ODE

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1 Solving Ordinary Differential Equations

The first set of problems we address are ordinary differntial equation of the form:

$$\frac{dy}{dt} = f(y,t) \qquad y(t_0) = y_0 \tag{1}$$

where y is the unknown function, t is the independent variable, and $f(y,t)$ is a given function. The initial condition $y(t_0) = y_0$ specifies the value of the solution at time t_0.

This is a quite general formulation that includes many problems of interest in science and engineering. This equation models, for example, simple physical problems such asradiocative decay.

Tabella 1: Some physical phenomena modeled by ordinary differential equations (ODEs).

Physical phenomena	**ODEs**
Radioactive decay	$\frac{dN}{dt} = -\lambda N$
Exponential growth	$\frac{dy}{dt} = ky$
Newton cooling law	$\frac{dT}{dt} = -k(T - T_{\text{amb}})$
Charge/Discharge plate (RC)	$\frac{dV}{dt} = \frac{1}{RC}(V_0 - V)$
Logistic growth	$\frac{dP}{dt} = rP(1 - P/K)$
First order kinetics	$\frac{d[A]}{dt} = -k[A]$

1.1 Euler algorithm

The simplest numerical method for solving ordinary differential equations is the Euler method. It is based on the idea of approximating the solution by a series of small steps, using a Taylor expantion up to the first order.

$$y(t_0 + \Delta t) \approx y(t_0) + \Delta t \frac{dy}{dt} = y_0 + \Delta t f(y_0, t_0) + O(\Delta t^2) \tag{2}$$

Iterating this equation and introducing a discrete time variable $t_n = t_0 + n\Delta t$, we obtain the Euler algorithm:

$$y_{n+1} = y_n + \Delta t f(y_n, t_n) + O(\Delta t^2) \tag{3}$$

In contrast to the analytical solution which was easy for the simple example given above, the Euler method retains its simplicity nomatter which differential equations it is applied to.