Homework #01

SMDS-2023-2024

Statstical Methods in Data Science II a.y. 2022-2023

M.Sc. in Data Science

deadline: April 26th, 2024

A. Simulation

1. Consider the following joint discrete distribution of a random vector (Y,Z) taking values over the bi-variate space:

$$\begin{aligned}\mathcal{S} = \mathcal{Y} \times \mathcal{Z} &= \{(1,1);(1,2);(1,3);\\ &\quad (2,1);(2,2);(2,3);\\ &\quad (3,1);(3,2);(3,3)\}\end{aligned}$$

The joint probability distribution is provided as a matrix `J` whose generic entry `J[y,z]` $= Pr\{Y=y, Z=z\}$

```
J
```

```
     1    2    3
1 0.06 0.17 0.10
2 0.10 0.12 0.11
3 0.14 0.02 0.18
```

```
S
```

```
      row col
(1,1)   1   1
(1,2)   1   2
(1,3)   1   3
(2,1)   2   1
(2,2)   2   2
(2,3)   2   3
(3,1)   3   1
(3,2)   3   2
(3,3)   3   3
```

You can load the matrix S of all the couples of the states in $\mathcal{S}$ and the matrix J containing the corresponding bivariate probability masses from the file "Hmwk.RData". How can you check that J is a probability distribution?

---

1. Answer:

We can check whether J is a *valid probability distribution* by verifying if it satisfies the two properties of *non-negativity* and *summation to 1*. Since J is discrete we will refer to the discrete versions of the properties.

The first states that all elements of the probability distribution must be non-negative:

$$Pr\{Y = y, Z = z\} \geq 0 \; \forall y, z \in \{1, 2, 3\}$$

We can write a simple conditional statement to check if this condition is satisfied:

```
# function to verify if a probability
# distribution is non-negative
verify_non_negativity <- function(A) {
    return(sum(A >= 0) == length(A))
}

verify_non_negativity(J)
```

```
[1] TRUE
```

As we can see the first property is satisfied. The property of summation to 1 instead states that the elements of the probability distribution must sum up to 1:

$$\sum_{y,z \in \{1,2,3\}} Pr\{Y = y, Z = z\} = 1$$

This also can be checked with a simple conditional statement:

```
# function to verify is a probability
# distribution has sum equal to 1
verify_sum_to_one <- function(A) {
    return(all.equal(sum(A), 1))
}
```

It's important to notice that, to account for *machine precision*, we used the method `all.equal` instead of the `==` operator.

```
verify_sum_to_one(J)
```

```
[1] TRUE
```

Also the second condition is satisfied, therefore we can say that J is indeed a valid probability distribution.

---

2. How many *conditional distributions* can be derived from the joint distribution J? Please list and derive them.

---

2. Answer:

We can derive 6 conditional probability distributions by fixing on either Y or Z a value in $\{1,2,3\}$; we have 2 random variables with 3 possible values each, therefore $3 * 2 = 6$ conditional distributions. To derive them we have just to implement the definition of conditional probability:

$$Pr(Y = y|Z = z) = \frac{Pr(Y = y, Z = z)}{Pr(Z = z)}$$

Where at the numerator we have the joint distribution, provided by the matrix J, and at the denominator the *marginal distribution* of the value we condition on, defined as:

$$Pr(Z = z) = \sum_y Pr(Y = y, Z = z)$$

The base elements of this second formula are the joint distributions that we already know, therefore we have derived all ingredients we need for the computation. In the previous definitions Y and Z can be swapped to get the formulas for Z conditioned on Y. Below we implement the functions to get the marginal and conditional distributions:

```
# support of y
Y = 1:3
# support of z
Z = 1:3

# marginal distribution of z
get_marginal_distro_Z <- function() {
    marginal = c()
    for (z in 1:3) {
        marginal = c(marginal, sum(J[, z]))
    }
    return(marginal)
}

# marginal distribution of y
get_marginal_distro_Y <- function() {
    marginal = c()
    for (y in 1:3) {
        marginal = c(marginal, sum(J[y, ]))
    }
    return(marginal)
}
```

```r
# marginal distribution of y given z
get_conditional_distro_Y <- function(y, z) {
    return(J[y, z]/get_marginal_distro_Z()[z])
}

# marginal distribution of z given y
get_conditional_distro_Z <- function(y, z) {
    return(J[y, z]/get_marginal_distro_Y()[y])
}
```

Here we call the functions to get for each possible conditioning the conditional probability distribution:

Pr(Y | Z = 1):

```r
get_conditional_distro_Y(Y, 1)
```

```
        1         2         3 
0.2000000 0.3333333 0.4666667 
```

Pr(Y | Z = 2):

```r
get_conditional_distro_Y(Y, 2)
```

```
         1          2          3 
0.54838710 0.38709677 0.06451613 
```

Pr(Y | Z = 3):

```r
get_conditional_distro_Y(Y, 3)
```

```
        1         2         3 
0.2564103 0.2820513 0.4615385 
```

Pr(Z | Y = 1):

```r
get_conditional_distro_Z(1, Z)
```

```
        1         2         3 
0.1818182 0.5151515 0.3030303 
```

Pr(Z | Y = 2):

```r
get_conditional_distro_Z(2, Z)
```

```
        1         2         3 
0.3030303 0.3636364 0.3333333 
```

Pr(Z | Y = 3):

```
get_conditional_distro_Z(3, Z)
```

```
         1          2          3 
0.41176471 0.05882353 0.52941176 
```

---

3. Make sure they are probability distributions.

---

3. Answer:

To make sure every conditional distribution we created is a probability distribution we can simply use the functions defined over point $A.1$ and check whether they give us a positive response. To reduce the amount of code required we create a new function that checks them both:

```
# function that verifies if a probability
# distribution satisfies both properties
is_valid_distro <- function(A) {
    return(verify_non_negativity(A) & verify_sum_to_one(A))
}
```

Pr(Y | Z = 1):

```
is_valid_distro(get_conditional_distro_Y(Y, 1))
```

```
[1] TRUE
```

Pr(Y | Z = 2):

```
is_valid_distro(get_conditional_distro_Y(Y, 2))
```

```
[1] TRUE
```

Pr(Y | Z = 1):

```
is_valid_distro(get_conditional_distro_Y(Y, 3))
```

```
[1] TRUE
```

Pr(Z | Y = 1):

```
is_valid_distro(get_conditional_distro_Z(1, Z))
```

```
[1] TRUE
```

Pr(Z | Y = 2):

```r
is_valid_distro(get_conditional_distro_Z(2, Z))
```

```
[1] TRUE
```

Pr(Z | Y = 3):

```r
is_valid_distro(get_conditional_distro_Z(3, Z))
```

```
[1] TRUE
```

---

4. Can you simulate from this `J` distribution? Please write down a working procedure with few lines of R code as an example. Can you conceive an alternative approach? In case write down an alternative working procedure with few lines of R

---

4. Answer:

We can simulate from the `J` by using the `sample` method. We will have to treat the matrix as a vector and turn back the indices of the vector into tuples:

```r
index_to_tuple <- function(n) {
    row = ((n - 1)%%3) + 1
    col = ((n - 1)%/%3) + 1

    return(c(row, col))
}

sample_from_J <- function() {
    indices = sample(length(J), size = 1, prob = J)
    tuples = index_to_tuple(indices)
    return(tuples)
}

sample_from_J()
```

```
[1] 1 2
```

We can think of an alternative method that exploits the previously defined marginal and conditional distributions. We first sample Z from its marginal distribution and then sample Y from the distribution conditioned on the result of the first sampling:

```r
# sampling from marginal Z
sample_from_marginal_Z <- function() {
    distro = get_marginal_distro_Z()
```

```r
    sample(length(distro), size = 1, prob = distro)
}

# sampling from Y conditioned on Z
sample_from_Y_conditioned <- function(z) {
    distro = get_conditional_distro_Y(Y, z)
    sample(length(distro), size = 1, prob = distro)
}

# sampling of tuple
sample_marginal_conditional_1 <- function() {
    z = sample_from_marginal_Z()
    y = sample_from_Y_conditioned(z)
    return(c(y, z))
}

sample_marginal_conditional_1()
```

```
[1] 3 1
```

Of curse we can also do the opposite, first sampling from the marginal distribution of Y and then from the conditional distribution of Z conditioned on the result of the first sampling:

```r
# sampling from marginal Y
sample_from_marginal_Y <- function() {
    distro = get_marginal_distro_Y()
    sample(length(distro), size = 1, prob = distro)
}

# sampling from Z conditioned on Y
sample_from_Z_conditioned <- function(y) {
    distro = get_conditional_distro_Z(y, Z)
    sample(length(distro), size = 1, prob = distro)
}

# sampling of tuple v.2
sample_marginal_conditional_2 <- function() {
    y = sample_from_marginal_Z()
    z = sample_from_Y_conditioned(y)
    return(c(y, z))
}

sample_marginal_conditional_2()
```

```
[1] 1 1
```

B. Bulb lifetime: a conjugate Bayesian analysis of exponential data

You work for Light Bulbs International. You have developed an innovative bulb, and you are interested in characterizing it statistically. You test 20 innovative bulbs to determine their lifetimes, and you observe the following data (in hours), which have been sorted from smallest to largest.

1, 13, 27, 43, 73, 75, 154, 196, 220, 297,
344, 610, 734, 783, 796, 845, 859, 992, 1066, 1471

Based on your experience with light bulbs, you believe that their lifetimes Y_i can be modeled using an exponential distribution conditionally on θ where $\psi = 1/\theta$ is the average bulb lifetime.

1. Write the main ingredients of the Bayesian model.

---

1. Answer:

The main ingredients of the Bayesian model are:

- **Prior distribution**: the prior distribution $\pi(\theta)$ is a probability distribution that represents the knowledge we have about the parameter of interest θ before any evidence gathered from observing the data is taken into account. The choice of the family of the prior distribution with respect to the distribution of the data is a determinant factor when we want to make a *Conjugate Bayesian Analysis*.
- **Likelihood function**: the likelihood function $f(y|\theta)$ is the probability of the observed data fixed the parameter of interest θ. It represents the information we gather from observations.
- **Prior-to-Posterior formula**: we can update the *uncertainty* we have about θ by "inserting" in our prior the knowledge we get from the data Y using the Bayes formula:

$$\pi(\theta|Y) \propto f(Y|\theta)\pi(\theta)$$

---

2. Choose a conjugate prior distribution $\pi(\theta)$ with mean equal to 0.003 and standard deviation 0.00173.

---

2. Answer:

Since our knowledge about the problem suggests us that the problem can be modeled using an Exponential distribution the choice for a conjugate prior ends on a *Gamma distribution*. The choice of a Gamma prior for an Exponential distribution can be justified, but we leave the reasoning to the point $4.$ of this exercise.

We indicate the shape and rate parameters of the Gamma respectively with α and β, while we use μ and σ^2 for mean and variance.

Since $\mu = \frac{\alpha}{\beta}$ and $\sigma^2 = \frac{\alpha}{\beta^2}$ we can easily derive:

$$\alpha = \frac{\mu^2}{\sigma^2} = \frac{0.003^2}{0.00173^2} = 3.007$$

$$\beta = \frac{\mu}{\sigma^2} = \frac{0.003}{0.00173^2} = 1002.372$$

Therefore:

$$\theta \sim \pi(\theta) = Gamma(\alpha = 3.007, \beta = 1002.372)$$

---

3. Argue why with this choice you are providing only a vague prior opinion on the average lifetime of the bulb.

---

3. Answer:

To answer this question we have to first to derive what our prior distribution $\pi(\theta)$ says about the average bulb lifetime ψ. The parameter θ has a prior Gamma distribution function and by definition $\psi = \frac{1}{\theta}$, therefore:

$$\psi \sim InvGamma(\alpha, \beta)$$

The parameters α and β of this distribution are the same of the prior distribution of θ. From this we can calculate the mean and the variance of ψ according to its prior distribution:

$$E[\psi] = \frac{\beta}{\alpha - 1} = \frac{1002.372}{3.007 - 1} = 499.438$$

$$Var[\psi] = \frac{\beta^2}{(\alpha-1)^2(\alpha-2)} = \frac{1002.372^2}{(3.007-1)^2(3.007-2)} = 247704.352$$

We can link this extremely high variance of the prior to the aforementioned vagueness of the prior opinion on the average lifetime of the bulb.

---

4. Show that this setup fits into the framework of the conjugate Bayesian analysis.

---

4. Answer:

We know from the text of the exercise that our data follow an Exponential distribution given θ:

$$f(y_i|\theta) = \theta e^{-\theta y_i}$$

From which we can write the likelihood function:

$$L_{\underline{Y}}(\theta) = \prod_{i=1}^{n} f(y_i|\theta) = \theta^n e^{-\theta \sum_{i=1}^{n} y_i}$$

Now introduce the prior as a Gamma distribution:

$$\pi(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}$$

In this way we get the following posterior:

$$\pi(\theta|\underline{Y}) \propto \pi(\theta) L_{\underline{Y}}(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \theta^n e^{-\theta \sum_{i=1}^{n} y_i} \propto \theta^{\alpha-1} e^{-\beta\theta} \theta^n e^{-\theta \sum_{i=1}^{n} y_i} = \theta^{(\alpha+n)-1} e^{-\theta(\beta + \sum_{i=1}^{n} y_i)}$$

The posterior stays in the same family of distributions, with updated parameters:

$$\alpha^* = \alpha + n, \qquad \beta^* = \beta + \sum_{i=1}^{n} y_i$$

Therefore the setup fits into the framework of the Conjugate Bayesian Analysis.

---

5. Based on the information gathered on the 20 bulbs, what can you say about the main characteristics of the lifetime of your innovative bulb? Argue that we have learnt some relevant information about the θ parameter and this can be converted into relevant information about the unknown average lifetime of the innovative bulb $\psi = 1/\theta$.

---

5. Answer:

We can implement into code the procedure described above to get the posterior distribution of θ:

```r
data = c(1, 13, 27, 43, 73, 75, 154, 196, 220, 297,
    344, 610, 734, 783, 796, 845, 859, 992, 1066, 1471)

mean = 0.003
variance = 0.00173^2

alpha = mean^2/variance
beta = mean/variance

cat("Original alpha parameter:", alpha, "\nOriginal beta parameter:",
    beta, "\nOriginal mean:", mean, "\nOriginal variance:",
    variance)
```

```
Original alpha parameter: 3.007117
Original beta parameter: 1002.372
Original mean: 0.003
Original variance: 2.9929e-06
```

```r
alpha_star = alpha + length(data)
beta_star = beta + sum(data)

mean_star = alpha_star/beta_star
variance_star = alpha_star/beta_star^2

cat("Updated alpha parameter:", alpha_star, "\nUpdated beta parameter:",
    beta_star, "\nUpdated mean:", mean_star, "\nUpdated variance:",
    variance_star)
```

```
Updated alpha parameter: 23.00712
Updated beta parameter: 10601.37
Updated mean: 0.002170202
Updated variance: 2.047095e-07
```

Keeping in mind the mean and variance of the prior distribution of ψ we can confront them with the values we obtain from the posterior distribution:

$$E[\psi^*] = \frac{\beta^*}{\alpha^* - 1} = \frac{10601.37}{23.007 - 1} = 481.727$$

$$Var[\psi^*] = \frac{\beta^{*2}}{(\alpha^* - 1)^2(\alpha^* - 2)} = \frac{10601.37^2}{(23.007 - 1)^2(23.007 - 2)} = 11046.845$$

What we can observe is that the mean of the distribution of ψ has gotten closer to the empirical mean of the observations and that the variance has decreased by more than 20 times.

---

6. However, your boss would be interested in the probability that the average bulb lifetime $1/\theta$ exceeds 550 hours. What can you say about that after observing the data? Provide her with a meaningful Bayesian answer.

---

6. Answer:

We can take our new α^* and β^* parameters, calculate the posterior distribution of ψ and use it to answer the question. In particular to get the probability that the average bulb lifetime exceeds 550 hours we will need to calculate the quantile of 550 and subtract it to 1:

```
library(invgamma)

quantile = pinvgamma(q = 550, shape = alpha_star, rate = beta_star)
answer = 1 - quantile

cat("The probability that the average bulb lifetime exceeds 550 hours is:",
    answer)
```

```
The probability that the average bulb lifetime exceeds 550 hours is: 0.2254117
```

We can notice that this result is very different from what we would get from using only the observed data using a Frequentist approach; in the set of samples that has been provided there are 9 out of 20 observations with a lifetime greater than 550 and this would lead a Frequentist approach to give a much higher result.

C. Exchangeability

Let us consider an infinitely exchangeable sequence of binary random variables $X_1, ..., X_n, ...$

1. Provide the definition of the distributional properties characterizing an infinitely echangeable binary sequence of random variables $X_1, ..., X_n, ...$. Consider the De Finetti representation theorem relying on a suitable distribution $\pi(\theta)$ on $[0, 1]$ and show that

$$\begin{aligned} E[X_i] &= E_\pi[\theta] \\ E[X_i X_j] &= E_\pi[\theta^2] \\ Cov[X_i X_j] &= Var_\pi[\theta] \end{aligned}$$

---

1. Answer:

We say that a infinite sequence of random variables $X_1, ..., X_n, ...$ is infinitely exchangeable if the joint distribution of any tuple is equal to the joint distribution of any permutation of the same tuple:

$$(X_1, ..., X_k) \stackrel{d}{=} (X_{\sigma_1}, ..., X_{\sigma_k}) \ \ \forall \text{ permutation } \sigma$$

Given a vector of k elements there are $k!$ permutations of the elements. As the main book of the course "A First Course in Bayesian Statistical Methods" simply puts, $X_1, ..., X_n$ *are exchangeable if the subscript labels convey no information about the outcomes*.

An important property that is consequence of the exchangeability it's what is expressed by *"De Finetti's Theorem"*. The theorem says that if a sequence of binary random variables $X_1, ..., X_n, ...$ is exchangeable there exist a distribution $\pi(\theta)$ on $[0, 1]$ such that:

$$Pr(X_1 = x_1, ..., X_n = x_n) = \int_{[0,1]} \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \pi(\theta) d\theta$$

Meaning that the observations are conditionally independent and identically distributed given a fixed parameter θ.

Showing $\mathbf{E}[\mathbf{X_i}] = \mathbf{E}_\pi[\theta]$:

From the definition of the expected value defined over a binary value we have that:

$$E[X_i] = \sum_{x_i=0}^{1} x_i Pr(X_i = x_i)$$

As starting hypothesis our sequence $X_1, ..., X_n, ...$ is composed of binary variables and it is infinitely exchangeable, therefore De Finetti's Theorem holds and we have:

$$Pr(X_1 = x_1, ..., X_n = x_n) = \int_{[0,1]} \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i}\pi(\theta)d\theta$$

This holds for any n, therefore we can fix $n = 1$ from which we get permutations consisting of a single element that we will call X_i, that is the same as the one presented in the formula of the expected value.

$$Pr(X_i = x_i) = \int_{[0,1]} \theta^{x_i}(1-\theta)^{1-x_i}\pi(\theta)d\theta$$

We can therefore rewrite the latter as:

$$E[X_i] = \sum_{x_i=0}^{1} x_i \int_{[0,1]} \theta^{x_i}(1-\theta)^{1-x_i}\pi(\theta)d\theta$$

When $x_i = 0$ the element of the sum is equal to zero, we can then rewrite the formula by just leaving the case where $x_i = 1$:

$$E[X_i] = 1 \int_{[0,1]} \theta^{1}(1-\theta)^{1-1}\pi(\theta)d\theta = \int_{[0,1]} \theta\pi(\theta)d\theta$$

The element on the right of the equation follows exactly the *definition of expected value* for continuous random variables, from which we conclude that:

$$E[X_i] = E_\pi[\theta]$$

Showing $\mathbf{E[X_iX_j] = E_\pi[\theta^2]}$:

We start in a similar fashion to the previous proof, with the formula of the expected value, this time of the product of two binary random variables:

$$E[X_iX_j] = \sum_{x_i=0}^{1}\sum_{x_j=0}^{1} x_ix_jPr(X_i = x_i, X_j = x_j)$$

Still as before we write the formula from De Finetti's Theorem, this time fixing $n = 2$. We also to impose $i \neq j$:

$$Pr(X_i = x_i, X_j = x_j) = \int_{[0,1]} \theta^{x_i}(1-\theta)^{1-x_i}\theta^{x_j}(1-\theta)^{1-x_j}\pi(\theta)d\theta = \int_{[0,1]} \theta^{x_i+x_j}(1-\theta)^{2-x_i-x_j}\pi(\theta)d\theta$$

Substituting this into the expected value formula get us:

$$E[X_iX_j] = \sum_{x_i=0}^{1}\sum_{x_j=0}^{1} x_ix_j \int_{[0,1]} \theta^{x_i+x_j}(1-\theta)^{2-x_i-x_j}\pi(\theta)d\theta$$

However the only element of those summations that is not zero is the one where $x_i = 1$ and $x_j = 1$:

$$E[X_iX_j] = \int_{[0,1]} \theta^2(1-\theta)^0\pi(\theta)d\theta = \int_{[0,1]} \theta^2\pi(\theta)d\theta$$

Which is the *definition of the second moment* of θ, therefore:

$$E[X_iX_j] = E_\pi[\theta^2]$$

Showing $\mathbf{Cov}[\mathbf{X_i}, \mathbf{X_j}] = \mathbf{Var}_\pi[\theta]$:

Let's recall the definition of covariance between two random variables X_i and X_j:

$$Cov[X_i, X_j] = E[X_iX_j] - E[X_i]E[X_j]$$

Let's also recall the definition of variance of a random variable A:

$$Var(A) = E[A^2] - E[A]^2$$

We have already found the results of the two components of the covariance, namely:

$$E[X_iX_j] = E_\pi[\theta^2], \quad E[X_i] = E[X_j] = E_\pi[\theta]$$

And by substituting them we understand they follow the *definition of variance* of θ:

$$Cov[X_i, X_j] = E_\pi[\theta^2] - E_\pi[\theta]E_\pi[\theta] = E_\pi[\theta^2] - E_\pi[\theta]^2 = Var_\pi[\theta]$$

---

2. Prove that any couple of random variabes in that sequence must be non-negatively correlated.

---

2. Answer:

Let's recall the definition of correlation between two random variables X_i and X_j, that is the Pearson Correlation Coefficient:

$$Cor[X_i, X_j] = \rho_{X_i,X_j} = \frac{Cov[X_i, X_j]}{\sigma_{X_i}\sigma_{X_j}}$$

Where σ is the standard deviation, the square root of the variance:

$$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{Var[X]} \geq 0$$

The denominator of the correlation formula is always non-negative as it is the product of two standard deviations and they are always non-negative. We have already studied the numerator in the previous points of this exercise and we concluded that:

$$Cov[X_i, X_j] = Var_\pi[\theta] \geq 0$$

We know that the variance is always non-negative, therefore also the numerator is non-negative. Since both numerator and denominator of the correlation formula are non-negative we can conclude that any couple of random variables of the sequence must be non-negatively correlated.

---

3. Find what are the conditions on the distribution $\pi(\cdot)$ so that $Cor[X_i X_j] = 1$.

---

3. Answer:

Recalling again the definition of correlation stated above, to get a value of correlation equal to 1 we need the numerator and the denominator of the formula to be the same:

$$Cor[X_i, X_j] = 1 \iff Cov[X_i, X_j] = \sigma_{X_i}\sigma_{X_j}$$

Since X_i and X_j are Bernoulli random variables with the same parameter θ they have the same variance $Var(X_i) = Var(X_j)$ and therefore we can simplify the condition as:

$$Cov[X_i, X_j] = \sigma_{X_i}\sigma_{X_j} = \sqrt{Var[X_i]}\sqrt{Var[X_j]} = Var[X_i]$$

To find the variance of X_i we must exploit the *Law of the Total Variance*:

$$Var[X_i] = E_\pi[Var[X_i|\theta]] + Var_\pi[E[X_i|\theta]] = E_\pi[\theta(1-\theta)] + Var_\pi[\theta] = E_\pi[\theta^2] - E_\pi[\theta] + Var_\pi[\theta]$$

We use this decomposition together with the previous result on the Covariance of two elements of the sequence:

$$Cov[X_i, X_j] = Var[X_i] \leftrightarrow Var_\pi[\theta] = E_\pi[\theta^2] - E_\pi[\theta] + Var_\pi[\theta] \leftrightarrow E_\pi[\theta] = E_\pi[\theta^2]$$

The condition becomes that *the first moment of the distribution of θ must be equal to its second moment.*

---

4. What do these conditions imply on the type and shape of $\pi(\cdot)$? (make an example).

---

4. Answer:

Exchangeability and De Finetti's Theorem do not impose conditions on the type and shape of $\pi(\cdot)$, other than the fact that it has to be a distribution on $[0, 1]$, since θ represents the common success chance of the random variable in the sequence.

Meanwhile the condition we impose to get $Cor[X_i X_j] = 1$ is very strong but somewhat difficult to understand. To get a clearer idea of what this condition means we can try to apply it into the definition of variance:

$$Var_\pi[\theta] = E_\pi[\theta^2] - E_\pi[\theta]^2 = E_\pi[\theta](1 - E_\pi[\theta])$$

When used in this shape it is immediately recognizable the fact that a variance formula of the same form can be found in the $Bernoulli$ distribution, for which the condition we found for the correlation to be equal to 1 is true no matter what parameter $p \in [0, 1]$ we choose, since:

$$\theta \sim Ber(p), \qquad E[\theta] = p, \qquad Var[\theta] = p(1 - p)$$

A Bernoulli distribution also satisfies the fact that the distribution of θ has to be inside $[0, 1]$, but foregoes the continuity of the distribution. We can try to impose the condition on a $Beta(\alpha, \beta)$ distribution, since it is defined in the interval $[0, 1]$:

$$\theta \sim Beta(\alpha, \beta), \qquad E[\theta] = \frac{\alpha}{\alpha + \beta}, \qquad Var[\theta] = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$

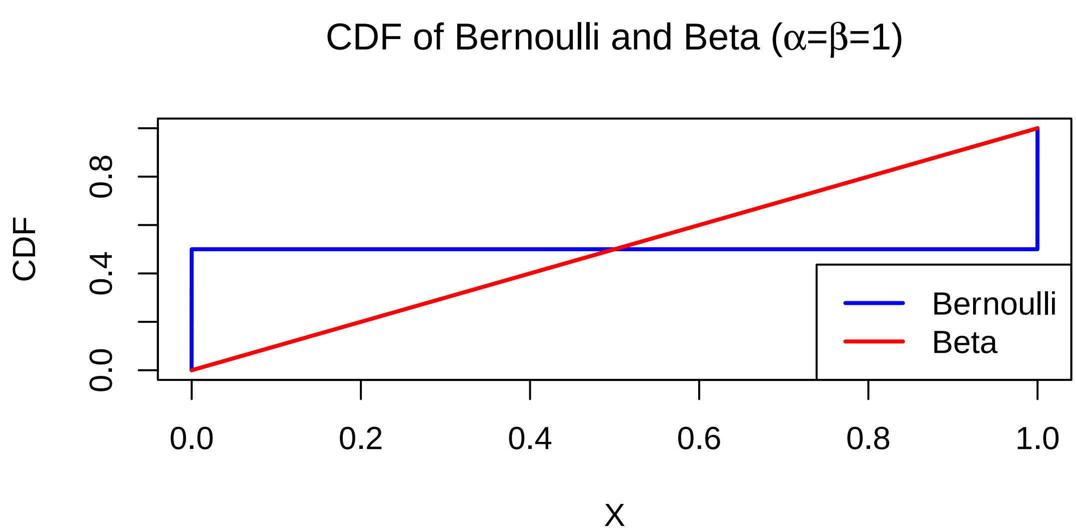
$$Var[\theta] = E[\theta](1 - E[\theta]) \rightarrow \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{\alpha}{\alpha + \beta}\left(1 - \frac{\alpha}{\alpha + \beta}\right) \rightarrow \frac{\beta}{\alpha + \beta + 1} = \beta \rightarrow \alpha = -\beta$$

But since $\alpha, \beta > 0$ this is impossible. It is however possible to prove that when α and β approach zero the $Beta$ distribution tends to the $Bernoulli$ distribution, reinforcing the point made above.

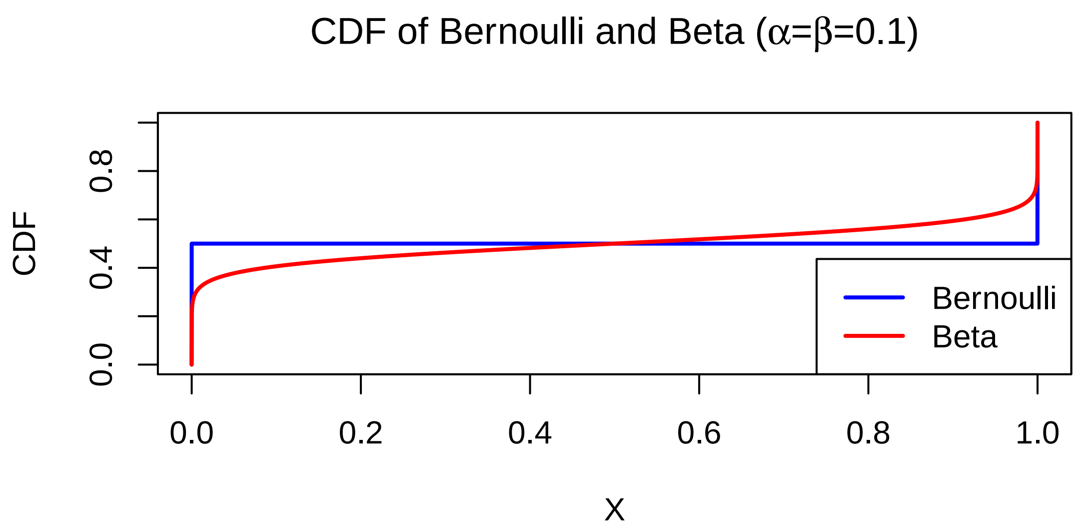
We provide below a visual proof by showing that for increasingly smaller values for the parameters α and β the CDF of the $Beta$ becomes more and more similar to the $Bernoulli$ one:

```r
plot_bernoulli_and_beta = function(value) {
    plot_title <- bquote("CDF of Bernoulli and Beta (" *
        alpha * "=" * beta * "=" * .(value) * ")")
    x <- seq(-1e-04, 1, by = 1e-04)

    cdf_bernoulli <- pbinom(x, size = 1, prob = 0.5)
    cdf_beta <- pbeta(x, shape1 = value, shape2 = value)

    plot(x, cdf_bernoulli, type = "l", lwd = 2, xlab = "X",
        ylab = "CDF", main = plot_title, xlim = c(0,
            1), ylim = c(0, 1), col = "blue")
    lines(x, cdf_beta, col = "red", lwd = 2)

    legend("bottomright", legend = c("Bernoulli", "Beta"),
        col = c("blue", "red"), lty = 1, lwd = 2)
}
```

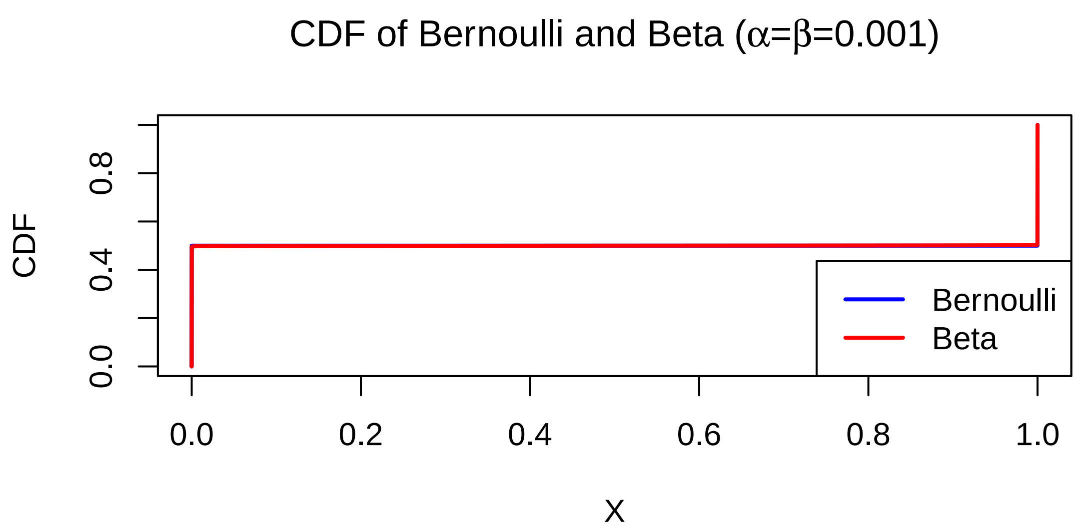
```r
plot_bernoulli_and_beta(1)
```




```r
plot_bernoulli_and_beta(0.1)
```




```r
plot_bernoulli_and_beta(0.001)
```



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