Aberrant notes on combinatorics

Chapter 1

Three minimax theorems

Let $\mathcal{U}$ and $\mathcal{V}$ be two (large) sets. Let $\varphi(x\,;z)$ be a relation symbol. We denote by $\varphi(\mathcal{U}\,;\mathcal{V})$ the set $\{\langle a,b\rangle \in \mathcal{U}\times\mathcal{V} \ :\ \varphi(a\,;b)\}$ which we call: the relation defined by $\varphi(x\,;z)$. Sets of the form $\varphi(\mathcal{U}\,;b) = \{a \in \mathcal{U} \ :\ \varphi(a\,;b)\}$, for some $b \in \mathcal{V}$, are called definable sets. If $A \subseteq \mathcal{U}$ we write $\varphi(A\,;b)$ for the trace of $\varphi(\mathcal{U}\,;b)$ on A, that is, the set $\varphi(\mathcal{U}\,;b)\cap A$. We call it a definable subset of A. It would be more appropriate to call these sets *global types*, respectively *types over* A. But this would make the terminology (if possible) more obscure.

We denote by $\varphi(\mathcal{U}\,;b)_{b\in\mathcal{V}}$ and $\varphi(A\,;b)_{b\in\mathcal{V}}$ the collection of definable sets, respectively definable subsets of A.

For $k \leq |A|$ we use following notation interchangeably

$$\binom{A}{k} \quad = \quad A^{(k)} \quad = \quad \Big\{A' \subseteq A \ :\ |A'| = k\Big\}$$

1 Hall's Marriage Theorem

Let $\varphi(x\,;z)$ be given. Let $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{V}$ be finite sets.

We say that $A' \subseteq A$ is a set of distinct representatives for $\varphi(A\,;B)$ if $|\varphi(A';b)| = 1$ for every $b \in B$.

1.1 Hall's Marriage Theorem *For every finite $B \subseteq \mathcal{V}$, the following are equivalent*

1. *$\varphi(A\,;B)$ has a set of distinct representatives;*

2. $|B'| \ \leq \ \Big|\bigcup\limits_{b\in B'} \varphi(A\,;b')\Big|$ *for every $B' \subseteq B$.*

Proof (1 $\Rightarrow$ 2) The following holds for any set of distinct representatives A' and $B' \subseteq B$

$$\begin{aligned} |B'| \quad &= \quad \Big|\bigcup_{b\in B'} \varphi(A';b')\Big| \\ &\subseteq \quad \Big|\bigcup_{b\in B'} \varphi(A\,;b')\Big|. \end{aligned}$$

(2 $\Rightarrow$ 1) Reason by induction on the cardinality of B. If B is empty, the claim is clear. Now assume $|B| > 0$ and consider two cases.

a. This is the case when the inequality in 2 is strict for all $B' \subseteq B$. Pick any pair a, b such that $\varphi(a,b)$. Then $\varphi(A \setminus \{a\}, B \setminus \{b\})$ still satisfy 2. By induction hypothesis, it has a set of distinct representatives A'. Then $A' \cup \{a\}$ is a set of distinct representatives for $\varphi(A\,;B)$.

b. Suppose instead that, for some $B' \subseteq B$, 2 holds with equality. Define

$$A' \quad = \quad \bigcup_{b\in B'} \varphi(A\,;b)$$

It is clear that 2 holds for $\varphi(A';B')$. Below we prove that 2 also holds for $\varphi(A\smallsetminus A'\,;\,B\smallsetminus B')$. Once this claim is proved, we apply the induction hypothesis to obtain sets of distinct representatives for these two relations and note that their union is a set of distinct representatives for $\varphi(A\,;B)$.

To prove the claim assume that there is a set $B'' \subseteq B$ that contradicts 2, then

$$\Big|\bigcup_{b\in B''} \varphi(A\smallsetminus A';\,b)\Big| \quad < \quad |B''|.$$

By the definition of A'

$$\begin{aligned} A' \,\cup\, \bigcup_{b\in B} \varphi(A\smallsetminus A';b) \quad &= \quad A' \cup \bigcup_{b\in B''} \varphi(A\,;b) \\ &= \quad \bigcup_{b\in B'\cup B''} \varphi(A\,;b) \end{aligned}$$

Hence

$$\Big|\bigcup_{b\in B'\cup B''} \varphi(A\,;b)\Big| \quad = \quad |A'|+|B''|$$

As $|A'| = |B'|$ by the choice of B', this contractics the assumption that B satisfies 2. This prove the claim and with it the theorem. □

2 Kőnig's Minimax Theorem

Let $\varphi(x\,;z)$ be given. Let $A \subseteq \mathcal{U}$ and $B \subseteq \mathcal{V}$ be finite sets.

A matching of $\varphi(A\,;B)$ is a pair of sets $A' \subseteq A$ and $B' \subseteq B$ such that $|\varphi(A'\,;b)| = |\varphi(a\,;B')| = 1$ for every $a,b \in A',B'$. In other words $\varphi(A'\,;B')$ is the graph of a bijection between A' and B'. Therefore $|A'| = |B'|$; this is called the cardinality of the matching. The matching number of $\varphi(A\,;B)$ is the maximal cardinality of a matching.

Note that is A' is a set of distinct representatives for $\varphi(A\,;B)$, then there is a $B' \subseteq B$ such that A',B'. Hence $\nu(\varphi)$ is less or equal than the cardinality of any set of distinct representatives.

A (vertex) cover of $\varphi(A\,;B)$ is a pair of sets $A' \subseteq A$ and $B' \subseteq B$ such that $\varphi(A\,;B)$ is contained in $(A' \times B) \cup (A \times B')$. We will mainly use the easy characterization given by the fact below.

1.2 Fact *The following are equivalent*

1. *A',B' is a cover;*

2. *$\varphi(A\,;b) \subseteq A'$ for every $b \in B \smallsetminus B'$;*

3. $\varphi(a\,;B) \subseteq B'$ *for every* $a \in A \smallsetminus A'$. $\square$

We call $|A'| + |B'|$ the cardinality of the cover. the cover number of $\varphi(A\,;B)$ is the minimal cardinality of a cover.

1.3 Kőnig's Minimax Theorem *For any given* $\varphi(A\,;B)$*, matching number = cover number. That is, the maximal cardinality of a matching equals the minimal cardinality of a cover.*

Proof ($\leq$) We prove that $|A''| \leq |A'| + |B'|$ for every cover A', B' and every matching A'', B''.

As $\varphi(A\,;b) \subseteq A'$ for every $b \in B \smallsetminus B'$, in particular we have that $\varphi(A''\,;b) \subseteq A'$ for every $b \in B'' \smallsetminus B'$. Therefore all elements of A'' are in A' but for at most $|B'|$ elements. Hence $|A''| \leq |A'| + |B'|$ is clear.

($\geq$) Let A', B' be a cover of minimal cardinality. We prove that there is a matching of cardinality at least $|A'| + |B'|$.

We break $\varphi(A\,;B)$ into two relations, find a matching of each of these and join them together to obtain a matching of cardinality $\geq \tau(\varphi)$. Precisely, first we show that $\varphi(A \smallsetminus A'\,;B')$ has a set of distinct representatives $A_1 \subseteq A \smallsetminus A'$. Hence A_1, B_1 is a matching for some $B_1 \subseteq B'$. Second, we apply the same argument shows that $\varphi(A'\,;B \smallsetminus B')^*$ has a set of distinct representatives $B_2 \subseteq B \smallsetminus B'$. Hence A_2, B_2 is a matching for some $A_2 \subseteq A'$. Then $(A_1 \cup A_2)$, $(B_1 \cup B_2)$ is a matching of $\varphi(A\,;B)$. The cardinality of this maching is $|A_1| + |A_2| = |B_1| + |A_2| \leq |B'| + |A'|$.

We use Hall's Marriage Theorem to prove the first claim above. The second is proved by the symmetric argument (using 3 for 2 of the fact above).

We need to check that $\varphi(A \smallsetminus A'\,;B')$ satisfies 2 of Theorem 1.1. Suppose not. Then there is a set $B'' \subseteq B'$ such that $|A''| < |B''|$, where

$$A'' \quad = \quad \bigcup_{b \in B''} \varphi(A \smallsetminus A', b)$$

Then $(A' \cup A''), (B' \smallsetminus B'')$ would be a cover of cardinality $< |A'| + |B'|$. This contradicts the minimality of A', B'. $\square$

3 Dilworth's Theorem

Dilworth's Theorem is minimax theorem essentially equivalent to Kőnig's Theorem. To highlight the connection we choose to prove it using Kőnig's Theorem. Alternatively we could have proved Dilworth's Theorem directly and derived Kőnig's and Hall's Theorem from it.

In this section we assume that $\mathcal{U} = \mathcal{V}$ and that $\varphi(x\,;z)$ defines a (strict) partial order on $\mathcal{U}$. An antichain is a set $A \subseteq \mathcal{U}$ such that $\neg\varphi(a\,;b)$ for every $a, b \in A$. A chain is a set A such that $\varphi(a\,;b) \vee \varphi(b\,;a)$ for every distinct $a, b \in A$.

1.4 Dilworth's Theorem *The maximal cardinality of an antichain* $A' \subseteq A$ *equals the minimal cardinality of a partition of* A *into chains.*

Proof ($\leq$) We prove that the cardinality of an antichain cannot exceed the cardinality of a partition of A into chains.

Let $A_1, \ldots, A_k$ be a partition of A into chains and let A' be an antichains. A chain can contain at most one element of A', hence $|A'| \leq k$.

($\geq$) Let $A \smallsetminus A' \subseteq A$ be an antichain (for uniformity with the notation in Kőnig's Theorem). We prove that there is a partition $A_1, \ldots, A_k$ into chains for some $k \leq |A \smallsetminus A'|$.

Let $B' \subseteq B \subseteq \mathcal{V}$ be a disjoint copies of $A' \subseteq A \subseteq \mathcal{U}$. Then $\varphi(A\,;b) \subseteq A'$ for every $b \in B \smallsetminus B'$. By Kőnig's Theorem there is a matching A'', B'' of cardinality $|A''| = |B''| \geq |A' \cup B'| = |A'|$.

We construct a chain-partition as follow. Pick an element of $a \in A''$ and construct the longest possible chain $a = a_0, a_1, \ldots$ where $\varphi(a_i; a_{i+1})$ and $a_{2i}, a_{2i+1} \in A'', B''$. When the construction halts, initiate a new one from some fresh element of A''. Finally, put each element of $A \smallsetminus (A'' \cup B'')$ in a chain on its own.

As no element of A'' belongs to a chain on its own, therefore the number of chains is $\leq |A| \smallsetminus |A''| \leq |A| \smallsetminus |A'|$. $\square$

Chapter 2

Set systems

1 Sperner's Theorem

We say that $\varphi(A\,;b)_{b\in\mathcal{V}}$ is an antichain if there is no pair of elements $b, b' \in \mathcal{V}$ such that $\varphi(A\,;b) \subset \varphi(A\,;b')$. Antichains are also called Sperner systems.

2.1 Sperner's Theorem *Let $A \subseteq \mathfrak{U}$ have cardinality n, finite. If $\varphi(A\,;b)_{b\in\mathcal{V}}$ is an antichain then*

$$\big|\varphi(A\,;b)_{b\in\mathcal{V}}\big| \quad \leq \quad \binom{n}{\lfloor n/2 \rfloor}.$$

Proof Clearly, $\varphi(A\,;b)_{b\in\mathcal{V}}$ is the disjoint union of the sets $\binom{A}{k} \cap \varphi(A\,;b)_{b\in\mathcal{V}}$ for k ranging over $\{0,\ldots,n\}$. Then

$$\big|\varphi(A\,;b)_{b\in\mathcal{V}}\big| \quad \leq \quad \sum_{k=0}^{n} \left|\binom{A}{k} \cap \varphi(A\,;b)_{b\in\mathcal{V}}\right|.$$

As for every $k \leq n$

$$\binom{n}{k} \quad \leq \quad \binom{n}{\lfloor n/2 \rfloor},$$

the theorem follows immediately from the LYM inequality that we prove below. □

The acronym LYM stands for Lubell-Yamamoto-Meshalkin.

2.2 Lemma (LYM inequality) *Let $A \subseteq \mathfrak{U}$ have cardinality n, finite. If $\varphi(A\,;b)_{b\in\mathcal{V}}$ is an antichain then*

$$\sum_{k=0}^{n} \left|\binom{A}{k} \cap \varphi(A\,;b)_{b\in\mathcal{V}}\right| \cdot \binom{n}{k}^{-1} \quad \leq \quad 1.$$

Proof Let Π be uniform random variable that ranges over the set of permutations of $A = \{a_1,\ldots,a_n\}$. For any $\varphi(A\,;b)$ of cardinality k

$$\Pr\Big(\Pi\{a_1,\ldots,a_k\} = \varphi(A\,;b)\Big) \quad = \quad \binom{n}{k}^{-1}.$$

The events above are disjoint for distinct sets $\varphi(A\,;b)$, hence

$$\Pr\Big(\Pi\{a_1,\ldots,a_k\} \in \varphi(A\,;b)_{b\in\mathcal{V}}\Big) \quad = \quad \left|\binom{A}{k} \cap \varphi(A\,;b)_{b\in\mathcal{V}}\right| \cdot \binom{n}{k}^{-1}.$$

As $\varphi(A\,;b)_{b\in\mathcal{V}}$ is an antichain, for distinct k the events above are disjoint, hence

$$\Pr\Big(\bigcup_{k=0}^{n} \Pi\{a_1,\ldots,a_k\} \in \varphi(A\,;b)_{b\in\mathcal{V}}\Big) \quad = \quad \sum_{k=0}^{n} \left|\binom{A}{k} \cap \varphi(A\,;b)_{b\in\mathcal{V}}\right| \cdot \binom{n}{k}^{-1}.$$

Now, the inequality is evident. □

Let $\Pr_k$ be the probability measure on the subsets of A that is concentrated and uniform on $A^{(k)}$. Namely, for $A' \subseteq A$

$$\Pr_k\bigl(\{A'\}\bigr) \quad = \quad \begin{cases} 0 & \text{if } \ |A'| \neq k \\ \binom{n}{k}^{-1} & \text{if } \ |A'| = k \end{cases}$$

Then the the LYM inequality asserts that if $\varphi(A\,;b)_{b\in\mathcal{V}}$ is an antichain then

$$\sum_{k=0}^{n} \Pr_k\bigl(\varphi(A\,;b)_{b\in\mathcal{V}}\bigr) \quad \leq \quad 1.$$

This inequality is strict when $\varphi(A\,;b)_{b\in\mathcal{V}} = A^{(k)}$ for some k. In the next section we show that these are the only cases.