

# Scombinatorics

<b>Notation and preliminaries</b>	<b>3</b>
<b>1 Three minimax theorems</b>	<b>4</b>
1 Hall's Marriage Theorem	4
2 Kőnig's Minimax Theorem	5
3 Dilworth's Theorem	6
<b>2 Set systems</b>	<b>8</b>
1 Sperner's Theorem	8
2 The Erdős-Ko-Rado Theorem	9
<b>3 Stability</b>	<b>11</b>
1 The order property	11
2 Approximable sets	15
<b>4 Vapnik-Chervonenkis theory</b>	<b>19</b>
1 The Vapnik-Chervonenkis dimension	19
2 The Sauer-Shelah lemma	21
<b>5 Law(s) of large numbers</b>	<b>23</b>
1 Inequalities	23
2 Two Weak Laws of Large Numbers	27
3 A Uniform Law of Large Numbers	29
4 A Uniform Law of Large Numbers, again	31
<b>6 Small transversals</b>	<b>35</b>
1 Transversals and packings	35
2 Helly-type properties	37
3 The $(p,q)$ -theorem	39
<b>7 Zarankiewicz problem(s)</b>	<b>41</b>
1 The Kővári-Sós-Turán Theorem	41
2 The NIP case	42
<b>8 Szemerédi regularity lemma</b>	<b>43</b>
1 The regularity lemma	43
2 Stable regularity lemma	43
3 Distal regularity lemma	43



## Notation and preliminaries

Let  $\mathcal{U}$  and  $\mathcal{V}$  be two infinite sets. Let  $\varphi(x; z)$  be a relation symbol, or a formula, whatever. We denote by  $\varphi(\mathcal{U}; \mathcal{V})$  the set  $\{\langle a, b \rangle \in \mathcal{U} \times \mathcal{V} : \varphi(a; b)\}$  which we call: the relation defined by  $\varphi(x; z)$ . Sets of the form  $\varphi(\mathcal{U}; b) = \{a \in \mathcal{U} : \varphi(a; b)\}$ , for some  $b \in \mathcal{V}$ , are called **definable** sets.

In the first chapters we always restrict the study to the **trace** of  $\varphi(\mathcal{U}; \mathcal{V})$  on some finite set  $A \times B$ , where  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$ . We write  $\varphi(A; B)$  for  $\varphi(\mathcal{U}; \mathcal{V}) \cap A \times B$ . Similarly, we write  $\varphi(A; b)$  for the trace of  $\varphi(\mathcal{U}; b)$  on  $A$ , that is, the set  $\varphi(\mathcal{U}; b) \cap A$ . We call it a definable subset of  $A$ .

We denote by  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  and  $\varphi(A; b)_{b \in \mathcal{V}}$  the collection of definable sets, respectively definable subsets of  $A$ .

A similar notations is used with the roles of  $\mathcal{U}$  and  $\mathcal{V}$  interchanged. I.e., we think of  $\mathcal{U}$  as the set of parameters defining subsets of  $\mathcal{V}$ . We write  $\varphi(x; z)^{\text{op}}$  to signal that the formula  $\varphi(x; z)$  is considered in this **dual** setting.

For the model theoretic minded, it would be more appropriate to call the definable sets *global types*, respectively *types over A*. But this would make the terminology (if possible) more obscure.

For  $k \leq |A|$  we use following notation interchangeably

$$\binom{A}{k} = A^{(k)} = \{A' \subseteq A : |A'| = k\}$$

For  $n$  a non negative integer we write

$$(n) = \{1, \dots, n\},$$

$$n = [n] = \{0, \dots, n-1\},$$

and  $[n] = \{0, \dots, n\}$



Note that the latter conflicts with the notation used in combinatorics.

# Chapter 1

## Three minimax theorems

Though apparently unrelated, the three theorems in this chapter can be derived one each other. We prove them in an arbitrary order.

As evident from the statement, the last two theorems are minimax theorems. The first theorem less so, hence the title is only approximately correct.

### 1 Hall's Marriage Theorem

Let  $\varphi(x; z)$  be given. Let  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  be finite sets.

We say that  $A' \subseteq A$  is a **set of distinct representatives** for  $\varphi(A; B)$  if

$$|\varphi(A'; b)| = |\varphi(a; B)| = 1 \quad \text{for every } a, b \in A', B,$$

or, in other words, if  $\varphi(A'; B)$  is the graph of a bijection.

**1.1 Hall's Marriage Theorem** For every finite  $B \subseteq \mathcal{V}$ , the following are equivalent

1.  $\varphi(A; B)$  has a set of distinct representatives;
2.  $|B'| \leq \left| \bigcup_{b \in B'} \varphi(A; b) \right|$  for every  $B' \subseteq B$ .

**Proof (1 $\Rightarrow$ 2)** The following holds for any set of distinct representatives  $A'$  and every set  $B' \subseteq B$

$$|B'| = \left| \bigcup_{b \in B'} \varphi(A'; b) \right| \subseteq \left| \bigcup_{b \in B'} \varphi(A; b) \right|.$$

**(2 $\Rightarrow$ 1)** Reason by induction on the cardinality of  $B$ . If  $|B| = 1$ , the claim is clear. Now assume  $|B| > 1$  and consider two cases.

- a. This is the case when the inequality in 2 is strict for all nonempty  $B' \subset B$ . Pick any pair  $a, b \in A, B$  such that  $\varphi(a; b)$ . Then  $\varphi(A \setminus \{a\}; B \setminus \{b\})$  still satisfy 2. By induction hypothesis, it has a set of distinct representatives  $A'$ . Then  $A' \cup \{a\}$  is a set of distinct representatives for  $\varphi(A; B)$ .
- b. Suppose instead that for some nonempty  $B' \subset B$  the inequality in 2 holds with equality. Define

$$A' = \bigcup_{b \in B'} \varphi(A; b)$$

It is clear that 2 holds for  $\varphi(A'; B')$ . Below we prove that 2 also holds for  $\varphi(A \setminus A'; B \setminus B')$ . Once this claim is proved, we apply the induction hypothesis to obtain sets of distinct representatives for these two relations and note that

their union is a set of distinct representatives for  $\varphi(A; B)$ .

To prove the claim assume that there is a set  $B'' \subseteq B \setminus B'$  that contradicts 2, then

$$\left| \bigcup_{b \in B''} \varphi(A \setminus A'; b) \right| < |B''|.$$

By the definition of  $A', B'$

$$\begin{aligned} \bigcup_{b \in B' \cup B''} \varphi(A; b) &= \bigcup_{b \in B'} \varphi(A; b) \cup \bigcup_{b \in B''} \varphi(A; b) \\ &= A' \cup \bigcup_{b \in B''} \varphi(A; b) \\ &= A' \cup \bigcup_{b \in B''} \varphi(A \setminus A'; b) \end{aligned}$$

The two sets above are disjoint, hence

$$\left| \bigcup_{b \in B' \cup B''} \varphi(A; b) \right| > |A'| + |B''|$$

As  $|A'| = |B'|$ , by the choice of  $A', B'$ , we obtain that  $B' \cup B''$  contradicts the inequality in 2. This prove the claim and with it the theorem.  $\square$

## 2 König's Minimax Theorem

Let  $\varphi(x; z)$  be given. Let  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  be finite sets.

A **matching** of  $\varphi(A; B)$  is a pair of sets  $A' \subseteq A$  and  $B' \subseteq B$  such that  $\varphi(A'; B')$  is the graph of a bijection between  $A'$  and  $B'$  in other words

$$|\varphi(A'; b)| = |\varphi(a; B')| = 1 \quad \text{for every } a, b \in A', B'.$$

Yet in other words,  $A'$  is a set of distinctive representatives for  $\varphi(A; B')$

We call  $|A'| = |B'|$  the cardinality of the matching. The **matching number** of  $\varphi(A; B)$  is the maximal cardinality of a matching.

Note that is  $A'$  is a set of distinct representatives for  $\varphi(A; B)$ , then there is a  $B' \subseteq B$  such that  $A', B'$ . Hence the matching number is less or equal than the cardinality of any set of distinct representatives (if it exists).

A **(vertex) cover** of  $\varphi(A; B)$  is a pair of sets  $A' \subseteq A$  and  $B' \subseteq B$  such that  $\varphi(A; B)$  is contained in  $(A' \times (B \setminus B')) \cup ((A \setminus A') \times B')$ . We will mainly use this property as characterized by the easy fact below.

**1.2 Fact** The following are equivalent

1.  $A', B'$  is a cover;
2.  $\varphi(A; b) \subseteq A'$  for every  $b \in B \setminus B'$ ;
3.  $\varphi(a; B) \subseteq B'$  for every  $a \in A \setminus A'$ .

$\square$

We call  $|A'| + |B'|$  the cardinality of the cover. The **cover number** of  $\varphi(A; B)$  is the minimal cardinality of a cover.

**1.3 Kőnig's Minimax Theorem** For any given  $\varphi(A; B)$ , matching number = cover number. That is, the maximal cardinality of a matching equals the minimal cardinality of a cover.

**Proof** ( $\leq$ ) We prove that  $|A''| \leq |A'| + |B'|$  for every cover  $A', B'$  and every matching  $A'', B''$ .

As  $\varphi(A; b) \subseteq A'$  for every  $b \in B \setminus B'$ , in particular we have that  $\varphi(A''; b) \subseteq A'$  for every  $b \in B'' \setminus B'$ . By the definition of matching, all these sets  $\varphi(A''; b)$  are distinct singletons. Hence  $|B''| - |B'| \leq |B \setminus B'| \leq |A'|$  is clear.

( $\geq$ ) Let  $A', B'$  be a cover of minimal cardinality. We prove that there is a matching of cardinality at least  $|A'| + |B'|$ .

We break  $\varphi(A; B)$  into two relations, find a matching of each of these and join them together to obtain a matching of cardinality  $\geq |A'| + |B'|$ . Precisely, first we show that  $\varphi(A \setminus A'; B')$  has a set of distinct representatives  $A_1 \subseteq A \setminus A'$ . Hence  $A_1, B'$  is a matching. Second, we apply the same argument shows that  $\varphi(A'; B \setminus B')^{\text{op}}$  has a set of distinct representatives  $B_2 \subseteq B \setminus B'$ . Hence  $A', B_2$  is a matching. Then  $(A_1 \cup A'), (B' \cup B_2)$  is a matching of  $\varphi(A; B)$ . The cardinality of this matching is  $|A_1| + |A'| = |B_2| + |A'| = |B'| + |A'|$ .

We use Hall's Marriage Theorem to prove the first claim above. The second is proved by the symmetric argument (using 3 of the fact above in place of 2).

We need to check that  $\varphi(A \setminus A'; B')$  satisfies 2 of Theorem 1.1. Suppose not. Then there is a set  $B'' \subseteq B'$  such that  $|A''| < |B''|$ , where

$$A'' = \bigcup_{b \in B''} \varphi(A \setminus A', b)$$

Then  $(A' \cup A''), (B' \setminus B'')$  would be a cover of cardinality  $< |A'| + |B'|$ . This contradicts the minimality of  $A', B'$ .  $\square$

### 3 Dilworth's Theorem

Dilworth's Theorem is minimax theorem essentially equivalent to Kőnig's Theorem. To highlight the connection we choose to prove it using Kőnig's Theorem. Alternatively we could have proved Dilworth's Theorem directly and derived Kőnig's and Hall's Theorem from it.

Let  $<$  be a strict partial order on  $\mathcal{U}$ . An **antichain** is a set  $A' \subseteq \mathcal{U}$  such that  $a < a'$  for every  $a, a' \in A'$ . A **chain** is a set  $A' \subseteq \mathcal{U}$  such that  $a < a' \vee a' < a$  for every distinct  $a, a' \in A'$ .

**1.4 Dilworth's Theorem** Let  $A \subseteq \mathcal{U}$  be finite. The maximal cardinality of an antichain  $A' \subseteq A$  equals the minimal cardinality of a partition of  $A$  into chains.

**Proof** ( $\leq$ ) We prove that the cardinality of an antichain cannot exceed the cardinality of a partition of  $A$  into chains.

Let  $A_1, \dots, A_k$  be a partition of  $A$  into chains and let  $A'$  be an antichain. A chain can contain at most one element of  $A'$ , hence  $|A'| \leq k$ .

( $\geq$ ) Let  $A' \subseteq A$  be an antichain of maximal cardinality. We prove that there is a partition  $A_1, \dots, A_k$  into chains for some  $k \leq |A'|$ .

Let  $\mathcal{V}$  be a disjoint copy of  $\mathcal{U}$ . Let  $f : \mathcal{U} \rightarrow \mathcal{V}$  the bijection that maps each element of  $\mathcal{U}$  to its copy in  $\mathcal{V}$ . For  $a, b \in \mathcal{U}$  such that  $a < b$  let  $\varphi(a; fb)$ . Let  $A_1, B_1$  be a cover of  $\varphi(A; f[A])$ . We claim that  $A \setminus (A_1 \cup f^{-1}[B_1])$  is an antichain. In fact, if  $a < b$  then either  $a \in A_1$  or  $fb \in B_1$ , by the definition of cover. This proves the claim.

As  $A'$  has maximal cardinality,  $|A| - |A'| \leq |A_1 \cup f^{-1}[B_1]| \leq |A_1 \cup B_1|$ . If we choose  $a$  over  $A_1, B_1$  of minimal cardinality, by König's Theorem there is a matching  $A'', B''$  of cardinality  $|A''| \geq |A_1 \cup B_1|$ . Hence  $|A''| \geq |A \setminus A'|$ .

We construct a chain-partition of  $A$  as follow. Pick an element of  $a_0 \in A''$  and construct the longest possible chain  $a_0, b_0, a_1, b_1, \dots, a_m, b_m, a_{m+1}$  where  $a_i \in A''$  for all  $i \leq m$ , and  $b_i \in B''$  is the (unique) element such that  $\varphi(a_i; b_i)$  and  $a_{i+1} \in A$  is the copy of  $b_i \in B''$ . The construction halts at the first  $a_{m+1} \notin A''$ . Then we start a new chain from some fresh element of  $A''$  until the chains  $a_0 < a_1 < \dots < a_m < a_{m+1}$  constructed in this way cover the whole of  $A''$ . Note that these chains are pairwise disjoint. Finally, put each element of  $A$  not covered by these chains in a chain on its own.

Notice that the elements of  $A''$  belongs to a chain of length at least 2. Therefore the number  $k$  of chains necessary to cover  $A$  is  $\leq |A| \setminus |A''| \leq |A'|$ .  $\square$

# Chapter 2

## Set systems

### 1 Sperner's Theorem

We say that  $\varphi(A; b)_{b \in \mathcal{V}}$  is an **antichain** if there is no pair of distinct elements  $b, b' \in \mathcal{V}$  such that  $\varphi(A; b) \subset \varphi(A; b')$ . Antichains are also called **Sperner systems**.

If all sets in  $\varphi(A; b)_{b \in \mathcal{V}}$  are distinct and of equal cardinality, then we clearly have an antichain. If  $|A| = n$ , the cardinality of a collection of subsets of  $A$ , all of cardinality  $k$ , is maximal when  $k = \lfloor n/2 \rfloor$  or  $k = \lceil n/2 \rceil$ . In this case

$$\begin{aligned} |\varphi(A; b)_{b \in \mathcal{V}}| &= \binom{n}{\lfloor n/2 \rfloor} \\ &= \binom{n}{\lceil n/2 \rceil}. \end{aligned}$$

By the following classical theorem, this bound holds for all antichain. This is one of the first results of external combinatorics (though the term has been coined a few years later).

**2.1 Sperner's Theorem** Let  $A \subseteq \mathcal{U}$  have cardinality  $n$ , finite. If  $\varphi(A; b)_{b \in \mathcal{V}}$  is an antichain then

$$|\varphi(A; b)_{b \in \mathcal{V}}| \leq \binom{n}{\lfloor n/2 \rfloor}.$$

**Proof** Clearly,  $\varphi(A; b)_{b \in \mathcal{V}}$  is the disjoint union of the sets  $\binom{A}{k} \cap \varphi(A; b)_{b \in \mathcal{V}}$  for  $k$  ranging over  $\{0, \dots, n\}$ . Then

$$|\varphi(A; b)_{b \in \mathcal{V}}| \leq \sum_{k=0}^n \left| \binom{A}{k} \cap \varphi(A; b)_{b \in \mathcal{V}} \right|.$$

As for every  $k \leq n$

$$\binom{n}{k} \leq \binom{n}{\lfloor n/2 \rfloor},$$

the theorem follows immediately from the LYM inequality that we prove below.  $\square$

The acronym LYM stands for Lubell-Yamamoto-Meshalkin.

**2.2 Lemma (LYM inequality)** Let  $A \subseteq \mathcal{U}$  have cardinality  $n$ , finite. If  $\varphi(A; b)_{b \in \mathcal{V}}$  is an antichain then

$$\sum_{k=0}^n \left| \binom{A}{k} \cap \varphi(A; b)_{b \in \mathcal{V}} \right| \cdot \binom{n}{k}^{-1} \leq 1.$$

**Proof** Let  $\Pi$  be uniform random variable that ranges over the set of permutations of  $A = \{a_1, \dots, a_n\}$ . For any  $\varphi(A; b)$  of cardinality  $k$



$$\mathbb{P}\left(\Pi\{a_1, \dots, a_k\} = \varphi(A; b)\right) = \binom{n}{k}^{-1}.$$

The events above are disjoint for distinct sets  $\varphi(A; b)$ , hence

$$\mathbb{P}\left(\Pi\{a_1, \dots, a_k\} \in \varphi(A; b)_{b \in \mathcal{V}}\right) = \left|\binom{A}{k} \cap \varphi(A; b)_{b \in \mathcal{V}}\right| \cdot \binom{n}{k}^{-1}.$$

As  $\varphi(A; b)_{b \in \mathcal{V}}$  is an antichain, for distinct  $k$  the events above are disjoint, hence

$$\mathbb{P}\left(\bigcup_{k=0}^n \Pi\{a_1, \dots, a_k\} \in \varphi(A; b)_{b \in \mathcal{V}}\right) = \sum_{k=0}^n \left|\binom{A}{k} \cap \varphi(A; b)_{b \in \mathcal{V}}\right| \cdot \binom{n}{k}^{-1}.$$

Now, the inequality is evident.  $\square$

Let  $\mathbb{P}_k$  be the probability measure on the subsets of  $A$  that is concentrated and uniform on  $A^{(k)}$ . Namely, for  $A' \subseteq A$

$$\mathbb{P}_k(\{A'\}) = \begin{cases} 0 & \text{if } |A'| \neq k \\ \binom{n}{k}^{-1} & \text{if } |A'| = k \end{cases}$$

Then the the LYM inequality asserts that if  $\varphi(A; b)_{b \in \mathcal{V}}$  is an antichain then

$$\sum_{k=0}^n \mathbb{P}_k(\varphi(A; b)_{b \in \mathcal{V}}) \leq 1.$$

This inequality is strict when  $\varphi(A; b)_{b \in \mathcal{V}} = A^{(k)}$  for some  $k$ . In the next section we show that these are the only cases.

## 2 The Erdős-Ko-Rado Theorem

**2.3 Lemma (Peter J. Cameron)** Let  $G$  be a 1-transitive finite graph. If  $G$  contains a clique of cardinality  $m$ , then every subgraph  $H \subseteq G$  contains a clique of cardinality

$$\geq m \frac{|H|}{|G|}.$$

**Proof** Let  $C$  be a clique in  $G$  of cardinality  $m$ . Let  $k$  the cardinality of the largest clique in  $H$ . Let  $n = |\text{Aut}(G)|$ . By 1-transitivity, the sets  $\{f \in \text{Aut}(G) : fa = b\}$ , for any fixed  $a \in G$  and  $b$  ranging over  $G$ , have all the same cardinality. Hence, for any given pair  $\langle a, b \rangle$ , they have cardinality  $n/|G|$ .

Count the pairs  $\langle a, f \rangle \in C \times \text{Aut}(G)$  such that  $fa \in H$ . For every  $a \in C$  there are  $n \cdot |H|$  automorphisms. So the number of pairs is  $m \cdot n \cdot |H|/|G|$

On the other hand for each  $f \in \text{Aut}(G)$  there are at most  $k$  choices of  $a \in C$ . So  $m \cdot n \cdot |H|/|G| \leq kn$ .  $\square$

**2.4 Erdős-Ko-Rado Theorem** Let  $A \subseteq \mathcal{U}$  be a finite set of cardinality  $n$ . Let  $k \leq n/2$ . Let  $\varphi(A; b)_{b \in \mathcal{V}}$  be an intersecting family of sets of cardinality  $k$ . Then

$$\left|\varphi(A; b)_{b \in \mathcal{V}}\right| \leq \binom{n-1}{k-1}.$$

**Proof** Let  $m = \left| \varphi(A; b)_{b \in \mathcal{V}} \right|$ . Consider the graph

$$G = \binom{A}{k},$$

$$E(G) = \left\{ \{A', A''\} : A' \cap A'' \neq \emptyset \right\}.$$

Enumerate the elements of  $A$ , say  $A = \{a_0, \dots, a_{n-1}\}$ . Consider the following subgraph of  $G$

$$H = \left\{ \{a_i, \dots, a_{i+k-1}\} : 0 \leq i < n \right\},$$

where the indices are intended modulo  $n$ . As  $k \leq n$ , the largest clique in  $H$  has cardinality  $k$ . As  $\varphi(A'; b)_{b \in \mathcal{V}}$  is a clique of  $G$ , by the lemma above,

$$k \geq m \frac{|H|}{|G|} = m \cdot n \cdot \binom{n}{k}^{-1}$$

therefore

$$m \leq \binom{n-1}{k-1} \quad \square$$

# Chapter 3

## Stability

### 1 The order property

The **chain index** of  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$ , or of  $\varphi(x; z)$  when  $\mathcal{U}$  and  $\mathcal{V}$  are clear, is the maximal length (that is,  $n + 1$ ) of a chain of the form

$$\text{ch} \quad \varphi(A; b_0) \subset \dots \subset \varphi(A; b_n)$$

for some set  $A \subseteq \mathcal{U}$  and some  $b_0, \dots, b_n \in \mathcal{V}$ . Note that we allow  $\varphi(A; b_0)$  to be empty. This choice produces a small asymmetry below in the definition of ladder; see also Fact 3.3.

**3.1 Example** If  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  consists of just one set, the chain index is 1. If it contains two distinct sets, the chain index is at least 2 and it is exactly 2 if there are no more two sets, or if all sets are disjoint.  $\square$

If a maximal length does not exist, we say that  $\varphi(x; z)$  is **unstable**, or that it has the **order-property**. Otherwise we say that it is **stable**.

In place of requiring the existence of the chain in  $\text{ch}$ , we could equivalently ask for a pair of tuples  $a_1, \dots, a_n \in \mathcal{U}$  and  $b_0, \dots, b_n \in \mathcal{V}$  such that

$$\text{ld} \quad \varphi(a_h; b_k) \Leftrightarrow h \leq k.$$

We call this pair of tuples a **ladder** of length  $n + 1$ . We may also say **ladder index** instead of chain index. Setting  $A = \{a_1, \dots, a_n\}$  we easily obtain a chain from a ladder, the converse is left as an easy exercise for the reader.

**3.2 Exercise** Let  $\varphi(x; z)$  have chain index  $n + 1$  or more. Let  $A \subseteq \mathcal{U}$  be a minimal set such that a chain as in  $\text{ch}$  obtains for some  $b_0, \dots, b_n \in \mathcal{V}$ . Prove that there is a ladder  $a_1, \dots, a_n$  and  $b_0, \dots, b_n$  such that  $A = \{a_1, \dots, a_n\}$  and conclude that  $\varphi(x; z)$  has ladder index  $n + 1$ .  $\square$

The following facts are obvious but worth noting.

**3.3 Fact** Let  $\varphi(x; z)$  have ladder index  $n + 1$ . Then  $\varphi(x; z)^{\text{op}}$  has ladder index  $\geq n$ .  $\square$

**3.4 Fact** Let  $\varphi(x; z)$  have ladder index  $n$ . Then  $\neg\varphi(x; z)$  has ladder index  $n$ .  $\square$

**Proof** If for all  $h \in [n]$  and  $k \in [n]$

$$\neg\varphi(a_h; b_k) \Leftrightarrow h \leq k$$

then  $a'_h = a_{n+1-h}$  and  $b'_k = b_{n-k}$  satisfy

$$\begin{aligned}
\varphi(a'_h; b'_k) &\Leftrightarrow \varphi(a_{n+1-h}; b_{n-h}) \\
&\Leftrightarrow n - k > n + 1 - h \\
&\Leftrightarrow h \leq k
\end{aligned}$$

□

The following definition is connected with those above, though in a less evident manner. We write  ${}^n2$  for the set of binary sequences of length  $n$  or, more precisely, the set of functions  $s : [n] \rightarrow [2]$ . We write  $s_h$  for the value of  $s$  at  $h$ , and  $s|_h$  for the restriction of  $s$  to  $[h]$ . We define  ${}^{<n}2 = \{r : r \in {}^h2, h \in [n]\}$ .

A **branching tree** of height  $n$  for the formula  $\varphi(x; z)$  is a function

$$\begin{aligned}
\bar{a} : {}^{<n}2 &\rightarrow \mathcal{U} \\
r &\mapsto a_r,
\end{aligned}$$

which we may also present by writing  $\bar{a} = \langle a_r : r \in {}^{<n}2 \rangle$ , such that

$$2r \quad \emptyset \neq \bigcap_{h=0}^{n-1} \neg^{s_h} \varphi(a_{s|_h}; \mathcal{V}) \quad \text{for all } s \in {}^n2.$$

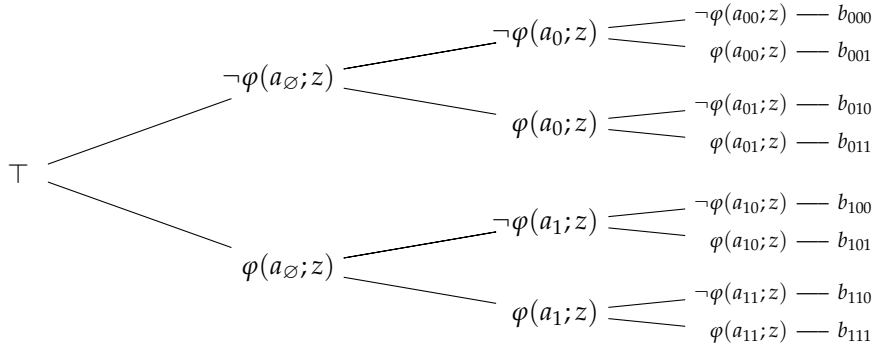
where  $\neg^i$ , for  $i$  a non negative integer, denotes a negation symbol repeated  $i$  times. In other words  $2r$  requires the existence of some  $\langle b_s : s \in {}^n2 \rangle$  such that

$$\varphi(a_{s|_h}; b_s) \Leftrightarrow s_h = 1 \quad \text{for all pairs } s \in {}^n2 \text{ and } h \in [n]$$

or, with slightly different notation,

$$\varphi(a_r; b_s) \Leftrightarrow r \cap 1 \subseteq s \quad \text{for all pairs } r \subset s \in {}^n2.$$

It helps to represent a branching tree as follows. For definiteness, fix  $n = 3$ . Consider a full binary tree of height  $n + 1$  and assign to each internal node (different from root and leaves) a formula as depicted below. Then  $2r$  requires that all formulas in each branch  $s \in 2^n$  are satisfied by some  $b_s \in \mathcal{V}$ .



The Shelah **2-rank** of  $\varphi(x; z)$  is the maximal height of a branching tree for  $\varphi(x; z)$ . If such a maximal integer does not exist, we say that the 2-rank is infinite.

**3.5 Example** If  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  consists of just one set, the only branching tree for  $\varphi(x; z)$  is the empty tree. Therefore the 2-rank is 1. If there are at least two distinct definable sets, then we can always find  $a_\emptyset, b_0, b_1$  such that  $\varphi(a_\emptyset, b_0) \wedge \varphi(a_\emptyset, b_1)$  hence the 2-rank is at least 2. □

A branching tree  $\bar{a}' = \langle a'_r : r \in {}^m 2 \rangle$  is a **subtree** of  $\bar{a}$  if there is an  $\subseteq$ -preserving map  $f : {}^m 2 \rightarrow {}^n 2$  such that  $a'_r = a_{fr}$ .

The theorem below needs the following Ramsey-like lemma.

**3.6 Lemma** Let  $\bar{a} = \langle a_r : r \in {}^{<n} 2 \rangle$  be a branching tree for  $\varphi(x; z)$ . Let  $k \in [n]$  be given. Then, for every red-blue coloring of  $\text{range}(\bar{a})$ , there is a monochromatic branching subtree, say  $\bar{a}' = \langle a'_r : r \in {}^{n'} 2 \rangle$ , and either all nodes of  $\bar{a}'$  are red and  $n' = k$ , or they are all blue and  $n' = n - k$ .

**Proof** Induction on  $n$ . First, note that the lemma is trivial when  $k = 0$  or  $k = n$ . In particular the lemma holds for  $n = 1$ .

Assume the lemma true for  $n$  and prove it for  $n + 1$ . Let  $\bar{a} = \langle a_r : r \in {}^{n+1} 2 \rangle$  be given. Fix some  $k \in [n + 1]$ . We want a branching tree  $\bar{a}'$  that is either red of height  $k$  or blue of height  $n + 1 - k$ .

If we discard the trivial cases, we can assume  $k \in (n]$ .

For  $i = 0, 1$  we define  $\bar{a}_i = \langle a_{i \smallfrown r} : r \in {}^{<n} 2 \rangle$ .

First suppose that  $a_\emptyset$  is blue. If, for either  $i = 0$  or  $i = 1$ , there is a red branching tree  $\bar{a}'_i$  of height  $k$ , we are done. Otherwise, for both  $i = 0, 1$  there is a blue branching tree  $\bar{a}'_i$  of height  $n - k$ . Then we graft these two trees on  $a_\emptyset$ , which is also blue, and obtain the required blue tree of height  $n - k + 1$ .

Now suppose that  $a_\emptyset$  is red. If, for either  $i = 0$  or  $i = 1$ , there is a blue branching tree  $\bar{a}'_i \subseteq \bar{a}_i$  of height  $n - (k - 1)$ , we are done. Otherwise, we graft on  $a_\emptyset$  two red trees of height  $k - 1$  to obtain a red tree of height  $k$ . □

We are now ready to characterize stability via the 2-rank. The proof is based on Hodges [7]. It is a direct proof which yields an explicit bound on the 2-rank given the ladder index (it yields also the converse, but this is easy). This bound may be far from optimal.

Shelah was the first to prove the equivalence  $1 \Leftrightarrow 2$  below. His proof is model theoretic and does not give explicit bounds. However it introduces some deep insight on stable formulas that we will present in the next section.

**3.7 Theorem** The following are equivalent

1.  $\varphi(x; z)$  is stable;
2.  $\varphi(x; z)$  has finite 2-rank.

Precisely, if  $n_{\text{ld}}$  and  $n_{2r}$  are the ladder index and the 2-rank, respectively, then

$$\begin{aligned} n_{\text{ld}} &< 2^{n_{2r}+1}; \\ n_{2r} &< 2^{n_{\text{ld}}+1} - 2. \end{aligned}$$

**Proof (2  $\Rightarrow$  1)** We prove the contrapositive. We show that if there is a ladder of length  $m = 2^n$ , say  $a_1, \dots, a_{m-1}$  and  $b_0, \dots, b_{m-1}$ , then there is a branching tree  $\bar{a}'$  of height  $n$ . This also proves the first inequality above. In fact, if  $n_{\text{ld}} \geq 2^{n_{2r}+1}$ , there

would exist a branching tree of height  $n_{2r} + 1$  which is a contradiction.

The branching tree  $\bar{a}' = \langle a'_r : r \in {}^{<n}2 \rangle$  is defined as follows

$$a'_r = a_h \quad \text{where } h \text{ is obtained reading } r \frown 1 \frown 0^{n-|r|-1} \text{ as an } n\text{-digit binary number.}$$

To verify  $2r$  we define for  $s \in {}^n2$

$$b'_s = b_k \quad \text{where } k \text{ is obtained reading } s \text{ as an } n\text{-digit binary number.}$$

Then it is easy to verify that for all pairs  $r \subset s \in {}^n2$

$$\begin{aligned} \varphi(a'_r; b'_s) &\Leftrightarrow \varphi(a_h; b_k) && \text{where } h \text{ and } k \text{ are like above} \\ &\Leftrightarrow h \leq k \\ &\Leftrightarrow r \frown 1 \frown 0^{n-|r|-1} \leq s && \text{as } n\text{-digit binary numbers} \\ &\Leftrightarrow r \frown 1 \subseteq s \end{aligned}$$

(1 $\Rightarrow$ 2) We prove the contrapositive. We claim that if there is a branching tree  $\bar{a}$  of height  $2^n - 2$  then there is a ladder  $a_1, \dots, a_{n-1}$  and  $b_0, \dots, b_{n-1}$ , of length  $n$ , such that  $a_1, \dots, a_{n-1} \in \text{range}(\bar{a})$  and  $b_0, \dots, b_{n-1} \in \{b_s : s \in {}^{2^n-2}2\}$ . This yields also the second inequality of the theorem. In fact, if  $n_{2r} \geq 2^{n_{\text{ld}}+1} - 2$ , there would exist a ladder of length  $n_{\text{ld}} + 1$  which is a contradiction.

As  $n_{\text{ld}} \geq 1$ , we start the induction from  $n = 2$ . In this case the claim is witnessed by ladder  $a_\emptyset$  and  $b_0, b_1$ . Now we assume the claim is true for  $n$  and prove it for  $n + 1$ .

Let  $\langle a_r : r \in {}^{<2m+2}2 \rangle$ , where  $m = 2^n - 2$ , be a branching tree of height  $2^{n+1} - 2$ . To each  $b \in \{b_s : s \in {}^{2m+2}2\}$  we associate a red-blue coloring of  $\text{range}(\bar{a})$  as follows. A node  $a \in \text{range}(\bar{a})$  is colored

$$\begin{aligned} \text{red} &\quad \text{if } \varphi(a; b) \text{ holds;} \\ \text{blue} &\quad \text{otherwise, that is, } \neg\varphi(a; b). \end{aligned}$$

We consider two cases that are exhaustive by Lemma 3.6. Note that we are applying the lemma only to the subtree  $\langle a_{1 \frown r} : r \in {}^{2m+1}2 \rangle$ .

Case 1: for some  $b$  there is a red subtree of  $\langle a_{1 \frown r} : r \in {}^{2m+1}2 \rangle$  of height  $m + 1$ . Let  $\bar{a}'$  be this red tree and consider its subtree  $\langle a'_{0 \frown r} : r \in {}^{<m}2 \rangle$ . By induction hypothesis, there are  $A \subseteq \{a'_{0 \frown r} : r \in {}^{<m}2\}$  and  $b_0, \dots, b_{n-1}$  such that

$$(1) \quad \varphi(A; b_0) \subset \dots \subset \varphi(A; b_{n-1})$$

Let  $A' = A \cup \{a'_\emptyset\}$  then

$$(2) \quad \varphi(A'; b_0) \subset \dots \subset \varphi(A'; b_{n-1})$$

In fact, as  $b_0, \dots, b_{n-1} \in \neg\varphi(a'_\emptyset; \mathcal{V})$ , this is the same chain as (1). Therefore, if we extend the chain on the right with  $\varphi(A'; b) = A'$ , we obtain the required chain of length  $n + 1$ .

Case 2: for every  $b$  there is a blue subtree of  $\langle a_{1 \frown r} : r \in {}^{2m+1}2 \rangle$  of height  $m$ . Pick any  $b \in \{b_s : s \in {}^{2m+2}2\}$  such that  $\neg\varphi(a_\emptyset, b)$  and let  $\bar{a}'$  the corresponding blue subtree. Apply the induction hypothesis to obtain  $A \subseteq \text{range}(\bar{a}')$  and  $b_0, \dots, b_{n-1}$  such that (1). We claim that (2) above holds with  $A' = A \cup \{a_\emptyset\}$ . In fact,  $b_0, \dots, b_{n-1} \in \varphi(a_\emptyset; \mathcal{V})$  so (2) is the chain in (1) with all sets augmented by  $a_\emptyset$ . We can extend the

chain on the left with  $\varphi(A'; b) = \emptyset$  and obtain the required chain of length  $n + 1$ .  $\square$

## 2 Approximable sets

The notion of approximable set is a combinatorial counterpart of the model theoretical notion of externally definable set.

**3.8 Definition** We say that  $\mathcal{A} \subseteq \mathcal{U}$  is **approximable** if for every finite set  $A \subseteq \mathcal{U}$  there is a  $b \in \mathcal{V}$  such that  $\varphi(A; b) = \mathcal{A} \cap A$ . If we also have that  $\varphi(\mathcal{U}; b) \subseteq \mathcal{A}$ , then we say that  $\mathcal{A}$  is approximable **from below**.  $\square$

The following is immediate.

**3.9 Fact** The following are equivalent for every  $\mathcal{A} \subseteq \mathcal{U}$

1.  $\mathcal{A}$  is approximable from below;
2. for every finite set  $A \subseteq \mathcal{A}$  there is a  $b \in \mathcal{V}$  such that  $A \subseteq \varphi(\mathcal{U}; b) \subseteq \mathcal{A}$ .  $\square$

Towards the main theorem of this section we prove three separated lemmas.

**3.10 Lemma** Let  $\varphi(x; z)$  be a stable formula. Every set  $\mathcal{A} \subseteq \mathcal{U}$  approximable by  $\varphi(x; z)$  is approximable from below by the formula

$$\psi(x; z_0, \dots, z_n) = \bigwedge_{i=0}^n \varphi(x; z_i)$$

where  $n + 1$  is the ladder index of  $\varphi(x; z)$ .

**Proof** Let  $A \subseteq \mathcal{A}$  be finite. We prove that  $A \subseteq \psi(\mathcal{U}; b_0, \dots, b_n) \subseteq \mathcal{A}$  for some  $b_0, \dots, b_n$ . To obtain the  $b_k$ , we construct a ladder inductively. Pick any  $b_0$  be such that  $A \subseteq \varphi(\mathcal{U}; b_0)$ , which we can do by approximability. Now, suppose that  $a_1, \dots, a_k$  and  $b_0, \dots, b_k$  have been defined. If

$$A \subseteq \bigcap_{i=0}^k \varphi(\mathcal{U}; b_i) \subseteq \mathcal{A}$$

we set  $b_{k+1} = \dots = b_n = b_k$  and stop as we already have the required parameters. Otherwise pick any element

$$a_{k+1} \in \bigcap_{i=0}^k \varphi(\mathcal{U}; b_i) \setminus \mathcal{A}$$

and some  $b_{k+1}$  such that

$$A \subseteq \varphi(\mathcal{U}; b_{k+1}) \subseteq \mathcal{U} \setminus \{a_1, \dots, a_{k+1}\}.$$

Such a parameter  $b_{k+1}$  exists because  $\mathcal{A}$  is approximable (apply Definition 3.8 with  $A \cup \{a_1, \dots, a_{k+1}\}$  for  $A$ ). Note that at stage  $n$  we have constructed a ladder of length  $n$  for  $\neg\varphi(x; z)$ . In fact for all  $h \in (n)$  and  $k \in [n]$  we have

$$\neg\varphi(a_h; b_k) \Leftrightarrow h \leq k.$$

As  $\varphi(x; z)$  has ladder index  $n + 1$ , by Fact 3.4 it is not possible to further prolong this chain, hence

$$A \subseteq \bigcap_{i=0}^n \varphi(\mathcal{U}; b_i) \subseteq \mathcal{A}$$

as required.  $\square$

We prove that the formula  $\psi(x; z_0, \dots, z_n)$  in the lemma above is itself stable, though with a larger ladder index.

**3.11 Lemma** Let  $\varphi_i(x; z)$ , where  $i = 1, \dots, m$ , be formulas with ladder index  $n_i$ . Let

$$\varphi(x; z) = \bigwedge_{i=1}^m \varphi_i(x; z)$$

Then  $\varphi(x; z)$  has ladder index  $< R(n_1 + 1, \dots, n_m + 1)$ , the Ramsey number for  $m$ -colorings.

**Proof** Suppose for a contradiction that there is a ladder  $a_1, \dots, a_n$  and  $b_0, \dots, b_n$ , where  $n = R(n_1 + 1, \dots, n_m + 1) - 1$ . Let  $C_i$  contains the pairs  $\{h, k\}$  such that  $\neg \varphi_i(a_h; b_k)$  and  $0 \leq k < h \leq n$ . Then from Id we obtain

$$\bigcup_{i=1}^m C_i = \binom{[n]}{2}$$

By the definition of  $n$ , for some  $i \in [m]$ , there is a set  $H$  of cardinality  $n_i + 1$  such that  $H^{(2)} \subseteq C_i$ . Assume  $i=1$  for definiteness.

Write  $a'_1, \dots, a'_{n_1}$  and  $b'_0, \dots, b'_{n_1}$  for the tuples obtained by restricting  $a_1, \dots, a_n$  and  $b_0, \dots, b_n$  to the indexes in  $H$ . These tuples witness that  $\varphi_1(x; z)$  has ladder index at least  $n_1 + 1$ , which contradicts the assumption of the lemma.  $\square$

To apply the proposition to the formula  $\psi(x; z_0, \dots, z_n)$  in the lemma above, let  $z = z_0, \dots, z_n$  and let  $\varphi_i(x; z) = \varphi(x; z_i)$ . That is,  $\varphi_i(x; z)$  defines a relation between  $\mathcal{U}$  and  $\mathcal{V}^{n+1}$  that depends only on the  $i$ -th coordinate. The ladder index of every  $\varphi_i(x; z)$  is the same as  $\varphi(x; z_0)$  as a relation between  $\mathcal{U}$  and  $\mathcal{V}$ .

**3.12 Lemma** If  $\mathcal{A}$  is approximated from below by a stable formula  $\varphi(x; z)$  then for some  $b_0, \dots, b_n \in \mathcal{V}$

$$\mathcal{A} = \bigcup_{i=0}^n \varphi(\mathcal{U}; b_i)$$

where  $n + 1$  is the ladder index of  $\varphi(x; z)$ .

**Proof** Let  $A \subseteq \mathcal{U}$  be finite. To obtain the  $b_k$ , we construct a ladder by stages. Pick any  $b_0$  be such that  $\varphi(\mathcal{U}; b_0) \subseteq \mathcal{A}$ . Now, suppose that  $a_1, \dots, a_k$  and  $b_0, \dots, b_k$  have been defined. If

$$\mathcal{A} \subseteq \bigcup_{i=1}^k \varphi(\mathcal{U}; b_i)$$

we set  $b_{k+1} = \dots = b_n = b_k$  and stop. As the construction guarantees the converse inclusion, we have the required parameters. Otherwise pick any element



$$a_{k+1} \in \mathcal{A} \setminus \bigcup_{i=1}^k \varphi(\mathcal{U}; b_i)$$

and some  $b_{k+1}$  such that

$$\{a_1, \dots, a_{k+1}\} \subseteq \varphi(\mathcal{U}; b_{k+1}) \subseteq \mathcal{A}.$$

Such a parameter  $b_{k+1}$  exists because  $\mathcal{A}$  is approximable from below. Note that at stage  $n$  we have constructed a ladder of length  $n$  for  $\varphi(x; z)$ . In fact for all  $h \in (n]$  and  $k \in [n]$  we have

$$\neg \varphi(a_h; b_k) \Leftrightarrow h \leq k.$$

As  $\varphi(x; z)$  has ladder index  $n + 1$ , it is not possible to prolonge this chain, hence

$$\mathcal{A} = \bigcup_{i=0}^n \varphi(\mathcal{U}; b_i)$$

as required.  $\square$

The main theorem of this section is an immediate corollary is the three lemmas above.

**3.13 Theorem** Let  $\varphi(x; z)$  be stable. Then there are  $m, n$  such every set  $\mathcal{A} \subseteq \mathcal{U}$  approximable by  $\varphi(x; z)$  is definable by  $\psi(x; \bar{b})$ , where

$$\psi(x; \bar{z}) = \bigvee_{j=0}^m \bigwedge_{i=0}^n \varphi(z; z_{i,j})$$

and  $\bar{b} \in \mathcal{V}^{n \times m}$  is some  $n \times m$ -tuple of parameters. The numbers  $m, n$  only depend on the ladder index of  $\varphi(x; z)$ .  $\square$

We conclude this section with the sketch of an argument that shows that stable formulas have finite 2-rank, that is,  $1 \Rightarrow 2$  in Theorem 3.7.

As a matter of fact, below we prove an apparently weaker claim

# if there is a branching tree of height  $\omega$ , say  $\{a_s : s \in 2^{<\omega}\}$ , the ladder index is infinite.

In general an infinite 2-rank does not always implies the existence of a branching tree of height  $\omega$ . Nevertheless, by standard model theoretic argument of compactness, this suffices to obtain  $1 \Rightarrow 2$  in Theorem 3.7. Model theory is beyond the scope of this notes, hence here we only present the argument that proves #.

First note that Theorem 3.13 remains valid when  $\mathcal{U}$  is replaced by a subset. In fact, the same formula  $\psi(x; \bar{z})$  works for all  $\mathcal{U}' \subseteq \mathcal{U}$  because the ladder index of  $\varphi(x; z)$  does not decrease when moving to a subset.

For  $s \in 2^\omega$ , let  $\mathcal{U}_s = \{a_{s \upharpoonright n} : n < \omega\} \subseteq \mathcal{U}$ . Let  $\mathcal{A}_s = \{a_{s \upharpoonright n} : n < \omega, s_n = 1\}$ . Then  $\mathcal{A}_s \subseteq \mathcal{U}_s$  is approximable with respect to the collection  $\varphi(\mathcal{U}_s; b)_{b \in \mathcal{V}}$ . Therefore, from the theorem above, we obtain  $\mathcal{A}_s = \psi(\mathcal{U}_s; \bar{b}_s)$  for some  $\bar{b}_s \in \mathcal{V}^{n \times m}$ , where the formula  $\psi(x; \bar{z})$  does non depend on  $s$ .

For  $s \neq s'$ , let  $n$  be least such that  $s_n \neq s'_n$ . Then  $a_{s \upharpoonright n} = a_{s' \upharpoonright n}$  belongs to  $\mathcal{U}_s \cap \mathcal{U}_{s'}$  and

to  $\mathcal{A}_s \triangle \mathcal{A}_{s'}$ . Therefore  $\bar{b}_s \neq \bar{b}_{s'}$ , hence there is a continuum of such tuples  $\bar{b}_s$ . On the other hand, without loss of generality we can assume that  $\mathcal{V}$  is a countable set. A contradiction.

# Chapter 4

## Vapnik-Chervonenkis theory

### 1 The Vapnik-Chervonenkis dimension

If all subsets of  $A \subseteq \mathcal{U}$  are definable, that is  $\mathcal{P}A = \varphi(A, b)_{b \in \mathcal{V}}$  we say that  $A$  is **shattered** by  $\varphi(x; z)$ . The following is called the **shatter function**

$$\pi_\varphi(n) = \max \left\{ |\varphi(A, b)_{b \in \mathcal{V}}| : A \in \binom{\mathcal{U}}{n} \right\}$$

So,  $\pi_\varphi(n)$  gives the maximal number of definable subsets that a set of cardinality  $n$  can have. Trivially,  $\pi_\varphi(n) \leq 2^n$  for all  $n$ . Moreover, if  $\pi_\varphi(n) = 2^n$  for some  $n$ , then  $\pi_\varphi(k) = 2^k$  for every  $k \leq n$ .

The **Vapnik-Chervonenkis dimension** of  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$ , or of  $\varphi(x; z)$ , abbreviated by **VC-dimension**, is the maximal cardinality of a finite set  $A \subseteq \mathcal{U}$  that is shattered by  $\varphi(x; z)$ . Equivalently, it is the maximal  $k$  such that  $\pi_\varphi(k) = 2^k$ . If such a maximum does not exist, we say that the VC-dimension is infinite or that  $\varphi(x; z)$  has **IP** (the independence property). Otherwise, we say that  $\varphi(x; z)$  has **NIP** (not the independence property). We may also say: *is IP*, or *is NIP*.

As  $\mathcal{U}$  and  $\mathcal{V}$  are usually clear from the context, we may say VC-dimension of  $\varphi(x; z)$  for the VC-dimension of  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$ .

- 4.1 Example** If  $\varphi(x; z)$  is either  $\top$  or  $\perp$ , then it shatters only the empty set, therefore it has VC-dimension 0. □
- 4.2 Example** If  $\varphi(x; z)$  has ladder dimension  $n$  then it has VC-dimension at most  $n$ . Hence stable formulas are NIP. □
- 4.3 Example** If  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  is a non trivial chain of sets, then its VC-dimension is 1. □
- 4.4 Example** Let  $\mathcal{U} = \mathbb{R}$  and  $\mathcal{V} = \mathbb{R}^2$ . Let  $\varphi(x; z_1, z_2)$  be the formula  $z_1 < x < z_2$ . Then its VC-dimension 2. □
- 4.5 Example** Let  $\mathcal{U} = \mathcal{V} = \mathbb{R}^2$ . Let  $\varphi(x_1, x_2; z_1, z_2)$  be the formula  $y < z_1 \cdot x + z_2$ . Then its VC-dimension 3 (by Radon's Theorem). □
- 4.6 Example** If  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  is the set of all subsets of  $\mathcal{U}$  of cardinality  $\leq k$ . Then its VC-dimension is  $k$  and

$$\pi_\varphi(n) = \sum_{i=0}^k \binom{n}{i}.$$

Incidentally, we note that this is also the shatter function of the collection of all subsets of  $\mathcal{U}$  of cardinality exactly  $k$ . In fact we always assume  $\mathcal{U}$  is infinite (or at least very large, in this case, size  $\geq 2k$  suffices).  $\square$

The VC-dimension of  $\varphi(x; z)^{\text{op}}$  is called the **dual VC-dimension** or **VC-codimension** of  $\varphi(x; z)$ .

**4.7 Proposition** If  $\varphi(x; z)$  has VC-dimension  $k$ , then its VC-codimension is  $\leq 2^{k+1} - 1$ .

**Proof** Suppose that the VC-dimension of  $\varphi(x; z)^{\text{op}}$  is  $\geq 2^{k+1}$ . We prove that the VC-dimension of  $\varphi(x; z)$  is at least  $k + 1$ . Let  $B = \{b_I : I \subseteq [k + 1]\}$  be a set of cardinality  $2^{k+1}$  shattered by  $\varphi(x; z)^{\text{op}}$ . That is, for every  $\mathcal{J} \subseteq \mathcal{P}[k + 1]$  there is  $a_{\mathcal{J}}$  such that

$$\varphi(a_{\mathcal{J}}, b_I) \Leftrightarrow I \in \mathcal{J} \quad \text{for all } I \subseteq [k + 1]$$

Let  $a_i = a_{\{I : i \in I\}}$ . Then from the equivalence above we obtain

$$\varphi(a_i, b_I) \Leftrightarrow i \in I$$

That is,  $\varphi(x; z)$  shatters  $A = \{a_i : i \in [k + 1]\}$ .  $\square$

We prove that the bound in the proposition above is optimal.

**4.8 Proposition** For every  $k$ , there is a formula  $\varphi(x; z)$  that has VC-dimension  $k$  and VC-codimension  $2^{k+1} - 1$ .

**Proof** Let  $k$  be given. We claim that there is a formula  $\varphi(x; z)$  with VC-dimension  $2^{k+1} - 1$  and VC-codimension  $k$ . As the dual of the dual is the primal, this claim is equivalent to the proposition. Let  $\mathcal{U}$  an infinite set, let  $\mathcal{V} = \mathcal{P}(\mathcal{U})$ . Fix an equivalence relation on  $\mathcal{U}$  with  $2^{k+1} - 1$  many classes. We say that  $b \in \mathcal{V}$  is compatible if it is union of equivalence classes. Define

$$\varphi(x; z) = x \in z \wedge z \text{ is compatible}$$

(Recall that in the preliminaries to this notes, we have agreed that  $\mathcal{U}$  and  $\mathcal{V}$  are some fixed infinite sets. Without this constraint we could have taken as  $\mathcal{U}$  a set of cardinality  $2^{k+1} - 1$ ; defined  $\varphi(x; z) = x \in z$ ; and forgot the equivalence relation altogether.)

Clearly,  $\varphi(x; z)$  has VC-dimension  $2^{k+1} - 1$ . Note that the dimension of  $\varphi(x; z)^{\text{op}}$  is at least  $k$ . Otherwise  $\varphi(x; z)$  would have dimension  $\leq 2^k - 1$  by Proposition 4.7.

So, it suffices to prove that the dimension of  $\varphi(x; z)^{\text{op}}$  is exactly  $k$ . Assume for a contradiction that some  $B = \{b_i : i \in [k + 1]\} \subseteq \mathcal{V}$  is shattered by  $\varphi(x; z)^{\text{op}}$ . Then there is some set  $A = \{a_I : I \subseteq [k + 1]\} \subseteq \mathcal{U}$  such that

$$\varphi(a_I, b_i) \Leftrightarrow i \in I$$

The  $b_i \in B$  are necessarily compatible sets, therefore any two distinct  $a_I \in A$  are non equivalent. But this is impossible by cardinality reasons.  $\square$

**4.9 Exercise** Let  $\varphi(x; z)$  have VC-dimension  $k$ . Assume that there is no  $A \subseteq \mathcal{U}$  of car-

dinality  $\leq k$  such that  $\varphi(a; \mathcal{V})_{a \in A}$  covers  $\mathcal{V}$ . Prove that  $\varphi(x; z)$  has VC-codimension  $\leq 2^{k+1} - 2$ . Prove that the bound is optimal. Hint: let  $\mathcal{U}$  be any infinite set  $\mathcal{V} = \mathcal{U}^{(\leq m)}$ , where  $m = 2^{k+1} - 2$ . Let  $\varphi(x; z) = x \in z$ . Prove that  $\varphi(x; z)$  has VC-codimension  $k$ .  $\square$

## 2 The Sauer-Shelah lemma

According to Gil Kalai in [8], Sauer-Shelah's Lemma can be described as an *eigen-theorem* because it is important in many different areas of mathematic (model theory, learning theory, probability theory, ergodic theory, Banach spaces, to name a few). No wonder it has been discovered and rediscovered many times.

It has been proved independently by Shelah [13], Sauer [12], and Vapnik-Chervonenkis [14] around 1970 (Shelah gives credit to Micha Perles). Saharon Shelah was working in model theory while Norbert Sauer, Vladimir Vapnik and Alexey Chervonenkis were in statistical learning theory.

**4.10 Sauer-Shelah Lemma** If  $\varphi(x; z)$  has VC-dimension  $k$  then for every  $n \geq k$

$$\pi_{\varphi}(n) \leq \sum_{i=0}^k \binom{n}{i}.$$

The set system presented in Example 4.6 shows that the bound is optimal.

An alternative proof of the Sauer-Shelah Lemma derives it as corollary of a lemma by Alain Pajor [11].

**4.11 Pajor's Lemma** Let  $A \subseteq \mathcal{U}$  be finite.

$$|\varphi(A, b)_{b \in \mathcal{V}}| \leq \left| \{C \subseteq A : C \text{ is shattered by } \varphi(x; z)\} \right|.$$

**Proof** If  $A$  is empty then  $|\varphi(A, b)_{b \in \mathcal{V}}| = 1$  and  $\emptyset$  is the only subset of  $A$  that  $\varphi$  shatters, so the inequality holds trivially. Otherwise, pick an  $a \in A$  and assume the lemma holds for  $A' = A \setminus \{a\}$ . Define

$$\psi(x; y) = \varphi(x; y) \wedge \neg \varphi(a; y) \wedge \exists y' [\varphi(a; y') \wedge \varphi(A'; y') = \varphi(A'; y)].$$

Notice that

$$|\varphi(A, b)_{b \in \mathcal{V}}| = \left| \varphi(A', b)_{b \in \mathcal{V}} \cup \left\{ \{a\} \cup \psi(A', b) : b \in \mathcal{V} \right\} \right|.$$

as the two sets in the r.h.s. are disjoint

$$|\varphi(A, b)_{b \in \mathcal{V}}| = |\varphi(A', b)_{b \in \mathcal{V}}| + |\psi(A', b)_{b \in \mathcal{V}}|.$$

By induction hypothesis,

$$|\varphi(A', b)_{b \in \mathcal{V}}| \leq \left| \{C \subseteq A' : C \text{ is shattered by } \varphi(x; z)\} \right| \tag{1}$$

and

$$|\psi(A', b)_{b \in \mathcal{V}}| \leq \left| \{C \subseteq A' : C \text{ is shattered by } \psi(x; z)\} \right|$$

$$= \left| \{C \subseteq A' : C \cup \{a\} \text{ is shattered by } \varphi(x; z)\} \right|. \quad (2)$$

In fact,  $C \subseteq A'$  is shattered by  $\varphi(x; y)$  if and only if  $C \cup \{a\}$  is shattered by  $\varphi(x; y)$ . Clearly,

$$(1) + (2) = \left| \{C \subseteq A : C \text{ is shattered by } \varphi(x; z)\} \right|,$$

so the lemma follows.  $\square$

**Proof of the Sauer-Shelah Lemma** Given  $\varphi(x; z)$  and  $n \geq k$  as in the lemma

$$\begin{aligned} \pi_\varphi(n) &= \max_{|A|=n} |\varphi(A, b)_{b \in \mathcal{V}}| \\ \pi_\varphi(n) &\leq \max_{|A|=n} |\{C \subseteq A : C \text{ shattered by } \varphi(x; z)\}| \quad \text{by Pajor's Lemma} \\ &\leq \sum_{i=0}^k \binom{n}{i} \quad \text{because } \varphi(x; z) \text{ has VC-dimension } k \end{aligned} \quad \square$$

We write  $f(n) = O(g(n))$  if there is a constant  $C$  such that  $|f(n)| \leq Cg(n)$  holds for all (sufficiently large)  $n$ .

The **VC-density** of  $\varphi(x; z)$  is the infimum over all real number  $r$  such that  $\pi_\varphi(n) = O(n^r)$ . It is infinite if no such  $r$  exist. The **dual VC-density** is defined accordingly.

By the Sauer-Shelah lemma the VC-density is at most as large as the VC-dimension. It could be smaller, however it is usually rather difficult to compute.

We conclude this section with a couple of inequalities that is useful to have at hand.

$$\begin{aligned} \sum_{i=0}^k \binom{n}{i} &= \sum_{i=0}^k \frac{n!}{i!(n-i)!} \\ &\leq \sum_{i=0}^k \frac{n^i}{i!} \\ &\leq \sum_{i=0}^k \frac{n^i k!}{i!(k-i)!} \\ &= \sum_{i=0}^k n^i \binom{k}{i} \\ &= (n+1)^k \quad \text{by the binomial theorem.} \end{aligned}$$

There is a second bound, which is better when  $k \geq 3$  and holds for  $n > k$

$$\begin{aligned} \sum_{i=0}^k \binom{n}{i} &\leq \left(\frac{n}{k}\right)^k \sum_{i=0}^k \left(\frac{k}{n}\right)^i \binom{n}{i} \quad \text{because } \frac{k}{n} < 1 \\ &\leq \left(\frac{n}{k}\right)^k \sum_{i=0}^n \left(\frac{k}{n}\right)^i \binom{n}{i} \\ &= \left(\frac{n}{k}\right)^k \left(1 + \frac{k}{n}\right)^n \quad \text{by the binomial theorem} \\ &\leq \left(\frac{ne}{k}\right)^k \quad \text{where } e \text{ is the base of the natural logarithm.} \end{aligned}$$

## Chapter 5

### Law(s) of large numbers

Quoting from some unpublished notes by Carlos C. Rodríguez

What is a Law of Large Numbers? I am glad you asked! The Laws of Large Numbers, or LLNs for short, come in three basic flavors: Weak, Strong and Uniform. They all state that the observed frequencies of events tend to approach the actual probabilities as the number of observations increases. Saying it in another way, the LLNs show that under certain conditions, we can asymptotically learn the probabilities of events from their observed frequencies. To add some drama we could say that if God is not cheating and S/he doesn't change the initial standard probabilistic model too much then, in principle, we (or other machines, or even the universe as a whole) could eventually find out the Truth, the whole Truth, and nothing but the Truth.

Bull! The Devil, is in the details.

I suspect that for reasons not too different in spirit to the ones above, famous minds of the past took the slippery slope of defining probabilities as the limits of relative frequencies. They became known as “frequentist”. They wrote books and indoctrinated generations of confused students.

#### 1 Inequalities

Throughout this and the next section we work with a given probability space  $\mathcal{U}, \mathbb{P}$ . For simplicity, the following two propositions are proved for discrete  $\mathbb{P}$ , but they are easily seen to hold in general.

**5.1 Definition** A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **convex** if for every tuples of real numbers  $p_i$  and  $x_i$  such that

$$\sum_{i=1}^n p_i = 1$$

we have

$$f\left(\sum_{i=1}^n p_i x_i\right) \leq \sum_{i=1}^n p_i f(x_i).$$

□

Note that, though the definition is usually given with  $n = 2$ , the general property above follows easily.

**5.2 Jensen's Inequality** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a convex function. Then

$$f(\mathbb{E}[X]) \leq \mathbb{E}[f(X)]$$

**Proof** For simplicity, assume that the sample space  $\mathcal{U}$  is finite. Then the claim is obvious from the definition.  $\square$

The following is arguably the most basic inequality in probability theory. Although it is almost trivial, it will be required several times in this chapter.

**5.3 Markov's Inequality** Let  $X$  be a nonnegative random variable with finite mean. Then for every  $\varepsilon > 0$

$$\mathbb{P}(X \geq \varepsilon) \leq \frac{\mathbb{E}[X]}{\varepsilon}$$

**Proof** For simplicity, assume that the sample space  $\mathcal{U}$  is finite. (The theorem holds in general, but we only need the finite case.) Define  $A = \{a \in \mathcal{U} : X(a) \geq \varepsilon\}$ .

$$\begin{aligned} \mathbb{E}[X] &= \sum_{a \in \mathcal{U}} \mathbb{P}(a) X(a) \\ &= \sum_{a \in A} \mathbb{P}(a) X(a) + \sum_{a \notin A} \mathbb{P}(a) X(a) \\ &\geq \sum_{a \in A} \mathbb{P}(a) X(a) \\ &\geq \varepsilon \sum_{a \in A} \mathbb{P}(a) \\ &= \varepsilon \mathbb{P}(X \geq \varepsilon) \end{aligned}$$

$\square$

**5.4 Corollary** Let  $X$  be a nonnegative random variable. If  $\mathbb{E}[X^k]$  exists, then for every  $\varepsilon > 0$

$$\mathbb{P}(X \geq \varepsilon) \leq \frac{\mathbb{E}[X^k]}{\varepsilon^k}$$

**Proof** By Markov's inequality, since  $\mathbb{P}(X \geq \varepsilon) = \mathbb{P}(X^k \geq \varepsilon^k)$ .  $\square$

Chebyshev's inequality (a.k.a. Chebysheff, Chebyshov, Tschebyscheff, Tschebycheff) is a special case of the corollary above.

**5.5 Chebyshev's Inequality** Let  $X$  be a random variable with finite mean and variance. Then for every  $\varepsilon > 0$

$$\mathbb{P}(|X - \mathbb{E}[X]| \geq \varepsilon) \leq \frac{\text{Var}[X]}{\varepsilon^2}$$

$\square$

To obtain exponential bounds, we frequently apply the following trick.

**5.6 Chernoff's method** Let  $X$  be a random variable with finite mean. Then for every  $t > 0$

$$\mathbb{P}(X \geq \varepsilon) \leq e^{-t\varepsilon} \mathbb{E}[e^{tX}]$$



**Proof** For every  $t > 0$

$$\begin{aligned} \mathbb{P}(X \geq \varepsilon) &= \mathbb{P}(e^{tX} \geq e^{t\varepsilon}) && \text{because } e^{tx} \text{ is increasing} \\ &\leq e^{-t\varepsilon} \mathbb{E}[e^{tX}], \end{aligned}$$

by Markov's inequality, which we may apply since  $e^{tX}$  is always positive.  $\square$

**5.7 Hoeffding's lemma** Let  $X$  be a bounded random variable, say  $a \leq X \leq b$ . Let  $\mathbb{E}[X] = \mu$  and  $d = b - a$ . Then

$$\mathbb{E}[e^{t(X-\mu)}] \leq \exp\left(\frac{t^2 d^2}{8}\right).$$

**Proof** For clarity, assume  $\mu = 0$ . The general result follows easily from this special case by centralization. Recall that, by convexity, for every  $x \in [a, b]$

$$e^{tx} \leq \frac{x-a}{d} e^{tb} + \frac{b-x}{d} e^{ta}$$

Then

$$e^{tX} \leq \frac{X-a}{d} e^{tb} + \frac{b-X}{d} e^{ta}$$

By the linearity of expectation,

$$\begin{aligned} \mathbb{E}[e^{tX}] &\leq \frac{b e^{ta} - a e^{tb}}{d} \\ \log \mathbb{E}[e^{tX}] &\leq \log \frac{b e^{ta} - a e^{tb}}{d} \end{aligned}$$

taking the Taylor series expansion of the r.h.s. at  $t = 0$  (the first and second derivatives vanish at 0; the second derivative is always  $\leq d^2/4$ ) we obtain

$$\log \mathbb{E}[e^{tX}] \leq \frac{t^2 d^2}{8}. \quad \square$$

**5.8 Hoeffding's Inequality** Let  $X_1, \dots, X_n$  be independent random variables with bounded range, say  $a \leq X_i \leq b$ . Define  $d = b - a$ .

$$M = \sum_{i=1}^n (X_i - \mathbb{E}[X_i])$$

Then for every  $\varepsilon > 0$

$$\begin{aligned} \mathbb{P}(M \geq \varepsilon) &\leq \exp\left(-\frac{2\varepsilon^2}{nd^2}\right), \\ \mathbb{P}(M \leq -\varepsilon) &\leq \exp\left(-\frac{2\varepsilon^2}{nd^2}\right). \end{aligned}$$

Clearly, the two inequalities above imply the following

$$\mathbb{P}(|M| \geq \varepsilon) \leq 2 \exp\left(-\frac{2\varepsilon^2}{nd^2}\right).$$

**Proof** Define  $\mathbb{E}[X_i] = \mu_i$ . Let  $t > 0$  be arbitrary.

$$\begin{aligned} \mathbb{P}(M \geq \varepsilon) &\leq e^{-t\varepsilon} \mathbb{E}[e^{tM}] && \text{by Chernoff's method (5.6)} \\ &= e^{-t\varepsilon} \prod_{i=1}^n \mathbb{E}[e^{t(X_i - \mu_i)}] && \text{by independence.} \end{aligned}$$

$$\begin{aligned}
&\leq e^{-t\epsilon} \prod_{i=1}^n \exp\left(\frac{t^2 d^2}{8}\right) && \text{by Hoeffding's Lemma (5.7).} \\
&= \exp\left(\frac{n t^2 d^2}{8} - t\epsilon\right)
\end{aligned}$$

Now substitute  $4\epsilon/nd^2$  for  $t$ . □

We prove Hoeffding's lemma with a slightly weaker bound (2 for 8). The purpose is to present two clever tricks *ghost sample* and *symmetrization* which in the following section is applied in a more complex setting.

First we need the following lemma. A **random sign variable** (a.k.a. Rademacher random variable) is a random variable  $\sigma \in \{-1, 1\}$  with mean 0.

**5.9 Lemma** Let  $\sigma$  be a random sign variable. Then for every  $t$

$$\mathbb{E}\left[e^{t\sigma}\right] \leq e^{t^2/2}$$

**Proof** Replace  $e^x$  with its Taylor expansion around  $x = 0$

$$\begin{aligned}
\mathbb{E}\left[e^{t\sigma}\right] &= \sum_{i=0}^{\infty} \frac{t^i \mathbb{E}[\sigma^i]}{i!} \\
&= \sum_{i=0}^{\infty} \frac{t^{2i}}{(2i)!} && \text{since } \mathbb{E}[\sigma^i] = \begin{cases} 1 & i \text{ even} \\ 0 & i \text{ odd} \end{cases} \\
&= \sum_{i=0}^{\infty} \frac{(t^2/2)^i}{i!} \\
&= e^{t^2/2}.
\end{aligned}$$
□

**5.10 Second proof of Hoeffding's Lemma** Recall that Hoeffding's Lemma claims that, if  $a \leq X \leq b$ , then

$$\mathbb{E}\left[e^{t(X-\mu)}\right] \leq \exp\left(\frac{t^2 d^2}{8}\right),$$

where  $\mathbb{E}[X] = \mu$  and  $d = b - a$ . Here we prove a marginally weaker bound (2 in place of 8).

Let  $X'$  be an independent copy of  $X$  (a.k.a. ghost sample). In particular  $\mu = \mathbb{E}(X')$ . Then

$$\begin{aligned}
\mathbb{E}\left[e^{t(X-\mu)}\right] &= \mathbb{E}\left[e^{t(X-\mathbb{E}[X'])}\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[e^{t(X-X')} \mid X\right]\right] && \text{by Jensen's inequality} \\
&\leq \mathbb{E}\left[e^{t(X-X')}\right]
\end{aligned}$$

Let  $\sigma$  be a random sign variable independent of  $X, X'$ . Then  $\sigma(X - X')$  has the same distribution of  $X - X'$ .

$$\begin{aligned}
&= \mathbb{E}\left[e^{t\sigma(X-X')}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{t\sigma(X-X')} \mid X, X'\right]\right]
\end{aligned}$$

$$\begin{aligned}
&\leq \mathbb{E}\left[e^{t^2(X-X')^2/2}\right] && \text{by Lemma 5.9} \\
&\leq e^{t^2 d^2/2} && \text{because } |X - X'| \leq d.
\end{aligned}$$

This yields the bound above with 2 in place of 8.  $\square$

## 2 Two Weak Laws of Large Numbers

A **sample**  $s$  is a sequence  $s_1, \dots, s_n$  of elements of  $\mathcal{U}$ . Its length  $|s| = n$  is also called **size** or **dimension**. We write  $\text{range}(s)$  for the set  $\{s_1, \dots, s_n\}$ . Note that this set may have cardinality  $< n$ .

To a sample  $s$  of size  $n$  we associate a finite probability measure on the subsets of  $\mathcal{U}$  namely, for any event  $A \subseteq \mathcal{U}$ , we define the empirical frequency of  $A$  given  $s$

$$\text{Fr}(s, A) = \frac{1}{n} \cdot |\{i : s_i \in A\}|.$$

It is convenient to rewrite it using indicator functions

$$\text{Fr}(s, A) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{s_i \in A}.$$

We are interested in the *existence* of samples that approximate the probability within  $\varepsilon$ . Suppose that, for a given event  $A$ , we can prove that

$$(1) \quad \mathbb{P}\left(s \in \mathcal{U}^n : |\text{Fr}(s, A) - \mathbb{P}(A)| \geq \varepsilon\right) \leq \text{some\_bound}(\varepsilon, n)$$

and that, for  $n$  large enough,  $\text{some\_bound}(\varepsilon, n) < 1$ . Then a sample of size  $\leq n$  that approximate the probability within  $\varepsilon$  is guaranteed to exist.

Random variables are convenient formalism to discuss these probabilities. We say **random element** of  $\mathcal{U}$  for a random variables  $S$  such that  $\mathbb{P}(S \in A) = \mathbb{P}(A)$  for every  $A \subseteq \mathcal{U}$ . A **random sample** from  $\mathcal{U}$  is a tuple  $S = S_1, \dots, S_n$  of independent random elements of  $\mathcal{U}$ . Then  $\mathbb{I}_{S_i \in A} = \mathbb{I}_A \circ S_i$  is a Bernoulli random variable with probability of success  $\mathbb{P}(A)$  and

$$\text{Fr}(S, A) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{S_i \in A}$$

is (up to the factor  $1/n$ ) a binomial random variable. This random variable is used below to find the bound in (1), in fact

$$\mathbb{P}\left(|\text{Fr}(S, A) - \mathbb{P}(A)| \geq \varepsilon\right)$$

equals the probability in (1) but is easier to estimate.

**5.11 Weak Law of Large Numbers** For every event  $A \subseteq \mathcal{U}$  and every tuple  $S = S_1, \dots, S_n$  of independent random elements of  $\mathcal{U}$

$$\mathbb{P}\left(|\text{Fr}(S, A) - \mathbb{P}(A)| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2}.$$

**Proof** The random variable  $\text{Fr}(S, A)$  has expected value  $\mathbb{P}(A)$  and variance  $\leq 1/n$ . By Chebyshev's inequality we obtain

$$\mathbb{P}\left(\left|\text{Fr}(S, A) - \mathbb{P}(A)\right| \geq \varepsilon\right) \leq \frac{1}{n\varepsilon^2}$$

which proves the theorem.  $\square$

Sometime we are interested in the minimal size of a sample that approximates the probability up to a given  $\varepsilon$ .

**5.12 Corollary** Assume  $\mathcal{U}$  is finite (of arbitrary cardinality, tough). For every  $A \subseteq \mathcal{U}$  and every  $\varepsilon > 0$  there is a sample  $s$  of size

$$|s| = \left\lfloor \frac{1}{\varepsilon^2} + 1 \right\rfloor$$

such that

$$\left|\text{Fr}(s, A) - \mathbb{P}(A)\right| < \varepsilon.$$

**Proof** By the Weak Law of Large Numbers above, a sample of size  $n$  exists if

$$\frac{1}{n\varepsilon^2} < 1$$

$\square$

In the following section we need a better bound for the Weak Law of Large Numbers. This is obtained with a similar proof.

**5.13 Weak Law of Large Numbers (with exponential bound)** For every event  $A \subseteq \mathcal{U}$  and every tuple  $S = S_1, \dots, S_n$  of independent random elements of  $\mathcal{U}$

$$\mathbb{P}\left(\left|\text{Fr}(S, A) - \mathbb{P}(A)\right| \geq \varepsilon\right) \leq 2e^{-2n\varepsilon^2}.$$

**Proof** Define

$$M = \sum_{i=1}^n \left( \mathbb{I}_{S_i \in A} - \mathbb{E}[\mathbb{I}_{S_i \in A}] \right)$$

As  $\mathbb{E}[\mathbb{I}_{S_i \in A}] = \mathbb{P}(A)$ , the inequality we have to prove can be rewritten as

$$\mathbb{P}\left(|M| \geq n\varepsilon\right) \leq 2e^{-2n\varepsilon^2}$$

and this follows immediately from Hoeffding inequality.  $\square$

Using the exponential bounds above, we can improve (by a constant factor) the size of the minimal sample size that approximates the probability obtained in Corollary 5.12.

**5.14 Corollary** For every  $A \subseteq \mathcal{U}$  and every  $\varepsilon > 0$  there is a sample  $s$  of size  $n$  where

$$n = \left\lfloor \frac{\log 2}{2\varepsilon^2} + 1 \right\rfloor$$

such that

$$\left|\text{Fr}(s, A) - \mathbb{P}(A)\right| < \varepsilon.$$

$\square$

### 3 A Uniform Law of Large Numbers

Throughout this section we work with a fixed family of definable subsets  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  that are events of the sample space  $\mathcal{U}, \mathbb{P}$ . It is convenient to introduce some abbreviations

$$\begin{aligned}\mathbb{P}(b) &= \mathbb{P}(\varphi(\mathcal{U}; b)) \\ \text{Fr}(s, b) &= \text{Fr}(s, \varphi(\mathcal{U}; b)) \\ \mathbb{I}_{s,b} &= \mathbb{I}_{\varphi(s;b)}\end{aligned}$$

An  **$\varepsilon$ -approximation** is a sample  $s$  such that

$$\left| \text{Fr}(s, b) - \mathbb{P}(b) \right| < \varepsilon \quad \text{for every } b \in \mathcal{V}.$$

We are interested in estimating the minimal size of an  $\varepsilon$ -approximation.

The main theorem of this section is this famous result of Vapnik-Chervonenkis [14].

**5.15 Vapnik-Chervonenkis Inequality** Let  $\pi_\varphi(n)$  be the shatter function of  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$ . Let  $S = S_1, \dots, S_n$  be a random sample from  $\mathcal{U}$ . Then, for every  $b \in \mathcal{V}$

$$\mathbb{P} \left( \left| \text{Fr}(S, b) - \mathbb{P}(b) \right| \geq \varepsilon \right) \leq 6 \pi_\varphi(n) \exp \left( - \frac{n\varepsilon^2}{32} \right).$$

N.B. Some technical measure-theoretical assumptions are necessary when  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  is uncountable. These have been omitted in the statement above, and will be discussed below.

**Proof** Let  $S' = S'_1, \dots, S'_n$  be an independent copy of  $S$ . Then, by the triangular inequality

$$\mathbb{P} \left( \left| \text{Fr}(S, b) - \mathbb{P}(b) \right| \geq \varepsilon \right) \leq \mathbb{P} \left( \left| \text{Fr}(S, b) - \text{Fr}(S', b) \right| \geq \frac{\varepsilon}{2} \right) + (*)$$

where,

$$(*) = \mathbb{P} \left( \left| \text{Fr}(S', b) - \mathbb{P}(b) \right| \geq \frac{\varepsilon}{2} \right)$$

which, by the Weak Law of Large Numbers 5.13,

$$\leq 2 e^{-n\varepsilon^2/2}$$

Let  $\sigma = \sigma_1, \dots, \sigma_n$  be a tuple of independent sign random variables. Then

$$\begin{aligned}\mathbb{P} \left( \left| \text{Fr}(S, b) - \text{Fr}(S', b) \right| \geq \frac{\varepsilon}{2} \right) &= \mathbb{P} \left( \left| \sum_{i=1}^n \mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b} \right| \geq \frac{n\varepsilon}{2} \right) \\ &= \mathbb{P} \left( \left| \sum_{i=1}^n \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \right| \geq \frac{n\varepsilon}{2} \right)\end{aligned}$$

Then, again by the triangular inequality

$$\leq 2 \mathbb{P} \left( \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{S_i, b} \right| \geq \frac{n\varepsilon}{4} \right)$$

Putting together the inequalities above we obtain

$$\mathbb{P} \left( \sup_{b \in \mathcal{V}} |\text{Fr}(S, b) - \mathbb{P}(b)| \geq \varepsilon \right) \leq 2 \mathbb{P} \left( \sup_{b \in \mathcal{V}} \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{S_i, b} \right| \geq \frac{n\varepsilon}{4} \right) + 2e^{-n\varepsilon^2/2}$$

Let  $s = s_1, \dots, s_n$  be a possible realization of  $S$ .

$$(1) \quad \mathbb{P} \left( \sup_{b \in \mathcal{V}} \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{s_i, b} \right| \geq \frac{n\varepsilon}{4} \right) = \mathbb{P} \left( \sup_{b \in \mathcal{V}} \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{s_i, b} \right| \geq \frac{n\varepsilon}{4} \mid S = s \right)$$

Finally, note that the r.h.s. of (1) only depends on  $\varphi(\{s_1, \dots, s_n\}; b)$ . Hence the  $\sup_{b \in \mathcal{V}}(\cdot)$  on the r.h.s. is actually a maximum among  $m = \pi_\varphi(n)$  events. Say, we can choose  $b_1, \dots, b_m \in \mathcal{V}$  such that

$$(2) \quad = \mathbb{P} \left( \max_{j \leq m} \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{s_i, b_j} \right| \geq \frac{n\varepsilon}{4} \right)$$

Note that, in general, for any real random variables  $X_1, \dots, X_m$  we have

$$\begin{aligned} \mathbb{P} \left( \max_{i \leq m} X_i \geq \varepsilon \right) &= \mathbb{P} \left( \bigcup_{i=1}^m X_i \geq \varepsilon \right) \\ &\leq \sum_{i=1}^m \mathbb{P}(X_i \geq \varepsilon) \end{aligned}$$

Hence, continuing from (2) we obtain

$$\begin{aligned} &\leq \sum_{j=1}^m \mathbb{P} \left( \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{s_i, b_j} \right| \geq \frac{n\varepsilon}{4} \right). \\ &\leq 2\pi_\varphi(n) \exp \left( -\frac{n\varepsilon^2}{32} \right), \end{aligned}$$

where the last inequality is obtained from Hoeffding's Inequality 5.8. In fact, Hoeffding's Inequality, applied to  $X_i = \sigma_i \mathbb{I}_{s_i, b}$  with  $n\varepsilon/4$  for  $\varepsilon$ , yields

$$\mathbb{P} \left( \left| \sum_{i=1}^n \sigma_i \mathbb{I}_{s_i, b} \right| \geq \frac{n\varepsilon}{4} \right) \leq 2 \exp \left( -\frac{n\varepsilon^2}{32} \right).$$

Unsurprisingly, the bound does not depend on  $b$ . Finally, proceeding from (1) we obtain

$$\begin{aligned} &\leq 4\pi_\varphi(n) \exp \left( -\frac{n\varepsilon^2}{32} \right) + 2 \exp \left( -\frac{n\varepsilon^2}{2} \right) \\ &\leq 6\pi_\varphi(n) \exp \left( -\frac{n\varepsilon^2}{32} \right), \end{aligned}$$

which finally proves the theorem.  $\square$

**5.16 Corollary** Let  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  have finite VC-dimension. For every  $\varepsilon > 0$  there is a finite sample  $s$  such that

$$\left| \text{Fr}(s, b) - \mathbb{P}(b) \right| < \varepsilon \quad \text{for every } b \in \mathcal{V}.$$

**Proof** It suffices to require that  $n = |s| = |\mathcal{S}|$  is large enough to guarantee

$$\mathbb{P} \left( \left| \text{Fr}(S, b) - \mathbb{P}(b) \right| \geq \varepsilon \right) < 1 \quad \text{for every } b \in \mathcal{V}.$$

By the Vapnik-Chervonenkis inequality 5.15, it suffices that

$$(3) \quad 6 \pi_\varphi(n) \exp\left(-\frac{n\varepsilon^2}{32}\right) < 1$$

By the Sauer-Shelah Lemma 4.10,  $\pi_\varphi(n)$  grows polynomially. Hence the inequality holds for  $n$  large enough.  $\square$

The corollary above is sufficient for our intended applications. For completeness, the following proposition gives an explicit bound.

**5.17 Proposition** There is a sample  $s$  as in the corollary above of size

$$n \leq c \frac{k}{\varepsilon^2} \log \frac{k}{\varepsilon}$$

where  $c$  is an absolute constant and  $k$  is the VC-dimension of  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$ .

**Proofsketch** By (3) in the proof above and the inequality proved after the Sauer-Shelah Lemma 4.10 it suffices that  $n = |s|$  satisfies

$$\log 6 + k \log(n+1) < \frac{n\varepsilon^2}{32},$$

which is the case if  $n$  satisfies the following inequality

$$c' \frac{k}{\varepsilon^2} < \frac{n}{\log n},$$

for some absolute constant  $c'$ . Finally, the latter inequality is satisfied if

$$c'' \frac{k}{\varepsilon^2} \log \frac{k}{\varepsilon^2} < n$$

for some absolute constant  $c''$ , see the exercise below.  $\square$

**5.18 Exercise** Prove that for all  $x, y > 1$

$$2x \log x < y \Rightarrow x < \frac{y}{\log y}$$

$\square$

## 4 A Uniform Law of Large Numbers, again

We prove a second version of the Vapnik-Chervonenkis Inequality. Which, I conjecture, is due to Devroye and Lugosi [5].

**5.19 Vapnik-Chervonenkis Inequality (2)** Let  $\pi_\varphi(n)$  be the shatter function of  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$ . Let  $S = S_1, \dots, S_n$  be a random sample from  $\mathcal{U}$ . Then, for every  $b \in \mathcal{V}$

$$\mathbb{E} \left| \text{Fr}(S, b) - \mathbb{P}(b) \right| \leq 2 \sqrt{\frac{\log(2 \pi_\varphi(n))}{n}}.$$

$\square$

The same caveat on measurability apply as for Inequality 5.15.

We note that the bound is not optimal, using a clever techniques called *chaining*, Dudley could prove that

$$\mathbb{E} \left| \text{Fr}(S, b) - \mathbb{P}(b) \right| < c \sqrt{\frac{k}{n}},$$

where  $k$  is the VC-dimension and  $c$  is absolute constant.

Before embarking in the proof of the theorem above, we prove the following (easy, although mysterious) lemma, which also has independent interest.

**5.20 Lemma** Let  $X_1, \dots, X_m$  be real valued random variables. Let  $c$  be such that

$$\mathbb{E}[e^{tX_i}] \leq e^{c^2 t^2 / 2} \quad \text{for every } i \leq m \text{ and every } t > 0.$$

Then

$$\mathbb{E}\left[\max_{i \leq m} X_i\right] \leq c\sqrt{2 \log m}.$$

If in addition

$$\mathbb{E}[e^{-tX_i}] \leq e^{c^2 t^2 / 2} \quad \text{for every } i \leq m \text{ and every } t > 0,$$

then

$$\mathbb{E}\left[\max_{i \leq m} |X_i|\right] \leq c\sqrt{2 \log(2m)}.$$

**Proof** By Jensen's inequality,

$$\begin{aligned} \exp\left(t \cdot \mathbb{E}\left[\max_{i \leq m} X_i\right]\right) &\leq \mathbb{E}\left[\exp\left(\max_{i \leq m} tX_i\right)\right] \\ &= \mathbb{E}\left[\max_{i \leq m} e^{tX_i}\right] \\ &\leq \mathbb{E}\left[\sum_{i \leq m} e^{tX_i}\right] \\ &= \sum_{i \leq m} \mathbb{E}[e^{tX_i}] \\ &\leq m e^{c^2 t^2 / 2} \end{aligned}$$

Taking the logarithm of both sides and replacing  $t$  with  $\frac{\sqrt{2 \log m}}{c}$ , we obtain the first inequality of the lemma.

To prove the second inequality, apply the first one to  $X_1, \dots, X_m, -X_1, \dots, -X_m$ . (N.B. note that independence is not assumed.)  $\square$

**Proof of the Vapnik-Chervonenkis inequality** Let  $S' = S'_1, \dots, S'_n$  be an independent copy of  $S$ . We claim that

$$(1) \quad \mathbb{E}\left[\sup_{b \in \mathcal{V}} |\text{Fr}(S, b) - \mathbb{P}(b)|\right] \leq \mathbb{E}\left[\sup_{b \in \mathcal{V}} |\text{Fr}(S, b) - \text{Fr}(S', b)|\right]$$

In fact,

$$\begin{aligned} \text{Fr}(S, b) - \mathbb{P}(b) &= \text{Fr}(S, b) - \mathbb{E}[\text{Fr}(S', b)] \\ &= \mathbb{E}[\text{Fr}(S, b) - \text{Fr}(S', b) \mid S]. \end{aligned}$$

Now, apply Jensen's inequality to the absolute value function, then use that

$$(2) \quad \sup_{b \in \mathcal{V}} \mathbb{E}[\dots] \leq \mathbb{E}[\sup_{b \in \mathcal{V}}(\dots)].$$

Write  $\mathbb{I}_b$  for the indicator function of  $\varphi(\mathcal{U}; b)$ . Then

$$|\text{Fr}(S, b) - \text{Fr}(S', b)| = \frac{1}{n} \left| \sum_{i=1}^n (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \right|$$



Let  $\sigma = \sigma_1, \dots, \sigma_n$  be a tuple of independent sign random variable. The random variable  $\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}$  has the same distribution of  $\sigma_i(\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b})$  hence

$$= \frac{1}{n} \mathbb{E} \left[ \sum_{i=1}^n \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \mid S, S' \right]$$

Inserting this into (1) we obtain

$$\mathbb{E} \left[ \sup_{b \in \mathcal{V}} | \text{Fr}(S, b) - \mathbb{P}(b) | \right] \leq \frac{1}{n} \mathbb{E} \left[ \sup_{b \in \mathcal{V}} \mathbb{E} \left[ \sum_{i=1}^n \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \mid S, S' \right] \right]$$

Let  $s, s'$  be a generic realization of  $S, S'$

$$\leq \frac{1}{n} \sup_{s, s'} \sup_{b \in \mathcal{V}} \mathbb{E} \left[ \sum_{i=1}^n \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \right]$$

and, by what remarked in (2)

$$\leq \frac{1}{n} \sup_{s, s'} \mathbb{E} \left[ \sup_{b \in \mathcal{V}} \left| \sum_{i=1}^n \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \right| \right]$$

Observe that once  $s, s'$  is fixed,  $\sup_{b \in \mathcal{V}}$  is actually a maximum among  $\pi_\varphi(2n)$  sets, in fact,  $\pi_\varphi(2n)$  is the number of definable subsets of  $A = \{s_1, \dots, s_n, s'_1, \dots, s'_n\}$ . Then, by Lemma 5.20 (the second inequality, with  $m = \pi_\varphi(2n)$  and  $i$  ranging over the definable subsets of  $A$ ), for an appropriate constant  $c$ ,

$$\leq \frac{1}{n} \sup_{s, s'} c \sqrt{2 \log (2 \pi_\varphi(2n))}.$$

As the r.h.s. does not depend on  $s, s'$ ,

$$\leq \frac{c}{n} \sqrt{2 \log (2 \pi_\varphi(2n))}$$

Finally, as  $\pi_\varphi(2n) \leq \pi_\varphi(n)^2$ ,

$$\leq \frac{2c}{n} \sqrt{\log (2 \pi_\varphi(n))}$$

The Vapnik-Chervonenkis inequality is proved if we can show the assumption of Lemma 5.20 holds with  $c = \sqrt{n}$ .

$$\mathbb{E} \left[ \exp \left( t \sum_{i=1}^n \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \right) \right] = \prod_{i=1}^n \mathbb{E} \left[ \exp \left( t \sigma_i (\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b}) \right) \right]$$

As  $\sigma_i(\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b})$  takes values in  $\{-1, 1\}$  with mean 0, by Lemma 5.9

$$\leq e^{nt^2/2}$$

and the same holds for  $-\sigma_i(\mathbb{I}_{S_i, b} - \mathbb{I}_{S'_i, b})$ . □

As an application we prove the Glivenko-Cantelli Theorem, an important theorem of mathematical statistics. The theorem says that the empirical cumulative distribution function converges uniformly to the true one. We prove an informative variant which gives the rate of convergence (though, this is not optimal).

**5.21 Glivenko-Cantelli Theorem** Let  $X = X_1, \dots, X_n$  be i.i.d. random variables. Let  $F(z) = \mathbb{P}(X_i \leq z)$  be their common cumulative distribution function. Let  $F_e(x)$  be the empirical cumulative distribution function, that is,

$$F_e(X, z) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}_{X_i \leq z}.$$

Then, for every  $z \in \mathbb{R}$ ,

$$\mathbb{E} \left| F_e(X, z) - F(z) \right| \leq 2 \sqrt{\frac{\log(2n+2)}{n}}. \quad \square$$

**Proof** Take  $\mathcal{U} = \mathcal{V} = \mathbb{R}$  and  $\varphi(x; z) = x \leq z$ . Assign to  $\varphi(\mathcal{U}; b) = (-\infty, b]$  probability  $\mathbb{P}(X_i \leq b)$ . Then, if  $S = S_1, \dots, S_n$  is a random sample from  $\mathcal{U}$

$$\mathbb{E} \left| F_e(X, z) - F(z) \right| = \mathbb{E} \left| \text{Fr}(S, b) - \mathbb{P}(b) \right|$$

hence, by the Vapnik-Chervonenkis inequality [5.15](#)

$$\leq \sqrt{2 \frac{\log(2 \pi_\varphi(n))}{n}}.$$

It is clear that  $\varphi(\mathcal{U}; b)_{b \in \mathcal{V}}$  has VC-dimension 1. By the Sauer-Shelah Lemma [4.10](#) and the inequalities proved thereafter,  $\pi_\varphi(n) \leq n + 1$ . The theorem follows.  $\square$

# Chapter 6

## Small transversals

Unlike the rest of these notes, this chapter is not self contained, as it relies on the duality of linear programming. The reader can use the result as a black box. Otherwise, we recommend [10, Chapter 6], a lively and conceptual introduction to linear programming (a rarity for an otherwise rather dry subject).

### 1 Transversals and packings

Let  $\varphi(x; z)$  be given. Let  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  be finite sets.

A subset  $A' \subseteq A$  is a **transversal** if  $\varphi(A', b) \neq \emptyset$  for every  $b \in B$ . Equivalently, if the sets  $\varphi(a, B)_{a \in A'}$  cover  $B$ , i.e.

$$B = \bigcup_{a \in A'} \varphi(a, B).$$

The **transversal number** is the smallest cardinality of a transversal  $A'$ . It denoted by  $\tau$ .

A subset  $B' \subseteq B$  is a **packing** if  $\varphi(a, b) \cap \varphi(a, b') = \emptyset$  for every distinct  $b, b' \in B'$ . Equivalently if  $|\varphi(a, B')| \leq 1$  for every  $a \in A$ . The **packing number** is the largest cardinality of a packing  $B'$ . It is denoted by  $\nu$ .

We may write  $\tau_{\varphi(A, B)}$  and  $\nu_{\varphi(A, B)}$  when ambiguity is of concern.

If  $A'$  is a transversal and  $B' \subseteq B$ , then the sets  $\varphi(a, B')_{a \in A'}$  cover  $B'$ . Now, suppose  $B'$  is a packing, then these sets contain at most one element, hence  $|B'| \leq |A'|$ . Therefore, we always have  $\nu \leq \tau$ . Very little can be said in general about the reverse direction.

**6.1 Example** Let  $\mathcal{U} = \mathbb{R}^2$  and  $\mathcal{V}$  is the set of lines in  $\mathbb{R}^2$ . Let  $\varphi(x; z)$  be the incidence (that is, membership) relation. Let  $A \subseteq \mathcal{U}$  and let  $B \subseteq \mathcal{V}$  be a set of  $n$  lines in generic position (any two lines intersect and every point is contained in at most two lines). Then  $\tau = \lceil n/2 \rceil$ , as each point belongs to at most two lines, while  $\nu = 1$ , as any two lines intersect.  $\square$

A (fractional) multiset over  $\mathcal{U}$  is a real-valued function  $A' : \mathcal{U} \rightarrow \mathbb{R}$ . The support **support** of a multiset  $A'$  is the set where it takes nonzero values. In this chapter will only consider nonnegative multisets with finite support. These can be interpreted as measures concentrated on a finite set.

If  $A'' : \mathcal{U} \rightarrow \mathbb{R}$  is another multiset, we write  $A' \cdot A''$  for the pointwise product of the two. We write  $A' \leq A''$  if  $A'(a) \leq A''(a)$  for every  $a \in \mathcal{U}$ .

We define the **size** of  $A'$  to be

$$|A'| = \sum_{a \in \mathcal{U}} A'(a)$$

We write  $\varphi(A', b)$  for the multiset  $A' \cdot \mathbb{I}_{\varphi(\mathcal{U}, b)}$ .

Multisets over  $\mathcal{V}$  are defined analogously.

A **fractional transversal** is a multiset  $A' \leq \mathbb{I}_A$  such that  $|\varphi(A', b)| \geq 1$  for every  $b \in B$ . The **fractional transversal number** of  $\varphi(x; z)$ , denoted by  $\tau^*$ , is the infimum of the size of the fractional transversals of  $\varphi(x; z)$ .

A fractional multiset  $B' \leq \mathbb{I}_B$  over  $\mathcal{V}$  is a **fractional packing** if  $|\varphi(a, B')| \leq 1$  for every  $a \in A$ . The **fractional packing number** of  $\varphi(x; z)$ , denoted by  $\nu^*$ , is the supremum of the size of the fractional packings of  $\varphi(x; z)$ .

**6.2 Example** The sets  $\mathcal{U}, \mathcal{V}$  and the relation  $\varphi(A; B)$  are as in the example 6.1. Let  $B'$  be a multiset that assigns  $1/2$  to every line in  $B$ . Then  $|\varphi(a, B')| \leq 1$  holds because each point is contained in at most two lines. Then  $\nu^* \geq |B'| = n/2$ . It is easy to see that  $\tau^* \geq n/2$ . We claim that  $\tau \leq n/2$ . If  $n$  is even, use the same transversal  $A'$  as in Example 6.1 is even. If  $n$  is odd, take 3 any lines, and assign  $1/2$  to the three intersection points. Proceed as in the even case with the other lines.  $\square$

**6.3 Exercise** Let  $\mathcal{U} = \mathbb{R}$  and  $\mathcal{V}$  is a set of finitely many closed intervals. Let  $\varphi(x; z)$  be the membership relation. Then  $\nu = \tau$ . Hint: use induction on  $\nu$ .  $\square$

**6.4 Theorem** For all  $\varphi(x; z)$  and all finite sets  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$ , we have  $\nu^* = \tau^*$  and this value is rational.

**Proof** Let  $A = \{a_1, \dots, a_m\}$  and  $B = \{b_1, \dots, b_n\}$ . Let  $F$  be the  $m \times n$ -matrix with entries  $\mathbb{I}_{\varphi(a_i, b_j)}$ . A multi-set over  $A$  is a naturally associated to a vector  $0 \leq x \in \mathbb{R}^m$ . A multi-set over  $B$  is associated to a vector  $0 \leq y \in \mathbb{R}^n$ . Then it is easy to verify that

$$\begin{aligned} \tau^* &= \inf \{ 1_m^T x : F^T x \geq 1_n, 0 \leq x \}; \\ \nu^* &= \sup \{ 1_n^T y : F y \leq 1_m, 0 \leq y \}. \end{aligned}$$

Therefore, by the duality of linear programming  $\nu^* = \tau^*$ .

As  $\tau^*$  is the minimum of the linear function  $x \mapsto 1_m^T x$  over a polyhedron, such minimum is attained at vertex. The inequalities describing the polyhedron have rational coefficients, so also the vertices have rational coordinates (elaborate on this).  $\square$

When  $\varphi(x; z)$  has finite VC-dimension, the transversal number  $\tau$  is bounded by a function of  $\tau^*$ .

**6.5 Proposition** Let  $\varphi(x; z)$  have VC-dimension  $k$ . Then for all finite sets  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$

$$\tau \leq c k (\tau^*)^2 \ln(k \tau^*)$$

where  $c$  is an absolute constant.

**Proof** Let  $A'$  be an optimal fractional transversal. After normalizing,  $A'$  defines

a probability measure  $\mathbb{P}$  on  $\mathcal{U}$ . Namely,  $\mathbb{P}(\{a\}) = A'(a)/\tau^*$  for  $a \in \mathcal{U}$ . By the definition of fractional transversal, every set  $\varphi(A; b)_{b \in B}$  has measure at least  $1/\tau^*$ . By Proposition 5.17, for every  $\varepsilon > 0$  there is a sample  $s$  of size

$$n \leq c \frac{k}{\varepsilon^2} \log \frac{k}{\varepsilon}$$

If we set  $\varepsilon = 1/\tau^*$ , then  $\text{range}(s)$  is a transversal and we obtain the required bound.  $\square$

The bound in the proposition above can be improved. One can replace  $(\tau^*)^2$  with  $\tau^*$  at the cost of a more difficult proof.

## 2 Helly-type properties

We now investigate methods of bounding  $\tau^* = \nu^*$ . As motivation we cite a classical theorem of Helly.

**6.6 Proposition (Helly Theorem)** Let  $\Phi$  be a finite family of convex sets in  $\mathbb{R}^d$ . Assume that any  $d + 1$  sets from  $\Phi$  have non-empty intersection. Then the whole family  $\Phi$  has non-empty intersection.  $\square$

Note that Helly's theorem does not hold for families of finite VC-dimension. A counter example of VC-dimension 2 can be constructed with a family containing sets that are unions of two finite intervals of the real line.

We will deal with the following property, which is more robust. It says that if there is *plenty of small* collections of sets with nonempty intersection, then there is a *large* collection with nonempty intersection.

**6.7 Definition** We say that  $\varphi(x; z)$  has fractional Helly number  $k$  if for all  $\alpha > 0$  there is a  $\beta > 0$  such that for every finite  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  the following holds (write  $n$  for  $|B|$ ):

$$(1) \quad \bigcap_{b \in B'} \varphi(A, b) \neq \emptyset \quad \text{for at least } \alpha \binom{n}{k} \text{ sets } B' \in \binom{B}{k}$$

then

$$(2) \quad \bigcap_{b \in B''} \varphi(A, b) \neq \emptyset \quad \text{for some } B'' \subseteq B \text{ of cardinality } \geq \beta n.$$

We say that  $\varphi$  has the fractional Helly property if it has fractional Helly number  $k$  for some finite  $k$ . The fractional Helly number of  $\varphi(x; z)$  is the smallest number  $k$  satisfying the property above.  $\square$

For further reference, we note that (2) in the definition above can be rewritten as

$$(2') \quad |\varphi(a, B)| \geq \beta n \quad \text{for some } a \in A.$$

The following theorem proves that NIP formulas have the fractionally Helly property (in a strong sense).

**6.8 Theorem (Matoušek)** Let  $\varphi(x; z)$  have VC-codimension  $< k$ . Then  $\varphi(x; z)$  has frac-

tional Helly number  $k$ . Moreover,  $\beta$  in the definition above only depends on  $\alpha$  and  $k$ .

**Proof** Let  $\alpha$  be arbitrary and set  $\beta = 1/2m$  where  $m$  is such that

$$\sum_{i=0}^{k-1} \binom{m}{i} < \frac{\alpha}{4} \binom{m}{k}.$$

Note that, by the Sauer-Shelah Lemma 4.10, the r.h.s. is strictly larger than  $\pi_\varphi^*(m)$  for all  $\varphi(x; z)$  with VC-codimension  $< k$ .

Assume for a contradiction that some finite  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  contradicts the definition above. That is,

$$(1) \quad \bigcap_{b \in B'} \varphi(A, b) \neq \emptyset \quad \text{for at least } \alpha \binom{n}{k} \text{ sets } B' \in \binom{B}{k}$$

and

$$(2) \quad |\varphi(a; B)| < \beta n \quad \text{for all } a \in A.$$

Note that we can assume that  $n > 2m$  otherwise  $\beta n < 1$  and (2) never occur. We will find a set  $B'' \subseteq B$  of cardinality  $m$  with more than  $\pi^*(m)$  distinct  $\varphi(x; z)^{\text{op}}$ -definable subsets, a contradiction.

Let  $P$  be the set of pairs  $B' \subseteq B'' \subseteq B$  such that  $|B'| = k$  and  $|B''| = m$ . We say that a pair  $B' \subseteq B''$  in  $P$  is *good* if there is  $a \in A$  such that  $B' = \varphi(a; B'')$ . That is,  $B'$  is a  $\varphi(x; z)^{\text{op}}$ -definable subset of  $B''$ .

Claim 1. Assume the uniform probability on  $P$ . Then the probability that a random pair is good is  $\geq \alpha/4$ .

Assume Claim 1 for now and continue with the proof. We can think that the random pair in  $P$  is chosen by first picking  $B'' \in \binom{B}{m}$  with the uniform distribution and then  $B' \in \binom{B''}{k}$  again with the uniform distribution. (To put it more pedantically, we are applying the theorem of total probability.) If the probability that a pair is good is  $\geq \alpha/4$ , then for at least one  $B'' \in \binom{B}{m}$  the probability of finding a good subset  $B'$  is  $\geq \alpha/4$ . Therefore,  $B''$  has  $\geq \frac{\alpha}{4} \binom{m}{k} > \pi^*(m)$  good subsets. A contradiction which proves the theorem given the claim.

We now prove the claim. There is another equivalent way to pick a random pair in  $P$ . First we choose at random  $B' \subseteq B$  of cardinality  $k$  then obtain  $B''$  by adding  $m - k$  random elements from  $B \setminus B'$ . By (1), the probability that  $B'$  is such that  $\bigcap_{b \in B'} \varphi(A, b) \neq \emptyset$  is at least  $\alpha$ . So, assume that  $B'$  is such, and fix any  $a$  in this intersection. By (2), there are  $|\varphi(a, B)| < \beta n$ . Then the probability that all  $b \in B'' \setminus B'$  are such that  $\neg \varphi(a, b)$  is at least

$$\begin{aligned} \binom{n - \beta n}{m - k} / \binom{n - k}{m - k} &= \prod_{i=0}^{m-k-1} \frac{n - \beta n - i}{n - k - i} && \text{we write } \beta n \text{ for } \lfloor \beta n \rfloor \\ &\geq \prod_{i=0}^{m-k-1} \frac{n - \beta n - m}{n - m} \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{n - \beta n - m}{n - m} \right)^m \\
&= \left( 1 - \frac{\beta n}{n - m} \right)^m \\
&\geq (1 - 2\beta)^m && \text{because } n > 2m \\
&\geq \left( 1 - \frac{1}{m} \right)^m && \text{because } \beta \leq 1/2m.
\end{aligned}$$

As we can assume  $m \geq 2$ , the probability that a random pair in  $P$  is good is at least  $\alpha/4$ . This proves Claim 1 and with it the theorem.  $\square$

### 3 The $(p,q)$ -theorem

For integers  $p \geq q$  we say that  $\varphi(x; z)$  has the  $(p, q)$ -property if for every finite  $A \subseteq \mathcal{U}$  and every  $B \subseteq \mathcal{V}$  of cardinality  $p$  there is some  $B' \subseteq B$  of cardinality  $q$  such that

$$\bigcap_{b \in B'} \varphi(A; b) \neq \emptyset$$

For ease of speaking we call  $\varphi(A; b)_{b \in B}$  a  $p$ -collection of definable subsets of  $A$ . Note that, strictly speaking, these is not collection of sets but collection of parameters defining sets (though not intrinsically relevant, it is convenient not to assume extensionality).

Hence, we can rephrase the  $(p, q)$ -property in plain English: out of any  $p$ -collection of definable sets there are at least  $q$  sets with nonempty intersection.

Helly's theorem says that any finite collection of convex sets in  $\mathbb{R}^d$  satisfying the  $(d+1, d+1)$ -property has non-empty intersection, i.e. admits a transversal of size 1. A generalization of this was conjectured by Hadwiger and Debrunner, and many years later proved by Alon and Kleitman [1], [2]. Subsequently, after proving Theorem 6.8, Matoušek [9] noted that the method in [2] applies also to collections of sets with finite VC-dimension. More precisely, Matoušek proved the existence of a bound to the cardinality of the transversal number  $\tau$  of NIP formulas with the  $(p, q)$ -property, and that this bound depends only on  $p, q$  and the VC-codimension of the formula. (Formally, the results [2] and [9] do not imply each other.)

**6.9 Theorem (Alon, Kleitman + Matoušek)** Let  $p \geq q > k$  be natural numbers. There is a number  $N = N(k, p, q)$  such that  $\tau_{\varphi(A; B)} \leq N$  for all formulas  $\varphi(x; z)$  with the  $(p, q)$ -property and VC-codimension  $< k$ , and every  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$ .

**Proof** As we are not trying to optimize  $N$ , we may prove the theorem for  $q = k + 1$ . By Proposition 6.5, the transversal number is bounded by a function of  $\tau^*$ , so it is enough to bound  $\tau^*$ . By Theorem 6.4, we can equivalently bound the fractional packing number  $\nu^*$  because it coincides with  $\tau^*$ .

Let  $B' \subseteq B$  be an optimal fractional packing. That is,  $\nu^* = |B'|$  where  $B' \leq \mathbb{I}_B$  is

such that  $|\varphi(a, B')| \leq 1$  for every  $a \in A$ . As we rather work with regular sets than with fractional multisets, we apply a trick that allows to replace  $B'$  with a regular set  $C'$ .

By Theorem 6.4 we may assume that  $B'$  is rational valued. Therefore  $B' = (1/m)C$  where  $m$  is a positive integer and  $C$  is a integral valued multiset over  $\mathcal{V}$ . Replace  $\mathcal{V}$  with  $\mathcal{V} \times [m]$ . Define  $\varphi_\times(a; b, i)$  to be the relation that holds if and only if  $\varphi(a; b)$  holds. Then we can replace the multiset  $C$  with a regular set  $C' \subseteq \mathcal{V} \times [m]$  such that  $|\varphi_\times(a; C')| \leq m$  for every  $a \in A$ .

If write  $n$  for  $|C'|$ , then  $v^* = n/m$ .

Claim 1.  $\varphi_\times(x; z, y)$  satisfies the  $(qp, q)$ -property.

Let  $D \subseteq \mathcal{V} \times [m]$  have cardinality  $qp$ . If the  $qp$ -collection  $\varphi_\times(A; b, i)_{b, i \in D}$  contains  $q$  copies of the same definable set  $\varphi(A; b)$ , then we immediately have the required  $q$ -collection with nonempty intersection. So, suppose not. Then  $\varphi_\times(A; b, i)_{b, i \in D}$  contains  $p$  distinct sets  $\varphi(A, b_1), \dots, \varphi(A, b_p)$ . Then the  $q$ -collection with nonempty intersection is obtained from the  $(p, q)$ -property of  $\varphi(x; z)$ .

Claim 2. There is an  $\alpha = \alpha(p, q) > 0$  such that

$$\bigcap_{b, i \in D} \varphi_\times(A; b, i) \neq \emptyset \quad \text{for at least } \alpha \binom{n}{q} \text{ sets } D \in \binom{C'}{q}.$$

By Claim 1, every  $qp$ -collection of  $\varphi_\times$ -definable sets contains at least one  $q$ -collection with non-empty intersection. Every  $q$ -collection is contained in  $\binom{n-q}{qp-q}$  many  $qp$ -collections. Therefore the number  $q$ -collections with non-empty intersection is at least

$$\binom{n}{qp} / \binom{n-q}{qp-q} = \binom{n}{q} / \binom{qp}{q}.$$

Therefore, the claim holds with  $1/\alpha = \binom{qp}{q}$ .

Now we can resume the proof of the theorem (recall that our goal is to bound  $v^*$  by a function of  $p, q$ , and  $k$ ). Let  $\beta = \beta(\alpha, k)$  be as in Theorem 6.8. As  $\varphi_\times(x; z, y)$  has the same VC-codimension as  $\varphi(x; z)$ , by Claim 2 there is an  $a \in A$  such that  $\varphi(a, C')$  has cardinality at least  $\beta n$ . So, from  $\beta n \leq |\varphi(a, C')| \leq m$  we obtain  $v^* \leq 1/\beta$ .  $\square$



## Chapter 7

### Zarankiewicz problem(s)

Let us start with presenting Zarankiewicz problem. Let  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  have cardinality  $m$  respectively  $n$ . Given two integers  $s, t$  (both at least 2 to avoid trivialities), what is the maximal cardinality of a graph  $\varphi(A; B)$  that does not contain  $A' \times B'$  for any  $A' \subseteq A$  and  $B' \subseteq B$  of cardinality  $s$  respectively  $t$ ? Denote this maximum by  $z(m, n; s, t)$ . Define also  $z(n; t) = z(m, n; s, t)$ . In 1951 Zarankiewicz posed the problem of determining  $z(n; 3)$  for  $n = 4, 5, 6$  and the general problem has also become known as the problem of Zarankiewicz.

#### 1 The Kővári-Sós-Turán Theorem

The theorem in this section is a classical result of Kővári, Sós, and Turán. They actually proved it for  $z(n; t)$  but the proof easily generalizes.

In the proof we need the following generalization of the binomial coefficient. For any positive integer  $t$  and any real  $x$  we define

$$\binom{x}{t} = \frac{x(x-1) \cdots (x-t+1)}{t!}.$$

It is easy to verify that for any fixed  $t$ , this is a convex and strictly increasing function of  $x$ .

**7.1 Kővári-Sós-Turán Theorem** For all  $2 \leq s \leq m$  and  $2 \leq t \leq n$

$$z(m, n; s, t) < (s-1)^{1/t}(n-t+1)m^{1-1/t} + (t-1)m$$

or also

$$< c(n m^{1-1/t} + m)$$

for some  $c = c(s, t)$ .

**Proof** Let  $A \subseteq \mathcal{U}$  and  $B \subseteq \mathcal{V}$  have cardinality  $m$  respectively  $n$ . Let  $P$  be the set of pairs  $\langle a, B' \rangle$  such that  $B' \subseteq \varphi(a; B)$  and  $B'$  has cardinality  $t$ . Writing  $d_a$  for the cardinality of  $\varphi(a; B)$ , we have

$$|P| = \sum_{a \in A} \binom{d_a}{t}$$

Write  $z$  for  $|\varphi(A; B)|$ , and note that

$$z = \sum_{a \in A} d_a.$$

As  $\binom{x}{t}$  is a convex function of  $x$ ,

$$(1) \quad \binom{z/m}{t} \leq \frac{1}{m} |P|.$$

Now, suppose that  $z = z(m, n; s, t)$ . Then for any fixed  $B' \subseteq B$  there are at most  $s - 1$  pairs  $\langle a, B' \rangle \in P$ . Hence

$$(2) \quad |P| \leq (s - 1) \binom{n}{t}.$$

Together, (1) and (2) yield

$$m \binom{z/m}{t} \leq (s - 1) \binom{n}{t}.$$

As  $\binom{x}{t}$  is strictly increasing function of  $x$ , and  $z/m < n$ ,

$$m \left( \frac{z}{m} - t + 1 \right)^t < (s - 1) (n - t + 1)^t$$

which easily yields the theorem. □

## 2 The NIP case

## **Chapter 8**

### **Szemerédi regularity lemma**

- 1 The regularity lemma**
- 2 Stable regularity lemma**
- 3 Distal regularity lemma**

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