form is:

$$\begin{bmatrix} D_1 B_g^2 + \Sigma_{r1} - \frac{\nu\Sigma_{f1}}{k} & -\frac{1}{k}\nu\Sigma_{f2} \\ -\Sigma_{s,1\to2} & D_2 B_g^2 + \Sigma_{a,2} \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \left(D_1 B_g^2 + \Sigma_{r1} - \frac{1}{k}\nu\Sigma_{f1}\right)\left(D_2 B_g^2 + \Sigma_{a2}\right) - \Sigma_{s,1\to2}\cdot\frac{1}{k}\nu\Sigma_{f2} = 0$$

$$\rightarrow D_1 D_2 [B_g^2]^2 + \left(D_1\Sigma_{a2} + \Sigma_{r1} D_2 - \frac{1}{k}\nu\Sigma_{f1} D_2\right)[B_g^2] + \left(\Sigma_{r1}\Sigma_{a2} - \frac{1}{k}\nu\Sigma_{f1}\Sigma_{a2} - \Sigma_{s,1\to2}\frac{1}{k}\nu\Sigma_{f2}\right) = 0$$

This is a second order eq. in B_g^2, we can solve it knowing that $k=1$.

It yields $B_g^2 = \begin{cases} -0.267 \rightarrow \text{NOT ACCEPT.} \\ 0.0104 \end{cases} \rightarrow B_g^2 = 0.010401$

But we know that $B_g^2 = \left(\frac{\pi}{R}\right)^2 \rightarrow \boxed{R = \frac{\pi}{B_g} = 30.8 \text{ cm}}$

c) The Dominance Ratio (DR) is defined as the ratio between the first mode eigenvalue (k_1) and the fundamental mode eigenvalue (k_0)

So $DR = \frac{k_1}{k_0}$

We know that:
- $B_{g,0} = B_g = \frac{\pi}{R}$
- $B_{g,1} = \frac{2\pi}{R} = 0.204 \ \frac{1}{cm}$

We can use the equation derived in point b) and solve for k_1: (we omit the subscript for B_g and k, we study the first mode)

$$\left(D_1 B_g^2 + \Sigma_{r1} - \frac{1}{k}\nu\Sigma_{f1}\right)\left(D_2 B_g^2 + \Sigma_{a2}\right) - \Sigma_{s,1\to2}\cdot\frac{1}{k}\nu\Sigma_{f2} = 0$$

$$D_1 D_2 B_g^4 + D_1 B_g^2 \Sigma_{a2} + D_2 B_g^2 \Sigma_{r1} + \Sigma_{a2}\Sigma_{r1} - \frac{1}{k}\nu\Sigma_{f1} D_2 B_g^2 - \frac{1}{k}\nu\Sigma_{f1}\Sigma_{a2} - \Sigma_{s,1\to2}\nu\Sigma_{f2}\cdot\frac{1}{k} = 0$$

$$k = \frac{\nu\Sigma_{f1} D_2 B_g^2 + \nu\Sigma_{f1}\Sigma_{a2} + \Sigma_{s,1\to2}\nu\Sigma_{f2}}{D_1 D_2 B_g^4 + D_1 B_g^2\Sigma_{a2} + D_2 B_g^2 \Sigma_{r1} + \Sigma_{a2}\Sigma_{r1}}$$

so $k_1 = 0.559$

Therefore (since $k_0 = 1$) we get $\boxed{DR = \frac{k_1}{k_0} = 0.559}$

QUESTION 3

a) Geometric data: $\begin{cases} R = 1m \\ H = 2m \end{cases}$

For a general 2-group diffusion problem the equations to solve are:

$$\begin{cases} -D_1 \nabla^2 \phi_1(\vec{r}) + \Sigma_{r1} \phi_1(\vec{r}) = \frac{\chi_1}{k}\left[\nu\Sigma_{f1}\phi_1(\vec{r}) + \nu\Sigma_{f2}\phi_2(\vec{r})\right] + \Sigma_{s,2\to1}\phi_2(\vec{r}) \\ -D_2 \nabla^2 \phi_2(\vec{r}) + \Sigma_{r2}\phi_2(\vec{r}) = \Sigma_{s,1\to2}\phi_1(\vec{r}) + \frac{\chi_2}{k}\left[\nu\Sigma_{f1}\phi_1(\vec{r}) + \nu\Sigma_{f2}\phi_2(\vec{r})\right] \end{cases}$$

where • $\Sigma_{r1} = \Sigma_{t1} - \Sigma_{s,1\to1} = \Sigma_{a,1} + \Sigma_{s,1\to2}$

• $\Sigma_{r2} = \Sigma_{t2} - \Sigma_{s,2\to2} = \Sigma_{a,2} + \Sigma_{s,2\to1}$

In our case : 1) $\chi_2 = 0$ $(\chi_1 = 1)$

2) $\Sigma_{a2} = 0.01 + 0.02e$

3) $\Sigma_{f2} = 0.02\,e$

$$\begin{cases} D_1 B_g^2 \phi_1(\vec{r}) + (\Sigma_{a1} + \Sigma_{s,1\to2})\phi_1(\vec{r}) = \nu\Sigma_{f1}\phi_1(\vec{r}) + \nu(0.02e)\phi_2(\vec{r}) \\ D_2 B_g^2 \phi_2(\vec{r}) + (0.01 + 0.02e + \Sigma_{s,2\to1})\phi_2(\vec{r}) = \Sigma_{s,1\to2}\phi_1(\vec{r}) \end{cases}$$

Matrix form:

$$\begin{bmatrix} D_1 B_g^2 + (\Sigma_{a1} + \Sigma_{s,1\to2}) - \nu\Sigma_{f1} & -\Sigma_{s,2\to1} - \nu(0.02e) \\ -\Sigma_{s,1\to2} & D_2 B_g^2 + (0.01 + 0.02e) + \Sigma_{s,2\to1} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

For this geometry $B_g^2 = \left(\frac{\nu_0}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 = 8.2505 \frac{1}{m^2} = 8.2505 \cdot 10^{-4} \frac{1}{cm^2}$

det = 0
to

$$\left[D_1 B_g^2 + \Sigma_{a1} + \Sigma_{s,1\to2} - \nu\Sigma_{f1}\right]\left[D_2 B_g^2 + (0.01 + 0.02e) + \Sigma_{s,2\to1}\right] - \Sigma_{s,1\to2}\left[\Sigma_{s,2\to1} + \nu(0.02e)\right] = 0$$

$\xrightarrow[\text{for } e]{\text{solving}}$ $\boxed{e = 25.9\%}$

b) There are two ways of interpreting this question:

① CONSERVATION OF MASS : we fix the amount of solution in the tank and we compute the change volume associated with termal expansion

② CRITICAL HEIGHT : find the height H' that makes the tank critical again given the reduced density

I will finish this exercise exploring both these interpretations.

① MASS CONSERVATION

b) The mass is conserved so $M = M'$ but $M = N \cdot V = N \cdot \pi R^2 \cdot H$

so $N \pi R^2 H = N' \pi R^2 H' \longrightarrow H' = \frac{N}{N'} \cdot H$

We know that $\begin{cases} N = 5 \cdot 10^{22} \\ N' = 4.5 \cdot 10^{22} \end{cases} \rightsquigarrow r = \frac{N'}{N} = 0.9 \rightsquigarrow \boxed{H' = \frac{H}{r} = 2.22\,m}$

c) Given the new height the new geom. buckling is $B_g^{2'} = 7.8 \cdot 10^{-4} \frac{1}{cm^2}$

we have to update the new cross sections and diff. coeff given by the new density:

① $\Sigma_x' = r\,\Sigma_x$

② $D' = \frac{D}{r}$

- $D_1' = D_1 / r = 1.\overline{4}$ cm
- $D_2' = D_2 / r = 0.\overline{4}$ cm
- $\Sigma_{a1}' = r\,\Sigma_{a1} = 0.0054\ \frac{1}{cm}$
- $\Sigma_{a2}' = r\,\Sigma_{a2} = 0.013662\ \frac{1}{cm}$ (same enrichment as point a)
- $\Sigma_{f1}' = r\,\Sigma_{f1} = 0.0036\ \frac{1}{cm}$
- $\Sigma_{f2}' = r\,\Sigma_{f2} = 0.004662\ \frac{1}{cm}$ (same enrichment as point a)
- $\Sigma_{s,1\to2}' = r\,\Sigma_{s,1\to2} = 0.018\ \frac{1}{cm}$
- $\Sigma_{s,2\to1}' = r\,\Sigma_{s,2\to1} = 0.0018\ \frac{1}{cm}$

We can then find the new ratio between the fluxes using the balance eq. for group 2:

$$\xi := \frac{\phi_2}{\phi_1} = \frac{\Sigma_{s,1\to2}'}{D_2' B_g^{2'} + \Sigma_{a2}' + \Sigma_{s,2\to1}'} = 1.1386$$

To condense a x.s. we can compute:

$$\Sigma = \frac{\phi_1 \Sigma_1 + \phi_2 \Sigma_2}{\phi_1 + \phi_2} = \frac{\phi_1}{\phi_1 + \phi_2}\Sigma_1 + \frac{\phi_2}{\phi_1 + \phi_2}\Sigma_2 = \frac{1}{1+\xi}\Sigma_1 + \frac{\xi}{1+\xi}\Sigma_2 = \frac{\Sigma_1 + \xi \Sigma_2}{1+\xi}$$

To compute the new eigenvalue k' we just need the following x.s. : $k' = \frac{\nu \Sigma_f}{D B_g^{'2} + \Sigma_a}$

And :

- $\Sigma_f = \frac{\Sigma_{f1}' + \xi \Sigma_{f2}'}{1+\xi} = 0.04165\ \frac{1}{cm}$
- $\Sigma_a = \frac{\Sigma_{a1}' + \xi \Sigma_{a2}'}{1+\xi} = 0.009798\ \frac{1}{cm}$
- $D = \frac{D_1' + \xi D_2'}{1+\xi} = 0.91204\ \frac{1}{cm}$

Therefore $k' = 0.99081$

We can finally compute the reactivity change : $\Delta\rho = \left(\frac{1}{k} - \frac{1}{k'}\right) = -927.5$ pcm

and the reactivity coefficient : $\boxed{\alpha = \frac{\Delta\rho}{\Delta T} = -18.55\ \frac{pcm}{K}}$

② **CRITICAL HEIGHT**

I can use the new cross sections computed above to find the critical height:

$$\begin{bmatrix} D_1' B_g'^2 + (\Sigma_{a1}' + \Sigma_{s,1\to 2}') - \nu\Sigma_{f1}' & -\Sigma_{s,2\to 1}' - \nu\Sigma_{f2}' \\ -\Sigma_{s,1\to 2}' & D_2' B_g'^2 + \Sigma_{a2}' + \Sigma_{s,2\to 1}' \end{bmatrix}$$

det = 0

$$\left(D_1' B_g'^2 + \Sigma_{r1}' - \nu\Sigma_{f1}'\right)\left(D_2' B_g'^2 + \Sigma_{r2}'\right) - \Sigma_{s,1\to 2}' \cdot \left(\Sigma_{s,2\to 1}' + \nu\Sigma_{f2}'\right) = 0$$

solve for B_g^2

$$B_g^2 = 6.68207 \cdot 10^{-4} \ \frac{1}{cm^2} = 6.68207 \ \frac{1}{m^2}$$

But $B_g^2 = \left(\frac{\nu_0}{R}\right)^2 + \left(\frac{\pi}{H}\right)^2 \rightarrow \frac{\pi}{H} = \sqrt{B_g^2 - \left(\frac{\nu_0}{R}\right)^2} \rightarrow H = \frac{\pi}{\sqrt{B_g^2 - \left(\frac{\nu_0}{R}\right)^2}} = 3.31336 \ m$

So the new height is $\boxed{H' = 3.31 \ m}$