22.211 Lecture 17

Advanced Nodal Methods

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Outline

Outline

Objectives

- Flare model
- 2nd order nodal review
- Transverse Integration
- NEM, ANM, SANM
- Discontinuity factors

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Why do we need nodal methods?

- First order finite difference methods require a mesh smaller than 1 cm to resolve the assembly homogenized reactor
- For a typical LWR modeled in 2 groups, this requires more than 128 million unknowns (about 1Gb)!
- Strong desire to come up with a method with a much smaller memory foot print. An ideal mesh size would be about that of an assembly.
- This would be represented by about 16,000 unknowns (about 15 Kb)
- The key is how to accurately represent the current at the interfaces using the provided unknowns.

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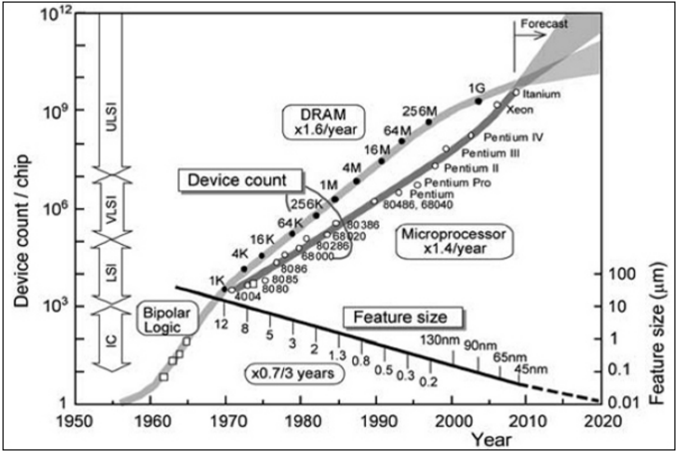
RAM Capacity by year



Expressing the current

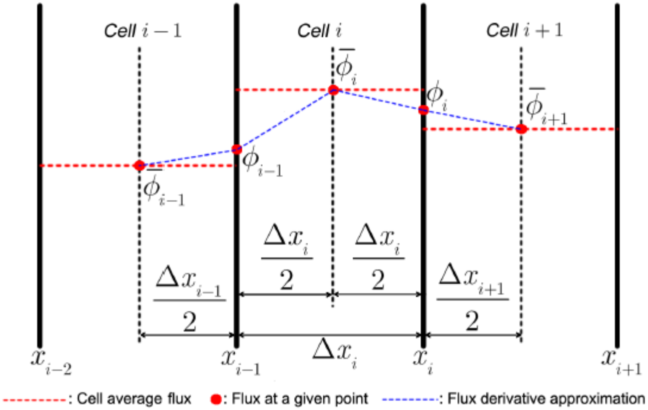
The goal of any discretization scheme is to find a form that allows us to express the net current.

- Finite Difference

$$J^k_{g,x^+} = -D^k_g \frac{\phi^k_g - \phi^k_{g,x+}}{\Delta/2}$$

The mesh required in finite difference is too small, thus nodal methods introduce a higher order approximation of the flux shape within an homogeneous node.

Finite Difference



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Flare model (Delp, 1964)

FLARE model provides a better representation of the leakage at the node edges with a better representation of the currents using partial currents.
The main variable that is solved for in the FLARE model is the power produced in a node.

$$s_g^k = \Delta x \Delta y \Delta z \Sigma_{f,g}^k \bar{\phi}_g^k$$

Coupling Coefficients

Coupling coefficients are defined as

$$w^{k,k+1} = \frac{k_{eff} J^{k,+}_{g,k+1}}{\nu\Sigma^k_{f,g}\bar{\phi}^k_g \Delta x^k \Delta y^k \Delta z^k}$$

$$w^{k+1,k} = \frac{k_{eff} J^{k+1,-}_{g,k}}{\nu\Sigma^{k+1}_{f,g}\bar{\phi}^{k+1}_g \Delta x^{k+1} \Delta y^{k+1} \Delta z^{k+1}}$$

where $J^{k,+}_{g,k+1}$ is an outgoing partial current to node k at the interface of node $k+1$. The coupling coefficients relate the leakage of neutrons to the fission source in the cell.

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Nodal equation

From these definitions we can write a balance equation in node k as

$$s^k = \frac{k_\infty^k}{k_{eff}} \left(w^{k,k} s^k + \sum_{l=1}^{6} w^{k+l,k} s^l \right) \tag{1}$$

where the sum over l represents a sum over all immediate neighbors, and

$$w^{k,k} = 1 - \sum_{l=1}^{6} w^{k,k+l} \tag{2}$$

with

$$k_\infty^k = \frac{\nu \Sigma_f^k}{\Sigma_a^k} \tag{3}$$

Approximation

The equation system forces continuity of the partial currents on each node surface. What remains is how to approximate the coupling coefficients? Using simple 1D models in transport and diffusion theory, the coupling coefficients can be approximated by:

$$w^{k,k+1} = (1-g)\frac{M^k}{2\Delta} + g\frac{(M^k)^2}{\Delta^2} \tag{4}$$

where $(M^k)^2$ is the migration area in node k, Δ is the node width in the appropriate direction and g is a free mixing parameter that is adjusted based on observed experimental results.

Only k_∞ and M^2 are needed!

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Diffusion equation

$$\frac{\partial}{\partial x} J^k_{g,x}(x,y,z) + \frac{\partial}{\partial y} J^k_{g,y}(x,y,z) + \frac{\partial}{\partial z} J^k_{g,z}(x,y,z) + \Sigma^k_{r,g}\phi^k_g(x,y,z) = Q^k_g(x,y,z)$$

with

$$J^k_{g,x}(x,y,z) = -D^k_g \frac{\partial}{\partial x}\phi^k_g(x,y,z)$$

Balance on a node

$$\frac{1}{\Delta x}\left[J^k_{g,x^+} - J^k_{g,x^-}\right] + \frac{1}{\Delta y}\left[J^k_{g,y^+} - J^k_{g,y^-}\right] + \frac{1}{\Delta z}\left[J^k_{g,z^+} - J^k_{g,z^-}\right] + \Sigma^k_{r,g}\bar{\phi}^k_g = \bar{Q}^k_g$$

where the node average flux is defined as

$$\bar{\phi}^k_g = \frac{1}{V}\int_{-\Delta x/2}^{\Delta x/2} dx \int_{-\Delta y/2}^{\Delta y/2} dy \int_{-\Delta z/2}^{\Delta z/2} dz \phi^k_g(x,y,z)$$

and the surface average current as

$$J^k_{g,x^+} = \frac{1}{\Delta y \Delta z}\int_{-\Delta y/2}^{\Delta y/2} dy \int_{-\Delta z/2}^{\Delta z/2} dz - D^k_g \frac{\partial}{\partial x}\phi^k_g(x,y,z)\Big|_{x=\Delta x/2}$$

Second order nodal methods

- First, let's assume that each mesh is normalized by its width, such that each mesh varies between $x = -1/2$ and $x = 1/2$.
- Define the flux shape as second order using polynomials that integrate to zero on the interval as to preserve the average flux.

$$\phi(x) = \bar{\phi} + a_1 x + a_2(3x^2 - 1/4)$$

Expressing net current at an interface

Our goal is to replace the net current in the balance equation by an expression that depends only on the cell average fluxes. For 2 neighboring node sharing an interface (in 1 group), we have 2 nodal balance equations, continuity of the current and continuity of the flux. These 4 conditions will allow us to determine the 4 coefficients and express the current as a function of cell average fluxes.

$$\frac{d\phi}{dx} = a_1 + 6a_2x$$

$$\frac{d^2\phi}{dx^2} = 6a_2$$

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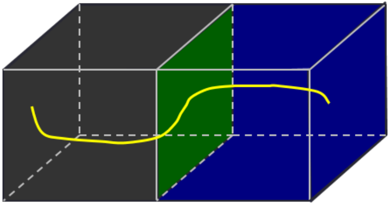
2 node system



We have 4 equations and 4 unknowns.

$$-D^k(6a_2) + \Sigma_a^k \bar{\phi}^k = \frac{1}{k}\nu\Sigma_f^k \bar{\phi}^k \Rightarrow -D^k(6a_2) = \bar{S}^k \bar{\phi}^k$$

$$-D^{k+1}(6b_2) + \Sigma_a^{k+1} \bar{\phi}^{k+1} = \frac{1}{k}\nu\Sigma_f^{k+1} \bar{\phi}^{k+1} \Rightarrow -D^{k+1}(6b_2) = \bar{S}^{k+1} \bar{\phi}^{k+1}$$

$$J^k(1/2) = J^{k+1}(-1/2) \Rightarrow D^k(a_1 + 3a_2) = D^{k+1}(b_1 - 3b_2)$$

$$\phi^k(1/2) = \phi^{k+1}(-1/2) \Rightarrow \bar{\phi}^k + \frac{1}{2}(a_1 + a_2) = \bar{\phi}^{k+1} - \frac{1}{2}(-b_1 + b_2)$$

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Solving for the coefficients

$$a_2 = \frac{-\bar{S}^k \bar{\phi}^k}{6D^k}$$

$$b_2 = \frac{-\bar{S}^{k+1} \bar{\phi}^{k+1}}{6D^{k+1}}$$

with

$$D^k(a_1 + 3a_2) = D^{k+1}(b_1 - 3b_2)$$

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from flux continuity

$$b_1 = 2\bar{\phi}^{k+1} - 2\bar{\phi}^k - a_1 - a_2 + b_2$$

replace in the previous equation

$$D^k\left(a_1 + 3a_2\right) = D^{k+1}\left(2\bar{\phi}^{k+1} - 2\bar{\phi}^k - a_1 - a_2 + b_2 - 3b_2\right)$$

thus

$$a_1 = \frac{\frac{D^{k+1}}{D^k}\left(2\bar{\phi}^{k+1} - 2\bar{\phi}^k - a_2 - 2b_2\right) - 3a_2}{1 + \frac{D^{k+1}}{D^k}}$$

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Expression for the current

$$J^k(1/2) = -D^k(a_1 + 3a_2)$$

with

$$a_1 = \frac{\frac{D^{k+1}}{D^k}\left(2\bar{\phi}^{k+1} - 2\bar{\phi}^k - a_2 - 2b_2\right) - 3a_2}{1 + \frac{D^{k+1}}{D^k}}$$

and

$$a_2 = \frac{-\bar{S}^k\bar{\phi}^k}{6D^k} \qquad b_2 = \frac{-\bar{S}^{k+1}\bar{\phi}^{k+1}}{6D^{k+1}}$$

Very similar to the finite difference equations, just different coefficients in front of the cell average fluxes. These coefficients provide a different representation of the current at the interface. You can thus use the same solver and just replace how you calculate the tri-diagonal terms.

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In 3D

The previous derivation was done for a 1D slab problem. Would the same process work in 3D? What would the polynomial look like?

The second key aspect of nodal methods is that the 3D problem is solved as a set of three 1D problems. For each 1D problem, the other two directions are represented through a transverse leakage term on which we must iterate.

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Transverse Integration

Integrate diffusion equation in 2 directions

$$\frac{1}{\Delta y \Delta z}\int_{-\Delta y/2}^{\Delta y/2} dy \int_{-\Delta z/2}^{\Delta z/2} dz$$

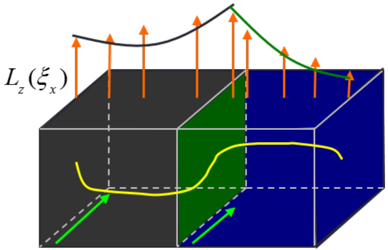
yields

$$\frac{d}{dx}J_{g,x}^{k}(x) + \Sigma_{r,g}^{k}\phi_{g}^{k}(x) = Q_{g}^{k}(x) - \frac{1}{\Delta y}L_{g,y}^{k}(x) - \frac{1}{\Delta z}L_{g,z}^{k}(x)$$

with

$$\phi_{g}^{k}(x) = \frac{1}{\Delta y \Delta z}\int_{-\Delta y/2}^{\Delta y/2} dy \int_{-\Delta z/2}^{\Delta z/2} dz \phi_{g}^{k}(x,y,z)$$

Transverse Leakage



$$L^k_{g,y}(x) = \frac{1}{\Delta z}\int_{-\Delta z/2}^{\Delta z/2} dz - D^k_g \frac{\partial}{\partial y}\phi^k_g(x,y,z)\Bigg|_{y=-\Delta y/2}^{y=-\Delta y/2}$$

$$L^k_{g,z}(x) = \frac{1}{\Delta y}\int_{-\Delta y/2}^{\Delta y/2} dz - D^k_g \frac{\partial}{\partial z}\phi^k_g(x,y,z)\Bigg|_{z=-\Delta z/2}^{z=-\Delta z/2}$$

Three 1D equations

$$-\frac{d}{dx}D_g^k\frac{d}{dx}\phi_g^k(x)+\Sigma_{r,g}^k\phi_g^k(x)=Q_g^k(x)-\frac{1}{\Delta y}L_{g,y}^k(x)-\frac{1}{\Delta z}L_{g,z}^k(x)$$

$$-\frac{d}{dy}D_g^k\frac{d}{dy}\phi_g^k(y)+\Sigma_{r,g}^k\phi_g^k(y)=Q_g^k(y)-\frac{1}{\Delta x}L_{g,x}^k(y)-\frac{1}{\Delta z}L_{g,z}^k(y)$$

$$-\frac{d}{dz}D_g^k\frac{d}{dz}\phi_g^k(z)+\Sigma_{r,g}^k\phi_g^k(z)=Q_g^k(z)-\frac{1}{\Delta x}L_{g,x}^k(z)-\frac{1}{\Delta y}L_{g,y}^k(z)$$

Three 1D equations that are coupled through the transverse leakage terms.

Transverse leakage approximation

Original approximation was made that leakages could be determined by the geometrical buckling

$$\left[DB^2\right]_x \bar{\phi}_g^k = \frac{1}{\Delta y}\bar{L}_{g,y}^k + \frac{1}{\Delta z}\bar{L}_{g,z}^k$$

A better approximation was made by Kord in his Master's thesis that has persisted since. He approximated the leakage term as a quadratic function that preserves the average of the two adjacent nodes.

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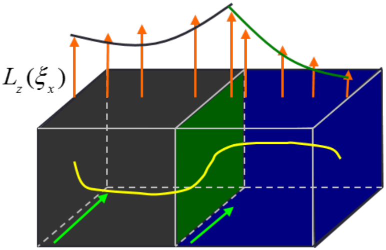
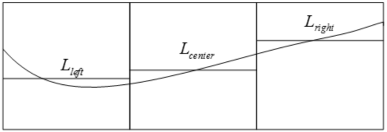
- Approximate the leakage as

$$L(x) = \bar{L} + l_1 P_1(x) + l_2 P_2(x)$$

- Use three nodes average leakage to determine coefficients
- Impose constraint that average leakage in each node is preserved





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Nodal Expansion Method (Finnemann, KWU, 1975)

The nodal expansion methods (NEM) is based on a low order polynomial expansion of the transverse integrated flux.

$$\phi^k(x) = \sum_{l=0}^{N} a_l^k \psi_l(x) \tag{5}$$

The functions are selected are not necessarily orthogonal and are selected to simplify downstream expressions.

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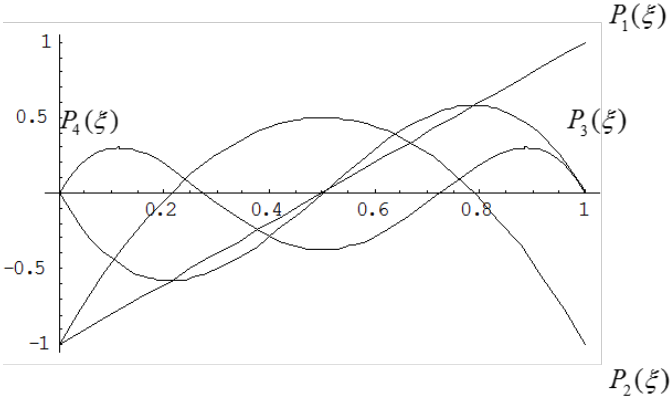
Polynomials

The typical NEM method uses a 4th order expansion with these polynomials:

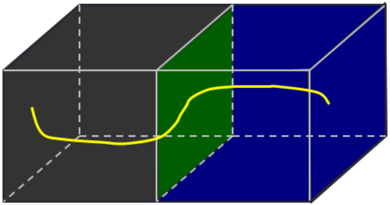
$$\begin{aligned}
P_0(\xi) &= 1 \\
P_1(\xi) &= 2\xi - 1 \\
P_2(\xi) &= 6\xi(1-\xi) - 1 \\
P_3(\xi) &= 6\xi(1-\xi)(2\xi - 1) \\
P_4(\xi) &= 6\xi(1-\xi)(5\xi^2 - 5\xi + 1)
\end{aligned} \tag{6}$$

where ξ varies between 0 and 1 ($\xi = (x - x_0)/\Delta x^k$), and the integration of all order except the 0th moment is zero.

Polynomials



2 node approach



Assuming that we know the transverse leakage terms and cell average fluxes, we have 16 unknowns per group (4 expansion coefficients per group and 2 nodes). The imposed conditions are:

1. Nodal balance in each node (4 equations)
2. Flux continuity at the interface (2 equations)
3. Current continuity at the interface (2 equations)

Weighted residual constraints

The last two relations are provided from a moment weighted residual using the 1st and 2nd moments.

$$\int_0^1 \psi_1(\xi)\Big(- \tilde{D}\frac{d}{d\xi^2}\phi(\xi) + \Sigma_r\phi(\xi) - Q(\xi)\Big) = 0 \tag{7}$$

and

$$\int_0^1 \psi_2(\xi)\Big(- \tilde{D}\frac{d}{d\xi^2}\phi(\xi) + \Sigma_r\phi(\xi) - Q(\xi)\Big) = 0 \tag{8}$$

where $\tilde{D} = D/\Delta^2$ which is an equivalent diffusion coefficient for the normalized mesh, and $Q(\xi)$ includes the fission source and transverse leakage terms. These provide the remaining 4 equations needed (2 per node).

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1st Moment

If we apply the expansion to ϕ and Q, we get

$$\int_0^1 (2\xi - 1)\Big(- \tilde{D}\frac{d}{d\xi^2}\sum_{l=0}^{N} a_l^k \psi_l(\xi) + \Sigma_r \sum_{l=0}^{N} a_l^k \psi_l(\xi) - \sum_{l=0}^{N} q_l^k \psi_l(\xi)\Big) = 0 \qquad (9)$$

By choice, the first three polynomials ψ_l are orthogonal to each other since they correspond to scaled Legendre polynomials, thus the integration will eliminate many of the cross terms. The 3rd and 4th order terms however were picked to enforce values of 0 at each end (reduces number of terms in surface current expression).

After integration

$$-\tilde{D}a_2(0) - \tilde{D}a_3(-12) - \tilde{D}a_4(0) + \Sigma_r a_1(1/3) + \Sigma_r a_3(1/5) + \Sigma_r a_4(0) \\ - q_1(1/3) - q_3(1/5) - q_4(0) = 0$$

which yields

$$a_3 = \frac{5q_1 + 3q_3 - 5\Sigma_r a_1}{3(60\tilde{D} + \Sigma_r)}$$

The second moment follows:

$$\int_0^1 (6\xi(1-\xi)-1)\Big(-\tilde{D}\frac{d}{d\xi^2}\sum_{l=0}^{N} a_l^k \psi_l(\xi) + \Sigma_r \sum_{l=0}^{N} a_l^k \psi_l(\xi) - \sum_{l=0}^{N} q_l^k \psi_l(\xi)\Big) = 0$$

After integration

$$-\tilde{D}a_2(0) - \tilde{D}a_3(0) - \tilde{D}a_4(12) + \Sigma_r a_2(1/5) + \Sigma_r a_3(0) + \Sigma_r a_4(-3/35)$$
$$- q_2(1/5) - q_3(0) - q_4(-3/35) = 0$$

which yields

$$a_4 = \frac{3q_4 - 7q_2 + 7\Sigma_r a_2}{420\tilde{D} + 3\Sigma_r}$$

From the other relations

Nodal balance in node k

$$-\tilde{D}^k \frac{d\phi^k}{d\xi}\Big|_{\xi=1} + \tilde{D}^k \frac{d\phi^k}{d\xi}\Big|_{\xi=0} + \Sigma_r a_o^k = q_0^k$$

which equals

$$\tilde{D}^k(12a_2^k + 12a_4^k) + \Sigma_r a_o^k = q_0^k$$

Continuity of the flux

$$\phi^{k-1}(1) = \phi^k(0)$$

which gives

$$a_0^{k-1} + a_1^{k-1} - a_2^{k-1} = a_0^k - a_1^k - a_2^k$$

and continuity of the current

$$-\tilde{D}^{k-1} \frac{d\phi^{k-1}}{d\xi}\Big|_{\xi=1} = -\tilde{D}^k \frac{d\phi^k}{d\xi}\Big|_{\xi=0}$$

which gives

$$\tilde{D}^{k-1} \times (2a_1^{k-1} - 6a_2^{k-1} - 6a_3^{k-1} - 6a_4^{k-1}) = \tilde{D}^k \times (2a_1^k + 6a_2^k - 6a_3^k + 6a_4^k)$$

2 node relations

$$12\tilde{D}^k(a_2^k + a_4^k) + \Sigma_r a_0^k = q_0^k$$
$$a_3^k = \frac{5q_1^k + 3q_3^k - 5\Sigma_r a_1^k}{3(60\tilde{D}^k + \Sigma_r)}$$
$$a_4^k = \frac{3q_4^k - 7q_2^k + 7\Sigma_r a_2^k}{420\tilde{D}^k + 3\Sigma_r}$$
$$12\tilde{D}^{k-1}(a_2^{k-1} + a_4^{k-1}) + \Sigma_r a_0^{k-1} = q_0^{k-1}$$
$$a_3^{k-1} = \frac{5q_1^{k-1} + 3q_3^{k-1} - 5\Sigma_r a_1^{k-1}}{3(60\tilde{D}^{k-1} + \Sigma_r)}$$
$$a_4^{k-1} = \frac{3q_4^{k-1} - 7q_2^{k-1} + 7\Sigma_r a_2^{k-1}}{420\tilde{D}^{k-1} + 3\Sigma_r}$$
$$a_0^{k-1} + a_1^{k-1} - a_2^{k-1} = a_0^k - a_1^k - a_2^k$$
$$\tilde{D}^{k-1} \times (2a_1^{k-1} - 6a_2^{k-1} - 6a_3^{k-1} - 6a_4^{k-1}) = \tilde{D}^k \times (2a_1^k + 6a_2^k - 6a_3^k + 6a_4^k)$$

Now what?

- The relations provide ways to connect all of the coefficients in order to represent the net current at the interface to a_0^k and a_0^{k-1}.
- You could technically build a large matrix system with all of these relations and all the coefficients as unknowns (similar to the finite difference matrix with different coupling coefficients).
- Typically, this is solved by coupling with a coarse mesh finite difference (CMFD) acceleration technique, where the finite difference fluxes are replaced by the 2 node interface currents.

Nonlinear correction

In order to preserve the transport neutron current across the surface, a correction is made to the current equation

$$\tilde{J}_g^{i+1/2,j} = -\tilde{D}_g^{i+1/2,j}\left(\phi_g^{i+1,j} - \phi_g^{i,j}\right) - \hat{D}_g^{i+1/2,j}\left(\phi_g^{i+1,j} + \phi_g^{i,j}\right)$$

where $\hat{D}$ is the nonlinear diffusion coefficient, $\tilde{J}$ is the surface average net current calculated from the transport sweep and $\tilde{D}$ is the commonly defined diffusion coefficient.

$$\hat{D}_g^{i+1/2,j} = \frac{-\tilde{D}_g^{i+1/2,j}\left(\phi_g^{i+1,j} - \phi_g^{i,j}\right) - \tilde{J}_g^{i+1/2,j}}{\left(\phi_g^{i+1,j} + \phi_g^{i,j}\right)}$$

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Analytical Nodal Method (MIT, 1978)

Instead of using a polynomial basis set, it was proposed to analytically solve the diffusion equation in the node.

$$-D_1\frac{d^2\phi_1(x)}{dx^2}+\Sigma_{r,1}\phi_1(x)-\frac{1}{k}\Big(\nu\Sigma_{f,1}\phi_1(x)+\nu\Sigma_{f,2}\phi_2(x)\Big)=-L_1(x)$$

$$-D_2\frac{d^2\phi_2(x)}{dx^2}+\Sigma_{r,2}\phi_2(x)-\Sigma_{s,1/\rightarrow 2}\phi_1(x)=-L_2(x)$$

with a solution of the form

$$\phi_g(x)=\phi_g^{hom}(x)+\phi_g^{p}(x)=\hat{\phi}_g^{hom}e^{iBx}+\phi_g^{p}(x)$$

Homogeneous system

$$\begin{bmatrix} D_1 B_g^2 + \Sigma_{r1} - \frac{1}{k}\nu\Sigma_{f1} & -\frac{1}{k}\nu\Sigma_{f2} \\ -\Sigma_{s1\rightarrow 2} & D_2 B_g^2 + \Sigma_{a2} \end{bmatrix} \begin{bmatrix} \hat{\phi}_1^{hom} \\ \hat{\phi}_2^{hom} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

We can solve for buckling eigenvalues B_2 by setting the determinant of the matrix to 0

$$(D_1 B_g^2 + \Sigma_{r1} - \frac{1}{k}\nu\Sigma_{f1})(D_2 B_g^2 + \Sigma_{a2}) - \Sigma_{s1\rightarrow 2}\frac{1}{k}\nu\Sigma_{f2} = 0$$

which yields

$$(B_g^2)^2 + \left(\frac{\Sigma_{r1}}{D_1} + \frac{\Sigma_{a2}}{D_2} - \frac{\nu\Sigma_{f1}}{kD_1}\right)B_g^2 + \frac{\Sigma_{r1}\Sigma_{a2} - \frac{1}{k}\nu\Sigma_{f1}\Sigma_{a2} - \Sigma_{s1\rightarrow 2}\frac{1}{k}\nu\Sigma_{f2}}{D_1 D_2} = 0$$

Buckling eigenvalues

$$(B_g^2)^2 + \left(\frac{\Sigma_{r1}}{D_1} + \frac{\Sigma_{a2}}{D_2} - \frac{\nu\Sigma_{f1}}{kD_1}\right)B_g^2 + \left(1 - \frac{k_\infty}{k}\right)\frac{\Sigma_{r1}}{D_1}\frac{\Sigma_{a2}}{D_2} = 0$$

which we re-write as

$$(B_g^2)^2 + 2bB_g^2 + c = 0$$

Roots of the equation

$$B_1^2 = b\left(-1 + \sqrt{1 - \frac{c}{b^2}}\right)$$

$$B_2^2 = b\left(-1 - \sqrt{1 - \frac{c}{b^2}}\right)$$

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$$k_\infty = \frac{\nu\Sigma_{f1}}{\Sigma_{r1}} + \frac{\Sigma_{s1\to 2}}{\Sigma_{r1}}\frac{\nu\Sigma_{f2}}{\Sigma_{a2}}$$

$$b = \left(\frac{\Sigma_{r1}}{D_1} + \frac{\Sigma_{a2}}{D_2} - \frac{\nu\Sigma_{f1}}{kD_1}\right)/2$$

$$c = \left(1 - \frac{k_\infty}{k}\right)\frac{\Sigma_{r1}}{D_1}\frac{\Sigma_{a2}}{D_2}$$

Homogeneous solutions

The homogeneous solution is thus a linear combination of the two eigenvectors and their associated buckling eigenvalues. The fundamental mode eigenvectors change form based on the value of k_∞

$$\phi_g^{hom,1} = a_{g,1} sin(B_1 x) + a_{g,2} cos(B_1 x) \quad k_\infty > k$$

$$\phi_g^{hom,1} = a_{g,1} sinh(B_1 x) + a_{g,2} cosh(B_1 x) \quad k_\infty < k$$

The second harmonic solution is

$$\phi_g^{hom,2} = a_{g,3} sinh(B_2 x) + a_{g,4} cosh(B_2 x)$$

Homogeneous solutions

For a cell with $k_\infty > k$, we get

$$\begin{bmatrix} \hat{\phi}_1^{hom} \\ \hat{\phi}_2^{hom} \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{2,1} sin(B_1 x) + a_{2,2} cos(B_1 x) \\ a_{2,3} sinh(B_2 x) + a_{2,4} cosh(B_2 x) \end{bmatrix}$$

where

$$r_m = \frac{a_{1,1}}{a_{2,1}} = \frac{a_{1,2}}{a_{2,2}} = \frac{D_2 B_m^2 + \Sigma_{a2}}{\Sigma_{s1\to 2}}$$

Particular solutions

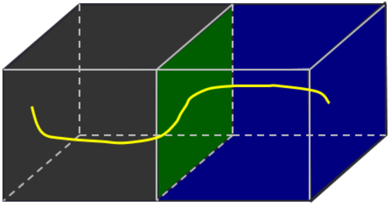
Depends only on the transverse leakage and can be solve readily when the leakage is known

$$\phi_g^p(x) = c_{0,g} + c_{1,g} P_1(x) + c_{2,g} P_2(x)$$

Combining with homogeneous solution we get

$$\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = \begin{bmatrix} r_1 & r_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{2,1} sin(B_1 x) + a_{2,2} cos(B_1 x) \\ a_{2,3} sinh(B_2 x) + a_{2,4} cosh(B_2 x) \end{bmatrix} + \begin{bmatrix} \phi_1^p \\ \phi_2^p \end{bmatrix}$$

2 node approach



Assuming that we know the transverse leakage terms, we have 8 unknowns (4 coefficients and 2 nodes). The imposed conditions are:

1. Nodal balance in each node (4 equations)
2. Flux continuity at the interface (2 equations)
3. Current continuity at the interface (2 equations)

How to solve?

- Knowing the flux solution in the node, we can now evaluate the average flux in the cell ($\bar{\phi}$).
- We can then define the net current at the interface as a function of node average fluxes.

Outline

Semi-analytical Nodal Method (Studsvik, 1985)

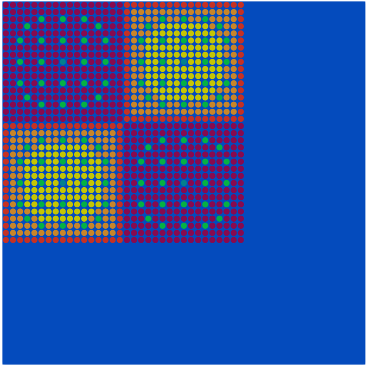
- NEM equations for the fast group
- ANM for the thermal group

Assuming that we know the transverse leakage terms, we have 12 unknowns (2 coefficients for the thermal group, 4 expansion coefficients for the fast group times 2 nodes). The imposed conditions are:

1. Nodal balance in each node (4 equations)
2. Flux continuity at the interface (2 equations)
3. Current continuity at the interface (2 equations)
4. Weighted residual equations in the fast group (4 equations)

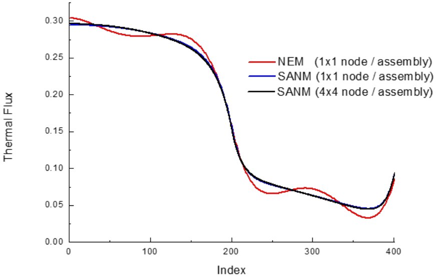
Results

Node Structure	Kernel	Eigenvalue Error,	Max. Power Errror, %
1x1	NEM	303	3.10
	ANM	-24	-0.01
	SANM	-25	-0.04
2x2	NEM	59	0.55
	ANM	-6	-0.04
	SANM	-6	-0.04

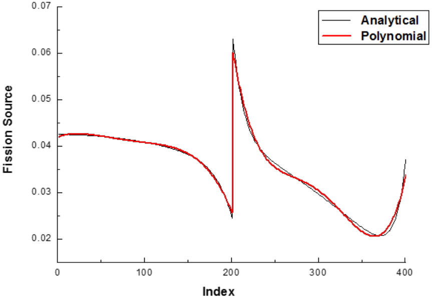


- SANM has the accuracy of ANM with the simplicity of NEM
- Nodal methods provide accurate solutions on meshes of 10-20cm

Results



Thermal Flux
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
100
200
300
400
Index
NEM (1x1 node / assembly)
SANM (1x1 node / assembly)
SANM (4x4 node / assembly)



Fission Source
0.07
0.06
0.05
0.04
0.03
0.02
0
100
200
300
400
Index
Analytical
Polynomial

References

- Lawrence, Progress in Nuclear Energy paper (1986)